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Random Sampling Scattered Data with Multivariate Bernstein Polynomials*

Feilong CAO[1] Sheng XIA[2]

Abstract In this paper, the multivariate Bernstein polynomials defined on a simplex are viewed as sampling operators, and a generalization by allowing the sampling operators to take place at scattered sites is studied. Both stochastic and deterministic aspects are applied in the study. On the stochastic aspect, a Chebyshev type estimate for the sampling operators is established. On the deterministic aspect, combining the theory of uniform distribution and the discrepancy method, the rate of approximating continuous function and L^p convergence for these operators are studied, respectively.

Keywords Approximation, Bernstein polynomials, Random sampling, Scattered data
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1 Introduction

Let $S := S_d$ be the simplex in $\mathbb{R}^d$ $(d \in \mathbb{N})$ defined by

$$S := \Big\{\mathbf{x} := (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d : x_i \geq 0,\ \|\mathbf{x}\|_1 := \sum_{i=1}^{d} |x_i| \leq 1\Big\}.$$

The Bernstein polynomials on S are given by

$$B_{n,d}f := B_{n,d}(f(\cdot), \mathbf{x}) := \sum_{\|\mu\|_1 \leq n} P_{n,\mu}(\mathbf{x}) f\Big(\frac{\mu}{n}\Big), \quad \mathbf{x} \in S,\ n \in \mathbb{N}, \tag{1.1}$$

where $\mu := (\mu_1, \mu_2, \cdots, \mu_d)$ with μ_i nonnegative integers, $\|\mu\|_1 := \sum\limits_{i=1}^{d} |\mu_i|$, and

$$P_{n,\mu}(\mathbf{x}) := \frac{n!}{\mu!(n - \|\mu\|_1)!} \mathbf{x}^{\mu} (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1}$$

with the convention $\mathbf{x}^{\mu} := x_1^{\mu_1} x_2^{\mu_2} \cdots x_d^{\mu_d}$, $\mu! := \mu_1! \mu_2! \cdots \mu_d!$. For $d = 1$, the multivariate Bernstein polynomials given in (1.1) reduce to the classical Bernstein polynomials:

$$B_n(f, x) := B_{n,1}(f, x) := \sum_{k=0}^{n} P_{n,k}(x) f\Big(\frac{k}{n}\Big), \quad x \in [0, 1].$$

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[1]Corresponding author. Department of Mathematics, China Jiliang University, Hangzhou 310018, China. E-mail: feilongcao@gmail.com
[2]Department of Mathematics, China Jiliang University, Hangzhou 310018, China. E-mail: xiashengjlxy@163.com
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Since Lorentz [1] first introduced the multivariate Bernstein polynomials in 1953, the polynomials have been extensively studied. In particular, the rate of convergence of the polynomials has been revealed in many literatures, such as [2–10]. On the other hand, the Bernstein polynomials have also been widely applied in many research fields, such as CAGD, approximation theory, probability, and so on. Recently, Wu, Sun, and Ma [11] viewed the classical Bernstein polynomials as sampling operators. The main motivation for this is as follows: In many real world problems, data at equally spaced sites are often unavailable, so are data collected from what are perceived to be equally spaced sites suffering from random errors due to signal delays, measurement inaccuracies, and other known or unknown factors. Therefore, they introduced a new version of classical Bernstein polynomials for which the sampling action takes place at scattered sites: $B_n^A(f,x) := \sum_{k=0}^{n} f(x_{n,k})P_{n,k}(x)$, where $A := \langle x_{n,k}\rangle$ is a triangular array and for each $n \in \mathbb{N}$, the numbers $x_{n,k}$ are arranged in the ascending order: $0 \le x_{n,0} < x_{n,1} < \cdots < x_{n,n} \le 1$. For the general version of the Bernstein polynomials, Wu, Sun, and Ma [11] contemplated from both probabilistic and deterministic perspectives and obtained some interesting results.

It is natural to introduce multivariate Bernstein polynomials in which the sampling action takes place at scattered sites $x_{n,\mu} \in S$:

$$\mathcal{B}_{n,d}(f,\mathbf{x}) := \sum_{\|\mu\|_1 \le n} f(x_{n,\mu})P_{n,\mu}(\mathbf{x}). \tag{1.2}$$

Of course, selecting $x_{n,\mu} = \frac{\mu}{n}$ takes us back to the classical multivariate Bernstein polynomials (1.1).

The main purpose of this paper is to address the multivariate Bernstein sampling operators (1.2). Firstly, for each fixed n, we consider $x_{n,\mu}$ as random variables that take values in S, and prove a Chebyshev type error estimate. Secondly, we study the approximation orders of the sampling operators for continuous or Lebesgue integrable function, respectively. Some results in [11] are extended to the case of higher dimension.

This paper is arranged as follows. A much more general setting for uniformly distributed, modulus of continuity, and the definition of star discrepancy in simplex S are introduced in Section 2. In Section 3, we estimate the Chebyshev type error for the sampling operators (1.2). By mean of the introduced star discrepancy, we discuss the order of approximating continuous function by such operators in Section 4. Finally, the L^p $(1 \le p < \infty)$ convergence of the operators is studied in Section 5.

2 Notation

For a Riemann integrable function f on the simplex S, we use the Quasi-Monto Carlo approximation $\int_S f(\mathbf{x})\mathrm{d}\mathbf{x} \approx \frac{1}{N}\sum_{k=1}^{N} f(\mathbf{x}_k)$ with $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_N \in S$. An idealized model is to replace the sequence of nodes $\mathbf{x}_1, \cdots, \mathbf{x}_N$ by an infinite sequence of points $\mathbf{x}_1, \mathbf{x}_2, \cdots$ in S, such that $\lim_{N\to\infty} \frac{1}{N}\sum_{k=1}^{N} f(\mathbf{x}_k) = \int_S f(\mathbf{x})\mathrm{d}\mathbf{x}$ holds. The resulting condition means that the sequence

$\mathbf{x}_1, \mathbf{x}_2, \cdots$ should be uniformly distributed in the simplex S.

A similar definition states that $\mathbf{x}_1, \mathbf{x}_2, \cdots$ are uniformly distributed in simplex S if

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{N} C_F(\mathbf{x}_k) = \lambda_d(F)$$

holds for all sub-domain F of S, where C_F is the characteristic function of F, and $\lambda_d(F)$ denotes the volume of sub-domain F.

For each fixed n, let $P := \langle x_{n,\mu} \rangle$ be a triangular array in S. Let $\mathcal{J}$ be a family of all sub-domain of S with the form:

$$J := \Big\{ (y_1, y_2, \cdots, y_d) : y_i \geq 0,\ \sum_{i=1}^{d} y_i \leq \alpha,\ 0 \leq \alpha < 1 \Big\}.$$

For arbitrary $J \in \mathcal{J}$, we define $A(J, P) := \sum_{\frac{\mu}{n} \in S} C_J(x_{n,\mu})$, where C_J is the characteristic function of J. Thus, $A(J, P)$ is the counting function that denotes the number of the points which belong to J.

The concept of discrepancy is an indispensable tool in the quantitative study of uniform distribution of a finite sequence. For fixed n, we denote $N = \#\{\mu : \|\mu\|_1 \leq n\} = \binom{n+d}{d}$, where # denotes the number of the points which belong to the set. We now introduce a general notion of the star discrepancy of a point set P, which is given by

$$D_N^*(P) := \sup_{J \in \mathcal{J}} \Big| \frac{A(J, P)}{N} - \lambda_d(J) \Big|. \tag{2.1}$$

According to this definition, a triangular array $P = \langle x_{n,\mu} \rangle$ is uniformly distributed in S if and only if $\lim_{N\to\infty} D_N^*(P) = 0$. We refer the readers to [12] for more details about the star discrepancy.

Let $C(S)$ denote the space of continuous function defined on S with uniform norm

$$\|f\|_C := \max_{\mathbf{x} \in S} |f(\mathbf{x})|.$$

The continuity modulus of function $f \in C(S)$ is defined as

$$\omega(f, \delta) := \max_{\|\mathbf{x}-\mathbf{y}\|_2 \leq \delta} |f(\mathbf{x}) - f(\mathbf{y})|,$$

where $\delta > 0$, and $\|\mathbf{x} - \mathbf{y}\|_2 := \big(\sum_{i=1}^{d} |x_i - y_i|^2 \big)^{\frac{1}{2}}$ is the Euclidean distance. We say that $f \in \mathrm{Lip}1$ if $\omega(f, \delta) = \mathcal{O}(\delta)$ $(\delta \to 0_+)$.

It is easy to see that $\lim_{\delta\to 0} \omega(f, \delta) = 0$ and

$$\omega(f, \lambda\delta) \leq (1 + \lambda)\omega(f, \delta), \quad \lambda > 0. \tag{2.2}$$

It is clear that the Bernstein polynomials $B_{n,d}(f, \mathbf{x})$ uniformly converge to $f(\mathbf{x})$ on S while n approaches infinity. We are delighted to mention the following result (see [13])

$$|B_{n,d}(f, \mathbf{x}) - f(\mathbf{x})| \leq 2\omega\Big(f, \frac{1}{\sqrt{n}}\Big), \tag{2.3}$$

which will be used in the following.

3 Chebyshev Type Error Estimate

In this section we study the following problem: Given $f \in C(S)$ and $\epsilon > 0$, draw points $x_{n,\mu}$ $(0 \le \|\mu\|_1 \le n)$ from S independently according to the distributions $F_{n,\mu}$, respectively, and estimate the probability

$$P\Big\{(x_{n,\mu}) : \Big\| \sum_{\|\mu\|_1 \le n} f(x_{n,\mu}) P_{n,\mu}(\mathbf{x}) - f(\mathbf{x}) \Big\|_C > \epsilon \Big\}.$$

To get such estimate, we need estimate the following quantities.

Lemma 3.1 *For each* μ $(0 \le \|\mu\|_1 \le n)$, *we have*

$$\int_S \Big\| \mathbf{x} - \frac{\mu}{n} \Big\|_2^{2j} P_{n,\mu}(\mathbf{x}) \mathrm{d}\mathbf{x} \le \frac{C_j}{(n+1)\cdots(n+d+j)}, \quad \mathbf{x} \in S, \ j \in \mathbb{N}_+,$$

where C_j *are positive constants independent of* n.

Proof It is easy to find out

$$\begin{aligned}
&\int_S \Big\| \mathbf{x} - \frac{\mu}{n} \Big\|_2^2 P_{n,\mu}(\mathbf{x}) \mathrm{d}\mathbf{x} \\
&= \frac{n!}{\mu!(n - \|\mu\|_1)!} \int_S \mathbf{x}^\mu (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1} \Big\| \mathbf{x} - \frac{\mu}{n} \Big\|_2^2 \mathrm{d}\mathbf{x} \\
&= \frac{n!}{\mu!(n - \|\mu\|_1)!} \underset{\substack{x_1, \cdots, x_d \ge 0 \\ x_1 + \cdots + x_d \le 1}}{\int \cdots \int} \prod_{i=1}^d x_i^{\mu_i} (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1} \sum_{i=1}^d \Big(x_i - \frac{\mu_i}{n} \Big)^2 \mathrm{d}x_1 \cdots \mathrm{d}x_d \\
&:= \mathrm{I}_1 - \mathrm{I}_2 + \mathrm{I}_3,
\end{aligned}$$

where

$$\mathrm{I}_1 = \frac{n!}{\mu!(n - \|\mu\|_1)!} \sum_{i=1}^d \underset{\substack{x_1, \cdots, x_d \ge 0 \\ x_1 + \cdots + x_d \le 1}}{\int \cdots \int} x_1^{\mu_1} \cdots x_i^{\mu_i + 2} \cdots x_d^{\mu_d} (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1} \mathrm{d}x_1 \cdots \mathrm{d}x_d,$$

$$\mathrm{I}_2 = \frac{n!}{\mu!(n - \|\mu\|_1)!} \sum_{i=1}^d \frac{2\mu_i}{n} \underset{\substack{x_1, \cdots, x_d \ge 0 \\ x_1 + \cdots + x_d \le 1}}{\int \cdots \int} x_1^{\mu_1} \cdots x_i^{\mu_i + 1} \cdots x_d^{\mu_d} (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1} \mathrm{d}x_1 \cdots \mathrm{d}x_d$$

and

$$\mathrm{I}_3 = \frac{n!}{\mu!(n - \|\mu\|_1)!} \sum_{i=1}^d \Big(\frac{\mu_i}{n} \Big)^2 \underset{\substack{x_1, \cdots, x_d \ge 0 \\ x_1 + \cdots + x_d \le 1}}{\int \cdots \int} x_1^{\mu_1} \cdots x_d^{\mu_d} (1 - \|\mathbf{x}\|_1)^{n - \|\mu\|_1} \mathrm{d}x_1 \cdots \mathrm{d}x_d.$$

With Liouville formula, we can write

$$\begin{aligned}
\mathrm{I}_1 &= \frac{n!}{\mu!(n - \|\mu\|_1)!} \sum_{i=1}^d \frac{\Gamma(\mu_1 + 1) \cdots \Gamma(\mu_i + 3) \cdots \Gamma(\mu_d + 1)}{\Gamma(\|\mu\|_1 + d + 2)} \int_0^1 (1 - u)^{n - \|\mu\|_1} u^{\|\mu\|_1 + d + 1} \mathrm{d}u \\
&= \frac{\Gamma(n+1)}{\Gamma(n+d+3)} \sum_{i=1}^d (\mu_i + 1)(\mu_i + 2).
\end{aligned}$$

Similarly,

$$\mathrm{I}_2 = 2\frac{\Gamma(n+1)}{n\Gamma(n+d+2)}\sum_{i=1}^{d}\mu_i(\mu_i+1)$$

and

$$\mathrm{I}_3 = \frac{\Gamma(n+1)}{n^2\Gamma(n+d+1)}\sum_{i=1}^{d}\mu_i^2.$$

Note that $\sum\limits_{i=1}^{d}\mu_i^2 = c\|\mu\|_1^2$, $\frac{1}{d} \le c \le 1$, then

$$\begin{aligned}\int_S \Big\|\mathbf{x}-\frac{\mu}{n}\Big\|_2^2 P_{n,\mu}(\mathbf{x})\mathrm{d}\mathbf{x} &= \frac{n!}{(n+d)!}\sum_{i=1}^{d}\Big(\frac{(\mu_i+1)(\mu_i+2)}{(n+d+2)(n+d+1)} - \frac{2\mu_i(\mu_i+1)}{n(n+d+1)} + \frac{\mu_i^2}{n^2}\Big)\\ &= \frac{n!}{(n+d)!}\Big(\frac{c\|\mu\|_1^2+3\|\mu\|_1+2d}{(n+d+2)(n+d+1)} - \frac{2c\|\mu\|_1^2+2\|\mu\|_1}{n(n+d+1)} + \frac{c\|\mu\|_1^2}{n^2}\Big)\\ &\le \frac{(1-c)n+c(d+1)(d+2)-4}{(n+1)\cdots(n+d+2)} \le \frac{C_1}{(n+1)\cdots(n+d+1)}.\end{aligned}$$

We have sufficient evidence to believe that there exists a constant C_j such that

$$\int_S \Big\|\mathbf{x}-\frac{\mu}{n}\Big\|_2^{2j} P_{n,\mu}(\mathbf{x})\mathrm{d}\mathbf{x} \le \frac{C_j}{(n+1)\cdots(n+d+j)}, \quad j \ge 2. \tag{3.1}$$

Lemma 3.2 *The random variable $x_{n,\mu}$ obeys the F_μ distribution, in which for each $\|\mu\|_1 \le n$, we denote by F_μ the distribution with density function:*

$$\mathbf{x} \to (n+1)\cdots(n+d)P_{n,\mu}(\mathbf{x}), \quad \mathbf{x} \in S. \tag{3.2}$$

Proof Assuming that $n \in \mathbb{N}$ and $\mathbf{x} \in S$ are given, we are enable to find a proper δ satisfying the following conditions: $D(\mathbf{x},\delta) := \{\mathbf{y} : \|\mathbf{x}-\mathbf{y}\|_1 \le \delta\} \subset S$, $N \cdot \lambda_d(D) := N \cdot \frac{(2\delta)^d}{d!} < 1$.

We can find the probability of the case that the point $x_{n,\mu}$ ($\|\mu\|_1 = k$) falls into the domain D is $N \cdot \frac{\lambda_d(D)}{\lambda_d(S)} = N \cdot (2\delta)^d$.

And the probability that k selected points turn out to be in the domain $[0, x_{n,\mu_1}-\delta) \times \cdots \times [0, x_{n,\mu_d}-\delta)$ can be figured out by the following formula: $d!\frac{n!}{\mu!(n-\|\mu\|_1)!}(\mathbf{x}-\mathbf{a})^\mu$, where $\mathbf{a} = \{\delta, \cdots, \delta\}$.

Further, the probability of the case that the remains appear in $\{\mathbf{y} : \mathbf{y} \in S, \mathbf{y} \ge \mathbf{x}+\mathbf{a}\}$ is

$$d!(1-\|\mathbf{x}\|_1-\delta)^{n-\|\mu\|_1}.$$

Therefore, the probabilities of all these three cases mentioned above are independent of each other, and the probability that all these cases happen simultaneously is

$$N \cdot (2\delta)^d(d!)^2\frac{n!}{\mu!(n-\|\mu\|_1)!}(\mathbf{x}-\mathbf{a})^\mu(1-\|\mathbf{x}\|_1-\delta)^{n-\|\mu\|_1} := F(\delta).$$

Then the density function of the random variable $x_{n,\mu}$ obeys

$$\lim_{\delta\to 0}\frac{F(\delta)}{\lambda_d(D)}=(n+1)\cdots(n+d)P_{n,\mu}(\mathbf{x}).$$

The following theorem gives a Chebyshev type error estimate of $\mathcal{B}_{n,d}(f)$.

Theorem 3.1 *Let $\varepsilon>0$ and $f\in C(S)$ be given. Suppose that $\omega(f,\frac{1}{\sqrt{n}})<\frac{\varepsilon}{6}$ and that $x_{n,\mu}$ ($\|\mu\|_1\le n$) are independently drawn from S according to the distributions F_μ ($\|\mu\|_1\le n$). Then there exists a positive constant C independent of n such that the following probability estimate holds:*

$$P\{(x_{n,\mu}):\|\mathcal{B}_{n,d}(f,\mathbf{x})-f(\mathbf{x})\|_C>\varepsilon\}\le C\frac{\omega^6\Big(f,\dfrac{1}{\sqrt{n}}\Big)}{\varepsilon^6}. \tag{3.3}$$

Proof Using (2.2)–(2.3), we have

$$\begin{aligned}\|\mathcal{B}_{n,d}(f,\mathbf{x})-f(\mathbf{x})\|_C &\le \|\mathcal{B}_{n,d}(f,\mathbf{x})-B_{n,d}(f,\mathbf{x})\|_C+\|B_{n,d}(f,\mathbf{x})-f(\mathbf{x})\|_C\\ &\le \Big\|\sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|P_{n,\mu}(\mathbf{x})\Big\|_C+2\omega\Big(f,\frac{1}{\sqrt{n}}\Big)\\ &\le \Big\|\sum_{\frac{\mu}{n}\in S}\omega\Big(f,\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2\Big)P_{n,\mu}(\mathbf{x})\Big\|_C+2\omega\Big(f,\frac{1}{\sqrt{n}}\Big)\\ &\le \Big\|\sum_{\frac{\mu}{n}\in S}\Big(1+\sqrt{n}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2\Big)\omega\Big(f,\frac{1}{\sqrt{n}}\Big)P_{n,\mu}(\mathbf{x})\Big\|_C+2\omega\Big(f,\frac{1}{\sqrt{n}}\Big)\\ &\le \sqrt{n}\omega\Big(f,\frac{1}{\sqrt{n}}\Big)\Big\|\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2P_{n,\mu}(\mathbf{x})\Big\|_C+3\omega\Big(f,\frac{1}{\sqrt{n}}\Big).\end{aligned}$$

For each fixed $\mathbf{x}\in S$, we have

$$\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2P_{n,\mu}(\mathbf{x})\le\max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2,$$

which implies that

$$\Big\|\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2P_{n,\mu}(\mathbf{x})\Big\|_C\le\max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2.$$

Therefore,

$$\begin{aligned}&P\{(x_{n,\mu}):\|\mathcal{B}_{n,d}(f,\mathbf{x})-f(\mathbf{x})\|_C>\epsilon\}\\ &\le P\Big\{(x_{n,\mu}):\frac{\omega\Big(f,\dfrac{1}{\sqrt{n}}\Big)}{\dfrac{1}{\sqrt{n}}}\max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2+3\omega\Big(f,\frac{1}{\sqrt{n}}\Big)>\epsilon\Big\}.\end{aligned}$$

By the assumption of the theorem, we have $3\omega\big(f, \frac{1}{\sqrt{n}}\big) \le \frac{\varepsilon}{2}$. Thus, in order that

$$\Big\|\mathcal{B}_{n,d}(f,\mathbf{x}) - f(\mathbf{x})\Big\|_C > \epsilon,$$

it is necessary that

$$\max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2 > \frac{1}{2}\varepsilon\frac{1}{\sqrt{n}\omega\Big(f, \dfrac{1}{\sqrt{n}}\Big)}.$$

Let $\alpha_n^{-1} = 2\sqrt{n}\omega(f, \frac{1}{\sqrt{n}})$, we have the following inequality:

$$P\{(x_{n,\mu}) : \|\mathcal{B}_{n,d}(f,\mathbf{x}) - f(\mathbf{x})\|_C > \varepsilon\} \le P\Big\{(x_{n,\mu}) : \max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2 > \alpha_n\varepsilon\Big\}.$$

Thus, for each $\frac{\mu}{n} \in S$, using Lemmas 3.1–3.2, we obtain

$$\begin{aligned}P\Big\{(x_{n,\mu}) : \Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2 > \alpha_n\varepsilon\Big\} &\le \frac{(n+d)!}{n!}\int_{\|\mathbf{x}-\frac{\mu}{n}\|_2 > \alpha_n\varepsilon} P_{n,\mu}(\mathbf{x})\frac{\Big\|\mathbf{x} - \dfrac{\mu}{n}\Big\|_2^6}{\alpha_n^6\varepsilon^6}\mathrm{d}\mathbf{x}\\ &\le \frac{(n+d)!}{n!}\int_S P_{n,\mu}(\mathbf{x})\frac{\Big\|\mathbf{x} - \dfrac{\mu}{n}\Big\|_2^6}{\alpha_n^6\varepsilon^6}\mathrm{d}\mathbf{x}\\ &\le C\frac{\omega^6\Big(f, \dfrac{1}{\sqrt{n}}\Big)}{\varepsilon^6}.\end{aligned}$$

The proof of Theorem 3.1 is completed.

4 Approximation Order

In this section, we will discuss the approximation behavior of $\mathcal{B}_{n,d}(f)$ by means of the property of D_N^*. So, we first give two lemmas.

Lemma 4.1 (see [14]) *Let $x, y \ge 0$. Then, for $1 \le p < \infty$, we have*

$$2^{1-p}|x-y|^p \le |x^p - y^p| \le p|x-y|(x^{p-1} + y^{p-1}). \tag{4.1}$$

Lemma 4.2 *Let $P = \langle x_{n,\mu}\rangle$, $Q = \langle y_{n,\mu}\rangle$ be triangular array in S. If there holds $\|x_{n,\mu} - y_{n,\mu}\|_1 \le \varepsilon$ for any given $\varepsilon > 0$ and any $x_{n,\mu} \in P$, $y_{n,\mu} \in Q$, then*

$$|D_N^*(P) - D_N^*(Q)| \le \frac{2}{(d-1)!}\varepsilon.$$

Proof Consider any domain

$$J_\alpha = \Big\{(x_{n,\mu_1}, \cdots, x_{n,\mu_d}) : x_{n,\mu_i} \ge 0,\ \sum_{i=1}^d x_{n,\mu_i} \le \alpha\Big\} \subseteq S.$$

Whenever $x_{n,\mu} \in J_\alpha$, then $\|x_{n,\mu} - y_{n,\mu}\|_1 \le \varepsilon$ implies $y_{n,\mu} \in J_{\alpha+\varepsilon} \cap S$. Hence, using the inequality (4.1), we have

$$\begin{aligned}\frac{A(J_\alpha, P)}{N} - \lambda_d(J_\alpha) &\le \frac{A(J_1, Q)}{N} - \lambda_d(J_1) + \frac{1}{d!}[(\alpha+\varepsilon)^d - \alpha^d] \\ &\le D_N^*(Q) + \frac{2}{(d-1)!}\varepsilon.\end{aligned}$$

Similarly,

$$\frac{A(J_\alpha, P)}{N} - \lambda_d(J_\alpha) \ge D_N^*(Q) - \frac{2}{(d-1)!}\varepsilon.$$

Therefore, we can deduce

$$|D_N^*(P) - D_N^*(Q)| \le \frac{2}{(d-1)!}\varepsilon.$$

Now we give an approximation behavior of $\mathcal{B}_{n,d}(f)$.

Theorem 4.1 *Let $P = \langle x_{n,\mu}\rangle$ be a triangular array in S. Then we have that for any $f \in C(S)$,*

$$|\mathcal{B}_{n,d}(f, \mathbf{x}) - f(\mathbf{x})| \le 2\max\Big\{(1+2d^{\frac{3}{2}})\omega(f, D_N^*(P)^{\frac{1}{d}}), 2\omega\Big(f, \frac{1}{\sqrt{n}}\Big)\Big\}. \tag{4.2}$$

Proof For $f \in C(S)$, according to the inequality (2.3),

$$\begin{aligned}|\mathcal{B}_{n,d}(f, \mathbf{x}) - f(\mathbf{x})| &\le |\mathcal{B}_{n,d}(f, \mathbf{x}) - B_{n,d}(f, \mathbf{x})| + |B_{n,d}(f, \mathbf{x}) - f(\mathbf{x})| \\ &\le |\mathcal{B}_{n,d}(f, \mathbf{x}) - B_{n,d}(f, \mathbf{x})| + 2\omega\Big(f, \frac{1}{\sqrt{n}}\Big).\end{aligned}$$

It suffices to show that

$$|\mathcal{B}_{n,d}(f, \mathbf{x}) - B_{n,d}(f, \mathbf{x})| \le (1 + C_d)\omega(f, D_N^*(P)^{\frac{1}{d}}).$$

Denote $\alpha = D_N^*(P)^{\frac{1}{d}}$, using the property of the continuity modulus, we have

$$\begin{aligned}|\mathcal{B}_{n,d}(f, \mathbf{x}) - B_{n,d}(f, \mathbf{x})| &= \Big|\sum_{\frac{\mu}{n}\in S}\Big(f(x_{n,\mu}) - f\Big(\frac{\mu}{n}\Big)\Big)P_{n,\mu}(\mathbf{x})\Big| \\ &\le \sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu}) - f\Big(\frac{\mu}{n}\Big)\Big|P_{n,\mu}(\mathbf{x}) \\ &\le \sum_{\frac{\mu}{n}\in S}\omega\Big(f, \Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2\Big)P_{n,\mu}(\mathbf{x}) \\ &\le \sum_{\frac{\mu}{n}\in S}\Big(1 + \frac{1}{\alpha}\Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2\Big)\omega(f, \alpha)P_{n,\mu}(\mathbf{x}) \\ &= \Big(1 + \frac{1}{\alpha}\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2 P_{n,\mu}(\mathbf{x})\Big)\omega(f, \alpha) \\ &\le \Big(1 + \frac{1}{\alpha}\max_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu} - \frac{\mu}{n}\Big\|_2\Big)\omega(f, \alpha).\end{aligned}$$

According to the inequality (4.1), we know

$$\begin{aligned}
\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2 &\le \sqrt{d}\sum_{i=1}^{d}\Big|x_{n,\mu_i}-\frac{\mu_i}{n}\Big| \\
&\le 2^{\frac{d-1}{d}}\sqrt{d}\sum_{i=1}^{d}\Big|x_{n,\mu_i}^{d}-\Big(\frac{\mu_i}{n}\Big)^{d}\Big|^{\frac{1}{d}} \\
&\le 2\sqrt{d}\sum_{i=1}^{d}\Big|x_{n,\mu_i}^{d}-\Big(\frac{\mu_i}{n}\Big)^{d}\Big|^{\frac{1}{d}} \\
&\le 2\sqrt{d}\big|\|x_{n,\mu}\|_1^{d}-\|\mu\|_1^{d}\big|^{\frac{1}{d}} \\
&\le 2\sqrt{d}\cdot(d!)^{\frac{1}{d}}\Big|\frac{\|x_{n,\mu}\|_1^{d}}{d!}-\frac{\|\mu\|_1^{d}}{(n+1)\cdots(n+d)}\Big|^{\frac{1}{d}} \\
&\le 2d^{\frac{3}{2}}D_N^{*}(P)^{\frac{1}{d}}.
\end{aligned}$$

The proof of Theorem 4.1 is completed.

5 The L^p Convergence

In this section, we will study the L^p $(1\le p<\infty)$ convergence for the multivariate Bernstein sampling operators.

Theorem 5.1 *Let $P=\langle x_{n,\mu}\rangle$ be a triangular array in S. Assume that*

$$\lim_{n\to\infty}\frac{1}{(n+1)\cdots(n+d)}\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2=0.$$

Then for each $f\in C(S)$, we have $\lim\limits_{n\to\infty}\|\mathcal{B}_{n,d}f-f\|_{L^1}=0$.

Proof It suffices to show that $\lim\limits_{n\to\infty}\|\mathcal{B}_{n,d}f-B_{n,d}f\|_{L^1}=0$. For this purpose, we find

$$\begin{aligned}
\|\mathcal{B}_{n,d}f-B_{n,d}f\|_{L^1} &= \Big\|\sum_{\frac{\mu}{n}\in S}\Big(f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big)P_{n,\mu}(\mathbf{x})\Big\|_{L^1} \\
&\le \sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|\cdot\Big\|P_{n,\mu}(\mathbf{x})\Big\|_{L^1} \\
&= \frac{1}{(n+1)\cdots(n+d)}\sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|.
\end{aligned}$$

Since $f\in C(S)$, for arbitrary $\varepsilon>0$, there exists $\eta>0$ such that

$$\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|\le\varepsilon$$

for $\|x_{n,\mu}-\frac{\mu}{n}\|_2\le\eta$. Thus,

$$\sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big| = \sum_{\|x_{n,\mu}-\frac{\mu}{n}\|_2\le\eta}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big| + \sum_{\|x_{n,\mu}-\frac{\mu}{n}\|_2>\eta}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|$$
$$\le \#\Big\{\mu:\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2\le\eta\Big\}\varepsilon+2\|f\|_C\#\Big\{\mu:\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2>\eta\Big\}.$$

For $\eta>0$, it is easy to write

$$\eta\#\Big\{\mu:\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2>\eta\Big\}\le\sum_{\|x_{n,\mu}-\frac{\mu}{n}\|_2>\eta}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2\le\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2.$$

For each $\varepsilon>0$, from the assumptions of theorem, there exists $N_1>0$ such that

$$\frac{1}{(n+1)\cdots(n+d)}\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2\le\varepsilon$$

for $n\ge N_1$. Denote $M=\sup_{\mathbf{x}\in S}|f(\mathbf{x})|$, thus

$$\|\mathcal{B}_{n,d}f-B_{n,d}f\|_{L^1}\le\frac{1}{(n+1)\cdots(n+d)}\sum_{\frac{\mu}{n}\in S}\Big|f(x_{n,\mu})-f\Big(\frac{\mu}{n}\Big)\Big|$$
$$\le\frac{\eta^d}{d!(n+1)\cdots(n+d)}\epsilon+\frac{2M\eta^{-1}}{(n+1)\cdots(n+d)}\sum_{\frac{\mu}{n}\in S}\Big\|x_{n,\mu}-\frac{\mu}{n}\Big\|_2$$
$$\le(1+2M\eta^{-1})\epsilon.$$

The proof of Theorem 5.1 is completed.

In order to discuss the case of $1<p<\infty$, we give the following lemma.

Lemma 5.1 *For $1<p<\infty$, there is a constant $C=C_{p,d}$ such that*

$$\|P_{n,\mu}(\mathbf{x})\|_{L_p}\le\frac{C(n+1)^{\frac{p-2d-1}{2p}}}{\prod_{i=1}^{d}(\mu_i+1)^{\frac{p-1}{2p}}(n-\|\mu\|_1+1)^{\frac{p-1}{2p}}}.$$

Proof With Liouville formula, we can write

$$\|P_{n,\mu}(\mathbf{x})\|_{L^p}=\frac{n!}{\mu!(n-\|\mu\|_1)!}\Big(\int_{\substack{x_1,\cdots,x_d\ge0\\ x_1+\cdots+x_d\le1}}\cdots\int\prod_{i=1}^{d}x_i^{p\mu_i}(1-|\mathbf{x}|)^{p(n-\|\mu\|_1)}\mathrm{d}x_1\cdots\mathrm{d}x_d\Big)^{\frac{1}{p}}$$
$$=\frac{n!}{\mu!(n-\|\mu\|_1)!}\Big(\frac{\prod_{i=1}^{d}\Gamma(p\mu_i+1)}{\Gamma(p\|\mu\|_1+d)}\int_0^1(1-u)^{p(n-\|\mu\|_1)}u^{p\|\mu\|_1+d-1}\mathrm{d}u\Big)^{\frac{1}{p}}$$
$$=\frac{n!}{\mu!(n-\|\mu\|_1)!}\Big(\frac{\prod_{i=1}^{d}\Gamma(p\mu_i+1)\Gamma(p(n-\|\mu\|_1)+1)}{\Gamma(pn+d+1)}\Big)^{\frac{1}{p}}.$$

Using Sterlings formula $\Gamma(z) \sim \mathrm{e}^{-z} z^{z-\frac{1}{2}} (2\pi)^{\frac{1}{2}}$, we have

$$\frac{n!}{\mu!(n-\|\mu\|_1)!} = \frac{\Gamma(n+1)}{\prod_{i=1}^{d} \Gamma(\mu_i+1)\Gamma(n-\|\mu\|_1+1)}$$
$$= \mathrm{e}^d (2\pi)^{-\frac{d}{2}} \frac{(n+1)^{n+\frac{1}{2}}}{\prod_{i=1}^{d} (\mu_i+1)^{\mu_i+\frac{1}{2}} (n-\|\mu\|_1+1)^{n-\|\mu\|_1+\frac{1}{2}}}$$

and

$$\frac{\prod_{i=1}^{d} \Gamma(p\mu_i+1)\Gamma(p(n-\|\mu\|_1)+1)}{\Gamma(pn+d+1)}$$
$$= (2\pi)^{\frac{d}{2}} \frac{\prod_{i=1}^{d} (p\mu_i+1)^{p\mu_i+\frac{1}{2}} (p(n-\|\mu\|_1)+1)^{p(n-\|\mu\|_1)+\frac{1}{2}}}{(pn+d+1)^{pn+d+\frac{1}{2}}}.$$

Thus, we can bound $\|P_{n,\mu}(\mathbf{x})\|_p$ as

$$\|P_{n,\mu}(\mathbf{x})\|_p \leq C_{p,d} \frac{(n+1)^{\frac{p-2d-1}{2p}}}{\prod_{i=1}^{d} (\mu_i+1)^{\frac{p-1}{2p}} (n-\|\mu\|_1+1)^{\frac{p-1}{2p}}}.$$

This completes the proof of Lemma 5.1.

Finally, we prove the L^p $(1 < p < \infty)$ convergence.

Theorem 5.2 *Let $1 < p < \infty$. Let $P = \langle x_{n,\mu} \rangle$ be a triangular array in S. Let*

$$W^{(p)}_{n,\mu,d} := \frac{(n+1)^{\frac{p-2d-1}{2p}}}{\prod_{i=1}^{d} (\mu_i+1)^{\frac{p-1}{2p}} (n-\|\mu\|_1+1)^{\frac{p-1}{2p}}}.$$

Assume that

$$\lim_{n\to\infty} \sum_{\frac{\mu}{n} \in S} W^{(p)}_{n,\mu,d} \Big\| x_{n,\mu} - \frac{\mu}{n} \Big\|_2 = 0.$$

Then for each function $f \in \mathrm{Lip}1$, we have $\lim_{n\to\infty} \|\mathcal{B}_{n,d} f - f\|_{L^p} = 0$.

Proof It suffices to show that $\lim_{n\to\infty} \|\mathcal{B}_{n,d} f - B_{n,d} f\|_{L^p} = 0$.
Using Lemma 5.1, we have

$$\|\mathcal{B}_{n,d} f - B_{n,d} f\|_{L^p} = \Big\| \sum_{\frac{\mu}{n} \in S} \Big(f(x_{n,\mu}) - f\Big(\frac{\mu}{n}\Big) \Big) P_{n,\mu}(\mathbf{x}) \Big\|_{L^p}$$
$$\leq \sum_{\frac{\mu}{n} \in S} \Big| f(x_{n,\mu}) - f\Big(\frac{\mu}{n}\Big) \Big| \|P_{n,\mu}(\mathbf{x})\|_{L^p}$$

$$\leq C_{p,d} \sum_{\frac{\mu}{n} \in S} W_{n,\mu,d}^{(p)} \left| f(x_{n,\mu}) - f\left(\frac{\mu}{n}\right) \right|.$$

$$\leq C_{p,d} \sum_{\frac{\mu}{n} \in S} W_{n,\mu,d}^{(p)} \left\| x_{n,\mu} - \frac{\mu}{n} \right\|_2.$$

This completes the proof of Theorem 5.2.

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