Chin. Ann. Math.
35B(4), 2014, 619–632
DOI: 10.1007/s11401-014-0843-y

Chinese Annals of Mathematics, Series B

The Periodic Solutions of a Nonhomogeneous String with Dirichlet-Neumann Condition*

Changqing TONG[1] Jing ZHENG[1]

Abstract This paper deals with the existence of periodic solutions of a nonhomogeneous string with Dirichlet-Neumann condition. The authors consider the case that the period is irrational multiple of space length and prove that for some irrational number, zero is not the accumulation point of the spectrum of the associated linear operator. This result can be used to prove the existence of the periodic solution avoid using Nash-Moser iteration.

Keywords Nonhomogeneous string, Periodic solutions, Weak solution, Continued fraction, Lagrange constant
2000 MR Subject Classification 35B10, 35L05, 35L20

1 Introduction

In this paper, we study the problem of time periodic solution of the nonhomogeneous string with Dirichlet-Neumann condition:

$$\begin{cases}\rho(x)u_{tt}-(\rho(x)u_x)_x+g(u)=f(x,t), & 0<x<\pi,\ t\in\mathbb{R},\\ u(0,t)=u_x(\pi,t)=0, & t\in\mathbb{R},\\ u(x,t+T)=u(x,t), & 0<x<\pi,\ t\in\mathbb{R}\end{cases}\tag{1.1}$$

under the following hypotheses: (H1)

$$\rho(x)\in H^2(0,\pi),\tag{1.2}$$

$$\rho(x)\geq 1,\quad \forall x\in[0,\pi],\tag{1.3}$$

$$\varrho=\operatorname{ess\,inf}\eta(x)>0,\tag{1.4}$$

where

$$\eta(x)=\frac{1}{2}\frac{\rho''}{\rho}-\frac{1}{4}\Big(\frac{\rho'}{\rho}\Big)^2;\tag{1.5}$$

(H2) The function $g:\mathbb{R}\to\mathbb{R}$ is continuous and nondecreasing, and

$$|g(y)-g(z)|\leq\gamma|y-z|$$

for some $\gamma\geq 0$.

Manuscript received October 13, 2012. Revised July 10, 2013.
[1]Department of Mathematics, Hangzhou Dianzi University, Hangzhou 310018, China.
E-mail: tongchangqing@hdu.edu.cn zhengjing@hdu.edu.cn
*This work was supported by the Zhejiang Provincial Department of Education Research Fund (No. Y201326873).

This problem was first studied by Barbu and Pavel [2], which describes the forced vibrations of a nonhomogeneous string and the propagation of waves in nonisotropic media. After that, Ji and Li [11–12] and Rudakov [13] considered this equation with various boundary conditions, but all these papers dealed with the case when the number $\omega = \frac{2\pi}{T}$ is rational, where T is the period of the solutions. When ω is irrational number, the spectrum of the associated linear operator with the system (1.1) may be accumulated to zero. This is a "small divisor problem". In order to solve the "small divisor problem", Baldi and Berti [1] used the technique of Lyapunov-Schmidt decomposition and Nash-Moser iteration and obtained the periodic solution for Dirichlet condition. This method was widely used by many people when $\rho(x)$ is a constant to deal irrational frequencies, even for higher spatial dimensions. About these results, one may consult Berti and Bolle [5–8], Berti, Bolle and Procesi [9]. But for Dirichlet-Neumann condition, this method seems to be difficult for solving the bifurcation equation. We will use the method of Berkovits and Mawhin [4] to prove that 0 is not the accumulation point of the spectrum of the associated linear operator for some special ω. Then by adapting the method of Barbu and Pavel [2], we can obtain the existence of the periodic solution. This method avoids the tedious Nash-Moser iteration, although our result is weaker than that of Baldi and Berti in [1].

This paper is arranged as follows. In Section 2, we will prove some results about the spectrum of the linear operator associated with the system (1.1), these results are essential for our proof. In Section 3, we will use the method similar to [2] to complete the proof of our main results. In Section 4, we will list some notions and properties about continued fractions used in Section 2.

2 Some Basic Properties of the Linear Operator

Before studying the system (1.1), we need to know the properties of the spectrum of the associated linear operator A, so we first recall some results from [2]. First, some adapted complete orthonormal system of eigenfunctions

$$\{\psi_m\varphi_n \mid m \in \mathbb{Z},\ n \in \mathbb{N}\}$$

of this linear operator A will be needed to be taken as a basis for functions space. In order to define the operator A and this space, some notions will be defined.

Let $\Omega = [0,\pi] \times [0,T]$ and set

$$\begin{aligned}\mathcal{D} = \{u(x,t) \in C^\infty(\Omega) \mid u(0,t) = u_x(\pi,t) = 0,\\ u(x,0) = u(x,T) \ \text{ and } \ u_t(x,0) = u_t(x,T)\}.\end{aligned}$$

For real number $r \geq 1$, we define

$$\|u\|_{L^r(\Omega)} = \Big\{\int_\Omega \rho(x)|u(x,t)|^r \mathrm{d}x\mathrm{d}t\Big\}^{\frac{1}{r}}, \quad \forall u \in \mathcal{D}.$$

The space $L^r(\Omega)$ is the closure of $\mathcal{D}$ with the norm $\|\cdot\|_{L^r(\Omega)}$. Suppose that the constant q satisfies the condition $\frac{1}{p} + \frac{1}{q} = 1$. For functions $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, we define

$$(u,\ v) = \int_\Omega \rho(x)u(x,t)v(x,t)\mathrm{d}x\mathrm{d}t.$$

Definition 2.1 *A function $u \in L^2(\Omega)$ is said to be a weak solution of the problem*

$$\begin{cases} \rho(x)u_{tt} - (\rho(x)u_x)_x = f(x,t), & in\ \Omega,\ f \in L^2(\Omega), \\ u(0,t) = u_x(\pi,t) = 0, & t \in [0,\ T], \\ u(x,0) = u(x,T), & u_t(x,0) = u_t(x,T),\ x \in [0,\ \pi], \end{cases} \tag{2.1}$$

if

$$\int_\Omega u(\rho\varphi_{tt} - (\rho\varphi_x)_x)\mathrm{d}x\mathrm{d}t = \int_\Omega f\varphi\mathrm{d}x\mathrm{d}t \tag{2.2}$$

for all $\varphi \in \mathcal{D}$.

Conversely, a weak solution of class $C^2(\Omega)$ satisfies (2.1) in classical sense.

Set

$$D(\widetilde{A}) = \big\{u \in L^2(\Omega) :\ \text{There exists } f \in L^2(\Omega) \text{ such that (2.2) holds}\big\}. \tag{2.3}$$

Define $\widetilde{A} : D(\widetilde{A}) \to L^2(\Omega)$ by

$$\widetilde{A}u = f, \quad u \in D(\widetilde{A}), \tag{2.4}$$

if and only if

$$\int_\Omega u(\rho\varphi_{tt} - (\rho\varphi_x)_x)\mathrm{d}x\mathrm{d}t = \int_\Omega \widetilde{A}u\varphi\mathrm{d}x\mathrm{d}t = \int_\Omega f\varphi\mathrm{d}x\mathrm{d}t, \tag{2.5}$$

and define A by

$$A = \rho^{-1}\widetilde{A}. \tag{2.6}$$

Clearly, $D(A) = D(\widetilde{A})$ contains the null function of $L^2(\Omega)$, and for each $u \in D(A)$ there exists precisely one $f \in L^2(\Omega)$ satisfying (2.2). Therefore the operator A defined by (2.4)–(2.6) is a linear operator $L^2(\Omega) \to L^2(\Omega)$ and (2.2) can be written as

$$\int_\Omega uA_0\varphi\mathrm{d}x\mathrm{d}t = \int_\Omega \widetilde{A}u\varphi\mathrm{d}x\mathrm{d}t = \int_\Omega \rho Au\varphi\mathrm{d}x\mathrm{d}t, \quad \varphi \in \mathcal{D},\ u \in D(A), \tag{2.7}$$

where

$$A_0\varphi = \rho\varphi_{tt} - (\rho\varphi_x)_x, \quad \varphi \in \mathcal{D}.$$

The operator $\widetilde{A}$ defined by (2.4)–(2.5) is said to be the linear operator associated with (2.1).

In the following, we consider the spectrum of the operator A on the functions $u \in L^2(\Omega)$ with the boundary condition

$$u(0,t) = u_x(\pi,t) = 0, \quad u(x,0) = u(x,T), \quad u_t(x,0) = u_t(x,T).$$

Using the classical method of separation of variables, we set $u(x,t) = \tau(t)\varphi(x)$ and derive that φ must satisfy the equation:

$$\begin{cases} -(\rho(x)\varphi')' = \rho(x)\lambda^2\varphi, \\ \varphi(0) = \varphi'(\pi) = 0. \end{cases} \tag{2.8}$$

We denote $\{\lambda_n, \varphi_n\}_{n\in\mathbb{N}}$ the eigenvalues and the eigenfunctions of the Sturm-Liouville problem (2.8). It was proved in [2] that if conditions (1.2)–(1.4) are satisfied, then there exist constants $b_0, b_1 > 0$ such that

$$\lambda_n = n + \frac{1}{2} + \theta_n, \tag{2.9}$$

where

$$0 < \frac{b_0}{2n+1} \le \theta_n \le \frac{b_1}{2n+1}, \quad \forall n \in \mathbb{N}. \tag{2.10}$$

Consider the complete orthonormal system of functions

$$\Big\{\frac{1}{\sqrt{2\pi}}\varphi_n(x)\psi_m(t)\Big\}_{n\in\mathbb{N}, m\in\mathbb{Z}}$$

of space $L^2(\Omega)$, where

$$\psi_m(t) = \sqrt{\omega}\mathrm{e}^{\mathrm{i}\mu_m t}, \quad \mu_m = m\omega. \tag{2.11}$$

Hence the spectrum of the linear operator A is

$$\sum_\omega = \{\sigma_{nm} \mid \sigma_{nm} = \lambda_n^2 - (\omega m)^2,\ n \in \mathbb{N},\ m \in \mathbb{Z}\}.$$

The set $\sum\limits_\omega$ has the following properties which is essential to the proof of our main results.

Theorem 2.1 *Assume that ω is irrational, and $M(\omega) < \infty$ (the definition of $M(\omega)$ is in Section 4). Set*

$$m_\omega = \min_{p,q\in\mathbb{Z}_+} pqM\Big(\frac{p}{q}\omega\Big). \tag{2.12}$$

If $\omega > 2b_1 m_\omega$, 0 is not an accumulation point of $\sum\limits_\omega$.

Remark 2.1 Notice that it obviously has

$$m_\omega \le M(\omega)$$

from the definition of m_ω. Since $M(\omega)$ is invariant under a translation through integers, the irrational number ω which satisfies the conditions $\omega > 2b_1 m_\omega$ exists.

Proof Assume that 0 is an accumulation point of $\sum\limits_\omega$. Then we can find a sequence $\{\sigma_k\}$ of eigenvalues such that $\sigma_k \to 0$ if $k \to \infty$. In other words,

$$\begin{aligned}\sigma_k = \lambda_{n_k}^2 - \mu_{m_k}^2 &= \Big(n_k + \frac{1}{2} + \theta_{n_k} - m_k\omega\Big)\Big(n_k + \frac{1}{2} + \theta_{n_k} + m_k\omega\Big)\\ &= \Big(n_k + \frac{1}{2}\Big)^2 + 2\Big(n_k + \frac{1}{2}\Big)\theta_{n_k} + \theta_{n_k}^2 - m_k^2\omega^2 \to 0,\end{aligned} \tag{2.13}$$

if $k \to \infty$. Because of $\lim\limits_{k\to\infty} \theta_{n_k} = 0$, it is equivalent to

$$\lim_{k\to\infty} \Big(n_k + \frac{1}{2} - m_k\omega\Big)\Big(n_k + \frac{1}{2} + m_k\omega\Big) + 2\Big(n_k + \frac{1}{2}\Big)\theta_{n_k} = 0. \tag{2.14}$$

We can write (2.14) in the form

$$m_k^2\Big(\frac{n_k+\dfrac{1}{2}}{m_k}-\omega\Big)\Big(\frac{n_k+\dfrac{1}{2}}{m_k}+\omega\Big)+2\Big(n_k+\frac{1}{2}\Big)\theta_{n_k}\to 0 \tag{2.15}$$

as $k\to\infty$. We may choose $m_k\geq 0$. Observing that $2\big(n_k+\frac{1}{2}\big)\theta_{n_k}$ is bounded and $n_k+m_k\omega$ is bounded below, then necessarily we have

$$\lim_{k\to\infty} m_k\Big(\frac{n_k+\dfrac{1}{2}}{m_k}-\omega\Big)=0, \tag{2.16}$$

and hence also

$$\lim_{k\to\infty}\frac{n_k+\dfrac{1}{2}}{m_k}=\omega. \tag{2.17}$$

Consequently, writing (2.15) in the form

$$m_k^2\Big(\frac{n_k+\dfrac{1}{2}}{m_k}-\omega\Big)^2+m_k^2\Big(\frac{n_k+\dfrac{1}{2}}{m_k}-\omega\Big)2\omega+2\Big(n_k+\frac{1}{2}\Big)\theta_{n_k}\to 0, \tag{2.18}$$

if $k\to\infty$, we deduce that

$$\lim_{k\to\infty} m_k^2\Big(\frac{n_k+\dfrac{1}{2}}{m_k}-\omega\Big)\omega+\Big(n_k+\frac{1}{2}\Big)\theta_{n_k}=0. \tag{2.19}$$

Consequently, for each $p,q\in\mathbb{Z}_+$, we have

$$\lim_{k\to\infty}(m_kq)^2\frac{1}{pq}\Big(\frac{\Big(n_k+\dfrac{1}{2}\Big)p}{m_kq}-\frac{p\omega}{q}\Big)\omega+\Big(n_k+\frac{1}{2}\Big)\theta_{n_k}=0. \tag{2.20}$$

Let $\varepsilon>0$, there exists $K\in\mathbb{N}$ such that

$$\begin{aligned}(m_kq)^2\Big|\frac{\Big(n_k+\dfrac{1}{2}\Big)p}{m_kq}-\frac{p\,\omega}{q}\Big| &\leq \frac{p\,q\Big(n_k+\dfrac{1}{2}\Big)}{\omega}\theta_{n_k}+\varepsilon\\ &\leq \frac{pq\Big(n_k+\dfrac{1}{2}\Big)}{\omega}\frac{b_1}{2n_k+1}+\varepsilon=\frac{p\,qb_1}{2\omega}+\varepsilon,\end{aligned}$$

whenever $k\geq K$. Write above as

$$\Big|\frac{(2n_k+1)p}{2m_kq}-\frac{p\,\omega}{q}\Big|\leq\frac{1}{(2m_kq)^2\dfrac{1}{\dfrac{2p\,qb_1}{\omega}+4\varepsilon}}.$$

With this result and the definition of the function $M(\omega)$ in Section 4, we see that

$$\frac{1}{2p\,q\dfrac{b_1}{\omega}+4\varepsilon}\leq M\Big(\frac{p}{q}\omega\Big) \tag{2.21}$$

for each $\varepsilon > 0$. Hence

$$\frac{\omega}{b_1} \leq 2p\,qM\Big(\frac{p}{q}\omega\Big), \quad \omega \leq 2b_1p\,qM\Big(\frac{p}{q}\omega\Big)$$

for all $p, q \in \mathbb{Z}_+$, and hence

$$\omega \leq 2b_1 m_\omega, \tag{2.22}$$

a contradiction.

Now, we can prove the main result of the linear operator A.

Proposition 2.1 *Let $T = \frac{2\pi}{\omega}$, and ω satisfy the condition of Theorem* 2.1. *Then $R(A)$ is closed in $L^2(\Omega)$, A is self-adjoint and $(A|_{R(A)})^{-1} \in L(R(A), R(A))$. For simplicity, we also denote $(A|_{R(A)})^{-1}$ by A^{-1}. Moreover, we have*

$$\|A^{-1}f\|_{L^2} \leq d^{-1}\|f\|_{L^2}, \quad \forall f \in R(A), \tag{2.23}$$

where $d = \inf\{|\lambda_l^2 - (\omega k)^2|; \lambda_l \neq |\omega k|\}$,

$$\langle A^{-1}f,\ f\rangle \geq -\alpha^{-1}\|f\|_{L^2}^2, \quad \forall f \in R(A), \tag{2.24}$$

$$\langle Ay,\ y\rangle \geq -\alpha^{-1}\|Ay\|_{L^2}, \quad \forall y \in D(A), \tag{2.25}$$

where $\alpha = \inf\{(\omega k)^2 - \lambda_l^2; \lambda_l < |\omega k|\}$,

$$\|A^{-1}f\|_{L^\infty} \leq C\|f\|_{L^2}, \quad \forall f \in R(A), \tag{2.26}$$

$$\|A^{-1}f\|_{H^1} \leq C\|f\|_{H^1}, \quad \forall f \in H^1(\Omega) \cap R(A), \tag{2.27}$$

and

$$R(A) = N(A)^\perp, \tag{2.28}$$

$$L^2(\Omega) = N(A) \oplus R(A). \tag{2.29}$$

Proof With respect to the orthonormal system $\{\psi_k\varphi_l\}$ defined by (2.8) and (2.11), the equation $Ay = f$ is equivalent to

$$(\lambda_l^2 - (\omega k)^2)y_{kl} = f_{kl}, \tag{2.30}$$

where $y = \sum y_{kl}\psi_k\varphi_l, f = \sum f_{kl}\psi_k\varphi_l$. This implies that the equation $Ay = f$ has a solution y only if $f \in N(A)^\perp$, i.e., $f_{kl} = 0$ for all (k,l) such that $\lambda_l = |\omega k|$. Indeed, this condition is also sufficient. For the equation $Ay = f$, if we set

$$y_{kl} = \frac{f_{kl}}{\lambda_l^2 - (\omega k)^2}, \quad \lambda_l \neq |\omega k|, \tag{2.31}$$

according to Theorem 2.1, 0 is not an accumulation point of $\sum_\omega$. So $d = \inf\{|\lambda_l^2 - (\omega k)^2|; \lambda_l \neq |\omega k|\} > 0$, and $\sum\limits_{\lambda_l \neq |\omega k|} |y_{kl}|^2$ is convergent. Moreover,

$$\sum |y_{kl}|^2 \leq \frac{1}{d^2}\sum |f_{kl}|^2 = \frac{1}{d^2}\|f\|_{L^2}^2. \tag{2.32}$$

By (2.30),

$$\langle A^{-1}f,\ f\rangle = \sum_{\lambda_l \neq |\omega k|} \frac{f_{kl}^2}{\lambda_l^2 - (\omega k)^2} \geq \sum_{\lambda_l < |\omega k|} \frac{f_{kl}^2}{\lambda_l^2 - (\omega k)^2}, \tag{2.33}$$

which yields (2.24). Let $y = A^{-1}f$ if $f \in H^1(\Omega) \cap R(A)$. So its weak derivative is

$$y_x = \sum_{\lambda_l \neq |\omega k|} y_{kl}\psi_k \varphi_l',$$

where $\{\varphi_l'\}$ is orthogonal in $L^2(0,\pi)$ and

$$\|\varphi_l'\|_{L^2}^2 = \int_0^\pi \rho(\varphi_l')^2 \mathrm{d}x = -\int_0^\pi \varphi_l \rho(\varphi_l')_x \mathrm{d}x = \lambda_l^2.$$

Therefore

$$\begin{aligned}\|y_x\|_{L^2}^2 &= \sum_{\lambda_l \neq |\omega k|} \lambda_l^2 |y_{kl}|^2 = \sum_{\lambda_l \neq |\omega k|} \frac{\lambda_l^2 |f_{kl}|^2}{(\lambda_l^2 - (\omega k)^2)^2} \\ &\leq \frac{1}{d^2} \sum_{\lambda_l \neq |\omega k|} \lambda_l^2 |f_{kl}|^2 = \frac{1}{d^2}\|f_x\|_{L^2}^2.\end{aligned}$$

Similarly, it also has

$$\|y_t\|_{L^2}^2 \leq \frac{1}{d^2} \sum_{\lambda_l \neq |\omega k|} |\omega k|^2 |f_{kl}|^2 = \frac{1}{d^2}\|f_t\|_{L^2}^2.$$

So (2.27) is proved.

In order to prove (2.26), notice that

$$|\lambda_l^2 - (\omega k)^2|^2 \geq d|\lambda_l^2 + (\omega k)^2|, \quad \lambda_l \neq |\omega k|.$$

So

$$\sum_{\substack{k,l \\ \lambda_l \neq |\omega k|}}^{\infty} \frac{1}{|\lambda_l - (\omega k)|^2 (\lambda_l + (\omega k))^2} \leq C,$$

where C is a constant independent of l. Then one has

$$\begin{aligned}\sum_{\substack{k,l \\ \lambda_l \neq |\omega k|}}^{\infty} \frac{|f_{kl}|}{|\lambda_l^2 - (\omega k)^2|} &\leq \Big(\sum_{k,l} |f_{kl}|^2\Big)^{\frac{1}{2}} \Big(\sum_{\substack{k,l \\ \lambda_l \neq |\omega k|}}^{\infty} \frac{1}{|\lambda_l - (\omega k)|^2 (\lambda_l + (\omega k))^2}\Big)^{\frac{1}{2}} \\ &\leq C\|f\|_{L^2}.\end{aligned}$$

So (2.26) is proved.

Finally, notice that $D(A)$ is densed in $L^2(\Omega)$ and A is symmetric and $R(A) = N(A)^{\perp}$. So A is self-adjoint.

3 Proof of the Main Result

Now we begin to consider the weak periodic solution of system (1.1). Recall that $u \in L^2(\Omega)$ is a weak solution of the problem (1.1) if and only if

$$\int_\Omega u(\rho\varphi_{tt} - (\rho\varphi_x)_x)\mathrm{d}x\mathrm{d}t + \int_\Omega g(u)\varphi\mathrm{d}x\mathrm{d}t = \int_\Omega f\varphi\mathrm{d}x\mathrm{d}t, \quad \forall\varphi \in \mathcal{D}. \tag{3.1}$$

In order to state our main results, we give an assumption on f and g first.

(H3) $f \in L^\infty(\Omega)$ and

$$g(-\infty) + \delta \le \rho(x)(P(\rho^{-1}f))(x,t) \le g(+\infty) - \delta \tag{3.2}$$

for some $\delta > 0$. Here $P : L^2(\Omega) \to N(A)$ is the projection operator on $N(A)$.

Now we state our main result of this paper.

Theorem 3.1 *Assume $T = \frac{2\pi}{\omega}$, where ω is an irrational number which satisfies the condition of Theorem* 2.1 *and the hypotheses* (H1)–(H3) *with $0 < \gamma < \alpha$, where $\alpha = \inf\{|\omega m|^2 - \lambda_n^2; \lambda_n < |\omega m|\}$. Then* (1.1) *has at least one weak solution $y \in L^\infty(\Omega)$.*

Proof Let

$$G(u) = \frac{g(u)}{\rho(x)}, \quad \text{a.e. } (x,t) \in \Omega,\ u \in L^2(\Omega).$$

In view of (H2), $G : L^2(\Omega) \to L^2(\Omega)$ is a continuous and monotone operator, i.e.,

$$\begin{aligned}
&\langle G(u) - G(v), u - v\rangle \ge 0, \quad \forall u, v \in L^2(\Omega),\\
&\|G(u) - G(v)\|^2_{L^2(\Omega)} \le \gamma\langle G(u) - G(v), u - v\rangle, \quad \forall u, v \in L^2(\Omega).
\end{aligned} \tag{3.3}$$

So u is a weak solution to (1.1) in Ω if and only if

$$Au + G(u) = \rho^{-1}f. \tag{3.4}$$

We first consider the following approximation of (3.4):

$$Au + (G + \varepsilon I)(u) = \rho^{-1}f, \quad u \in L^2(\Omega). \tag{3.5}$$

The proof will be divided into four steps.

Setp 1 To prove the existence of the solution of (3.5)

Letting $G_\varepsilon(u) = G(u) + \varepsilon u$, and according to the hypothesis (H2)

$$\begin{aligned}
&\langle G_\varepsilon(u) - G_\varepsilon(v), u - v\rangle \ge \varepsilon\|u - v\|^2_{L^2}, && \forall u, v \in L^2(\Omega),\\
&\|G_\varepsilon(u) - G_\varepsilon(v)\|^2_{L^2(\Omega)} \le (\gamma + \varepsilon)\langle G_\varepsilon(u) - G_\varepsilon(v), u - v\rangle, && \forall u, v \in L^2(\Omega),
\end{aligned} \tag{3.6}$$

so

$$\langle G_\varepsilon^{-1}(u) - G_\varepsilon^{-1}(v), u - v\rangle \ge \frac{1}{\gamma + \varepsilon}\|u - v\|^2_{L^2}, \quad \forall u, v \in L^2(\Omega). \tag{3.7}$$

Furthermore, it obviously has

$$R(G_\varepsilon) = L^2(\Omega). \tag{3.8}$$

Using the idea of Brezis [10], (3.5) can be equivalently written as

$$A^{-1}v + (G+\varepsilon I)^{-1}(\rho^{-1}f + v) \in N(A), \quad v \in R(A). \tag{3.9}$$

Indeed, if u is a solution of (3.5), we write $u = u_1 + u_2$, $u_1 \in N(A)$, $u_2 \in R(A)$, then

$$Au_2 + G_\varepsilon u = \rho^{-1}f, \quad G_\varepsilon u = \rho^{-1}f - Au_2.$$

Let $v = -Au_2$. Then

$$u = G_\varepsilon^{-1}(\rho^{-1}f + v) = u_1 + A^{-1}v,$$

which shows that (3.5) and (3.9) are equivalent.

On the other hand, (3.9) is equivalent to

$$A^{-1}v + (G+\varepsilon I)^{-1}(\rho^{-1}f + v) + \partial J(v) \ni 0, \quad v \in R(A), \tag{3.10}$$

where J is the indicator function of $R(A)$, and ∂J is the subdifferential of J. Taking into account that $\partial J(v)$ is the cone of the normals to $R(A)$ at v, it follows that $\partial J(v) = N(A)$ for all $v \in R(A)$.

Finally, (2.24) shows that $A^{-1} + \alpha^{-1}I$ is monotone on $R(A)$. So (3.10) can be written in the equivalent form

$$(A^{-1} + \alpha^{-1})v + G_\alpha(v) + \partial J(v) \ni 0, \quad v \in R(A) \tag{3.11}$$

with $G_\alpha v = (G+\varepsilon I)^{-1}(\rho^{-1}f + v) - \alpha^{-1}v$. In view of (3.11), G_α satisfies

$$\langle G_\alpha v_1 - G_\alpha v_2,\ v_1 - v_2\rangle \geq ((\gamma+\varepsilon)^{-1} - \alpha^{-1})|v_1 - v_2|^2, \quad v_1, v_2 \in R(A). \tag{3.12}$$

We now prove that (3.11) has a solution v_ε for each $\varepsilon < \alpha - \gamma$.

On the basis of (3.12), for $\varepsilon < \alpha - \gamma$, G_α is coercive and maximal monotone in $L^2(\Omega)$.

A key step now is to prove that the monotone operator $v \to A_\alpha + \partial J(v)$ with $A_\alpha = A^{-1} + \alpha^{-1}I$, $D(A_\alpha) = R(A)$ and $\partial J = N(A)$ is maximal monotone in $L^2(\Omega)$, i.e., for every $h \in L^2(\Omega)$ the equation

$$v + A_\alpha v + \partial J(v) \ni h \tag{3.13}$$

has a solution $v \in R(A)$. Indeed, this equation is equivalent to

$$v + A_\alpha v = (I-P)h, \quad v \in R(A), \tag{3.14}$$

which has a unique solution $v \in R(A)$. It follows that $A_\alpha + \partial J + G_\alpha$ is maximal monotone in $L^2(\Omega)$. Moreover, as G_α is coercive, $A_\alpha + \partial J + G_\alpha$ is onto. Therefore (3.11) has a solution $v_\varepsilon \in R(A)$ which is a solution of (3.10). This means that there exists $y_\varepsilon^1 \in N(A)$ such that

$$A^{-1}v_\varepsilon + (G+\varepsilon I)^{-1}(\rho^{-1} + v_\varepsilon) = y_\varepsilon^1.$$

Set

$$y_\varepsilon^2 = A^{-1}v_\varepsilon.$$

Then $y_\varepsilon = y_\varepsilon^1 - y_\varepsilon^2$ is a solution of (3.5).

Setp 2 Estimate the solution $\boldsymbol{y_\varepsilon}$

In order to estimate the solution y_ε of

$$\varepsilon y_\varepsilon + Ay_\varepsilon + G(y_\varepsilon) = \rho^{-1} f, \tag{3.15}$$

we note that by the assumption (H3), there exists $\xi = \xi(x,t)$ with $|\xi| \le C$, such that

$$\rho(x)(P(\rho^{-1}f))(x,t) + \delta w = g(\xi) \quad \text{a.e. } (x,t) \in \Omega$$

for all $\delta > 0$ sufficiently small and $|w| = 1$. Then the monotonicity of g yields

$$(g(y_\varepsilon) - \rho(x)P(\rho^{-1}f) - \delta w)(y_\varepsilon - \xi) \ge 0 \quad \text{a.e. } (x,t) \in \Omega$$

with $g(y_\varepsilon(x,t)) = \rho(x,t)G(y_\varepsilon)(x,t)$. So

$$\delta w y_\varepsilon \le (g(y_\varepsilon) - \rho(x)P(\rho^{-1}f))y_\varepsilon - \xi(g(y_\varepsilon) - g(\xi)) \quad \text{a.e. } (x,t) \in \Omega,$$

which implies $\big($for $w = \frac{y_\varepsilon(x,t)}{|y_\varepsilon(x,t)|}\big)$ that

$$\delta \|y_\varepsilon\|_{L^1(\Omega)} \le \langle Gy_\varepsilon - P(\rho^{-1}f),\ y_\varepsilon\rangle + C\|G(y_\varepsilon)\|_{L^2} + C_1 \quad \text{a.e. } (x,t) \in \Omega \tag{3.16}$$

for some positive constants C and C_1.

On the other hand, in view of $L^2(\Omega) = N(A) \oplus R(A)$, there exists $y_1 \in D(A)$ such that $\rho^{-1}f = P(\rho^{-1}f) + Ay_1$ and $\rho P(\rho^{-1}f) = g(z) = \rho G(z)$ for some $z = z(x,t)$ in $L^\infty(\Omega)$. Therefore, (3.15) can be written as

$$\varepsilon y_\varepsilon + A(y_\varepsilon - y_1) + G(y_\varepsilon) - G(z) = 0 \tag{3.17}$$

with $G(z) = P(\rho^{-1}f)$. Now we begin to prove that $\|G(y_\varepsilon)\|_{L^2}$ is bound. By (3.3), (3.17) and (2.24), we have

$$\begin{aligned}
&\gamma^{-1}\|Gy_\varepsilon - Gz\|_{L^2}^2 \le \langle G(y_\varepsilon) - G(z),\ y_\varepsilon - z\rangle \\
&= -\varepsilon\langle y_\varepsilon,\ y_\varepsilon - z\rangle - \langle A(y_\varepsilon - y_1),\ y_\varepsilon - y_1\rangle + \langle A(y_\varepsilon - y_1),\ z - y_1\rangle \\
&\le -\varepsilon\|y_\varepsilon\|_{L^2}^2 + \varepsilon\langle y_\varepsilon,\ z\rangle + \alpha^{-1}\|A(y_\varepsilon - y_1)\|_{L^2}^2 + \langle A(y_\varepsilon - y_1),\ z - y_1\rangle.
\end{aligned} \tag{3.18}$$

Substituting $A(y_\varepsilon - y_1) = G(z) - G(y_\varepsilon) - \varepsilon y_\varepsilon$ into (3.18) and with the following inequality

$$ab \le \varepsilon a^2 + (4\varepsilon)^{-1}b^2, \quad \forall \varepsilon > 0,\ a, b \in \mathbb{R},$$

it can be obtained

$$\begin{aligned}
&\gamma^{-1}\|Gy_\varepsilon - Gz\|_{L^2}^2 \le \langle G(y_\varepsilon) - G(z),\ y_\varepsilon - z\rangle \\
&\le -\varepsilon\|y_\varepsilon\|_{L^2}^2 + \varepsilon\langle y_\varepsilon,\ z\rangle + \alpha^{-1}\|G(z) - G(y_\varepsilon) - \varepsilon y_\varepsilon\|_{L^2}^2 + \langle G(z) - G(y_\varepsilon) - \varepsilon y_\varepsilon,\ z - y_1\rangle \\
&= \Big(\frac{\varepsilon^2}{\alpha} - \varepsilon\Big)\|y_\varepsilon\|_{L^2}^2 + \varepsilon\Big\langle y_\varepsilon,\ y_1 + \frac{2}{\alpha}(G(y_\varepsilon) - G(z))\Big\rangle \\
&\quad + \alpha^{-1}\|G(y_\varepsilon) - G(z)\|_{L^2}^2 + \langle G(y_\varepsilon) - G(z),\ y_1 - z\rangle \\
&\le \alpha^{-1}\|G(y_\varepsilon) - G(z)\|_{L^2}^2 + C\|G(y_\varepsilon) - G(z)\|_{L^2}^2 + \Big(\frac{\varepsilon^2}{\alpha} - \varepsilon\Big)\|y_\varepsilon\|_{L^2}^2
\end{aligned}$$

$$
\begin{aligned}
&+\varepsilon\Big(k\|y_\varepsilon\|_{L^2}^2+\frac{1}{4k}\Big\|y_1+\frac{2}{\alpha}(G(y_\varepsilon)-G(z))\Big\|_{L^2}^2\Big)\\
=\ &\alpha^{-1}\|G(y_\varepsilon)-G(z)\|_{L^2}^2+C\|G(y_\varepsilon)-G(z)\|_{L^2}^2+\Big(\frac{\varepsilon^2}{\alpha}-\varepsilon+\varepsilon k\Big)\|y_\varepsilon\|_{L^2}^2\\
&+\frac{\varepsilon}{4k}\|y_1+\frac{2}{\alpha}(G(y_\varepsilon)-G(z))\|_{L^2}^2. \qquad (3.19)
\end{aligned}
$$

Let $k=1-\frac{\varepsilon}{\alpha}$ in (3.19), it can be obtained

$$
\begin{aligned}
\gamma^{-1}\|G(y_\varepsilon)-G(z)\|_{L^2}^2 &\le \alpha^{-1}\|G(y_\varepsilon)-G(z)\|_{L^2}^2+C\|G(y_\varepsilon)-G(z)\|_{L^2}^2\\
&\quad+\frac{\alpha\varepsilon}{4(\alpha-\varepsilon)}\|y_1+\frac{2}{\alpha}(G(y_\varepsilon)-G(z))\|_{L^2}^2\\
&\le \frac{1}{\alpha-\varepsilon}\|G(y_\varepsilon)-G(z)\|_{L^2}^2+C\|G(y_\varepsilon)-G(z)\|_{L^2}^2+C. \qquad (3.20)
\end{aligned}
$$

So for ε small enough, we have

$$
\|G(y_\varepsilon)\|_{L^2}\le C. \qquad (3.21)
$$

By (3.21), it is easy to obtained the boundedness of $|Ay_\varepsilon|$. In fact, with (3.17) and (2.24), we have

$$
\begin{aligned}
&\|A(y_\varepsilon-y_1)\|_{L^2}^2=-\langle A(y_\varepsilon-y_1),\ \varepsilon y_\varepsilon+G(y_\varepsilon)-G(z)\rangle\\
&\le -\varepsilon\langle A(y_\varepsilon-y_1),\ y_\varepsilon-y_1\rangle-\varepsilon\langle A(y_\varepsilon-y_1),\ y_1\rangle-\langle A(y_\varepsilon-y_1),\ G(y_\varepsilon)-G(z)\rangle\\
&\le \frac{\varepsilon}{\alpha}\|A(y_\varepsilon-y_1)\|_{L^2}^2+\varepsilon C\|A(y_\varepsilon-y_1)\|_{L^2}+C\|A(y_\varepsilon-y_1)\|_{L^2}. \qquad (3.22)
\end{aligned}
$$

For ε small enough, $\|A(y_\varepsilon-y_1)\|_{L^2}$ is bounded, hence $\|Ay_\varepsilon\|_{L^2}$ is bounded.

Note that

$$
\begin{aligned}
\langle G(y_\varepsilon)-P(\rho^{-1}f),y_\varepsilon\rangle &= \langle G(y_\varepsilon)-G(z),\ y_\varepsilon\rangle\\
&=\langle -\varepsilon y_\varepsilon-A(y_\varepsilon-y_1),\ y_\varepsilon\rangle\\
&\le -\langle A(y_\varepsilon-y_1),\ y_\varepsilon-y_1\rangle-\langle A(y_\varepsilon-y_1),\ y_1\rangle\\
&\le \alpha^{-1}\|A(y_\varepsilon-y_1)\|_{L^2}^2+C\|A(y_\varepsilon-y_1)\|_{L^2}.
\end{aligned}
$$

With (3.16), we get

$$
\|y_\varepsilon\|_{L^1}\le C. \qquad (3.23)
$$

Sept 3 Estimate $\|y_\varepsilon\|_{L^\infty}$

It is now easy to prove that $\|y_\varepsilon\|_{L^\infty(\Omega)}$ is bounded. To this goal, write $y_\varepsilon=y_\varepsilon^1+y_\varepsilon^2$ with $y_\varepsilon^1\in N(A)$ and $y_\varepsilon^2\in R(A)$. Since $Ay_\varepsilon=Ay_\varepsilon^2$ is bounded in $L^2(\Omega)$, y_ε^2 is bounded in $L^\infty(\Omega)$. Consequently $y_\varepsilon^1=y_\varepsilon-y_\varepsilon^2$ is bounded in $L^1(\Omega)$. So its Fourier coefficients

$$
y_{\varepsilon mn}^1=\int_\Omega y_\varepsilon^1(x,t)\rho(x)\varphi_n(x)\psi_m(t)\mathrm{d}x\mathrm{d}t
$$

are bounded as $|\varphi_n(x)|\le C$, $|\psi_m(t)|\le C$ for some C independent of m,n,x and t. Therefore $|y_{\varepsilon mn}^1|\le C\|y_\varepsilon^1\|_{L^1(\Omega)}\le C_1$. Taking into account that $N(A)$ is finite dimensional, it follows that y_ε^1 is bounded in $L^\infty(\Omega)$, and hence $\|y_\varepsilon\|_{L^\infty}\le C$.

Setp 4 Taking limit as $\varepsilon \to 0$

We first show that $\{Ay_\varepsilon\}$ and $\{Gy_\varepsilon\}$ are Cauchy sequence in $L^2(\Omega)$. Set $z_{\varepsilon\lambda} = \varepsilon y_\varepsilon - \lambda y_\lambda$, it is obviously that $z_{\varepsilon\lambda} \to 0$ in $L^2(\Omega)$ as $\lambda, \varepsilon \to 0$. On the other hand, from (3.15) we have

$$\langle A(y_\varepsilon - y_\lambda),\ y_\varepsilon - y_\lambda\rangle + \langle G(y_\varepsilon) - G(y_\lambda),\ y_\varepsilon - y_\lambda\rangle \le C\|z_{\varepsilon\lambda}\|_{L^2}. \tag{3.24}$$

Combination of (3.24), (3.3) and (2.24), it leads to

$$\gamma^{-1}\|G(y_\varepsilon) - G(y_\lambda)\|_{L^2}^2 \le C\|z_{\varepsilon\lambda}\|_{L^2} + \alpha^{-1}\|A(y_\varepsilon - y_\lambda)\|_{L^2}^2. \tag{3.25}$$

Substituting

$$A(y_\varepsilon - y_\lambda) = G(y_\varepsilon) - G(y_\lambda) - z_{\varepsilon\lambda}$$

into (3.25) and noticing that $\gamma\alpha^{-1} < 1$, we have that $|G(y_\varepsilon) - G(y_\lambda)| \to 0$ as $\lambda, \varepsilon \to 0$, and therefore $A(y_\varepsilon - y_\lambda)$ is also a Cauchy sequence in $L^2(\Omega)$. The sequence $\{y_\varepsilon\}$ is bounded in $L^2(\Omega)$, so it contains a weakly convergent subsequence (denoted it again by $\{y_\varepsilon\}$ for simplicity). Taking into account that $G(y_\varepsilon)$ is strongly convergent in $L^2(\Omega)$, it follows that $G(y_\varepsilon) \to G(y)$ (strongly) in $L^2(\Omega)$. Finally, it follows that $y \in D(A)$, $Ay_\varepsilon \to Ay$, and letting $\varepsilon \to 0$, (3.15) implies (3.4).

We now can prove that actually $y_\varepsilon \to y$ strongly in $L^2(\Omega)$. Indeed, $Ay_\varepsilon^2 = Ay_\varepsilon \to Ay$ strongly in $L^2(\Omega)$. So $y_\varepsilon^2 = A^{-1}(Ay_\varepsilon)$ is also strongly convergent in $L^2(\Omega)$ (say $y_\varepsilon^2 \to y^2$). Then $y^2 \in R(A)$. As $y_\varepsilon^1 = y_\varepsilon - y_\varepsilon^2 \to y - y^2$ and $N(A)$ is finite dimensional, it follows that $y_\varepsilon^1 \to y - y^2 = y^1$ and $y^1 \in N(A)$. The conclusion is that $y_\varepsilon \to y$ is strongly in $L^2(\Omega)$. On the other hand, y_ε is bounded in $L^\infty(\Omega)$, so $y \in L^\infty(\Omega)$.

4 Appendix

In this appendix, some basic properties about continued fractions will be listed and one can consult [3–4] for the proof of these results.

Let α be real number, and put $a_0 = [\alpha]$, where $[\cdot]$ denotes the integer part. Then

$$\alpha = a_0 + \frac{1}{\alpha_1} \tag{4.1}$$

with some $\alpha_1 > 1$ if $\alpha > a_0$. Put $a_1 = [\alpha_1]$ and continue the above process. Then, we obtain the continued decomposition of α. This process does not terminate if and only if α is an irrational number. Then we obtain the continued fraction decomposition of

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{4.2}$$

and generally denote it as

$$\alpha = [a_0, a_1, a_2, a_3, \cdots], \tag{4.3}$$

where $a_0, a_1, a_2, a_3, \cdots$ are integers and are called the complete quotients of α. Generally, we denote

$$\frac{p_n}{q_n} = [a_0, a_1, a_2, a_3, \cdots, a_n] \tag{4.4}$$

with p_n, q_n relatively prime integers, which are the convergent of α such that $\frac{p_n}{q_n} \to \alpha$ as $n \to \infty$. It is well known that the p_n, q_n are recursively defined by the following relations:

$$p_0 = a_0, \quad q_0 = 1, \quad p_1 = a_0 a_1 + 1, \quad q_1 = a_1,$$
$$p_n = a_n p_{n-1} + p_{n-2}, \quad q_n = a_n q_{n-1} + q_{n-2}.$$

About these p_n, q_n, the following theorems were proved by Ben-Naoum and Mawhin [3].

Theorem 4.1 *Each irrational number α corresponds to a unique (extended) number $M(\alpha) \in [\sqrt{5}, \infty]$ having the following properties:*

(1) *For each positive number $\mu < M(\alpha)$, there exist infinitely many pairs (p_i, q_i) with $q_i \neq 0$, such that*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

(2) *If $M(\alpha)$ is finite, then, for each $\mu > M(\alpha)$, there exist only finitely many pairs (p_i, q_i) satisfying the inequality*

$$\left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{\mu q_i^2}.$$

The extended real number $M(\alpha)$ is called the Lagrange or the Markov constant of α. If we set

$$\mathcal{M}(\alpha) = \left\{ M \in \mathbb{R}_0^+ : \text{There exist infinitely many } (p_i, q_i) \text{ satisfy } \left|\alpha - \frac{p_i}{q_i}\right| \leq \frac{1}{M q_i^2} \right\},$$

then $\mathcal{M}(\alpha)$ is an interval and Theorem 4.1 says that $M(\alpha) = \sup \mathcal{M}(\alpha)$.

Theorem 4.2 *$M(\alpha)$ is finite if and only if the partial quotients sequence $(a_i)_{i\in\mathbb{N}}$ of α is bounded.*

Any α with bounded partial quotients sequence $(a_i)_{i\in\mathbb{N}}$ is said to have bounded partial quotients. Borel and Bernstein have proved that the set of irrational numbers having bounded partial quotients is a dense uncountable and null subset of the real line.

If α is an irrational number, we need some properties on the behavior of the function $M(\alpha)$ under the action of the group of transformations T defined by

$$\beta = T(\alpha) = \frac{a\alpha + b}{c\alpha + d}, \tag{4.5}$$

where $a, b, c, d \in \mathbb{Z}$ are such that $ad - bc \neq 0$. Notice that then

$$\alpha = T^{-1}(\beta) = \frac{-d\beta + b}{c\beta - a}$$

and

$$(-d)(-a) - b\,c = ad - b\,c.$$

About this transformation, it has the following results which is proved in [4].

Theorem 4.3 *If $\beta = \frac{a\alpha+b}{c\alpha+d}$ for some $a, b, c, d \in \mathbb{Z}$ such that $ad - bc \neq 0$, then*

$$M(\alpha) \leq |ad - bc| M(\beta),$$
$$M(\beta) \leq |ad - bc| M(\alpha).$$

The following results are immediately from Theorem 4.3.

Corollary 4.1 *If $\beta = \frac{a\alpha+b}{c\alpha+d}$ for some $a, b, c, d \in \mathbb{Z}$ such that $ad-bc \neq 0$, then β has bounded partial quotients if and only if α has bounded partial quotients.*

Corollary 4.2 *If p and $q \in \mathbb{Z}$, with $p, q \neq 0$, then*

$$M\Big(\frac{p}{q}\alpha\Big) \leq |p\,q| M(\alpha).$$

The modular group is the group of transformations defined by (4.5) with $|ad - bc| = 1$. Theorem 4.3 shows that $M(\alpha)$ is invariant under the action of the modular group. In particular, when $c = 0,\ d = 1,\ T(\alpha)$ is a translation through integers. So the Lagrange constant is invariant under translations through integers, and if $\{\alpha\} = \alpha - [\alpha]$, one has

$$M(\alpha) = M(\{\alpha\}).$$

Acknowledgement The authors would like to thank the anonymous referee for their helpful comments and suggestions which lead to much improvement of the earlier version of this paper.

References

[1] Baldi, P. and Berti, M., Forced vibrations of a nonhomogeneous string, *SIAM J M Analysis*, **40**, 2008, 382–412.

[2] Barbu, V. and Pavel, N. H., Periodic solutions to nonlinear one-dimensional wave equation with x-dependent coefficients, *Trans. Amer. Math. Soc.*, **349**, 1997, 2035–2048.

[3] Ben-Naoum, A. K. and Mawhin, J., The periodic Dirichlet problem for some semilinear wave equations, *J. Differential Equations*, **96**, 1992, 340–354.

[4] Berkovits, J. and Mawhin, J., Diophantine approximation, Bessel functions and radially symmetric periodic solutions of semilinear wave equations in a ball, *Trans. Amer. Math. Soc.*, **353**, 2001, 5041–5055.

[5] Berti, M. and Bolle, P., Periodic solutions of nonlinear wave equations with general nonlinearities, *Comm. Math. Phys.*, **243**(2), 2003, 315–328.

[6] Berti, M. and Bolle, P., Multiplicity of periodic solutions of nonlinear wave equations, *Nonlinear Analysis TMA.*, **56**(7), 2004, 1011–1046.

[7] Berti, M. and Bolle, P., Cantor families of periodic solutions for completely resonant nonlinear wave equations, *Duke Mathematical J.*, **134**(2), 2006, 359–419.

[8] Berti, M. and Bolle, P., Sobolev periodic solutions of nonlinear wave equations in higher spatial dimensions, *Archive for Rational Mechanics and Analysis*, **195**, 2010, 609–642.

[9] Berti, M., Bolle, P. and Procesi, M., An abstract Nash-Moser theorem with parameters and applications to PDEs, *Ann. Inst. Henri Poincaré Anal.*, **27**, 2010, 377–399.

[10] Brézis, H., Periodic solutions of nonlinear vibrating strings and duality principles, *Bull. AMS*, **8**, 1983, 409–426.

[11] Ji, S. and Li, Y., Periodic solutions to one-dimensional wave equation with x-dependent coefficients, *J. Differential Equations*, **229**, 2006, 466–493.

[12] Ji, S. and Li, Y., Time periodic solutions to one-dimensional wave equation with periodic or anti-periodic boundary conditions, *Proc. Roy. Soc. Edinburgh Sect. Ser. A*, **137**, 2007, 349–371.

[13] Rudakov, I. A., Periodic solutions of a nonlinear wave equation with nonconstant coefficients, *J. Differential Equations*, **229**, 2006, 466–493.