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On a Spectral Sequence for Twisted Cohomologies*

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Abstract Let $(\Omega^*(M), d)$ be the de Rham cochain complex for a smooth compact closed manifolds M of dimension n. For an odd-degree closed form H, there is a twisted de Rham cochain complex $(\Omega^*(M), d + H_\wedge)$ and its associated twisted de Rham cohomology $H^*(M, H)$. The authors show that there exists a spectral sequence $\{E_r^{p,q}, d_r\}$ derived from the filtration $F_p(\Omega^*(M)) = \bigoplus\limits_{i\geq p} \Omega^i(M)$ of $\Omega^*(M)$, which converges to the twisted de Rham cohomology $H^*(M, H)$. It is also shown that the differentials in the spectral sequence can be given in terms of cup products and specific elements of Massey products as well, which generalizes a result of Atiyah and Segal. Some results about the indeterminacy of differentials are also given in this paper.

Keywords Spectral sequence, Twisted de Rham cohomology, Massey product, Differential
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1 Introduction

Let M be a smooth compact closed manifold of dimension n, and $\Omega^*(M)$ be the space of smooth differential forms over $\mathbb{R}$ on M. We have the de Rham cochain complex $(\Omega^*(M), d)$, where $d : \Omega^p(M) \to \Omega^{p+1}(M)$ is the exterior differentiation, and its cohomology $H^*(M)$ (the de Rham cohomology). The de Rham cohomology with coefficients in a flat vector bundle is an extension of the de Rham cohomology.

The twisted de Rham cohomology was first studied by Rohm and Witten [13] for the antisymmetric field in superstring theory. By analyzing the massless fermion states in the string sector, Rohm and Witten obtained the twisted de Rham cochain complex $(\Omega^*(M), d + H_3)$ for a closed 3-form H_3, and mentioned the possible generalization to a sum of odd closed forms. A key feature in the twisted de Rham cohomology is that the theory is not integer-graded but (like K-theory) is filtered with the grading mod 2. This has a close relation with the twisted K-theory and the Atiyah-Hirzebruch spectral sequence (see [1]).

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Let H be $\sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$, where H_{2i+1} is a closed $(2i+1)$-form. Then one can define a new operator $D = d + H$ on $\Omega^*(M)$, where H is understood as an operator acting by exterior multiplication (for any differential form w, $H(w) = H \wedge w$). As in [1, 13], there is a filtration on $(\Omega^*(M), D)$ as follows:

$$K_p = F_p(\Omega^*(M)) = \bigoplus_{i \geq p} \Omega^i(M). \tag{1.1}$$

This filtration gives rise to a spectral sequence

$$\{E_r^{p,q}, d_r\} \tag{1.2}$$

converging to the twisted de Rham cohomology $H^*(M, H)$ with

$$E_2^{p,q} \cong \begin{cases} H^p(M), & q \text{ is even}, \\ 0, & q \text{ is odd}. \end{cases} \tag{1.3}$$

For convenience, we first fix some notations in this paper. The notation $[r]$ denotes the greatest integer part of $r \in \mathbb{R}$. In the spectral sequence (1.2), for any $[y_p]_k \in E_k^{p,q}$, $[y_p]_{k+l}$ represents its class to which $[y_p]_k$ survives in $E_{k+l}^{p,q}$. In particular, as in Proposition 3.2, for $x_p \in E_1^{p,q}$, $[x_p]_2 = [x_p]_3 \in E_2^{p,q} = E_3^{p,q}$ represents the de Rham cohomology class $[x_p]$. $d_r[x_p]$ represents a class in $E_2^{p+r,q-r+1}$, which survives to $d_r[x_p]_r \in E_r^{p+r,q-r+1}$.

In [13, Appendix I], Rohm and Witten first gave a description of the differentials d_3 and d_5 for the case $D = d + H_3$. Atiyah and Segal [1] showed a method about how to construct the differentials in terms of Massey products, and gave a generalization of Rohm and Witten's result: The iterated Massey products with H_3 give (up to sign) all the higher differentials of the spectral sequence for the twisted cohomology (see [1, Proposition 6.1]). Mathai and Wu [9, p. 5] considered the general case of $H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$ and claimed, without proof, that $d_2 = d_4 = \cdots = 0$, while d_3, d_5, $\cdots$ are given by the cup products with H_3, H_5, $\cdots$ and the higher Massey products with them. Motivated by the method in [1], we give an explicit description of the differentials in the spectral sequence (1.2) in terms of Massey products.

We now describe our main results. Let A denote a defining system for the n-fold Massey product $\langle x_1, x_2, \cdots, x_n\rangle$, and $c(A)$ denote its related cocycle (see Definition 5.1). Then

$$\langle x_1, x_2, \cdots, x_n\rangle = \{c(A) \mid A \text{ is a defining system for } \langle x_1, x_2, \cdots, x_n\rangle\} \tag{1.4}$$

by Definition 5.2. To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for Massey products (see [14]). By Theorems 4.1–4.2 in this paper, there are defining systems for the two Massey products that we need (see Lemma 5.1). The notation $\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle_A$ in Theorem 1.1 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where A is a defining system obtained by Theorem 4.1 (see Definition 5.3). Similarly, the notation $\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p\rangle_A$ in Theorem 1.2 below denotes a cohomology class in $H^*(M)$ represented by $c(A)$, where A is a defining system obtained by Theorem 4.2 (see Definition 5.3).

Theorem 1.1 *For* $H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$ *and* $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$*, the differential of the spectral sequence* (1.2)*, i.e.,* $d_{2t+3} : E_{2t+3}^{p,q} \to E_{2t+3}^{p+2t+3,q-2t-2}$*, is given by*

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t [\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p \rangle_A]_{2t+3},$$

and $[\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p \rangle_A]_{2t+3}$ *is independent of the choice of the defining system* A *obtained from Theorem* 4.1*.*

Specializing Theorem 1.1 to the case $H = H_{2s+1}$ $(s \geq 2)$, we obtain

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t [\langle \underbrace{0, \cdots, 0}_{t+1}, x_p \rangle_A]_{2t+3}. \tag{1.5}$$

Obviously, much information has been concealed in the above expression. In particular, we give a more explicit expression of differentials for this special case, which is compatible with Theorem 1.1 (see Remark 5.6).

Theorem 1.2 *For* $H = H_{2s+1}$ $(s \geq 1)$ *only and* $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$*, the differential of the spectral sequence* (1.2)*, i.e.,* $d_{2t+3} : E_{2t+3}^{p,q} \to E_{2t+3}^{p+2t+3,q-2t-2}$*, is given by*

$$d_{2t+3}[x_p]_{2t+3} = \begin{cases} [H_{2s+1} \wedge x_p]_{2t+3}, & t = s-1, \\ (-1)^{l-1} [\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B]_{2t+3}, & t = ls - 1 \ (l \geq 2), \\ 0, & \text{otherwise}, \end{cases}$$

and $[\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B]_{2t+3}$ *is independent of the choice of the defining system* B *obtained from Theorem* 4.2*.*

Atiyah and Segal [1] gave the differential expression in terms of Massey products when $H = H_3$ (see [1, Proposition 6.1]). Obviously, the result of Atiyah and Segal is a special case of Theorem 1.2.

Some of the results above are known to experts in this field, but there is a lack of mathematical proof in the literature.

This paper is organized as follows. In Section 2, we recall some backgrounds about the twisted de Rham cohomology. In Section 3, we consider the structure of the spectral sequence converging to the twisted de Rham cohomology, and give the differentials d_i $(1 \leq i \leq 3)$ and d_{2k} $(k \geq 1)$. With the formulas of the differentials in $E_{2t+3}^{p,q}$ in Section 4, Theorems 1.1 and 1.2 are proved in Section 5. In Section 6, we discuss the indeterminacy of differentials of the spectral sequence (1.2).

2 Twisted de Rham Cohomology

For completeness, in this section, we recall some knowledge about the twisted de Rham cohomology. Let M be a smooth compact closed manifold of dimension n, and $\Omega^*(M)$ be the space of smooth differential forms on M. We have the de Rham cochain complex $(\Omega^*(M), d)$ with the exterior differentiation $d: \Omega^p(M) \to \Omega^{p+1}(M)$, and its cohomology $H^*(M)$ (the de Rham cohomology).

Let H denote $\sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$, where H_{2i+1} is a closed $(2i+1)$-form. Define a new operator $D = d + H$ on $\Omega^*(M)$, where H is understood as an operator acting by exterior multiplication (for any differential form w, $H(w) = H \wedge w$, also denoted by $H_\wedge$). It is easy to show that

$$D^2 = (d+H)^2 = d^2 + dH + Hd + H^2 = 0.$$

However, D is not homogeneous on the space of smooth differential forms $\Omega^*(M) = \bigoplus_{i\geq 0} \Omega^i(M)$.

Define $\Omega^*(M)$ to be a new (mod 2) grading as follows:

$$\Omega^*(M) = \Omega^o(M) \oplus \Omega^e(M), \tag{2.1}$$

where

$$\Omega^o(M) = \bigoplus_{\substack{i\geq 0 \\ i\equiv 1 \pmod 2}} \Omega^i(M), \quad \Omega^e(M) = \bigoplus_{\substack{i\geq 0 \\ i\equiv 0 \pmod 2}} \Omega^i(M). \tag{2.2}$$

Then D is homogenous for this new (mod 2) grading,

$$\Omega^e(M) \xrightarrow{D} \Omega^o(M) \xrightarrow{D} \Omega^e(M).$$

Define the twisted de Rham cohomology groups of M as follows:

$$H^o(M,H) = \frac{\ker[D: \Omega^o(M) \to \Omega^e(M)]}{\mathrm{im}[D: \Omega^e(M) \to \Omega^o(M)]}, \tag{2.3}$$

$$H^e(M,H) = \frac{\ker[D: \Omega^e(M) \to \Omega^o(M)]}{\mathrm{im}[D: \Omega^o(M) \to \Omega^e(M)]}. \tag{2.4}$$

Remark 2.1 (i) The twisted de Rham cohomology groups $H^*(M,H)$ $(* = o, e)$ depend on the closed form H but not just on its cohomology class. If H and H' are cohomologous, then $H^*(M,H) \cong H^*(M,H')$ (see [1, Section 6]).

(ii) The twisted de Rham cohomology is also an important homotopy invariant (see [9, Section 1.4]).

Let E be a flat vector bundle over M, and $\Omega^i(M,E)$ be the space of smooth differential i-forms on M with values in E. A flat connection on E gives a linear map

$$\nabla^E : \Omega^i(M,E) \to \Omega^{i+1}(M,E),$$

such that for any smooth function f on M and any $\omega \in \Omega^i(M,E)$,

$$\nabla^E(f\omega) = df \wedge \omega + f \cdot \nabla^E \omega, \quad \nabla^E \circ \nabla^E = 0.$$

Similarly, define $\Omega^*(M,E)$ to be a new (mod 2) grading as follows:

$$\Omega^*(M,E) = \Omega^o(M,E) \oplus \Omega^e(M,E), \tag{2.5}$$

where

$$\Omega^o(M,E) = \bigoplus_{\substack{i\geq 0 \\ i\equiv 1 \pmod 2}} \Omega^i(M,E), \quad \Omega^e(M,E) = \bigoplus_{\substack{i\geq 0 \\ i\equiv 0 \pmod 2}} \Omega^i(M,E). \tag{2.6}$$

Then $D^E = \nabla^E + H_\wedge$ is homogenous for the new (mod 2) grading,

$$\Omega^e(M,E) \xrightarrow{D^E} \Omega^o(M,E) \xrightarrow{D^E} \Omega^e(M,E).$$

Define the twisted de Rham cohomology groups of E as follows:

$$H^o(M,E,H) = \frac{\ker[D^E : \Omega^o(M,E) \to \Omega^e(M,E)]}{\operatorname{im}[D^E : \Omega^e(M,E) \to \Omega^o(M,E)]}, \tag{2.7}$$

$$H^e(M,E,H) = \frac{\ker[D^E : \Omega^e(M,E) \to \Omega^o(M,E)]}{\operatorname{im}[D^E : \Omega^o(M,E) \to \Omega^e(M,E)]}. \tag{2.8}$$

Results proved in this paper are also true for the twisted de Rham cohomology groups $H^*(M,E,H)$ $(* = o,e)$ with twisted coefficients in E without any change.

3 A Spectral Sequence for Twisted de Rham Cohomology and Its Differentials d_i $(1 \leq i \leq 3)$, d_{2k} $(k \geq 1)$

Recall $D = d + H$ and $H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$, where H_{2i+1} is a closed $(2i+1)$-form. Define the usual filtration on the graded vector space $\Omega^*(M)$ to be

$$K_p = F_p(\Omega^*(M)) = \bigoplus_{i\geq p} \Omega^i(M),$$

and $K = K_0 = \Omega^*(M)$. The filtration is bounded and complete,

$$K \equiv K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_n \supset K_{n+1} = \{0\}. \tag{3.1}$$

We have $D(K_p) \subset K_p$ and $D(K_p) \subset K_{p+1}$. The differential $D(= d + H)$ does not preserve the grading of the de Rham complex. However, it does preserve the filtration $\{K_p\}_{p\geq 0}$.

The filtration $\{K_p\}_{p\geq 0}$ gives an exact couple (with bidegree) (see [12]). For each p, K_p is a graded vector space with

$$K_p = (K_p \cap \Omega^o(M)) \oplus (K_p \cap \Omega^e(M)) = K_p^o \oplus K_p^e,$$

where $K_p^o = K_p \cap \Omega^o(M)$ and $K_p^e = K_p \cap \Omega^e(M)$. The cochain complex (K_p, D) is induced by $D : \Omega^*(M) \longrightarrow \Omega^*(M)$. In a way similar to (2.4), there are two well-defined cohomology groups $H_D^e(K_p)$ and $H_D^o(K_p)$. Note that a cochain complex with grading

$$K_p/K_{p+1} = (K_p^o/K_{p+1}^o) \oplus (K_p^e/K_{p+1}^e)$$

derives cohomology groups $H_D^o(K_p/K_{p+1})$ and $H_D^e(K_p/K_{p+1})$. Since $D(K_p) \subset K_{p+1}$, we have $D = 0$ in the cochain complex $(K_p/K_{p+1}, D)$.

Lemma 3.1 *For the cochain complex* $(K_p/K_{p+1}, D)$*, we have*

$$H_D^o(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M), & p \text{ is odd}, \\ 0, & p \text{ is even}, \end{cases}$$
$$H_D^e(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M), & p \text{ is even}, \\ 0, & p \text{ is odd}. \end{cases}$$

Proof If p is odd, then

$$K_p \cap \Omega^e(M) = K_{p+1} \cap \Omega^e(M) \text{ and } (K_p \cap \Omega^e(M))/(K_{p+1} \cap \Omega^e(M)) = 0.$$

We have

$$\begin{aligned} &(K_p \cap \Omega^o(M))/(K_{p+1} \cap \Omega^o(M)) = K_p^o/K_{p+1}^o \cong \Omega^p(M), \\ &H_D^o(K_p/K_{p+1}) \cong \Omega^p(M), \quad H_D^e(K_p/K_{p+1}) = 0. \end{aligned}$$

Similarly, for even p, we have

$$H_D^e(K_p/K_{p+1}) \cong \Omega^p(M), \quad H_D^o(K_p/K_{p+1}) = 0.$$

By the filtration (3.1), we obtain a short exact sequence of cochain complexes

$$0 \longrightarrow K_{p+1} \xrightarrow{i} K_p \xrightarrow{j} K_p/K_{p+1} \longrightarrow 0, \tag{3.2}$$

which gives rise to a long exact sequence of cohomology groups

$$\begin{aligned} \cdots \longrightarrow H_D^{p+q}(K_{p+1}) &\xrightarrow{i^*} H_D^{p+q}(K_p) \xrightarrow{j^*} H_D^{p+q}(K_p/K_{p+1}) \\ &\xrightarrow{\delta} H_D^{p+q+1}(K_{p+1}) \xrightarrow{i^*} H_D^{p+q+1}(K_p) \xrightarrow{j^*} \cdots. \end{aligned} \tag{3.3}$$

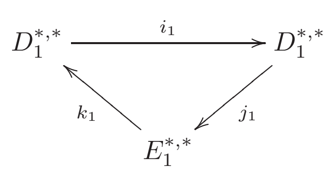
Note that in the exact sequence above,

$$H_D^i(K_p) = \begin{cases} H_D^e(K_p), & i \text{ is even}, \\ H_D^o(K_p), & i \text{ is odd}, \end{cases}$$
$$H_D^i(K_p/K_{p+1}) = \begin{cases} H_D^e(K_p/K_{p+1}), & i \text{ is even}, \\ H_D^o(K_p/K_{p+1}), & i \text{ is odd}. \end{cases}$$

Let

$$E_1^{p,q} = H_D^{p+q}(K_p/K_{p+1}), \quad D_1^{p,q} = H_D^{p+q}(K_p), \quad i_1 = i^*,\ j_1 = j^*,\ k_1 = \delta. \tag{3.4}$$

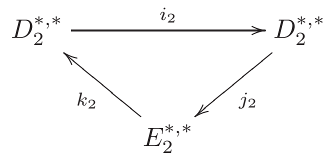
We get an exact couple from the long exact sequence (3.3)



(3.5)

with i_1 of bidegree $(-1,1)$, j_1 of bidegree $(0,0)$ and k_1 of bidegree $(1,0)$.

We have $d_1 = j_1k_1 : E_1^{*,*} \longrightarrow E_1^{*,*}$ with bidegree $(1,0)$, and $d_1^2 = j_1k_1j_1k_1 = 0$. By (3.5), we have the derived couple



(3.6)

by the following:

(1) $D_2^{*,*} = i_1D_1^{*,*}$, $E_2^{*,*} = H_{d_1}(E_1^{*,*})$.

(2) $i_2 = i_1|_{D_2^{*,*}}$, also denoted by i_1.

(3) If $a_2 = i_1a_1 \in D_2^{*,*}$, define $j_2(a_2) = [j_1a_1]_{d_1}$, where $[\ \ ]_{d_1}$ denotes the cohomology class in $H_{d_1}(E_1^{*,*})$.

(4) For $[b]_{d_1} \in E_2^{*,*} = H_{d_1}(E_1^{*,*})$, define $k_2([b]_{d_1}) = k_1b \in D_2^{*,*}$.

The derived couple (3.6) is also an exact couple, and j_2 and k_2 are well defined (see [6, 12]).

Proposition 3.1 (i) *There exists a spectral sequence* $(E_r^{p,q}, d_r)$ *derived from the filtration* $\{K_n\}_{n\geq 0}$*, where* $E_1^{p,q} = H_D^{p+q}(K_p/K_{p+1})$*,* $d_1 = j_1k_1$*, and* $E_2^{p,q} = H_{d_1}(E_1^{p,q})$*,* $d_2 = j_2k_2$*. The bidegree of* d_r *is* $(r, 1-r)$*.*

(ii) *The spectral sequence* $\{E_r^{p,q}, d_r\}$ *converges to the twisted de Rham cohomology*

$$\bigoplus_{p+q=1} E_\infty^{p,q} \cong H^o(M,H), \qquad \bigoplus_{p+q=0} E_\infty^{p,q} \cong H^e(M,H). \tag{3.7}$$

Proof Since the filtration is bounded and complete, the proof follows from the standard algebraic topology method (see [12]).

Remark 3.1 (1) Note that

$$H_D^i(K_p) = \begin{cases} H_D^e(K_p), & i \text{ is even},\\ H_D^o(K_p), & i \text{ is odd},\end{cases}$$

$$H_D^i(K_p/K_{p+1}) = \begin{cases} H_D^e(K_p/K_{p+1}), & i \text{ is even},\\ H_D^o(K_p/K_{p+1}), & i \text{ is odd}.\end{cases}$$

Then we have that $H_D^i(K_p)$ and $H_D^i(K_p/K_{p+1})$ are 2-periodic on i. Consequently, the spectral sequence $\{E_r^{p,q}, d_r\}$ is 2-periodic on q.

(2) There is also a spectral sequence converging to the twisted cohomology $H^*(M,E,H)$ for a flat vector bundle E over M.

Proposition 3.2 *For the spectral sequence in Proposition* 3.1*,*

(i) *The* $E_1^{*,*}$*-term is given by*

$$E_1^{p,q} = H_D^{p+q}(K_p/K_{p+1}) \cong \begin{cases} \Omega^p(M), & q \textit{ is even},\\ 0, & q \textit{ is odd},\end{cases}$$

and $d_1x_p = dx_p$ *for any* $x_p \in E_1^{p,q}$*.*

(ii) *The $E_2^{*,*}$-term is given by*

$$E_2^{p,q} = H_{d_1}(E_1^{p,q}) \cong \begin{cases} H^p(M), & q \text{ is even}, \\ 0, & q \text{ is odd}, \end{cases}$$

and $d_2 = 0$.

(iii) *$E_3^{p,q} = E_2^{p,q}$ and $d_3[x_p] = [H_3 \wedge x_p]$ for $[x_p]_3 \in E_3^{p,q}$.*

Proof (i) By Lemma 3.1, we have the $E_1^{*,*}$-term as desired, and by definition, we obtain $d_1 = j_1 k_1 : E_1^{p,q} \to E_1^{p+1,q}$. We only need to consider the case when q is even, otherwise $d_1 = 0$. By (3.2) for odd p (the case, when p is even, is similar), we have a large commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & K_{p+1}^e & \xrightarrow{i} & K_p^e & \xrightarrow{j} & 0 & \longrightarrow & 0 \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & K_{p+1}^o & \xrightarrow{i} & K_p^o & \xrightarrow{j} & \Omega^p(M) & \longrightarrow & 0 \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
0 & \longrightarrow & K_{p+1}^e & \xrightarrow{i} & K_p^e & \xrightarrow{j} & 0 & \longrightarrow & 0 \\
 & & \uparrow D & & \uparrow D & & \uparrow D & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{3.8}$$

where the rows are exact and the columns are cochain complexes.

Let $x_p \in \Omega^p(M) \cong H_D^{p+q}(K_p/K_{p+1}) \cong E_1^{p,q}$, and

$$x = \sum_{i=0}^{[\frac{n-p}{2}]} x_{p+2i} \tag{3.9}$$

be an (inhomogeneous) form, where x_{p+2i} is a $(p+2i)$-form $(0 \le i \le [\frac{n-p}{2}])$. Then $x \in K_p^o$, $jx = x_p$ and $Dx \in K_p^e$. Also $Dx \in K_{p+1}^e$. By the definition of the homomorphism δ in (3.3), we have

$$k_1 x_p = [Dx]_D, \tag{3.10}$$

where $[\ \ ]_D$ is the cohomology class in $H_D^*(K_{p+1})$. The class $[Dx]_D$ is well defined and independent of the choices of x_{p+2i} $(1 \le i \le [\frac{n-p}{2}])$ (see [3, p. 116]).

Choose $x_{p+2i} = 0$ $(1 \le i \le [\frac{n-p}{2}])$. Then we have

$$\begin{aligned} k_1 x_p &= [Dx]_D \\ &= [dx_p + H \wedge x_p]_D \\ &= \Big[dx_p + \sum_{l=1}^{[\frac{n-1}{2}]} H_{2l+1} \wedge x_p\Big]_D \in H_D^{p+q+1}(K_{p+1}). \end{aligned}$$

Thus, one obtains

$$d_1 x_p = (j_1 k_1) x_p = j_1(k_1(x_p)) = j_1 \Big[dx_p + \sum_{l=1}^{[\frac{n-1}{2}]} H_{2l+1} \wedge x_p \Big]_D = dx_p.$$

(ii) By the definition of the spectral sequence and (i), one obtains that $E_2^{p,q} \cong H^p(M)$ when q is even, and $E_2^{p,q} = 0$ when q is odd. Note $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$. It follows that $d_2 = 0$ by degree reasons.

(iii) Note that $[x_p]_3 \in E_3^{p,q}$ implies $dx_p = 0$. Choosing $x_{p+2i} = 0$ for $1 \leq i \leq [\frac{n-p}{2}]$, we get

$$[Dx]_D = [H \wedge x_p]_D = \Big[\sum_{l=1}^{[\frac{n-1}{2}]} H_{2l+1} \wedge x_p \Big]_D \in H_D^{p+q+1}(K_{p+1}),$$

where x is given in the proof of (i). Note

$$\begin{array}{ccccc} H_D^{p+q+1}(K_{p+1}) & \xleftarrow{i_1^2} & H_D^{p+q+1}(K_{p+3}) & \xrightarrow{j_1} & H_D^{p+q+1}(K_{p+3}/K_{p+4}) \\ [Dx]_D & \xrightarrow{(i_1^{-1})^2} & [Dx]_D & \xrightarrow{j_1} & H_3 \wedge x_p. \end{array} \tag{3.11}$$

It follows that

$$d_3[x_p]_3 = j_3 k_3 [x_p]_3 = j_3(k_1 x_p) = j_3 [Dx]_D = [j_1((i_1^{-1})^2 [Dx]_D)]_3 = [H_3 \wedge x_p]_3, \tag{3.12}$$

where the first, second and fourth identities follow from the definitions of d_3, k_3 and j_3, respectively, and the third and last identities follow from (3.10) and (3.11), respectively. By (ii), $d_2 = 0$, so $E_3^{p,q} = E_2^{p,q}$. Then we have

$$d_3[x_p] = [H_3 \wedge x_p].$$

Corollary 3.1 *$d_{2k} = 0$ for $k \geq 1$. Therefore, for $k \geq 1$,*

$$E_{2k+1}^{p,q} = E_{2k}^{p,q}. \tag{3.13}$$

Proof Note $d_{2k} : E_{2k}^{p,q} \longrightarrow E_{2k}^{p+2k,q+1-2k}$. By Proposition 3.2(ii), if q is odd, then $E_2^{p,q} = 0$, which implies that $E_{2k}^{p,q} = 0$. By degree reasons, we have $d_{2k} = 0$ and $E_{2k+1}^{p,q} = E_{2k}^{p,q}$ for $k \geq 1$.

The differential d_3 for the case $H = H_3$ is shown in [1, Section 6], and the $E_2^{p,q}$-term is also known.

4 Differentials d_{2t+3} $(t \geq 1)$ in Terms of Cup Products

In this section, we will show that the differentials d_{2t+3} $(t \geq 1)$ can be given in terms of cup products.

We first consider the general case of $H = \sum\limits_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$. For $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$, we let $x = \sum\limits_{j=0}^{[\frac{n-p}{2}]} x_{p+2j} \in F_p(\Omega^*(M))$. Then we have

$$\begin{aligned} Dx &= \Big(d + \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}\Big)\Big(\sum_{j=0}^{[\frac{n-p}{2}]} x_{p+2j}\Big) \\ &= dx_p + \sum_{j=0}^{[\frac{n-p}{2}]-1} \Big(dx_{p+2j+2} + \sum_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2}\Big). \end{aligned} \tag{4.1}$$

Denote $y = Dx = \sum\limits_{j=0}^{[\frac{n-p}{2}]} y_{p+2j+1}$, where

$$\begin{cases} y_{p+1} = dx_p, \\ y_{p+2j+3} = dx_{p+2j+2} + \sum\limits_{i=1}^{j+1} H_{2i+1} \wedge x_{p+2(j-i)+2}, \quad 0 \le j \le \Big[\dfrac{n-p}{2}\Big] - 1. \end{cases} \tag{4.2}$$

Theorem 4.1 *For $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \ge 1)$, there exist $x_{p+2i} = x_{p+2i}^{(t)}$ $(1 \le i \le t)$, such that $y_{p+2j+1} = 0$ $(0 \le j \le t)$ and*

$$d_{2t+3}[x_p]_{2t+3} = \Big[\sum_{i=1}^{t} H_{2i+1} \wedge x^{(t)}_{p+2(t-i)+2} + H_{2t+3} \wedge x_p\Big]_{2t+3},$$

where the $(p+2i)$-form $x^{(t)}_{p+2i}$ depends on t.

Proof The theorem is shown by mathematical induction on t.

When $t = 1$, $[x_p]_{2t+3} = [x_p]_5$. $[x_p]_5 \in E_5^{p,q}$ implies that $dx_p = 0$ and $d_3[x_p] = [H_3 \wedge x_p] = 0$ by Proposition 3.2. Thus there exists a $(p+2)$-form v_1, such that $H_3 \wedge x_p = d(-v_1)$. We can choose $x^{(1)}_{p+2} = v_1$ to get $y_{p+3} = dx^{(1)}_{p+2} + H_3 \wedge x_p = dv_1 + H_3 \wedge x_p = 0$ from (4.2). Noting

$$\begin{array}{ccccc} H_D^{p+q+1}(K_{p+1}) & \xleftarrow{i_1^4} & H_D^{p+q+1}(K_{p+5}) & \xrightarrow{j_1} & H_D^{p+q+1}(K_{p+5}/K_{p+6}) \\ [Dx]_D & \xrightarrow{(i_1^{-1})^4} & [Dx]_D & \xrightarrow{j_1} & y_{p+5}, \end{array} \tag{4.3}$$

we obtain

$$d_5[x_p]_5 = j_5 k_5 [x_p]_5 = j_5(k_1 x_p) = j_5[Dx]_D = [j_1(i_1^{-1})^4[Dx]_D]_5 = [y_{p+5}]_5. \tag{4.4}$$

The reasons for the identities in (4.4) are similar to those of (3.12). Thus, we have

$$d_5[x_p]_5 = [dx_{p+4} + H_3 \wedge x^{(1)}_{p+2} + H_5 \wedge x_p]_5 = [H_3 \wedge x^{(1)}_{p+2} + H_5 \wedge x_p]_5,$$

where the first identity follows from (4.4) and the definition of y_{p+5} in (4.2), and the second one follows from the fact that dx_{p+4} vanishes in $E_5^{*,*}$. Hence the result holds for $t = 1$.

Suppose that the result holds for $t \le m-1$. Now we show that the theorem also holds for $t = m$.

From $[x_p]_{2m+3} \in E_{2m+3}^{p,q}$, we have $[x_p]_{2m+1} \in E_{2m+1}^{p,q}$ and $d_{2m+1}[x_p]_{2m+1} = 0$. By induction, there exist $x_{p+2i}^{(m-1)}$ $(1 \le i \le m-1)$, such that

$$\begin{cases} y_{p+1}^{(m-1)}(x_p) = dx_p = 0, \\ y_{p+3}^{(m-1)}(x_p) = dx_{p+2}^{(m-1)} + H_3 \wedge x_p = 0, \\ y_{p+2i+1}^{(m-1)}(x_p) = dx_{p+2i}^{(m-1)} + \displaystyle\sum_{j=1}^{i-1} H_{2j+1} \wedge x_{p+2(i-j)}^{(m-1)} + H_{2i+1} \wedge x_p \\ \qquad\qquad = 0 \quad (2 \le i \le m-1), \\ d_{2m+1}[x_p]_{2m+1} = \Big[\displaystyle\sum_{i=1}^{m-1} H_{2i+1} \wedge x_{p+2(m-i)}^{(m-1)} + H_{2m+1} \wedge x_p\Big]_{2m+1} = 0. \end{cases} \tag{4.5}$$

By $d_{2m} = 0$ and the last equation in (4.5), there exists a $(p+2)$-form w_{p+2}, such that

$$\Big[\sum_{i=1}^{m-1} H_{2i+1} \wedge x_{p+2(m-i)}^{(m-1)} + H_{2m+1} \wedge x_p\Big]_{2m-1} = d_{2m-1}[w_{p+2}]_{2m-1}. \tag{4.6}$$

By induction and $[w_{p+2}]_{2m-1} \in E_{2m-1}^{p+2,q-2}$, there exist $w_{p+2(i+1)}^{(m-2)}$ $(1 \le i \le m-2)$, such that

$$\begin{cases} y_{p+3}^{(m-2)}(w_{p+2}) = dw_{p+2} = 0, \\ y_{p+5}^{(m-2)}(w_{p+2}) = dw_{p+4}^{(m-2)} + H_3 \wedge w_{p+2} = 0, \\ y_{p+2i+3}^{(m-2)}(w_{p+2}) = dw_{p+2(i+1)}^{(m-2)} + \displaystyle\sum_{j=1}^{i-1} H_{2j+1} \wedge w_{p+2(i-j+1)}^{(m-2)} + H_{2i+1} \wedge w_{p+2} \\ \qquad\qquad = 0 \quad (2 \le i \le m-2), \\ d_{2m-1}[w_{p+2}]_{2m-1} = \Big[\displaystyle\sum_{i=1}^{m-2} H_{2i+1} \wedge w_{p+2(m-i)}^{(m-2)} + H_{2m-1} \wedge w_{p+2}\Big]_{2m-1}. \end{cases} \tag{4.7}$$

By (4.6) and the last equation in (4.7), we obtain

$$\Big[\sum_{i=1}^{m-2} H_{2i+1} \wedge (x_{p+2(m-i)}^{(m-1)} - w_{p+2(m-i)}^{(m-2)}) + H_{2m-1} \wedge (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \wedge x_p\Big]_{2m-1} = 0.$$

Note that $d_{2m-2} = 0$, and it follows that there exists a $(p+4)$-form w_{p+4}, such that

$$\Big[\sum_{i=1}^{m-2} H_{2i+1} \wedge (x_{p+2(m-i)}^{(m-1)} - w_{p+2(m-i)}^{(m-2)}) + H_{2m-1} \wedge (x_{p+2}^{(m-1)} - w_{p+2}) + H_{2m+1} \wedge x_p\Big]_{2m-3}$$
$$= d_{2m-3}[w_{p+4}]_{2m-3}.$$

Keeping the same iteration process as mentioned above, we have

$$\Big[\sum_{i=1}^{2} \Big(H_{2i+1} \wedge \Big(x_{p+2(m-i)}^{(m-1)} - \sum_{j=1}^{m-3} w_{p+2(m-i)}^{(m-1-j)}\Big)\Big)$$

$$+\sum_{i=3}^{m-1}\Big(H_{2i+1}\wedge\Big(x^{(m-1)}_{p+2(m-i)}-\sum_{j=1}^{m-1-j}w^{(m-1-j)}_{p+2(m-i)}-w_{p+2(m-i)}\Big)\Big)+H_{2m+1}\wedge x_p\Big]_7=0.$$

By $d_6=0$, it follows that there exists a $(p+2(m-2))$-form $w_{p+2(m-2)}$, such that

$$\begin{aligned}&\Big[\sum_{i=1}^{2}\Big(H_{2i+1}\wedge\Big(x^{(m-1)}_{p+2(m-i)}-\sum_{j=1}^{m-3}w^{(m-1-j)}_{p+2(m-i)}\Big)\Big)+\sum_{i=3}^{m-1}\Big(H_{2i+1}\wedge\Big(x^{(m-1)}_{p+2(m-i)}\\&-\sum_{j=1}^{m-i-1}w^{(m-1-j)}_{p+2(m-i)}-w_{p+2(m-i)}\Big)\Big)+H_{2m+1}\wedge x_p\Big]_5=d_5[w_{p+2(m-2)}]_5.\end{aligned}\tag{4.8}$$

By induction and $[w_{p+2(m-2)}]_5\in E_5^{p+2(m-2),q-2(m-2)}$, there exists $w^{(1)}_{p+2(m-1)}$, such that

$$\begin{cases}y^{(1)}_{p+2m-3}(w_{p+2(m-2)})=dw_{p+2(m-2)}=0,\\ y^{(1)}_{p+2m-1}(w_{p+2(m-2)})=dw^{(1)}_{p+2(m-1)}+H_3\wedge w_{p+2(m-2)}=0,\\ d_5[w_{p+2(m-2)}]_5=[H_3\wedge w^{(1)}_{p+2(m-1)}+H_5\wedge w_{p+2(m-2)}]_5.\end{cases}\tag{4.9}$$

By (4.8), the last equation in (4.9) and $d_4=0$, it follows that there exists a $(p+2(m-1))$-form $w_{p+2(m-1)}$, such that

$$\begin{aligned}&\Big[\Big(H_3\wedge\Big(x^{(m-1)}_{p+2(m-1)}-\sum_{j=1}^{m-2}w^{(m-1-j)}_{p+2(m-1)}\Big)\Big)+\sum_{i=2}^{m-1}\Big(H_{2i+1}\wedge\Big(x^{(m-1)}_{p+2(m-i)}\\&-\sum_{j=1}^{m-i-1}w^{(m-1-j)}_{p+2(m-i)}-w_{p+2(m-i)}\Big)\Big)+H_{2m+1}\wedge x_p\Big]=d_3[w_{p+2(m-1)}]=[H_3\wedge w_{p+2(m-1)}]\end{aligned}$$

and $y^{(0)}_{p+2m-1}(w_{p+2(m-1)})=dw_{p+2(m-1)}=0$. Thus there exists a $(p+2m)$-form w_{p+2m}, such that

$$\begin{aligned}&\sum_{i=1}^{m-1}\Big(H_{2i+1}\wedge\Big(x^{(m-1)}_{p+2(m-i)}-\sum_{j=1}^{m-i-1}w^{(m-1-j)}_{p+2(m-i)}-w_{p+2(m-i)}\Big)\Big)\\&+H_{2m+1}\wedge x_p=dw_{p+2m}.\end{aligned}\tag{4.10}$$

Comparing (4.10) with (4.2), we choose at this time

$$\begin{cases}x_{p+2}=x^{(m)}_{p+2}=x^{(m-1)}_{p+2}-w_{p+2},\\ x_{p+2i}=x^{(m)}_{p+2i}=x^{(m-1)}_{p+2i}-\displaystyle\sum_{j=1}^{i-1}w^{(m-1-j)}_{p+2i}-w_{p+2i}\quad(2\le i\le m-1),\\ x_{p+2m}=x^{(m)}_{p+2m}=-w_{p+2m}.\end{cases}\tag{4.11}$$

From (4.2), by a direct computation, we have

$$\begin{cases}y_{p+1}=y^{(m-1)}_{p+1}(x_p)=0,\\ y_{p+2i-1}=y^{(m-1)}_{p+2i-1}(x_p)-\displaystyle\sum_{j=1}^{i-1}y^{(m-1-j)}_{p+2i-1}(w_{p+2j})=0\quad(2\le i\le m),\\ y_{p+2m+1}=0.\end{cases}\tag{4.12}$$

Note

$$
\begin{array}{ccccc}
H_D^{p+q+1}(K_{p+1}) & \xleftarrow{i_1^{2(m+1)}} & H_D^{p+q+1}(K_{p+2m+3}) & \xrightarrow{j_1} & H_D^{p+q+1}(K_{p+2m+3}/K_{p+2m+4}) \\
[Dx]_D & \xrightarrow{(i_1^{-1})^{2(m+1)}} & [Dx]_D & \xrightarrow{j_1} & y_{p+2m+3}.
\end{array}
\tag{4.13}
$$

By the similar reasons as in (3.12), the following identities hold:

$$
\begin{aligned}
d_{2m+3}[x_p]_{2m+3} &= j_{2m+3}k_{2m+3}[x_p]_{2m+3} \\
&= j_{2m+3}(k_1 x_p) \\
&= j_{2m+3}[Dx]_D \\
&= [j_1(i_1^{-1})^{2(m+1)}[Dx]_D]_{2m+3} \\
&= [y_{p+2m+3}]_{2m+3}.
\end{aligned}
\tag{4.14}
$$

So we have

$$
\begin{aligned}
d_{2m+3}[x_p]_{2m+3} &= [y_{p+2m+3}]_{2m+3} \\
&= \Big[dx_{p+2m+2} + \sum_{i=1}^{m} H_{2i+1} \wedge x^{(m)}_{p+2(m-i+1)} + H_{2m+3} \wedge x_p\Big]_{2m+3} \quad \text{(by (4.2))} \\
&= \Big[\sum_{i=1}^{m} H_{2i+1} \wedge x^{(m)}_{p+2(m-i+1)} + H_{2m+3} \wedge x_p\Big]_{2m+3},
\end{aligned}
$$

showing that the result also holds for $t = m$.

The proof of the theorem is completed.

Remark 4.1 Note that $x^{(t)}_{p+2i}$ $(1 \le i \le t)$ depend on t, and that $x^{(t_1)}_{p+2i} \neq x^{(t_2)}_{p+2i}$ depend on the condition $t_1 \neq t_2$ generally. $x^{(t)}_{p+2i}$ $(1 \le i \le t)$ are related to $x^{(t-1)}_{p+2j}$ $(1 \le j \le t-1,\ j \le i)$.

Now we consider the special case in which $H = H_{2s+1}$ $(s \ge 1)$ only. For this special case, we will give a more explicit result which is stronger than Theorem 4.1.

For $x = \sum\limits_{j=0}^{[\frac{n-p}{2}]} x_{p+2j}$, we have

$$
Dx = (d + H_{2s+1})\Big(\sum_{j=0}^{[\frac{n-p}{2}]} x_{p+2j}\Big) = \sum_{j=0}^{s-1} dx_{p+2j} + \sum_{j=s}^{[\frac{n-p}{2}]} (dx_{p+2j} + H_{2s+1} \wedge x_{p+2(j-s)}).
$$

Denote

$$
\begin{cases}
y_{p+2j+1} = dx_{p+2j}, & 0 \le j \le s-1, \\
y_{p+2j+3} = dx_{p+2j+2} + H_{2s+1} \wedge x_{p+2(j-s)+2}, & s-1 \le j \le \Big[\dfrac{n-p}{2}\Big] - 1.
\end{cases}
\tag{4.15}
$$

Then $Dx = \sum\limits_{j=0}^{[\frac{n-p}{2}]} y_{p+2j+1}$.

Theorem 4.2 *For $H = H_{2s+1}$ $(s \geq 1)$ only and $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$, there exist $x_{p+2is} = x_{p+2is}^{([\frac{t}{s}])}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2[\frac{t}{s}]s+2k} = 0$ for $1 \leq i \leq [\frac{t}{s}]$, $1 \leq j \leq s-1$ and $1 \leq k \leq t - [\frac{t}{s}]s$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq t)$ and*

$$d_{2t+3}[x_p]_{2t+3} = \begin{cases} [H_{2s+1} \wedge x_p]_{2s+1}, & t = s-1, \\ [H_{2s+1} \wedge x_{p+2(l-1)s}^{(l-1)}]_{2t+3}, & t = ls - 1 \ (l \geq 2), \\ 0, & otherwise, \end{cases}$$

where the $(p+2is)$-form $x_{p+2is}^{([\frac{t}{s}])}$ depend on $[\frac{t}{s}]$.

Proof We prove the theorem by mathematical induction on s.

When $s = 1$, the result follows from Theorem 4.1.

When $s \geq 2$, we prove the result by mathematical induction on t. We first show that the result holds for $t = 1$. Note that $[x_p]_5 \in E_5^{p,q}$ implies $y_{p+1} = dx_p = 0$. Choose $x_{p+2} = 0$ and make $y_{p+3} = 0$.

(i) When $s = 2$, by (4.4), we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = [dx_{p+4} + H_5 \wedge x_p]_5 = [H_5 \wedge x_p]_5.$$

(ii) When $s \geq 3$, by (4.4), we have

$$d_5[x_p]_5 = [y_{p+5}]_5 = [dx_{p+4}]_5 = 0.$$

Combining (i) and (ii), we have that the theorem holds for $t = 1$.

Suppose that the theorem holds for $t \leq m-1$. Now we show that the theorem also holds for $t = m$.

Case 1 $2 \leq m \leq s-1$.

By induction, the theorem holds for $1 \leq t \leq m-1$. Choosing $x_{p+2i} = 0$ $(1 \leq i \leq m)$, from (4.15), we easily get that $y_{p+2j+1} = 0$ $(0 \leq j \leq m)$. By (4.14)–(4.15), we have

$$\begin{aligned} d_{2m+3}[x_p]_{2m+3} &= [y_{p+2m+3}]_{2m+3} \\ &= \begin{cases} [dx_{p+2(m+1)}]_{2m+3}, & 2 \leq m \leq s-2, \\ [dx_{p+2(m+1)} + H_{2s+1} \wedge x_p]_{2m+3}, & m = s-1 \end{cases} \\ &= \begin{cases} 0, & 2 \leq m \leq s-2, \\ [H_{2s+1} \wedge x_p]_{2s+1}, & m = s-1. \end{cases} \end{aligned}$$

Case 2 $m = ls - 1$ $(l \geq 2)$.

By induction, the theorem holds for $t = m-1 = ls-2$. Thus, there exist $x_{p+2is} = x_{p+2is}^{([\frac{m-1}{s}])} = x_{p+2is}^{(l-1)}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq l-1$, $1 \leq j \leq s-1$ and $1 \leq k \leq s-2$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq ls-2)$. Choosing $x_{p+2(ls-1)} = 0$, by (4.15), we get

$$y_{p+2(ls-1)+1} = dx_{p+2(ls-1)} + H_{2s+1} \wedge x_{p+2(l-1)s-2} = 0 + H_{2s+1} \wedge 0 = 0.$$

Then we have

$$\begin{aligned} d_{2(ls-1)+3}[x_p]_{2(ls-1)+3} &= [y_{p+2ls+1}]_{2(ls-1)+3} \quad (\text{by } (4.14)) \\ &= [dx_{p+2ls} + H_{2s+1} \wedge x^{(l-1)}_{p+2(l-1)s}]_{2(ls-1)+3} \quad (\text{by } (4.15)) \\ &= [H_{2s+1} \wedge x^{(l-1)}_{p+2(l-1)s}]_{2(ls-1)+3}. \end{aligned}$$

Case 3 $m = ls$ $(l \geq 1)$.

By induction, there exist $x_{p+2is} = x^{([\frac{ls-1}{s}])}_{p+2is} = x^{(l-1)}_{p+2is}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq l-1$, $1 \leq j \leq s-1$ and $1 \leq k \leq s-1$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq ls-1)$. By the same method as in Theorem 4.1, one has that there exist $x_{p+2is} = x^{(l)}_{p+2is}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq l$, $1 \leq j \leq s-1$ and $1 \leq k \leq s-1$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq ls)$. By (4.14)–(4.15) and $x_{p+2ls-2s+2} = 0$, we have

$$\begin{aligned} d_{2ls+3}[x_p]_{2ls+3} &= [y_{p+2ls+3}]_{2ls+3} \\ &= [dx_{p+2ls+2} + H_{2s+1} \wedge x_{p+2ls-2s+2}]_{2ls+3} \\ &= 0. \end{aligned}$$

Case 4 $ls < m < (l+1)s - 1$ $(l \geq 1)$.

By induction, there exist $x_{p+2is} = x^{([\frac{m-1}{s}])}_{p+2is} = x^{(l)}_{p+2is}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2ls+2k} = 0$ for $1 \leq i \leq l$, $1 \leq j \leq s-1$ and $1 \leq k \leq m-ls-1$, such that $y_{p+2u+1} = 0$ $(0 \leq u \leq m-1)$. Choose $x_{p+2m} = 0$ and make $y_{p+2m+1} = 0$. By (4.14)–(4.15) and $x_{p+2m-2s+2} = 0$, we have

$$\begin{aligned} d_{2m+3}[x_p]_{2m+3} &= [y_{p+2m+3}]_{2m+3} \\ &= [dx_{p+2m+2} + H_{2s+1} \wedge x_{p+2m-2s+2}]_{2m+3} \\ &= 0. \end{aligned}$$

Combining Cases 1–4, we have that the result holds for $t = m$, and the proof is completed.

Remark 4.2 (1) Theorems 4.1–4.2 show that the differentials in the spectral sequence (1.2) can be computed in terms of cup products with H_{2i+1}'s. The existence of $x^{(t)}_{p+2i}$'s and $x^{([\frac{t}{s}])}_{p+2is}$'s in Theorems 4.1–4.2 plays an essential role in proving Theorems 1.1–1.2, respectively. Theorems 4.1–4.2 give a description of the differentials at the level of $E^{p,q}_{2t+3}$ for the spectral sequence (1.2), which was ignored in the previous studies of the twisted de Rham cohomology in [1, 9].

(2) Note that Theorem 4.2 is not a corollary of Theorem 4.1, and it can not be obtained from Theorem 4.1 directly.

5 Differentials d_{2t+3} $(t \geq 1)$ in Terms of Massey Products

The Massey product is a cohomology operation of higher order introduced in [8], which generalizes the cup product. May [10] showed that the differentials in the Eilenberg-Moore spectral sequence associated with the path-loop fibration of a path connected, simply connected space are completely determined by higher order Massey products. Kraines and Schochet [5] also described the differentials in Eilenberg-Moore spectral sequence by Massey products. In order to describe the differentials d_{2t+3} $(t \geq 1)$ in terms of Massey products, we first recall briefly the definition of Massey products (see [4, 10–12]). Then the main theorems in this paper will be shown.

Because of different conventions in the literature used to define Massey products, we present the following definitions. If $x \in \Omega^p(M)$, the symbol $\overline{x}$ will denote $(-1)^{1+\deg x}x = (-1)^{1+p}x$. We first define the Massey triple product.

Let x_1, x_2, x_3 be closed differential forms on M of degrees r_1, r_2, r_3 with $[x_1][x_2] = 0$ and $[x_2][x_3] = 0$, where $[\ ]$ denotes the de Rham cohomology class. Thus, there are differential forms v_1 of degree $r_1 + r_2 - 1$ and v_2 of degree $r_2 + r_3 - 1$, such that $dv_1 = \overline{x}_1 \wedge x_2$ and $dv_2 = \overline{x}_2 \wedge x_3$. Define the $(r_1 + r_2 + r_3 - 1)$-form

$$\omega = \overline{v}_1 \wedge x_3 + \overline{x}_1 \wedge v_2. \tag{5.1}$$

Then ω satisfies

$$\begin{aligned} d(\omega) &= (-1)^{r_1+r_2} dv_1 \wedge x_3 + (-1)^{r_1} \overline{x}_1 \wedge dv_2 \\ &= (-1)^{r_1+r_2} \overline{x}_1 \wedge x_2 \wedge x_3 + (-1)^{r_1+r_2+1} \overline{x}_1 \wedge x_2 \wedge x_3 \\ &= 0. \end{aligned}$$

Hence a set of all the cohomology classes $[\omega]$ obtained by the above procedure is defined to be the Massey triple product $\langle x_1, x_2, x_3 \rangle$ of x_1, x_2 and x_3. Due to the ambiguity of v_i, $i = 1, 2$, the Massey triple product $\langle x_1, x_2, x_3 \rangle$ is a representative of the quotient group

$$H^{r_1+r_2+r_3-1}(M)/([x_1]H^{r_2+r_3-1}(M) + H^{r_1+r_2-1}(M)[x_3]).$$

Definition 5.1 *Let $(\Omega^*(M), d)$ be de Rham complex, and x_1, x_2, $\cdots$, x_n be closed differential forms on M with $[x_i] \in H^{r_i}(M)$. A collection of forms, $A = (a_{i,j})$ for $1 \leq i \leq j \leq k$ and $(i, j) \neq (1, n)$, is said to be a defining system for the n-fold Massey product $\langle x_1, x_2, \cdots, x_n \rangle$ if*

(1) $a_{i,j} \in \Omega^{r_i + r_{i+1} + \cdots + r_j - j + i}(M)$,

(2) $a_{i,i} = x_i$ *for* $i = 1, 2, \cdots, k$,

(3) $d(a_{i,j}) = \sum\limits_{r=i}^{j-1} \overline{a}_{i,r} \wedge a_{r+1,j}$.

The $(r_1 + \cdots + r_n - n + 2)$-dimensional cocycle, $c(A)$, defined by

$$c(A) = \sum_{r=1}^{n-1} \overline{a}_{1,r} \wedge a_{r+1,n} \in \Omega^{r_1+\cdots+r_n-n+2}(M) \tag{5.2}$$

is called the related cocycle of the defining system A.

Remark 5.1 There is a unique matrix associated to each defining system A as follows:

$$\begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & \cdots & a_{1,n-2} & a_{1,n-1} & \\ & a_{2,2} & a_{2,3} & \cdots & a_{2,n-2} & a_{2,n-1} & a_{2,n} \\ & & a_{3,3} & \cdots & a_{3,n-2} & a_{3,n-1} & a_{3,n} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & a_{n-2,n-2} & a_{n-2,n-1} & a_{n-2,n} \\ & & & & & a_{n-1,n-1} & a_{n-1,n} \\ & & & & & & a_{n,n} \end{pmatrix}_{n\times n}.$$

Definition 5.2 *The n-fold Massey product $\langle x_1, x_2, \cdots, x_n\rangle$ is said to be defined, if there is a defining system for it. If it is defined, then $\langle x_1, x_2, \cdots, x_n\rangle$ consists of all classes $w \in H^{r_1+r_2+\cdots+r_n-n+2}(M)$ for which there exists a defining system A, such that c(A) represents w.*

Remark 5.2 There is an inherent ambiguity in the definition of the Massey product arising from the choices of defining systems. In general, the n-fold Massey product may or may not be a coset of a subgroup, but its indeterminacy is a subset of a matrix Massey product (see [10, Section 2]).

Based on Theorems 4.1–4.2, we have the following lemma on defining systems for the two Massey products we consider in this paper.

Lemma 5.1 (1) *For $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$ $(t \geq 1)$, there are defining systems for $\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle$ obtained from Theorem* 4.1*.*

(2) *For $[x_p]_{2t+3} \in E_{2t+3}^{p,q}$, when $t = ls - 1$ $(l \geq 2)$, there are defining systems for $\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p\rangle$ obtained from Theorem* 4.2*.*

Proof (1) From Theorem 4.1, there exist $x_{p+2j}^{(t)}$ $(1 \leq j \leq t)$, such that $y_{p+2i+1} = 0$ $(0 \leq i \leq t)$ and $d_{2t+3}[x_p]_{2t+3} = \big[\sum\limits_{i=1}^{t} H_{2i+1} \wedge x_{p+2(t-i+1)}^{(t)} + H_{2t+3} \wedge x_p\big]_{2t+3}$. By Theorem 4.1 and (4.2), there exists a defining system $A = (a_{i,j})$ for $\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle$ as follows:

$$\begin{cases} a_{t+2,t+2} = x_p, & \\ a_{i,i+k} = (-1)^k H_{2k+3}, & 1 \leq i \leq t+1-k,\ 0 \leq k < t, \\ a_{i,t+2} = (-1)^{t+2-i} x_{p+2(t+2-i)}^{(t)}, & 2 \leq i \leq t+1, \end{cases} \tag{5.3}$$

to which the matrix associated is given by

$$\begin{pmatrix} H_3 & -H_5 & H_7 & \cdots & (-1)^{t-1}H_{2t+1} & (-1)^t H_{2t+3} & \\ & H_3 & -H_5 & \cdots & (-1)^{t-2}H_{2t-1} & (-1)^{t-1}H_{2t+1} & (-1)^t x_{p+2t}^{(t)} \\ & & H_3 & \cdots & (-1)^{t-3}H_{2t-3} & (-1)^{t-2}H_{2t-1} & (-1)^{t-1} x_{p+2t-2}^{(t)} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & H_3 & -H_5 & (-1)^2 x_{p+4}^{(t)} \\ & & & & & H_3 & -x_{p+2}^{(t)} \\ & & & & & & x_p \end{pmatrix}_{(t+2)\times(t+2)}. \tag{5.4}$$

The desired result follows.

(2) By Theorem 4.2, there exist $x_{p+2is} = x_{p+2is}^{(l-1)}$, $x_{p+2(i-1)s+2j} = 0$ and $x_{p+2(l-1)s+2k} = 0$ for $1 \leq i \leq l-1$, $1 \leq j \leq s-1$ and $1 \leq k \leq s-1$, such that $y_{p+2i+1} = 0$ $(0 \leq i \leq t)$ and $d_{2t+3}[x_p]_{2t+3} = [H_{2s+1} \wedge x_{p+2(l-1)s}^{(l-1)}]_{2t+3}$. By Theorem 4.2 and (4.15), there also exists a defining

system $A=(a_{i,j})$ for $\langle\underbrace{H_{2s+1},\cdots,H_{2s+1}}_{l},x_p\rangle$ as follows:

$$\begin{cases} a_{i,j}=0, & 1\le i<j\le l,\\ a_{i,i}=H_{2s+1}, & 1\le i\le l,\\ a_{l+1,l+1}=x_p, & \\ a_{i,l+1}=(-1)^{l+1-i}x^{(l-1)}_{p+2(l+1-i)s}, & 2\le i\le l, \end{cases} \tag{5.5}$$

to which the matrix associated is given by

$$\begin{pmatrix} H_{2s+1} & 0 & 0 & \cdots & 0 & 0 & \\ & H_{2s+1} & 0 & \cdots & 0 & 0 & (-1)^{l-1}x^{(l-1)}_{p+2(l-1)s}\\ & & H_{2s+1} & \cdots & 0 & 0 & (-1)^{l-2}x^{(l-1)}_{p+2(l-2)s}\\ & & & \ddots & \vdots & \vdots & \vdots\\ & & & & H_{2s+1} & 0 & (-1)^2x^{(l-1)}_{p+4s}\\ & & & & & H_{2s+1} & (-1)x^{(l-1)}_{p+2s}\\ & & & & & & x_p \end{pmatrix}_{(l+1)\times(l+1)}. \tag{5.6}$$

The desired result follows.

To obtain our desired theorems by specific elements of Massey products, we restrict the allowable choices of defining systems for the two Massey products in Lemma 5.1 (see [14]). By Lemma 5.1, we give the following definitions.

Definition 5.3 (1) *Given a class* $[x_p]_{2t+3}\in E^{p,q}_{2t+3}$ $(t\ge 1)$*, a specific element of* $(t+2)$*-fold Massey product* $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle$*, denoted by* $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_A$*, is a class in* $H^{p+2t+3}(M)$ *represented by* $c(A)$*, where* A *is a defining system obtained from Theorem* 4.1*. We define the* $(t+2)$*-fold allowable Massey product* $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_\star$ *to be the set of all the cohomology classes* $w\in H^{p+2t+3}(M)$ *for which there exists a defining system* A *obtained from Theorem* 4.1*, such that* $c(A)$ *represents* w*.*

(2) *Similarly, given a class* $[x_p]_{2t+3}\in E^{p,q}_{2t+3}$ $(t\ge 1)$*, when* $t=ls-1$ $(l\ge 2)$*, we define the specific element of* $(l+1)$*-fold Massey product* $\langle\underbrace{H_{2s+1},\cdots,H_{2s+1}}_{l},x_p\rangle$ *and the* $(l+1)$*-fold allowable Massey product* $\langle\underbrace{H_{2s+1},\cdots,H_{2s+1}}_{l},x_p\rangle_\star$ *by replacing Theorem* 4.1 *by Theorem* 4.2 *in* (1)*.*

Remark 5.3 (1) From Definition 5.3, we can get the following:

$$\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_\star\subseteq\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle.$$

(2) The allowable Massey product $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_\star$ is less ambiguous than the general Massey product $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle$. Take $\langle H_3,H_3,x_p\rangle_\star$ in Definition 5.3 for example. Suppose

$H = \sum_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$. By Theorem 4.1 and (4.2), there exist $x^{(1)}_{p+2j}$, such that $y_{p+2i+1} = 0$ $(0 \le i \le 1)$ and $d_5[x_p]_5 = [H_3 \wedge x^{(1)}_{p+2} + H_5 \wedge x_p]_5$. By Lemma 5.1, we get a defining system A for $\langle H_3, H_3, x_p\rangle$ and its related cocycle $c(A) = -H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p$. Thus, we have

$$\langle H_3, H_3, x_p\rangle_A = [-H_3 \wedge x^{(1)}_{p+2} - H_5 \wedge x_p]. \tag{5.7}$$

Obviously, the indeterminacy of the allowable Massey product $\langle H_3, H_3, x_p\rangle_\star$ is $[H_3]H^{p+2}(M)$. However, in the general case, the indeterminacy of the Massey product $\langle H_3, H_3, x_p\rangle$ is $[H_3]H^{p+2}(M) + H^5(M)[x_p]$.

Similarly, the allowable Massey product $\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p\rangle_\star$ is less ambiguous than the general Massey product $\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p\rangle$.

Now we begin to prove our main theorems.

Proof of Theorem 1.1 By Lemma 5.1(1), there exist defining systems for $\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle$ given by Theorem 4.1. For any defining system $A = (a_{i,j})$ given by Theorem 4.1, by (5.4), we have

$$c(A) = (-1)^t \Big(\sum_{i=1}^{t} H_{2i+1} \wedge x^{(t)}_{p+2(t-i+1)} + H_{2t+3} \wedge x_p \Big).$$

By Definition 5.3, we have

$$\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle_A = [c(A)]. \tag{5.8}$$

Then by Theorem 4.1, we have

$$\begin{aligned} d_{2t+3}[x_p]_{2t+3} &= \Big[\sum_{i=1}^{t} H_{2i+1} \wedge x^{(t)}_{p+2(t-i+1)} + H_{2t+3} \wedge x_p \Big]_{2t+3} \\ &= (-1)^t [\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle_A]_{2t+3}. \end{aligned}$$

Thus, we have $d_{2t+3}[x_p]_{2t+3} = (-1)^t [\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle_A]_{2t+3}$.

By the arbitrariness of A, we have that $[\langle \underbrace{H_3, \cdots, H_3}_{t+1}, x_p\rangle_A]_{2t+3}$ is independent of the choice of the defining system A obtained by Theorem 4.1.

Example 5.1 For formal manifolds, which are manifolds with vanishing Massey products, it is easy to get

$$E_4^{p,q} \cong E_\infty^{p,q}$$

by Theorem 1.1. Note that simply connected compact Kähler manifolds are an important class of formal manifolds (see [2]).

Remark 5.4 (1) From the proof of Theorem 1.1, we have that the specific element

$$\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_A$$

represents a class in $E^{*,*}_{2t+3}$. For two different defining systems A_1 and A_2 given by Theorem 4.1, we have

$$\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_{A_1}\neq\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_{A_2}$$

generally. However, in the spectral sequence (1.2), we have

$$[\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_{A_1}]_{2t+3}=[\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_{A_2}]_{2t+3}.$$

(2) Since the indeterminacy of $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_\star$ does not affect our results, we will not analyze the indeterminacy of Massey products in this paper.

(3) By Theorem 1.1, $d_{2t+3}[x_p]_{2t+3}=(-1)^t[\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_A]_{2t+3}$ for $t\geq 1$, which is expressed only by H_3 and x_p. From the proof of Theorem 1.1, we know that the above expression conceals some information, because the other H_{2i+1}'s affect the result implicitly.

We have the following corollary (see [1, Proposition 6.1]).

Corollary 5.1 *For $H=H_3$ only and $[x_p]_{2t+3}\in E^{p,q}_{2t+3}$ $(t\geq 1)$, we have that in the spectral sequence* (1.2),

$$d_{2t+3}[x_p]_{2t+3}=(-1)^t[\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_A]_{2t+3},$$

and $[\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle_A]_{2t+3}$ is independent of the choice of the defining system A obtained from Theorem 4.1.

Remark 5.5 (1) Because the definition of Massey products is different from the definition in [1], the expression of differentials in Corollary 5.1 differs from the one in [1, Proposition 6.1].

(2) The two specific elements of $\langle\underbrace{H_3,\cdots,H_3}_{t+1},x_p\rangle$ in Theorem 1.1 and Corollary 5.1 are completely different, and equal $[c(A_1)]$ and $[c(A_2)]$, respectively, where $c(A_i)$ $(i=1,2)$ are related cocycles of the defining systems A_i $(i=1,2)$ obtained from Theorem 4.1. The matrices associated to the two defining systems are given by

$$\begin{pmatrix}
H_3 & -H_5 & H_7 & \cdots & (-1)^{t-1}H_{2t+1} & (-1)^tH_{2t+3} & \\
 & H_3 & -H_5 & \cdots & (-1)^{t-2}H_{2t-1} & (-1)^{t-1}H_{2t+1} & (-1)^t x^{(t)}_{p+2t}\\
 & & H_3 & \cdots & (-1)^{t-3}H_{2t-3} & (-1)^{t-2}H_{2t-1} & (-1)^{t-1}x^{(t)}_{p+2t-2}\\
 & & & \ddots & \vdots & \vdots & \vdots\\
 & & & & H_3 & -H_5 & (-1)^2x^{(t)}_{p+4}\\
 & & & & & H_3 & (-1)x^{(t)}_{p+2}\\
 & & & & & & x_p
\end{pmatrix}_{(t+2)\times(t+2)}$$

and

$$\begin{pmatrix} H_3 & 0 & 0 & \cdots & 0 & 0 & \\ & H_3 & 0 & \cdots & 0 & 0 & (-1)^t x^{(t)}_{p+2t} \\ & & H_3 & \cdots & 0 & 0 & (-1)^{t-1} x^{(t)}_{p+2t-2} \\ & & & \ddots & \vdots & \vdots & \vdots \\ & & & & H_3 & 0 & (-1)^2 x^{(t)}_{p+4} \\ & & & & & H_3 & (-1) x^{(t)}_{p+2} \\ & & & & & & x_p \end{pmatrix}_{(t+2)\times(t+2)},$$

respectively. Here $x^{(t)}_{p+2i}$ $(1 \leq i \leq t)$ in the first matrix are different from those in the second one.

For $H = H_{2s+1}$ $(s \geq 2)$ only (i.e., in the case $H_i = 0$, $i \neq 2s+1$) and $[x_p]_{2t+3} \in E^{p,q}_{2t+3}$ $(t \geq 1)$, we make use of Theorem 1.1 to get

$$d_{2t+3}[x_p]_{2t+3} = (-1)^t [\langle \underbrace{0, \cdots, 0}_{t+1}, x_p \rangle_A]_{2t+3}. \tag{5.9}$$

Obviously, some information has been concealed in the expression above. Another description of the differentials for this special case is shown in Theorem 1.2.

Proof of Theorem 1.2 When $t = s-1$, the result follows from Theorem 4.2.

When $t = ls - 1$ $(l \geq 2)$, from Lemma 5.1(2), we know that there exist defining systems for $\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle$ obtained from Theorem 4.2. For any defining system B given by Theorem 4.2, by (5.6), we get $c(B) = (-1)^{l-1} H_{2s+1} \wedge x^{(l-1)}_{p+2(l-1)s}$. By Definition 5.3,

$$\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B = [c(B)]. \tag{5.10}$$

Then by Theorem 4.2, we have

$$\begin{aligned} d_{2t+3}[x_p]_{2t+3} &= [H_{2s+1} \wedge x^{(l-1)}_{p+2(l-1)s}]_{2t+3} \\ &= (-1)^{l-1} [\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B]_{2t+3}. \end{aligned}$$

Thus

$$d_{2t+3}[x_p]_{2t+3} = (-1)^{l-1} [\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B]_{2t+3}.$$

By the arbitrariness of B, we have that $[\langle \underbrace{H_{2s+1}, \cdots, H_{2s+1}}_{l}, x_p \rangle_B]_{2t+3}$ is independent of the choice of the defining system B obtained from Theorem 4.2.

For the rest cases of t, the results follow from Theorem 4.2.

The proof of this theorem is completed.

Remark 5.6 We now use the special case $H = H_5$ and $d_9[x_p]_9$ to illustrate the compatibility between Theorems 1.1 and 1.2 for $s = 2$ and $t = 3$.

Note that in this case, we have $H_3 = 0$ and $H_i = 0$ for $i > 5$. By Theorem 1.1, we get the corresponding matrix associated to the defining system A for $\langle 0, 0, 0, 0, x_p\rangle_A$ is

$$\begin{pmatrix} 0 & -H_5 & 0 & 0 & \\ & 0 & -H_5 & 0 & -x_{p+6}^{(3)} \\ & & 0 & -H_5 & x_{p+4}^{(3)} \\ & & & 0 & -x_{p+2}^{(3)} \\ & & & & x_p \end{pmatrix}_{5\times 5} \tag{5.11}$$

and

$$\widetilde{d}_9[x_p]_9 = -[\langle 0, 0, 0, 0, x_p\rangle_A]_9. \tag{5.12}$$

By Theorem 1.2, in this case, the matrix associated to the defining system B for $\langle H_5, H_5, x_p\rangle_B$ is

$$\begin{pmatrix} H_5 & 0 & \\ & H_5 & -x_{p+4}^{(1)} \\ & & x_p \end{pmatrix}_{3\times 3} \tag{5.13}$$

and

$$\overline{d}_9[x_p]_9 = -[\langle H_5, H_5, x_p\rangle_B]_9. \tag{5.14}$$

We claim that $\langle H_5, H_5, x_p\rangle_\star = \langle 0, 0, 0, 0, x_p\rangle_\star$. For any defining system B above, there is a defining system $\widetilde{B}$

$$\begin{pmatrix} 0 & -H_5 & 0 & 0 & \\ & 0 & -H_5 & 0 & 0 \\ & & 0 & -H_5 & x_{p+4}^{(1)} \\ & & & 0 & 0 \\ & & & & x_p \end{pmatrix}_{5\times 5}$$

for $\langle 0, 0, 0, 0, x_p\rangle$, which can be obtained from Theorem 4.1, such that

$$\langle 0, 0, 0, 0, x_p\rangle_{\widetilde{B}} = \langle H_5, H_5, x_p\rangle_B.$$

Hence $\langle H_5, H_5, x_p\rangle_\star \subseteq \langle 0, 0, 0, 0, [x_p]\rangle_\star$. On the other hand, for any defining system A above, there also exists a defining system $\overline{A}$

$$\begin{pmatrix} H_5 & 0 & \\ & H_5 & -x_{p+4}^{(3)} \\ & & x_p \end{pmatrix}_{3\times 3}$$

for $\langle H_5, H_5, x_p\rangle$, which can be obtained from Theorem 4.2, such that

$$\langle H_5, H_5, x_p\rangle_{\overline{A}} = \langle 0, 0, 0, 0, x_p\rangle_A.$$

Therefore $\langle 0,0,0,0,x_p\rangle_\star \subseteq \langle H_5,H_5,x_p\rangle_\star$, and thus the claim follows.

By Theorem 1.1 and Remark 5.3, we have

$$\widetilde{d}_5[y_p]_5 = -[\langle 0,0,y_p\rangle_A]_5 = -[-H_5\wedge y_p]_5 = [H_5\wedge y_p]_5.$$

By Theorem 1.2, $\overline{d}_5[y_p]_5 = [H_5\wedge y_p]_5$. By Proposition 3.4, $\widetilde{d}_1 = \overline{d}_1 = d$ and $\widetilde{d}_3 = \overline{d}_3 = 0$. It follows that $\widetilde{d}_5 = \overline{d}_5$.

By Theorems 1.1 and 4.1, we have

$$\widetilde{d}_7[z_p]_7 = [\langle 0,0,0,z_p\rangle_A]_7 = [-H_5\wedge z_{p+2}^{(2)}]_7,$$

where $z_{p+2}^{(2)}$ is an arbitrary $(p+2)$-form satisfying $d(z_{p+2}^{(2)}) = 0\wedge z_p$. By Remark 5.4(2), we take $z_{p+2}^{(2)} = 0$. Then we have $\widetilde{d}_7[z_p]_7 = 0$, i.e., $\widetilde{d}_7 = 0$. At the same time, we also have $\overline{d}_7 = 0$ from Theorem 1.2. Thus $\widetilde{d}_7 = \overline{d}_7 = 0$.

By $\widetilde{E}_1^{p,q} = \overline{E}_1^{p,q}$, $\widetilde{d}_i = \overline{d}_i$ for $1\le i\le 7$ and $\langle H_5,H_5,x_p\rangle_\star = \langle 0,0,0,0,x_p\rangle_\star$, we can conclude that $\widetilde{d}_9 = \overline{d}_9$ from (5.12) and (5.14).

6 The Indeterminacy of Differentials in the Spectral Sequence (1.2)

Let $[x_p]_r \in E_r^{p,q}$. The indeterminacy of $[x_p]$ is a normal subgroup G of $H^*(M)$, which means that if there is another element $[y_p]\in H^p(M)$, which also represents the class $[x_p]_r\in E_r^{p,q}$, then $[y_p]-[x_p]\in G$.

In this section, we will show that for $H = \sum\limits_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$ and $[x_p]_{2t+3}$, the indeterminacy of the differential $d_{2t+3}[x_p]\in E_2^{p+2t+3,q-2t-2}$ is a normal subgroup of $H^*(M)$.

From the long exact sequence (3.3), we have a commutative diagram

$$\begin{array}{ccccccccc}
 & & \vdots & & & & \vdots & & \\
 & & \downarrow{\scriptstyle i^*} & & & & \downarrow{\scriptstyle i^*} & & \\
\cdots & \xrightarrow{\delta} & H_D^{p+q}(K_{p+1}) & \xrightarrow{j^*} & H_D^{p+q}(K_{p+1}/K_{p+2}) & \xrightarrow{\delta} & H_D^{p+q+1}(K_{p+2}) & \xrightarrow{j^*} & \cdots \\
 & & \downarrow{\scriptstyle i^*} & & & & \downarrow{\scriptstyle i^*} & & \\
\cdots & \xrightarrow{\delta} & H_D^{p+q}(K_{p}) & \xrightarrow{j^*} & H_D^{p+q}(K_{p}/K_{p+1}) & \xrightarrow{\delta} & H_D^{p+q+1}(K_{p+1}) & \xrightarrow{j^*} & \cdots \\
 & & \downarrow{\scriptstyle i^*} & & & & \downarrow{\scriptstyle i^*} & & \\
\cdots & \xrightarrow{\delta} & H_D^{p+q}(K_{p-1}) & \xrightarrow{j^*} & H_D^{p+q}(K_{p-1}/K_{p}) & \xrightarrow{\delta} & H_D^{p+q+1}(K_{p}) & \xrightarrow{j^*} & \cdots \\
 & & \downarrow{\scriptstyle i^*} & & & & \downarrow{\scriptstyle i^*} & & \\
 & & \vdots & & & & \vdots & &
\end{array} \tag{6.1}$$

in which any sequence consisting of a vertical map i^* followed by two horizontal maps j^* and δ and then a vertical map i^* followed again by j^*, δ, and iteration of this is exact. From this diagram, there is a spectral sequence, in which $E_1^{p,q} = H_D^{p+q}(K_p/K_{p+1})$ and for $r\ge 2$, $E_r^{p,q}$ is defined to be the quotient $Z_r^{p,q}/B_r^{p,q}$, where

$$\begin{aligned} Z_r^{p,q} &= \delta^{-1}(i^{*r-1}H_D^{p+q+1}(K_{p+r})), \\ B_r^{p,q} &= j^*(\ker[i^{*r-1}: H_D^{p+q}(K_p) \to H_D^{p+q}(K_{p-r+1})]). \end{aligned} \tag{6.2}$$

We also have a sequence of inclusions

$$B_2^{p,q} \subset \cdots \subset B_r^{p,q} \subset B_{r+1}^{p,q} \subset \cdots \subset Z_{r+1}^{p,q} \subset Z_r^{p,q} \subset \cdots \subset Z_2^{p,q}. \tag{6.3}$$

By [6–7], the $E_r^{*,*}$-term defined above is the same as the one in the spectral sequence (1.2). A similar argument about a homology spectral sequence is given in [15, p. 472–473].

Theorem 6.1 *Let* $H = \sum\limits_{i=1}^{[\frac{n-1}{2}]} H_{2i+1}$ *and* $[x_p]_r \in E_r^{p,q}$ $(r \geq 3)$. *Then the indeterminacy of* $[x_p] \in E_2^{p,q} \cong H^p(M)$ *is the following normal subgroup of* $H^p(M)$:

$$\frac{\mathrm{im}[\overline{\delta}: H_D^{p+q-1}(K_{p-r+1}/K_p) \to H_D^{p+q}(K_p/K_{p+1})]}{\mathrm{im}[d: \Omega^{p-1}(M) \to \Omega^p(M)]},$$

where d *is just the exterior differentiation, and* $\overline{\delta}$ *is the connecting homomorphism of the long exact sequence induced by the short exact sequence of cochain complexes*

$$0 \longrightarrow K_p/K_{p+1} \xrightarrow{\overline{i}} K_{p-r+1}/K_{p+1} \xrightarrow{\overline{j}} K_{p-r+1}/K_p \longrightarrow 0.$$

Proof From the above tower (6.3), we get a tower of subgroups of $E_2^{p,q}$

$$\begin{aligned} B_3^{p,q}/B_2^{p,q} &\subset \cdots \subset B_r^{p,q}/B_2^{p,q} \subset \cdots \subset Z_r^{p,q}/B_2^{p,q} \\ &\subset Z_3^{p,q}/B_2^{p,q} \subset Z_2^{p,q}/B_2^{p,q} = E_2^{p,q}. \end{aligned}$$

Note

$$E_r^{p,q} \cong (Z_r^{p,q}/B_2^{p,q})/(B_r^{p,q}/B_2^{p,q}).$$

It follows that the indeterminacy of $[x_p]$ is the normal subgroup $B_r^{p,q}/B_2^{p,q}$ of $H^p(M)$.

From the short exact sequences of cochain complexes

$$\begin{aligned} &0 \longrightarrow K_p \xrightarrow{i'} K_{p-r+1} \xrightarrow{j'} K_{p-r+1}/K_p \longrightarrow 0, \\ &0 \longrightarrow K_p/K_{p+1} \xrightarrow{\overline{i}} K_{p-r+1}/K_{p+1} \xrightarrow{\overline{j}} K_{p-r+1}/K_p \longrightarrow 0, \end{aligned}$$

we can get the following long exact sequence of cohomology groups:

$$\begin{aligned} &\cdots \xrightarrow{\delta'} H_D^s(K_p) \xrightarrow{i'^*} H_D^s(K_{p-r+1}) \xrightarrow{j'^*} H_D^s(K_{p-r+1}/K_p) \xrightarrow{\delta'} \cdots, \\ &\cdots \xrightarrow{\overline{\delta}} H_D^s(K_p/K_{p+1}) \xrightarrow{\overline{i}^*} H_D^s(K_{p-r+1}/K_{p+1}) \xrightarrow{\overline{j}^*} H_D^s(K_{p-r+1}/K_p) \xrightarrow{\overline{\delta}} \cdots, \end{aligned} \tag{6.4}$$

where δ' and $\overline{\delta}$ are the connecting homomorphisms.

Combining (3.3) and (6.4), we have the following commutative diagram of long exact sequences:

$$\begin{array}{ccccc} H_D^{p+q-1}(K_{p-r+1}/K_p) & \xrightarrow{\delta'} & H_D^{p+q}(K_p) & \xrightarrow{i'^*} & H_D^{p+q}(K_{p-r+1}) \\ & \searrow^{\overline{\delta}} & \downarrow{j^*} & & \\ & & H_D^{p+q}(K_p/K_{p+1}) & & \\ & & \downarrow{\delta} & \searrow^{\overline{i}^*} & \\ & & H_D^{p+q+1}(K_{p+1}) & & H_D^{p+q}(K_{p-r+1}/K_{p+1}) \end{array} \tag{6.5}$$

Using the above commutative diagram and the fact that $i^{*^{r-1}} = i'^*$, we have

$$\begin{aligned} B_r^{p,q} &= j^*(\ker[i^{*^{r-1}} : H_D^{p+q}(K_p) \to H_D^{p+q}(K_{p-r+1})]) \\ &= j^*(\ker[\ i'^* : H_D^{p+q}(K_p) \longrightarrow H_D^{p+q}(K_{p-r+1})]) \\ &\cong j^*(\mathrm{im}[\delta' : H_D^{p+q-1}(K_{p-r+1}/K_p) \to H_D^{p+q}(K_p)]) \\ &\cong \mathrm{im}[\overline{\delta} : H_D^{p+q-1}(K_{p-r+1}/K_p) \to H_D^{p+q}(K_p/K_{p+1})]. \end{aligned}$$

When $r = 2$, from (6.5), we have

$$\overline{\delta} = \delta' j^* : \quad H_D^{p+q-1}(K_{p-1}/K_p) \to H_D^{p+q}(K_p/K_{p+1}).$$

From (3.4), it follows that $\overline{\delta} = d_1$. By Proposition 3.2, $\overline{\delta} = d$. Thus, we have

$$\begin{aligned} B_2^{p,q} &\cong \mathrm{im}[\overline{\delta} : H_D^{p+q-1}(K_{p-1}/K_p) \to H_D^{p+q}(K_p/K_{p+1})] \\ &\cong \mathrm{im}[d : \Omega^{p-1}(M) \to \Omega^p(M)]. \end{aligned}$$

The desired result follows.

By Theorem 6.1, we obtain the following corollary.

Corollary 6.1 *In Theorem* 1.1, *for* $d_{2t+3}[x_p]_{2t+3} \in E_{2t+3}^{p+2t+3,q-2t-2}$, *we have that the indeterminacy of* $d_{2t+3}[x_p]$ *is a normal subgroup of* $H^{p+2t+3}(M)$

$$\frac{\mathrm{im}[\overline{\delta} : H_D^{p+q}(K_{p+1}/K_{p+2t+3}) \to H_D^{p+q+1}(K_{p+2t+3}/K_{p+2t+4})]}{\mathrm{im}[d : \Omega^{p+2t+2}(M) \to \Omega^{p+2t+3}(M)]},$$

where d *is just the exterior differentiation, and* $\overline{\delta}$ *is the connecting homomorphism of the long exact sequence induced by the short exact sequence of cochain complexes*

$$0 \longrightarrow K_{p+2t+3}/K_{p+2t+4} \xrightarrow{\overline{i}} K_{p+1}/K_{p+2t+4} \xrightarrow{\overline{j}} K_{p+1}/K_{p+2t+3} \longrightarrow 0.$$

Proof In Theorem 6.1, r, p and q are replaced by $2t+3, p+2t+3$ and $q-2t-2$, respectively. Then the desired result follows.

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