Chin. Ann. Math.
35B(4), 2014, 659–668
DOI: 10.1007/s11401-014-0841-0

Chinese Annals of Mathematics, Series B

Global Existence and Pointwise Estimates of Solutions to Generalized Benjamin-Bona-Mahony Equations in Multi Dimensions*

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Abstract This paper is concerned with the global existence and pointwise estimates of solutions to the generalized Benjamin-Bona-Mahony equations in all space dimensions. By using the energy method, Fourier analysis and pseudo-differential operators, the global existence and pointwise convergence rates of the solution are obtained. The decay rate is the same as that of the heat equation and one can see that the solution propagates along the characteristic line.

Keywords Cauchy problem, Generalized Benjamin-Bona-Mahony equations, Multi dimensions, Global existence, Pointwise estimates
2000 MR Subject Classification 35L05

1 Introduction

In this paper, we are interested in the global existence and time-asymptotic behavior of solutions to generalized Benjamin-Bona-Mahony (GBBM) equations in all space dimensions. The GBBM equation is defined as

$$\partial_t u - \triangle\partial_t u - \eta\triangle u + (\beta\cdot\triangledown)u + \operatorname{div} f(u) = 0, \tag{1.1}$$

where $u \in \mathbb{R}^1$, η is a positive constant, and β is a real constant vector. $f(u)=(f_1(u),\cdots,f_n(u))^{\mathrm{T}}$, and $f_i(u) = u^2$, where n is the space dimension. In this paper, $n \geq 1$. The initial data is given by

$$u|_{t=0} = u_0(x). \tag{1.2}$$

The well-known Benjamin-Bona-Mahony (BBM) equation is of the form

$$u_t - u_{xxt} + u_x + uu_x = 0, \quad -\infty < x < \infty,\ t > 0. \tag{1.3}$$

It was proposed and studied in [1] by Benjamin, Bona and Mahony for the special physical situations in the long wave limit for nonlinear dispersive media. Since then, the existence and uniqueness of solutions to various generalized BBM equaitons have been proved by many authors (see [1–4]). The decays of solutions were also studied in [5–9]. However, most of these studies are in low space dimensions and the decay estimates are in L_p norm. The aim of this paper is to give the global existence and pointwise decay rates of solutions to Cauchy problems of the GBBM equation in all space dimensions.

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*This work was supported by the National Natural Science Foundation of China (No. 11101121).

First we introduce some notations. As usual, Fourier transformation to the variable $x \in \mathbb{R}^n$ is

$$\widehat{f}(\xi,t) = \int_{\mathbb{R}^n} f(x,t) \mathrm{e}^{-\mathrm{i}x\cdot\xi} \mathrm{d}x,$$

and the inverse Fourier transformation to the variable ξ is

$$f(x,t) = (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{f}(\xi,t) \mathrm{e}^{\mathrm{i}x\cdot\xi} \mathrm{d}\xi.$$

We also use $F^{-1}(\widehat{f})$ to denote the inverse Fourier transformation of function $\widehat{f}$. $D^\alpha f = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} \cdots \partial_{x_n}^{\alpha_n} f$ for multi-index $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. $W^{s,p}(\mathbb{R}^n),\ s \in Z_+,\ p \in [1,\infty]$, denotes the usual Sobelev space with the norm

$$\|f\|_{W^{s,p}} := \sum_{|\alpha|=0}^{s} \|D^\alpha f\|_{L_p}.$$

In particular, $W^{s,2} = H^s$. We denote the generic constant by C. All the convolutions are about the spatial variable x in this paper.

We arrange this paper as follows. In Section 2, we derive the solution formula of the Cauchy problem. We need many inequalities in our analysis. We list the inequalities and their proofs in Section 3. We construct a solution sequence due to the solution formula of (1.1)–(1.2), and then prove that the sequence is a Cauchy sequence in a Banach space. Thus it converges to the solution of our problem. We leave these treatment processes to Section 4 and Section 5. Finally, we give our conclusion in Section 6.

2 Solution Formula

The aim of this section is to derive the solution formula from the problem (1.1)–(1.2). The linearized form of (1.1) is

$$\partial_t u - \triangle \partial_t u - \eta \triangle u + (\beta \cdot \bigtriangledown) u = 0. \tag{2.1}$$

Taking Fourier transform to variable x of (2.1), we have

$$\partial_t \widehat{u} + \partial_t |\xi|^2 \widehat{u} + \eta |\xi|^2 \widehat{u} + \mathrm{i}(\beta \cdot \xi) \widehat{u} = 0. \tag{2.2}$$

The corresponding initial data is given by

$$\widehat{u}|_{t=0} = \widehat{u}_0(\xi). \tag{2.3}$$

The solution to the problem (2.2)–(2.3) is given by

$$\widehat{u}(\xi,t) = \mathrm{e}^{-\frac{\eta|\xi|^2}{1+|\xi|^2}t - \frac{\mathrm{i}\beta\cdot\xi t}{1+|\xi|^2}} \widehat{u}_0. \tag{2.4}$$

Set $\widehat{G}(\xi,t) = \mathrm{e}^{-\frac{\eta|\xi|^2}{1+|\xi|^2}t - \frac{\mathrm{i}\beta\cdot\xi t}{1+|\xi|^2}}$. By the Duhamel principle, we get the solution formula for (1.1)–(1.2):

$$u(x,t) = G * u_0 - \int_0^t G(t-s) * (I - \triangle)^{-1} \mathrm{div} f(u)(s) \mathrm{d}s.$$

Set $\widehat{H}(\xi,t) = \frac{\widehat{G}}{1+|\xi|^2}$. Then

$$u(x,t) = G * u_0 - \int_0^t H(t-s) * \mathrm{div}\, f(u)(s) \mathrm{d}s. \tag{2.5}$$

Due to (2.5), we define a solution sequence $\{u^{(m)}(x,t)\}$ satisfying

$$\begin{cases} u^{(m)}(x,t) = G * u_0 - \displaystyle\int_0^t H(t-s) * \operatorname{div} f(u^{(m-1)})(s)\mathrm{d}s, \\ u^{(m)}(x,0) = u_0(x), \quad u^{(0)}(x,t) = 0, \end{cases} \tag{2.6}$$

where $m \geq 1$. Next we will prove that $\{u^{(m)}(x,t)\}$ is a Cauchy sequence in a Banach space, and then it converges to the solution to (1.1)–(1.2). To do so, we shall need many inequalities. We collect them in the following section.

3 Preliminaries

In order to estimate $u^{(m)}(x,t)$, we must analyse the decay property for G, H first. Set

$$\chi_1(\xi) = \begin{cases} 1, & |\xi| \leq R, \\ 0, & |\xi| > R+1, \end{cases} \qquad \chi_2(\xi) = \begin{cases} 1, & |\xi| \geq R+1, \\ 0, & |\xi| < R, \end{cases}$$

where χ_1, χ_2 are smooth cut-off functions and $\chi_1(\xi) + \chi_2(\xi) = 1$.

Set $\widehat{G}_i = \chi_i \widehat{G}$, $\widehat{H}_i = \chi_i \widehat{H}$ for $i = 1, 2$.

For G_1, H_1, we have the decay property as follows.

Lemma 3.1 *There exists a constant $C_{N,\alpha}$ depending on N, α such that*

$$|D_x^\alpha G_1| \leq C_{N,\alpha}(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{2}} \frac{1}{(1+\frac{|x-\beta t|^2}{1+t})^N}, \tag{3.1}$$

$$|D_x^\alpha H_1| \leq C_{N,\alpha}(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{2}} \frac{1}{(1+\frac{|x-\beta t|^2}{1+t})^N}, \tag{3.2}$$

where N is a positive integer. Throughout this paper $N > \frac{n}{2}$.

Conveniently, we next denote $B_N(x,t) = \frac{1}{(1+\frac{|x|^2}{1+t})^N}$.

Proof When $|\xi|$ is bounded, using the Taylor expansion, we have

$$-\frac{\eta|\xi|^2}{1+|\xi|^2} = -\eta|\xi|^2 + O(|\xi|)^4,$$

$$-\frac{\mathrm{i}\beta\cdot\xi}{1+|\xi|^2} = -\mathrm{i}\beta\cdot\xi(1-|\xi|^2+O(|\xi|^4)).$$

Then

$$\widehat{G}_1(\xi,t) = \mathrm{e}^{-\eta|\xi|^2 t - \mathrm{i}\beta\cdot\xi t + O(|\xi|^3)t}, \tag{3.3}$$

$$\widehat{H}_1(\xi,t) = \mathrm{e}^{-\eta|\xi|^2 t - \mathrm{i}\beta\cdot\xi t + O(|\xi|^3)t}(1-|\xi|^2+O(|\xi|^3)). \tag{3.4}$$

We know that $\mathrm{e}^{-\mathrm{i}\beta\cdot\xi t}$ is a parallel operator. It can not contribute to the decay factor, but it has a physical meaning. So next we will not neglect the effect of the operator. From (3.3)–(3.4), we get

$$|D_\xi^\beta \xi^\alpha(\widehat{G}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t})| \leq C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|}t^{\frac{|\beta|}{2}})(1+|\xi|^2 t)^{1+|\beta|}\mathrm{e}^{-\eta|\xi|^2 t},$$

$$|D_\xi^\beta \xi^\alpha(\widehat{H}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t})| \leq C(|\xi|^{(|\alpha|-|\beta|)_+} + |\xi|^{|\alpha|}t^{\frac{|\beta|}{2}})(1+|\xi|^2 t)^{1+|\beta|}\mathrm{e}^{-\eta|\xi|^2 t},$$

where $(|\alpha|-|\beta|)_+ := \begin{cases} |\alpha|-|\beta|, & \text{if } |\alpha|\ge|\beta|,\\ 0, & \text{if } |\alpha|<|\beta|. \end{cases}$ From [10, Lemma 3.1], we have

$$\begin{aligned}|D_x^\alpha F^{-1}(\widehat{G}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t})| &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_N(x,t),\\ |D_x^\alpha F^{-1}(\widehat{H}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t})| &\le C(1+t)^{-\frac{n+|\alpha|}{2}}B_N(x,t).\end{aligned}$$

Then

$$\begin{aligned}|D_x^\alpha G_1(x,t)| &= |D_x^\alpha F^{-1}(\widehat{G}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t}\mathrm{e}^{-\mathrm{i}\beta\cdot\xi t})|\\ &= |D_x^\alpha F^{-1}(\widehat{G}_1(\xi,t)\mathrm{e}^{\mathrm{i}\beta\cdot\xi t})|(x-\beta t,t)\\ &\le C_{N,\alpha}(1+t)^{-\frac{|\alpha|}{2}-\frac{n}{2}}B_N(x-\beta t,t).\end{aligned}$$

Similarly we get (3.2).

G_2, H_2 have the construction as follows.

Lemma 3.2 *There exists a positive constant b and distributions $g_1^i(x), g_2^i(x)$ for $i=1,2$ such that*

$$\begin{aligned}|G_2(x,t)| &\le C\mathrm{e}^{-bt}(g_1^1(x)+g_2^1(x)+C_0\delta(x)),\\ |\bigtriangledown H_2(x,t)| &\le C\mathrm{e}^{-bt}(g_1^2(x)+g_2^2(x)+C_1\delta(x)),\end{aligned}$$

where $\delta(x)$ is Dirac function and

$$|D_x^\alpha g_1^i(x)| \le C_{N,\alpha}(1+|x|^2)^{-N}, \tag{3.5}$$

$$\|g_2^i\|_{L^1} \le C, \quad \operatorname{supp} g_2^i(x) \subset \{x, |x|<2\epsilon\} \tag{3.6}$$

with ϵ being sufficiently small.

Proof When $|\xi|$ is large enough, using Taylor expansions, we have

$$\begin{aligned}-\frac{\eta|\xi|^2}{1+|\xi|^2} &= -\eta\Big(1-\frac{1}{|\xi|^2}+O\Big(\frac{1}{|\xi|^4}\Big)\Big),\\ \frac{\mathrm{i}\beta\cdot\xi}{1+|\xi|^2} &= \mathrm{i}\beta\cdot\xi\frac{1}{|\xi|^2}\Big(1-\frac{1}{|\xi|^2}+O\Big(\frac{1}{|\xi|^4}\Big)\Big).\end{aligned}$$

Thus we have

$$\begin{aligned}\widehat{G}_2(\xi,t) &= \mathrm{e}^{-\eta t}\mathrm{e}^{\eta(\frac{1}{|\xi|^2}+O(\frac{1}{|\xi|^4}))t}\mathrm{e}^{-\mathrm{i}\beta\cdot\xi\frac{1}{|\xi|^2}(1-\frac{1}{|\xi|^2}+O(\frac{1}{|\xi|^4}))t},\\ \xi\widehat{H}_2(\xi,t) &= \mathrm{e}^{-\eta t}\mathrm{e}^{\eta(\frac{1}{|\xi|^2}+O(\frac{1}{|\xi|^4}))t}\mathrm{e}^{-\mathrm{i}\beta\cdot\xi\frac{1}{|\xi|^2}(1-\frac{1}{|\xi|^2}+O(\frac{1}{|\xi|^4}))t}\frac{\xi}{|\xi|^2}\Big(1-\frac{1}{|\xi|^2}+O\Big(\frac{1}{|\xi|^4}\Big)\Big).\end{aligned} \tag{3.7}$$

Then there exists a positive constant b such that

$$\begin{aligned}|\widehat{G}_2(\xi,t)| &\le C, & |\xi\widehat{H}_2(\xi,t)| &\le C,\\ |D_\xi^\gamma\widehat{G}_2| &\le C|\xi|^{-1-|\gamma|}\mathrm{e}^{-bt}, & |D_\xi^\gamma(\xi\widehat{H}_2)| &\le C|\xi|^{-1-|\gamma|}\mathrm{e}^{-bt}\end{aligned}$$

with $|\gamma|\ge 1$. From [10, Lemma 3.2], we get our results.

When dealing with the convolution with the nonlinearized part, we need the following four lemmas.

Lemma 3.3 *For positive constants b, N, when t is large enough, we have*

$$\mathrm{e}^{-bt}(1+|x|^2)^{-N} \le C(1+t)^{-\frac{n+|\alpha|}{2}}B_N(x-\beta t,t).$$

Proof Noticing that $|x-\beta t|^2 \le 2|x|^2 + 2(|\beta|t)^2$, if N, t are large enough, we have

$$B_N(x-\beta t,t) \ge \frac{C}{(1+\frac{|x|^2+(|\beta|t)^2}{1+t})^N} \ge \frac{C}{(1+|x|^2+|\beta|^2t)^N} \ge C(1+|x|^2)^{-N}|\beta|^{-2N}t^{-N}.$$

Thus we get our lemma.

Lemma 3.4 *When $n_1, n_2 > \frac{n}{2}$ and $n_3 = \min\{n_1, n_2\}$, we have*

$$\int_{\mathbb{R}^n} \Big(1+\frac{|x-y|^2}{1+t}\Big)^{-n_1}(1+|y|^2)^{-n_2}\mathrm{d}y \le C\Big(1+\frac{|x|^2}{1+t}\Big)^{-n_3}, \tag{3.8}$$

$$\int_{\mathbb{R}^n} (1+|x-y|^2)^{-n_1}(1+|y|^2)^{-n_2}\mathrm{d}y \le C(1+|x|^2)^{-n_3}. \tag{3.9}$$

Proof We just prove (3.8). The proof of (3.9) is similar.
When $|x-y| \ge \frac{|x|}{2}$, (3.8) is easily got.
When $|x-y| < \frac{|x|}{2}$, we have $|y| \ge \frac{|x|}{2}$. Thus

$$(1+|y|^2)^{-n_2} \le C(1+|x|^2)^{-n_3}.$$

If $|x| \le \sqrt{t}$, we have $B_{n_3}(x,t) \ge C$. (3.8) is easily got.
If $|x| > \sqrt{t}$, we have

$$\frac{1+|x|^2}{1+t} \ge \frac{1}{2}\frac{1+t+|x|^2}{1+t}.$$

Thus

$$\begin{aligned}\Big(1+\frac{|x-y|^2}{1+t}\Big)^{-n_1}(1+|y|^2)^{-n_2} &\le C\Big(\frac{1+|x-y|^2}{1+t}\Big)^{-n_3}(1+|x|^2)^{-n_2}\\ &\le C(1+|x-y|^2)^{-n_3}(1+t)^{n_3}(1+|x|^2)^{-n_3}\\ &\le C(1+|x-y|^2)^{-n_3}\Big(1+\frac{|x|^2}{1+t}\Big)^{-n_3}.\end{aligned}$$

Then (3.8) is proved.

Set

$$\begin{aligned}\theta &= (1+t-s)^{-\frac{n+1}{2}}(1+s)^{-n},\\ P &= B_N(x-y-\beta(t-s),t-s)B_{2[\frac{n}{2}]+2}(y-\beta s,s).\end{aligned}$$

Lemma 3.5

$$\int_0^t \Big[\int_{\mathbb{R}^n} \theta P \mathrm{d}y\Big]\mathrm{d}s \le C(1+t)^{-\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).$$

Proof We divide the proof into two different cases.
Case 1 $|x-\beta t| \le \sqrt{t}$.
In this case, $B_{[\frac{n}{2}]+1}(x-\beta t,t) \ge C$. From [11, Lemma 5.2], we have

$$\begin{aligned}\int_0^t\int_{\mathbb{R}^n} \theta P\mathrm{d}y\mathrm{d}s &\le \int_0^{\frac{t}{2}} \theta(1+s)^{\frac{n}{2}}\mathrm{d}s + \int_{\frac{t}{2}}^t \theta(1+t-s)^{\frac{n}{2}}\mathrm{d}s\\ &\le C(1+t)^{-\frac{n}{2}}\\ &\le C(1+t)^{-\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).\end{aligned}$$

Case 2 $|x-\beta t| \ge \sqrt{t}$.

We also divide the proof into two different cases.

Case 2.1 $s \le \frac{t}{2}$.

If $|y-\beta s| \ge \frac{|x-\beta t|}{2}$, then

$$B_{2[\frac{n}{2}]+2}(y-\beta s, s) \le \frac{1}{(\frac{|y-\beta s)|^2}{1+s})^{2[\frac{n}{2}]+2}} \le C\frac{1}{(\frac{|x-\beta t|^2}{1+t})^{[\frac{n}{2}]+1}}\Big(\frac{1+s}{1+t}\Big)^{\frac{n}{2}} \le CB_{[\frac{n}{2}]+1}(x-\beta t, t)\Big(\frac{1+s}{1+t}\Big)^{\frac{n}{2}}.$$

From [11, Lemma 5.2], we have

$$\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}\theta P \mathrm{d}y\mathrm{d}s \le C\int_0^{\frac{t}{2}}[\theta(1+t-s)^{\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t, t)]\Big(\frac{1+s}{1+t}\Big)^{\frac{n}{2}}\mathrm{d}s \le C(1+t)^{-\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t, t).$$

If $|y-\beta s| \le \frac{|x-\beta t|}{2}$, then $|x-y-\beta(t-s)| \ge \frac{|x-\beta t|}{2}$. We have

$$\int_0^{\frac{t}{2}}\int_{\mathbb{R}^n}\theta P \mathrm{d}y\mathrm{d}s \le C\int_0^{\frac{t}{2}}\Big[\int_{\mathbb{R}^n}\theta(1+s)^{\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t, t)\mathrm{d}y\Big]\mathrm{d}s \le C(1+t)^{-\frac{n}{2}}B_{[\frac{n}{2}]+1}(x-\beta t, t).$$

Case 2.2 $s \ge \frac{t}{2}$.

The proof of this part is similar to that of Case 2.1, so we omit it here.

Lemma 3.6 *There exists a constant depending only on n such that*

$$B_n(x-\beta s, s) \le CB_n(x-\beta t, t)(1+t-s)^n.$$

Proof Since $|x-\beta t|^2 \le 2(|x-\beta s|^2 + |\beta|^2(t-s)^2)$, we have

$$B_n(x-\beta t, t) \ge C\Big(1+\frac{|x-\beta s|^2+|\beta|^2(t-s)^2}{1+t}\Big)^{-n} \ge C\Big(1+\frac{|x-\beta s|^2}{1+t}+|\beta|^2(t-s)\Big)^{-n} \ge C\min\{(t-s)^{-n}, 1\}B_n(x-\beta s, s).$$

Thus we get our lemma.

We can now enter into the estimate of the sequence $\{u^{(m)}(x,t)\}$.

4 Estimate of the Sequence

We first give an estimate for $u^{(1)}(x,t)$, and then use mathematical induction to get the estimate for $\{u^{(m)}(x,t)\}$.

Lemma 4.1 *If $u_0 \in H^l$, $l > 1+[\frac{n}{2}]$, and $\|u_0\|_{H^l} = E$, $|u_0| \le CE(1+|x|^2)^{-1-[\frac{n}{2}]}$ with E small enough, then we have*

$$|D_x^\alpha u^{(1)}(x,t)| \le CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t, t) \quad \text{for } |\alpha| < l-1-\Big[\frac{n}{2}\Big].$$

Proof From (2.6), we have

$$D_x^\alpha u^{(1)}(x,t) = D_x^\alpha G_1(x,t) * u_0 + D_x^\alpha G_2(x,t) * u_0. \tag{4.1}$$

From (3.1) and Lemma 3.4, we have

$$\begin{aligned}|D_x^\alpha G_1(x,t) * u_0| &\le CE \int (1+t)^{-\frac{n+|\alpha|}{2}} \frac{1}{(1+\frac{|x-y-\beta t|^2}{1+t})^N} (1+|y|^2)^{-[\frac{n}{2}]-1} \mathrm{d}y \\ &\le CE(1+t)^{-\frac{n+|\alpha|}{2}} \Big(1+\frac{|x-\beta t|^2}{1+t}\Big)^{-[\frac{n}{2}]-1}. \end{aligned}\tag{4.2}$$

From (3.7), when $|\gamma| > \frac{n}{2}$, we have

$$\begin{aligned}|x^\gamma D_x^\alpha G_2(x,t) * u_0| &\le C \int_{\mathbb{R}^n \cap \{\xi; |\xi|>R\}} |D_\xi^\gamma \xi^\alpha \widehat{G}_2 \widehat{u}_0| \mathrm{d}\xi \\ &\le C\mathrm{e}^{-\frac{\eta t}{2}} \int_{\mathbb{R}^n \cap \{\xi; |\xi|>R\}} |\xi|^{|\alpha|-|\gamma|} |\widehat{u}_0| \mathrm{d}\xi \\ &\le C\mathrm{e}^{-\frac{\eta t}{2}} \Big(\int |\xi|^{2|\alpha|} |\widehat{u}_0|^2 \mathrm{d}\xi\Big)^{\frac{1}{2}} \Big(\int |\xi|^{-2|\gamma|} \mathrm{d}\xi\Big)^{\frac{1}{2}} \\ &\le CE\mathrm{e}^{-\frac{\eta t}{2}}. \end{aligned}\tag{4.3}$$

If $|\alpha| < l - 1 - \left[\frac{n}{2}\right]$, we have

$$\|D_x^\alpha u_0\|_{L_\infty} \le C\|u_0\|_{H^l}. \tag{4.4}$$

From Lemma 3.2 and (4.4), we have

$$\begin{aligned}|D_x^\alpha G_2(x,t) * u_0| &= |G_2 * D_x^\alpha u_0| \le \|G_2\|_{L_1} \|D_x^\alpha u_0\|_{L_\infty} \\ &\le C\mathrm{e}^{-bt} \|D_x^\alpha u_0\|_{L_\infty} \le C\mathrm{e}^{-bt} \|u_0\|_{H^l}. \end{aligned}\tag{4.5}$$

Taking $|\gamma| = 2N$ in (4.3), from (4.3) and (4.5), we have

$$|D_x^\alpha G_2(x,t) * u_0| \le CE\mathrm{e}^{-bt}(1+|x|^2)^{-N}.$$

From Lemma 3.3, we get

$$|D_x^\alpha G_2(x,t) * u_0| \le CE(1+t)^{-\frac{n+|\alpha|}{2}} B_{[\frac{n}{2}]+1}(x-\beta t, t). \tag{4.6}$$

From (4.2) and (4.6), we get our result.

Lemma 4.2 *For $m > 1$, if*

$$|D_x^\alpha u^{(m-1)}| \le CE(1+t)^{-\frac{n+|\alpha|}{2}} B_{[\frac{n}{2}]+1}(x-\beta t, t),$$

then

$$|D_x^\alpha u^{(m)}| \le CE(1+t)^{-\frac{n+|\alpha|}{2}} B_{[\frac{n}{2}]+1}(x-\beta t, t).$$

Proof Because $f_i(u) = u^2$, we have

$$\begin{aligned}|D_x^\alpha f_i(u^{(m-1)})| &\le \sum_{|\alpha_1|+|\alpha_2|=|\alpha|} |D_x^{\alpha_1} u^{(m-1)} D_x^{\alpha_2} u^{(m-1)}| \\ &\le CE(1+t)^{-n-\frac{|\alpha|}{2}} B_{2[\frac{n}{2}]+2}(x-\beta t, t). \end{aligned}\tag{4.7}$$

We still denote

$$\theta=(1+t-s)^{-\frac{n+1}{2}}(1+s)^{-n},$$
$$P=B_N(x-y-\beta(t-s),t-s)B_{2[\frac{n}{2}]+2}(y-\beta s,s).$$

From Lemmas 3.1, 3.5 and (4.7), we have

$$\begin{aligned}&\Big|\int_0^t D_x^\alpha H_1(t-s)*\mathrm{div}f(u^{(m-1)})(s)\mathrm{d}s\Big|\\ \le&\Big|\int_0^{\frac{t}{2}}\nabla D_x^\alpha H_1(t-s)*f(u^{(m-1)})(s)\mathrm{d}s\Big|+\Big|\int_{\frac{t}{2}}^t\nabla H_1(t-s)*D_x^\alpha f(u^{(m-1)})(s)\mathrm{d}s\Big|\\ \le&CE\Big|\int_0^{\frac{t}{2}}\theta(1+t-s)^{-\frac{|\alpha|}{2}}P\mathrm{d}s\Big|+CE\Big|\int_{\frac{t}{2}}^t\theta P(1+s)^{-\frac{|\alpha|}{2}}\mathrm{d}s\Big|\\ \le&CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).\end{aligned}\tag{4.8}$$

From Lemma 3.2, we have

$$\begin{aligned}&\Big|\int_0^t D_x^\alpha H_2(t-s)*\mathrm{div}\,f(u^{(m-1)})(s)\mathrm{d}s\Big|\\ \le&\Big|\int_0^t\nabla H_2(t-s)*D_x^\alpha f(u^{(m-1)})(s)\mathrm{d}s\Big|\\ \le&\Big|\int_0^t\mathrm{e}^{-b(t-s)}(g_1^2+g_2^2+C_1\delta(x))*D_x^\alpha f(u^{(m-1)})(s)\mathrm{d}s\Big|\\ \le&\Big|\int_0^t\mathrm{e}^{-b(t-s)}(g_1^2*D_x^\alpha f(u^{(m-1)})(s)+g_2^2*D_x^\alpha f(u^{(m-1)})(s)\\ &+C_1D_x^\alpha f(u^{(m-1)})(x,s))\mathrm{d}s\Big|\\ :=&\mathrm{R}_1+\mathrm{R}_2+\mathrm{R}_3.\end{aligned}\tag{4.9}$$

From Lemma 3.6 and (4.7), we have

$$\begin{aligned}\mathrm{R}_3\le&\int_0^t\mathrm{e}^{-b(t-s)}(1+s)^{-n-\frac{|\alpha|}{2}}B_{2[\frac{n}{2}]+2}(x-\beta s,s)\mathrm{d}s\\ \le&CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).\end{aligned}\tag{4.10}$$

From (3.5), (4.7), Lemmas 3.4 and 3.6, we have

$$\begin{aligned}\mathrm{R}_1\le&\int_0^t\int_{\mathbb{R}^n}CE\mathrm{e}^{-b(t-s)}(1+s)^{-n-\frac{|\alpha|}{2}}(1+|y|^2)^{-N}B_{2[\frac{n}{2}]+2}(x-y-\beta s,s)\mathrm{d}y\mathrm{d}s\\ \le&CE\int_0^t\mathrm{e}^{-b(t-s)}(1+s)^{-n-\frac{|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta s,s)\mathrm{d}s\\ \le&CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).\end{aligned}\tag{4.11}$$

From (3.6), (4.7) and Lemma 3.6, we have

$$\begin{aligned}\mathrm{R}_2\le&\int_0^t\int_{\{y;|y-x|\le2\epsilon\}}CE\mathrm{e}^{-b(t-s)}(1+s)^{-n-\frac{|\alpha|}{2}}g_2^2(y)B_{2[\frac{n}{2}]+2}(x-y-\beta s,s)\mathrm{d}y\mathrm{d}s\\ \le&CE\int_0^t\mathrm{e}^{-b(t-s)}(1+s)^{-n-\frac{|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta s,s)\|g_2^2\|_{L_1}\mathrm{d}s\\ \le&CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).\end{aligned}\tag{4.12}$$

Together with (4.9)–(4.12), we get

$$\Big| \int_0^t D_x^\alpha H_2(t-s) * \operatorname{div} f(u^{(m-1)})(s)\mathrm{d}s \Big| \\ \le CE(1+t)^{-\frac{n+|\alpha|}{2}} B_{[\frac{n}{2}]+1}(x-\beta t, t). \tag{4.13}$$

From (2.6), we know that

$$D_x^\alpha u^{(m)} = D_x^\alpha u^{(1)} - \int_0^t D_x^\alpha H(t-s) * \operatorname{div} f(u^{(m-1)})(s)\mathrm{d}s. \tag{4.14}$$

From (4.8), (4.13)–(4.14) and Lemma 4.1, we get our result.

From Lemmas 4.1 and 4.2, using mathematical induction, we know that for all $m \ge 1$ and $|\alpha| < l - 1 - \frac{n}{2}$,

$$|D_x^\alpha u^{(m)}(x,t)| \le CE(1+t)^{-\frac{n+|\alpha|}{2}} B_{[\frac{n}{2}]+1}(x-\beta t, t). \tag{4.15}$$

5 Convergence of the Sequence

In this section, we will prove that $\{u^{(m)}(x,t)\}$ is a Cauchy sequence in a Banach space, and thus it converges to the solution of (1.1)–(1.2).

From [11, Lemma 5.2], it follows that

$$\Big(\int_{\mathbb{R}^n} B_{2[\frac{n}{2}]+2}(x-\beta t, t)\mathrm{d}x \Big)^{\frac{1}{2}} \le C(1+t)^{\frac{n}{4}}. \tag{5.1}$$

From (4.15) and (5.1), we know $u^{(m)}(x,t) \in L_\infty(0,\infty; H^{l-2-[\frac{n}{2}]})$. Thus $\{u^{(m)}\}$ is in a Banach space. We next prove that it is a Cauchy sequence.

Lemma 5.1 *$\{u^{(m)}\}$ is a Cauchy sequence in $L_\infty(0,\infty; H^{l-2-[\frac{n}{2}]})$.*

Proof From (2.6), $\{u^{(m)}\}$ satisfies the following equation

$$\partial_t u^{(m)} - \triangle \partial_t u^{(m)} - \eta \triangle u^{(m)} + (\beta \cdot \bigtriangledown) u^{(m)} = -\operatorname{div} f(u^{(m-1)}). \tag{5.2}$$

Thus

$$\partial_t D_x^\alpha u^{(m)} - \triangle \partial_t D_x^\alpha u^{(m)} - \eta \triangle D_x^\alpha u^{(m)} + (\beta \cdot \bigtriangledown) D_x^\alpha u^{(m)} = -D_x^\alpha \operatorname{div} f(u^{(m-1)}).$$

Set $v^{(m)}(x,t) = u^{(m)}(x,t) - u^{(m-1)}(x,t)$. Then

$$\partial_t D_x^\alpha v^{(m)} - \triangle \partial_t D_x^\alpha v^{(m)} - \eta \triangle D_x^\alpha v^{(m)} + (\beta \cdot \bigtriangledown) D_x^\alpha v^{(m)} \\ = -D_x^\alpha \operatorname{div} f(u^{(m-1)}) + D_x^\alpha \operatorname{div} f(u^{(m-2)}). \tag{5.3}$$

Multiplying $D_x^\alpha v^{(m)}$ in the two sides of (5.3) and integrating with respect to x in $\mathbb{R}^n$, we get

$$\frac{1}{2}\frac{\partial}{\partial t}\|D_x^\alpha v^{(m)}\|_{L_2}^2 + \frac{1}{2}\frac{\partial}{\partial t}\|D_x^\alpha \bigtriangledown v^{(m)}\|_{L_2}^2 + \eta \|\bigtriangledown D_x^\alpha v^{(m)}\|_{L_2}^2 \\ = -\int D_x^\alpha \operatorname{div}\big(f(u^{(m-1)}) - f(u^{(m-2)})\big) D_x^\alpha v^{(m)}\mathrm{d}x.$$

When $m \ge 2$, from (2.6) we have $v^{(m)}(x,0) = 0$. Thus for $m \ge 3$, we have

$$\|D_x^\alpha v^{(m)}\|_{L_2}^2 + \|D_x^\alpha \bigtriangledown v^{(m)}\|_{L_2}^2 + \int_0^t \eta \|\bigtriangledown D_x^\alpha v^{(m)}\|_{L_2}^2 \mathrm{d}s \\ \le \sum_{|\alpha_1|+|\alpha_2|=|\alpha|+1} \int_0^t \int_{\mathbb{R}^n} |D_x^{\alpha_1} v^{(m-1)} D_x^{\alpha_2}(u^{(m-1)} + u^{(m-2)}) D_x^\alpha v^{(m)}|\mathrm{d}x\mathrm{d}s \\ := \mathrm{R}.$$

From (4.15), we know

$$\begin{aligned}\mathrm{R} &\leq CE\int_0^t\int_{\mathbb{R}^n}(1+s)^{-n-\frac{1}{2}}B_{[\frac{n}{2}]+1}(x-\beta s,s)|D_x^\alpha v^{(m)}|\mathrm{d}x\mathrm{d}s \\ &\leq CE\int_0^t(1+s)^{-n-\frac{1}{2}}(1+s)^{\frac{n}{4}}\|D_x^\alpha v^{(m)}\|_{L_2}\mathrm{d}s \\ &\leq CE\|v^{(m)}\|_{L_\infty(0,\infty;H^{l-2-[\frac{n}{2}]})}. \end{aligned} \tag{5.4}$$

If $CE<1$, from (5.4) we know that $\{u^{(m)}\}$ is a Cauchy sequence in Banach space $L_\infty(0,\infty;H^{l-2-[\frac{n}{2}]})$, and thus it converges to the solution to (1.1)–(1.2).

6 Conclusion

Theorem 6.1 *If* $\|u_0\|_{H^l}=E,\ l>1+\left[\frac{n}{2}\right], \|u_0\|_{L_\infty}\leq CE(1+|x|^2)^{-1-[\frac{n}{2}]}$ *with* E *small enough, then* (1.1)–(1.2) *have a global solution in time* $u(x,t)$*, for* $|\alpha|\leq l-[\frac{n}{2}]-2$*, satisfying*

$$|D_x^\alpha u(x,t)|\leq CE(1+t)^{-\frac{n+|\alpha|}{2}}B_{[\frac{n}{2}]+1}(x-\beta t,t).$$

Remark 6.1 The solution has the same decay rate as the heat kernel, so our estimation must be optimal.

Remark 6.2 The solution decays much faster away along the characteristic line $x=\beta t$, so we can say that the solution propagates along the characteristic line. It coincides with the physical phenomenon.

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