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Lower Bounds on the (Laplacian) Spectral Radius of Weighted Graphs*

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Abstract The weighted graphs, where the edge weights are positive numbers, are considered. The authors obtain some lower bounds on the spectral radius and the Laplacian spectral radius of weighted graphs, and characterize the graphs for which the bounds are attained. Moreover, some known lower bounds on the spectral radius and the Laplacian spectral radius of unweighted graphs can be deduced from the bounds.

Keywords Weighted graphs, Adjacency matrix, Laplacian matrix, Spectral radius, Lower bounds
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1 Introduction

In this paper, we consider a simple connected weighted graph in which the edge weights are positive numbers. Let $G = (V, E)$ be a simple connected weighted graph with a vertex set $V = \{v_1, v_2, \cdots, v_n\}$. We denote by w_{ij} the weight of the edge v_iv_j and assume $w_{ij} = w_{ji}$. For short, we write $i \sim j$ if the vertices v_i and v_j are adjacent. For $v_i \in V$, let $w_i = w(v_i) = \sum\limits_{j\sim i} w_{ij}$. If G is a weighted graph with $w_i = w_j$ for any $v_i, v_j \in V$, then G is called a regular weighted graph. If $G = (X \cup Y, E)$ is a weighted bipartite graph with $w_i = w_j$ for any $v_i, v_j \in X$ and $w_k = w_l$ for any $v_k, v_l \in Y$, then G is called a semiregular weighted bipartite graph.

For $v_i \in V$, let $\gamma_i = \gamma(v_i) = \sum\limits_{j\sim i} w_{ij}w_j$. If G is a weighted graph with $\frac{\gamma_i}{w_i} = \frac{\gamma_j}{w_j}$ for any $v_i, v_j \in V$, then G is called a pseudo-regular weighted graph. If $G = (X \cup Y, E)$ is a weighted bipartite graph with $\frac{\gamma_i}{w_i} = \frac{\gamma_j}{w_j}$ for any $v_i, v_j \in X$ and $\frac{\gamma_k}{w_k} = \frac{\gamma_l}{w_l}$ for any $v_k, v_l \in Y$, then G is called a pseudo-semiregular weighted bipartite graph. Obviously, any regular weighted graph is a pseudo-regular weighted graph and any semiregular weighted bipartite graph is a pseudo-semiregular weighted bipartite graph.

The adjacency matrix $A(G)$ of a weighted graph G is defined as $A(G) = (a_{ij})_{n\times n}$, where

$$a_{ij} = \begin{cases} w_{ij}, & \text{if } i \sim j, \\ 0, & \text{otherwise.} \end{cases}$$

Let $W(G) = \mathrm{diag}(w_1, w_2, \cdots, w_n)$. Then the Laplacian matrix $L(G)$ of a weighted graph G is $L(G) = W(G) - A(G)$. The signless Laplacian matrix $Q(G)$ of a weighted graph G is

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$Q(G) = W(G)+A(G)$. Clearly, $A(G)$, $L(G)$ and $Q(G)$ are real symmetric matrices. Hence their eigenvalues are real numbers. We denote by $\lambda_1(M)$ the largest eigenvalue of a real symmetric matrix M. For a weighted graph G, we denote by $\lambda_1(G)$, $\mu_1(G)$ and $q_1(G)$ the largest eigenvalues of $A(G)$, $L(G)$ and $Q(G)$, respectively, and call them the spectral radius, the Laplacian spectral radius and the signless Laplacian spectral radius of G, respectively. When G is connected, $A(G)$ and $Q(G)$ are irreducible matices and so by Perron-Frobenius Theorem, $\lambda_1(G)$ and $q_1(G)$ are simple with the positive eigenvectors.

If $w_{ij} = 1$ for all edges v_iv_j, then G is an unweighted graph. For an unweighted graph, $w_i = w(v_i) = d_i$ is the degree of $v_i \in V(G)$, and γ_i is the 2-degree of v_i. There exists a vast literature that studies the bounds of the spectral radius, the Laplacian spectral radius and the signless spectral radius. We refer the reader to [1, 7–8, 10–13, 15–16, 21] for more information.

For weighted graphs, Yang, Hu and Hong [19] gave the upper and lower bounds of the spectral radius of the weighted trees; Das and Bapat [6] and Sorgun and Büyükköse [17] gave some upper bounds of the spectral radius; Rojo [14] and Das [4–5] gave some upper bounds of the Laplacian spectral radius.

The remainder of this paper is organized as follows. In Section 2, we give some useful lemmas. In Section 3, we present some lower bounds of the spectral radius of weighted graphs. In Section 4, we give some lower bounds of the signless Laplacian spectral radius of weighted graphs, from which we can get some lower bounds of the Laplacian spectral radius of weighted graphs. From these bounds, we can deduce some known lower bounds on the spectral radius and the Laplacian spectral radius of unweighted graphs.

2 Some Lemmas

The following are some useful lemmas.

Lemma 2.1 (see [10]) *Let A be a nonnegative symmetric matrix and x be a unit vector of $\mathcal{R}^n$. If $\lambda_1(A) = x^{\mathrm{T}}Ax$, then $Ax = \lambda_1(A)x$.*

Lemma 2.2 (see [18]) *Let G be a simple connected weighted bipartite graph. Then $\mu_1(G) = q_1(G)$.*

Proof Let $G = (X \cup Y, E)$ be a connected weighted bipartite graph with n vertices and suppose that $X = \{v_1, v_2, \cdots, v_k\}$, $Y = \{v_{k+1}, v_{k+2}, \cdots, v_n\}$. Let $U = (u_{ij})$ be the $n \times n$ diagonal matrix with $u_{ii} = 1$ if $1 \le i \le k$ and $u_{ii} = -1$ if $k + 1 \le i \le n$. It is easy to see that $U^{-1}L(G)U = Q(G)$, which implies that $L(G)$ and $Q(G)$ have the same spectrum. Hence $\mu_1(G) = q_1(G)$.

Lemma 2.3 (see [2]) *Let $M = (m_{ij})$ be an $n \times n$ irreducible nonnegative matrix with the spectral radius $\rho(M)$, and $R_i(M)$ be the i-th row sum of M for $1 \le i \le n$. Then*

$$\min\{R_i(M) : 1 \le i \le n\} \le \rho(M) \le \max\{R_i(M) : 1 \le i \le n\}.$$

Moreover, either equality holds if and only if the row sums of M are all equal.

By Lemma 2.3, the following result holds immediately.

Lemma 2.4 *Let G be a simple connected weighted graph. Then*

$$\min\{2w_i : 1 \le i \le n\} \le q_1(G) \le \max\{2w_i : 1 \le i \le n\}.$$

Moreover, either equality holds if and only if G is a regular weighted graph.

Lemma 2.5 (see [5]) *Let G be a simple connected weighted graph. Then*

$$\mu_1(G) \le \max\{w_i + w_j : i \sim j,\ 1 \le i, j \le n\},$$

where the equality holds if and only if G is a regular weighted bipartite graph or a semiregular weighted bipartite graph.

3 Lower Bounds of the Spectral Radius

The following theorem is one of our main results.

Theorem 3.1 *Let G be a simple connected weighted graph of order n. Then*

$$\lambda_1(G) \ge \sqrt{\frac{\sum\limits_{i=1}^{n} {\gamma_i}^2}{\sum\limits_{i=1}^{n} {w_i}^2}}, \tag{3.1}$$

where the equality holds if and only if G is a pseudo-regular weighted graph or a pseudo-semiregular weighted bipartite graph.

Proof Let $A(G)$ be the adjacency matrix of G and $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ be the unit positive eigenvector of $A(G)$ corresponding to $\lambda_1(A(G))$. For short, we write $A(G)$ as A in the following proof. Take

$$C = \sqrt{\frac{1}{\sum\limits_{i=1}^{n} {w_i}^2}}\,(w_1, w_2, \cdots, w_n)^{\mathrm{T}}.$$

Noting that C is a unit positive vector, we have

$$\lambda_1(G) = \lambda_1(A) = \sqrt{\lambda_1(A^2)} = \sqrt{X^{\mathrm{T}} A^2 X} \ge \sqrt{C^{\mathrm{T}} A^2 C}.$$

Since

$$AC = \sqrt{\frac{1}{\sum\limits_{i=1}^{n} {w_i}^2}}\Big(\sum_{j=1}^{n} w_{1j} w_j, \cdots, \sum_{j=1}^{n} w_{nj} w_j\Big)^{\mathrm{T}} = \sqrt{\frac{1}{\sum\limits_{i=1}^{n} {w_i}^2}}(\gamma_1, \cdots, \gamma_n)^{\mathrm{T}}, \tag{$*$}$$

we have

$$\lambda_1(G) = \lambda_1(A) = \sqrt{\lambda_1(A^2)} \ge \sqrt{C^{\mathrm{T}} A^2 C} = \sqrt{\frac{\sum\limits_{i=1}^{n} {\gamma_i}^2}{\sum\limits_{i=1}^{n} {w_i}^2}}.$$

If the equality holds, then

$$\lambda_1(A^2) = C^{\mathrm{T}} A^2 C.$$

By Lemma 2.1, $A^2 C = \lambda_1(A^2) C$. If the multiplicity of $\lambda_1(A^2)$ is one, then $X = C$, which implies $\gamma_i = \lambda_1(G) w_i$ $(1 \le i \le n)$. Hence G is a pseudo-regular weighted graph. Otherwise, the multiplicity of $\lambda_1(A^2) = (\lambda_1(A))^2$ is two, which implies that $-\lambda_1(A)$ is also an eigenvalue of G. Then G is a connected bipartite graph by a theorem of Frobenius (see, for example, [3, Theorem 0.3]). Without loss of generality, we assume

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

where $B = (b_{i,j})$ is an $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Let

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

and

$$C = \sqrt{\frac{1}{\sum_{i=1}^{n} {w_i}^2}} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix},$$

where $X_1 = (x_1, x_2, \cdots, x_{n_1})^{\mathrm{T}}$, $X_2 = (x_{n_1+1}, x_{n_1+2}, \cdots, x_n)^{\mathrm{T}}$, $C_1 = (w_1, w_2, \cdots, w_{n_1})^{\mathrm{T}}$ and $C_2 = (w_{n_1+1}, w_{n_1+2}, \cdots, w_n)^{\mathrm{T}}$. Since

$$A^2 = \begin{pmatrix} BB^{\mathrm{T}} & 0 \\ 0 & B^{\mathrm{T}}B \end{pmatrix},$$

we have

$$BB^{\mathrm{T}}C_1 = \lambda_1(A^2)C_1, \quad B^{\mathrm{T}}BC_2 = \lambda_1(A^2)C_2$$

and

$$BB^{\mathrm{T}}X_1 = \lambda_1(A^2)X_1, \quad B^{\mathrm{T}}BX_2 = \lambda_1(A^2)X_2.$$

Noting that BB^{T} and $B^{\mathrm{T}}B$ have the same nonzero eigenvalues, $\lambda_1(A^2)$ is the spectral radius of BB^{T} and its multiplicity is one. So $X_1 = p_1C_1$ (p_1 is a constant), which implies $\frac{\gamma_i}{w_i} = \frac{\gamma_j}{w_j}$ $(1 \le i < j \le n_1)$. Similarly, $X_2 = p_2C_2$ (p_2 is a constant), which implies $\frac{\gamma_i}{w_i} = \frac{\gamma_j}{w_j}$ $(n_1 + 1 \le i < j \le n)$. Hence G is a pseudo-semiregular weighted graph.

Conversely, if G is a pseudo-regular weighted graph, then $\frac{\gamma_i}{w_i} = p$ $(1 \le i \le n)$ is a constant, which implies $AC = pC$. By Perron-Frobenius Theorem (see [2]), for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. Hence $\lambda_1(G) = p = \sqrt{\frac{\sum_{i=1}^{n} {\gamma_i}^2}{\sum_{i=1}^{n} {w_i}^2}}$.

If G is a pseudo-semiregular weighted bipartite graph, we assume

$$A = \begin{pmatrix} 0 & B \\ B^{\mathrm{T}} & 0 \end{pmatrix},$$

$\frac{\gamma_i}{w_i} = p_1$ $(1 \le i \le n_1)$ and $\frac{\gamma_i}{w_i} = p_2$ $(n_1 + 1 \le i \le n)$, where $B = (b_{i,j})$ is an $n_1 \times n_2$ matrix with $n_1 + n_2 = n$. Let $C_1 = (w_1, w_2, \cdots, w_{n_1})^{\mathrm{T}}$ and $C_2 = (w_{n_1+1}, w_{n_1+2}, \cdots, w_n)^{\mathrm{T}}$. So for each i $(1 \le i \le n_1)$, the i-th element of $BB^{\mathrm{T}}C_1$ is

$$r_i(BB^{\mathrm{T}}C_1) = \sum_{j=1}^{n_1}\sum_{k=1}^{n_2} b_{ik}b_{jk}w_j = \sum_{k=1}^{n_2} b_{ik} \sum_{j=1}^{n_1} b_{jk}w_j = \sum_{k=1}^{n_2} b_{ik}p_2w_{n_1+k} = p_1p_2w_i.$$

Similarly, $r_j(B^{\mathrm{T}}BC_2) = p_1p_2w_{n_1+j}$ for each j $(1 \le j \le n_2)$. Hence $A^2C = p_1p_2C$, where $C = \sqrt{\frac{1}{\sum_{i=1}^{n} {w_i}^2}}(w_1, w_2, \cdots, w_n)^{\mathrm{T}}$. It is known that for any positive eigenvector of a nonnegative matrix, the corresponding eigenvalue is the spectral radius of that matrix. So

$$\lambda_1(A^2) = p_1p_2 = C^{\mathrm{T}}A^2C.$$

From the equality $(*)$, we have

$$\lambda_1(A^2) = p_1 p_2 = \frac{\sum_{i=1}^{n} {\gamma_i}^2}{\sum_{i=1}^{n} {w_i}^2}.$$

It follows that

$$\lambda_1(G) = \sqrt{\frac{\sum_{i=1}^{n} {\gamma_i}^2}{\sum_{i=1}^{n} {w_i}^2}}.$$

This completes the proof of Theorem 3.1.

Corollary 3.1 (1) *Let G be a pseudo-regular weighted graph with $\gamma(v) = p\,w(v)$ for each $v \in V(G)$. Then $\lambda_1(G) = p$.*

(2) *Let G be a pseudo-semiregular weighted bipartite graph with the bipartition (X, Y). If $\gamma(v) = p_x w(v)$ for each $v \in X$ and $\gamma(v) = p_y w(v)$ for each $v \in Y$, then $\lambda_1(G) = \sqrt{p_x p_y}$.*

Since a regular weighted graph must be a pseudo-regular weighted graph and a semiregular weighted bipartite graph must be a pseudo-semiregular weighted bipartite graph, we have the following results immediately from Corollary 3.1.

Corollary 3.2 (1) *Let G be a regular weighted graph with $w(v) = a$ for each $v \in V(G)$. Then $\lambda_1(G) = a$.*

(2) *Let G be a semiregular weighted bipartite graph with the bipartition (X, Y). If $w(v) = a$ for each $v \in X$ and $w(v) = b$ for each $v \in Y$, then $\lambda_1(G) = \sqrt{ab}$.*

Corollary 3.3 *Let G be a simple connected weighted graph of order n. Then*

$$\lambda_1(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2}, \tag{3.2}$$

where the equality holds if and only if G is a regular weighted graph or a semiregular weighted bipartite graph.

Proof By Theorem 3.1 and the Cauchy-Schwarz inequality,

$$\lambda_1(G) \geq \sqrt{\frac{{\gamma_1}^2 + {\gamma_2}^2 + \cdots + {\gamma_n}^2}{{w_1}^2 + {w_2}^2 + \cdots + {w_n}^2}} \geq \sqrt{\frac{(\gamma_1 + \gamma_2 + \cdots + \gamma_n)^2}{n({w_1}^2 + {w_2}^2 + \cdots + {w_n}^2)}}\,.$$

Since

$$\gamma_1 + \gamma_2 + \cdots + \gamma_n = {w_1}^2 + {w_2}^2 + \cdots + {w_n}^2,$$

we have

$$\lambda_1(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2}\,.$$

If the equality holds, G is a pseudo-regular weighted graph or a pseudo-semiregular weighted bipartite graph (by Theorem 3.1) with $\gamma_i = \gamma_j$ for all $1 \leq i < j \leq n$. Thus G is a regular weighted graph or a semiregular weighted bipartite graph. Conversely, if G is a regular weighted graph, the equality holds immediately. If G is a semiregular weighted bipartite graph, we assume that $w(v_1) = \cdots = w(v_{n_1}) = a$ and $w(v_{n_1+1}) = \cdots = w(v_n) = b$. Since $n_1 a = (n - n_1)b$, $\sqrt{\frac{1}{n}\sum\limits_{i=1}^{n} {w_i}^2} = \sqrt{ab}$. By Corollary 3.2, we have $\lambda_1(G) = \sqrt{ab}$. Thus the equality holds.

Corollary 3.4 *Let G be a simple connected weighted graph of order n. Then*

$$\lambda_1(G) \geq \min\{w_i : 1 \leq i \leq n\}. \tag{3.3}$$

Proof By Corollary 3.3 and the Cauchy-Schwarz inequality,

$$\lambda_1(G) \geq \sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2} \geq \sqrt{\frac{\Big(\sum\limits_{i=1}^{n} w_i\Big)^2}{n^2}} \geq \min\{w_i : 1 \leq i \leq n\}.$$

Remark 3.1 If G is a simple connected unweighted graph of order n with the degree sequence $d_1, d_2, \cdots, d_n$, the minimum degree δ, and $t_i = \sum\limits_{j\sim i} d_i$, then the inequalities (3.1), (3.2) and (3.3) become

$$\lambda_1(G) \geq \sqrt{\frac{\sum\limits_{i=1}^{n} {t_i}^2}{\sum\limits_{i=1}^{n} {d_i}^2}}, \tag{3.4}$$

$$\lambda_1(G) \geq \sqrt{\frac{\sum\limits_{i=1}^{n} {d_i}^2}{n}}, \tag{3.5}$$

$$\lambda_1(G) \geq \delta, \tag{3.6}$$

respectively. The inequality (3.4) is one of the main results in [20], and the inequality (3.5) is one of the main results in [9].

4 Lower Bounds of the (Signless) Laplacian Spectral Radius

Theorem 4.1 *Let G be a simple connected weighted graph of order n. Then*

$$q_1(G) \geq \sqrt{\frac{\sum\limits_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum\limits_{i=1}^{n} {w_i}^2}}, \tag{4.1}$$

where the equality holds if and only if G is a regular weighted graph or a semiregular weighted bipartite graph.

Proof Let $W(G) + A(G)$ be the signless Laplacian matrix of G and $X = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ be the unit positive eigenvector of $W(G) + A(G)$ corresponding to $q_1(G)$. For short, we write $W(G) + A(G)$ as $W + A$ in the following proof. Take

$$C = \sqrt{\frac{1}{\sum\limits_{i=1}^{n} {w_i}^2}}(w_1, w_2, \cdots, w_n)^{\mathrm{T}}.$$

Then

$$q_1(G) = \sqrt{\lambda_1((W+A)^2)} = \sqrt{X^{\mathrm{T}}(W+A)^2X} \geq \sqrt{C^{\mathrm{T}}(W+A)^2C}.$$

Since

$$\begin{aligned}(W+A)C &= \sqrt{\frac{1}{\sum_{i=1}^{n} {w_i}^2}}\Big({w_1}^2 + \sum_{j=1}^{n} w_{1j}w_j, \cdots, {w_n}^2 + \sum_{j=1}^{n} w_{nj}w_j\Big)^{\mathrm{T}} \\ &= \sqrt{\frac{1}{\sum_{i=1}^{n} {w_i}^2}}({w_1}^2 + \gamma_1, \cdots, {w_n}^2 + \gamma_n)^{\mathrm{T}},\end{aligned}$$

we have

$$q_1(G) \geq \sqrt{C^{\mathrm{T}}(W+A)^2C} = \sqrt{\frac{\sum_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum_{i=1}^{n} {w_i}^2}}.$$

If the equality holds, then

$$\lambda_1((W+A)^2) = C^{\mathrm{T}}(W+A)^2C,$$

which implies that $(W+A)^2C = \lambda_1((W+A)^2)C$ (by Lemma 2.1). Since $W+A$ is a nonnegative irreducible positive semidefinite matrix, all eigenvalues of $W+A$ are nonnegative. By Perron-Frobenius Theorem, the multiplicity of $\lambda_1(W+A)$ is one. Since $\lambda_1((W+A)^2) = (\lambda_1(W+A))^2$, we have the multiplicity of $\lambda_1((W+A)^2)$ is one. Hence, if the equality holds, then $X = C$. By $\lambda_1(W+A)C = (W+A)C$, we have $\lambda_1(W+A)w_i = {w_i}^2 + \gamma_i$ for $i = 1, 2, \cdots, n$. Thus $w_i + \frac{\gamma_i}{w_i} = w_j + \frac{\gamma_j}{w_j}$ for all $i \neq j$. Assume, without loss of generality, that $w_1 = a = \max\{w_i : 1 \leq i \leq n\}$, $w_2 = b = \min\{w_i : 1 \leq i \leq n\}$ and $a \neq b$. Then we have

$$a + \frac{\gamma_1}{a} = b + \frac{\gamma_2}{b}.$$

Since $\gamma_1 \geq ab$ and $\gamma_2 \leq ab$,

$$a + b \leq a + \frac{\gamma_1}{a} = b + \frac{\gamma_2}{b} \leq a + b.$$

Thus we must have $\gamma_1 = ab = \gamma_2$. This implies $w(v) = a$ or $w(v) = b$ for all $v \in V(G)$, since G is a connected weighted graph. Hence G is a regular weighted graph or a semiregular weighted bipartite graph.

Conversely, if G is a regular weighted graph with $w(v) = a$ for each $v \in V$, then

$$\sqrt{\frac{\sum_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum_{i=1}^{n} {w_i}^2}} = 2a.$$

By Lemma 2.4, $q_1(G) = 2a$ and so the equality holds.

If G is a semiregular connected bipartite graph with $w(v_1) = \cdots = w(v_{n_1}) = a$ and $w(v_{n_1+1}) = \cdots = w(v_n) = b$, noting that $n_1a = (n-n_1)b$, we have

$$\sqrt{\frac{\sum_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum_{i=1}^{n} {w_i}^2}} = \sqrt{\frac{n_1(a^2+ab)^2 + (n-n_1)(b^2+ab)^2}{n_1a^2 + (n-n_1)b^2}} = a + b.$$

By Lemmas 2.2 and 2.5, $q_1(G) = \mu_1(G) = a + b$ and so the equality holds.

Corollary 4.1 *Let G be a simple connected weighted graph of order n. Then*

$$q_1(G) \geq 2\sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2}, \tag{4.2}$$

where the equality holds if and only if G is a regular weighted graph.

Proof By Theorem 4.1 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} q_1(G) &\geq \sqrt{\frac{\sum\limits_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum\limits_{i=1}^{n} {w_i}^2}} \\ &\geq \sqrt{\frac{({w_1}^2 + \gamma_1 + {w_2}^2 + \gamma_2 + \cdots + {w_n}^2 + \gamma_n)^2}{n({w_1}^2 + {w_2}^2 + \cdots + {w_n}^2)}} \\ &= \sqrt{\frac{(2{w_1}^2 + 2{w_2}^2 + \cdots + 2{w_n}^2)^2}{n({w_1}^2 + {w_2}^2 + \cdots + {w_n}^2)}} \\ &= 2\sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2}\,. \end{aligned}$$

If the equality holds, G is a regular weighted graph or a semiregular bipartite weighted graph (by Theorem 4.1) with ${w_i}^2 + \gamma_i = {w_j}^2 + \gamma_j$ for $1 \leq i < j \leq n$. If G is a semiregular bipartite weighted graph, without loss of generality, we assume that $w_1 = a = \max\{w_i : 1 \leq i \leq n\}$ and $w_2 = b = \min\{w_i : 1 \leq i \leq n\}$. Then we have $a^2 + ab = b^2 + ab$, which implies $a = b$. Hence G is a regular bipartite weighted graph. Conversely, if G is a regular weighted graph, by Lemma 2.4, the equality holds immediately.

Corollary 4.2 *Let G be a simple connected weighted graph. Then*

$$q_1(G) \geq \min\{2w_i : 1 \leq i \leq n\}. \tag{4.3}$$

Proof By Corollary 4.1 and the Cauchy-Schwarz inequality,

$$q_1(G) \geq 2\sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2} \geq 2\sqrt{\frac{\left(\sum\limits_{i=1}^{n} w_i\right)^2}{n^2}} \geq \min\{2w_i : 1 \leq i \leq n\}.$$

Remark 4.1 Let G be a simple connected unweighted graph with the degree sequence $d_1, d_2, \cdots, d_n$, the minimum degree δ, and $t_i = \sum\limits_{j\sim i} d_j$. Then the inequalities (4.1), (4.2) and

(4.3) become

$$q_1(G) \geq \sqrt{\frac{\sum_{i=1}^{n} ({d_i}^2 + t_i)^2}{\sum_{i=1}^{n} {d_i}^2}},$$

$$q_1(G) \geq 2\sqrt{\frac{\sum_{i=1}^{n} {d_i}^2}{n}},$$

$$q_1(G) \geq 2\delta,$$

respectively.

By Lemma 2.2, for a simple connected weighted bipartite graph G, its Laplacian spectral radius $\mu_1(G)$ is equal to its signless Laplacian spectral radius $q_1(G)$. So by Theorem 4.1 and Corollaries 4.1–4.2, the following results hold immediately.

Theorem 4.2 *Let G be a simple connected bipartite weighted graph of order n. Then*

$$\mu_1(G) \geq \sqrt{\frac{\sum_{i=1}^{n} ({w_i}^2 + \gamma_i)^2}{\sum_{i=1}^{n} {w_i}^2}}, \tag{4.4}$$

where the equality holds if and only if G is a regular weighted bipartite graph or a semiregular weighted bipartite graph.

Corollary 4.3 *Let G be a simple connected bipartite weighted graph of order n. Then*

$$\mu_1(G) \geq 2\sqrt{\frac{1}{n}\sum_{i=1}^{n} {w_i}^2}, \tag{4.5}$$

where the equality holds if and only if G is a regular weighted bipartite graph.

Corollary 4.4 *Let G be a simple connected bipartite weighted graph. Then*

$$\mu_1(G) \geq \min\{2w_i : 1 \leq i \leq n\}. \tag{4.6}$$

Remark 4.2 Let G be a simple connected unweighted graph with the degree sequence $d_1, d_2, \cdots, d_n$, the minimum degree δ, and $t_i = \sum_{j\sim i} d_j$. Then the inequalities (4.4), (4.5) and (4.6) become

$$\mu_1(G) \geq \sqrt{\frac{\sum_{i=1}^{n} ({d_i}^2 + t_i)^2}{\sum_{i=1}^{n} {d_i}^2}}, \tag{4.7}$$

$$\mu_1(G) \geq 2\sqrt{\frac{\sum_{i=1}^{n} {d_i}^2}{n}}, \tag{4.8}$$

$$\mu_1(G) \geq 2\delta, \tag{4.9}$$

respectively. The inequality (4.7) is one of the main results in [20], and the inequality (4.8) is one of the main results in [10].

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