

C*-Algebraic Intertwiners for Degenerate Principal Series of Special Linear Groups

Pierre CLARE¹

Abstract The author constructs unitary intertwiners for degenerate C*-algebraic universal principal series of $SL(n+1)$ over a local field by explicitly normalizing standard intertwining integrals at the level of Hilbert modules.

Keywords Group C*-algebras, Hilbert modules, Semisimple Lie groups, Principal series representations, Intertwining operators

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1 Introduction

The relation between the theory of unitary representations of topological groups and C*-algebras is a classical topic (see [5]) that has been developed over the years. An example of particular relevance here is the work of Rieffel who gave in [16] a description of induced representations in terms of Hilbert modules.

In an effort to obtain elements of a C*-algebraic formulation of the representation theory of semisimple Lie groups, Rieffel's construction was adapted in [3] to describe parabolic induction, which resulted in the construction of Hilbert modules related to the principal series induced from a given association class of parabolic subgroups. After the results of generic irreducibility obtained in [3] in the spirit of F. Bruhat's theory, the next step in the direction of an accurate description of the C*-algebras related to a Lie group consists in developing the theory of intertwining operators. These objects were originally introduced by Knapp and Stein in [10–11] (see also [19] and recently [4]) and play a central role in the theory, as they not only allow to decide when principal series representations are reducible but also relate to the densities in the Plancherel formula and the construction of complementary series among other things.

The central piece in the classical study of intertwining operators is the normalization process, allowing to go from integral transformations analogous to Radon transforms and called standard intertwining integrals to unitary intertwiners that can often be expressed in terms of classical geometric transforms (see for instance [2, 13, 15] for recent examples).

Standard intertwining integrals were studied in the context of Hilbert modules in [3] and a result of normalization was obtained in this framework for $SL(2)$ in [1], using Fourier transforms twisted by the non-trivial Weyl element. The point of the present paper is to extend this result to the case of maximally degenerate principal series of special linear groups: We will show how

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¹Department of Mathematics, Pennsylvania State University, McAllister Building, University Park, PA-16802, USA. E-mail: clare@math.psu.edu

a Fourier transform defined on an appropriate space extends to a unitary intertwining operator between Hilbert modules and normalizes the standard intertwining integral in a sense that will be made precise.

The paper is organised as follows: Section 2 is devoted to the description of degenerate principal series of $\mathrm{SL}(n+1)$ in terms of Hilbert modules. More precisely, universal C^* -algebraic principal series are introduced in the discussion concluded by Definition 2.1 and Proposition 2.2 characterises a useful submodule of functions. Section 3 starts with the study of the Radon transform defining the standard intertwining integral. Then we introduce a Fourier transform adapted to the situation and prove in Theorem 3.1 that it extends to a unitary operator between Hilbert modules that normalizes the standard integral at the level of C^* -algebraic principal series. Finally, we explain in Section 4 how the results extend to the case of non-archimedean local fields.

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2 C^* -Algebraic Universal Principal Series

2.1 Structure and notations

Let F be the field $\mathbb{R}$ or $\mathbb{C}$ of real or complex numbers. If x is a matrix with coefficients in F , the matrix obtained by conjugating all the entries of x will be denoted by $\overline{x}$. The real matrix $x = \frac{1}{2}(x + \overline{x})$ will be called the real part of x and denoted by $\mathrm{Re}(x)$.

Troughout the article, G_F will denote the group $\mathrm{SL}(n+1, F)$ of matrices of size $n+1$ with determinant 1, for $n \geq 1$. Let θ be the Cartan involution of G_F defined by $\theta(g) = {}^t\overline{g}^{-1}$. The subgroup P_F of upper block-triangular matrices of type $(n, 1)$ is a maximal parabolic subgroup of G_F . It admits a Langlands decomposition $P_F = L_F N_F$ with θ -stable Levi component

$$L_F = \left\{ \left[\begin{array}{c|c} & 0 \\ \hline a & \vdots \\ 0 & 0 \\ \hline 0 & \dots & 0 & \det(a)^{-1} \end{array} \right], a \in \mathrm{GL}(n, F) \right\}$$

and unipotent radical

$$N_F = \left\{ \left[\begin{array}{c|c} I_n & X \\ \hline 0 & \dots & 0 & 1 \end{array} \right], X \in F^n \right\},$$

where I_n denotes the identity matrix of size n , so that P_F identifies with the semi-direct product $\mathrm{GL}(n, F) \ltimes F^n$. The opposite parabolic subgroup $\overline{P}_F$ is the image of P_F under θ . It decomposes as $\overline{P}_F = L_F \overline{N}_F$, with $\overline{N}_F = {}^t N_F$.

The group $L_F \simeq \mathrm{GL}(n, F)$ is endowed with the Haar measure $d^\times a$ defined by

$$d^\times a = |\det(a)|^{-n} da,$$

where da denotes the restriction of the Lebesgue measure of F^{n^2} , and $|\cdot|$ is the usual absolute value in the real case and is defined by $|z| = \overline{z}z$ in the complex case.

2.2 The homogeneous spaces G/N and $G/\overline{N}$

The central result of this paper will rely on some analysis on the quotient space G/N described in this section. To avoid confusion, group actions will be denoted using dots (e.g. $a.b$), whereas matrix multiplications will be written without any symbol (e.g. ab).

Notation 2.1 The vector space of matrices with p lines, q columns and coefficients in F is denoted by $M_{p,q}(F)$ (simply $M_p(F)$ if $p = q$) and we let $X_F = M_{n+1,n}(F)$.

The groups G_F and L_F act on X_F from the left and the right respectively by $g.x = gx$ and $x.a = xa$ for g in G_F , a in $GL(n, F)$ and x in X_F .

Notation 2.2 The set of elements of rank n in X_F is denoted by $M_{n+1,n}^\times(F)$.

This space is dense in X_F and inherits the actions of G_F and $GL(n, F)$. We denote by dx the G_F -invariant measure obtained by restricting the Lebesgue measure of $M_{n+1,n}(F) \simeq F^{n(n+1)}$. The right action of $GL(n, F)$ on X_F transforms the measure according to

$$\int_{X_F} f(x.a)dx = |\det_F(a)|^{-(n+1)} \int_{X_F} f(x)dx. \quad (2.1)$$

Proposition 2.1 *The homogeneous space G_F/N_F identifies with $M_{n+1,n}^\times(F)$ as a topological and measured space. The identification is (G_F, L_F) -equivariant.*

Proof The stabiliser subgroup in G_F of $x_0 = \begin{bmatrix} I_n \\ 0 \dots 0 \end{bmatrix}$ is N_F , so the map $b : G_F \rightarrow X_F$, $g \mapsto g.x_0$ identifies G_F/N_F with $b(G) = G.x_0$ in a G_F -equivariant way. Since L_F normalizes N_F and

$$\left[\begin{array}{c|c} a & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 & \det(a)^{-1} \end{array} \right] x_0 = \left[\begin{array}{c} a \\ 0 \dots 0 \end{array} \right] = x_0.a$$

for any a , it is also L -equivariant.

To identify the image of b , we observe that it is the map that extracts the left block of size $(n+1, n)$ of a matrix in $SL(n+1, F)$, that is,

$$b : \left[\begin{array}{c|c} u & \begin{matrix} * \\ \vdots \\ * \end{matrix} \end{array} \right] \mapsto u$$

for $u \in X_F$, so $b(g)$ has the maximal rank for all g . Surjectivity follows from the fact that any linearly independant family may be completed into a basis with determinant 1. Finally, b is compatible with the topologies induced on G_F and $M_{n+1,n}^\times(F)$ by the standard ones on $M_{n+1}(F)$ and X_F , and G_F/N_F admits essentially one G_F -invariant measure, which corresponds via b to the restriction of the Lebesgue measure to X_F .

From now on, we will use freely the identification established in the above result and identify any element of L_F with the corresponding element of $GL(n, F)$.

An analogous description can be obtained for the “opposite” quotient $G_F/\overline{N}_F$ identified with $\overline{X}_F = M_{n,n+1}(F)$ through the map $\overline{b}: g \mapsto y_0 g^{-1}$, where the matrix $y_0 = \begin{bmatrix} & 0 \\ & \vdots \\ I_n & \\ & 0 \end{bmatrix}$ has stabiliser $\overline{N}_F$ and $\overline{X}_F$ carries actions of G_F and L_F respectively given by $g.y = yg^{-1}$ and $y.a = a^{-1}y$ for g in G_F and a in L_F .

2.3 The modules and $\mathcal{E}(G/N)$ and $\mathcal{E}(G/\overline{N})$

From now on, we will systematically omit the subscript F . The set of smooth functions with compact support on a space T will be denoted by $C_c^\infty(T)$, and the Schwartz space of functions on T , all of whose derivatives are rapidly decreasing, by $\mathcal{S}(T)$.

In order to study intertwining operators in a C^* -algebraic context, we will work in the framework of universal principal series, introduced in [3]. The basic objects will be Hilbert modules defined as completions of $C_c^\infty(G/N)$ and $C_c^\infty(G/\overline{N})$ respectively.

Notation 2.3 If f is a function of the variable x and m is a group element such that mx makes sense, the right translation of f by m is denoted by $f^m: x \mapsto f(xm)$. Similarly, ${}^m f(x) = f(mx)$.

The actions of G and L can be defined for functions on the larger space X . Namely, for a function f defined on X , and elements g of G and a of L , we let

$$g.f(x) = f(g^{-1}x), \quad (2.2)$$

$$f.a(x) = |\det(a)|^{-\frac{n+1}{2}} f(xa^{-1}) = |\det(a)|^{-\frac{n+1}{2}} f^{a^{-1}}(x), \quad (2.3)$$

which integrates to

$$f.\varphi(x) = \int_{\mathrm{GL}(n,F)} f(xa^{-1})\varphi(a) |\det(a)|^{-\frac{n+1}{2}} d^\times a \quad (2.3')$$

for any φ integrable over $\mathrm{GL}(n, F)$.

Finally we consider the pairing

$$\langle f, h \rangle_X(a) = |\det(a)|^{\frac{n+1}{2}} \int_X \overline{f(x)} h(xa) dx = |\det(a)|^{\frac{n+1}{2}} \langle f, h^a \rangle_{L^2} \quad (2.4)$$

for suitable functions f and h defined on X , where $\langle \cdot, \cdot \rangle_{L^2}$ denotes the usual inner product for square-integrable functions. If f and h are smooth and compactly supported on $G/N \subset X$, then $\langle f, h \rangle_X$ is smooth and compactly supported on L and the above formulas define on $C_c^\infty(G/N)$ a structure of the pre-Hilbert module over $C_c^\infty(L)$. The scalar factor involved in (2.3) shows that f is actually considered as a half-density over G/N , and (2.4) then appears as a natural pairing.

The above formulas are special cases of the ones involved in the general construction of parabolic induction modules described in [3]. Therefore, they can be used to define a Hilbert module structure as follows.

Definition 2.1 *The Hilbert module obtained from $C_c^\infty(G/N)$ by extending the action (2.3') to the reduced C^* -algebra of $\mathrm{GL}(n, F)$, and then completing with respect to the norm induced by*

(2.4) is called the reduced C*-algebraic universal principal series associated to the pair (G, P) and is denoted by $\mathcal{E}(G/N)$.

As explained in [3], this construction generalizes Rieffel's construction of induction modules [16]. It provides a global picture of principal series induced from P in the sense that there exists a G -equivariant unitary specialisation map between the tensor product of $\mathcal{E}(G/N)$ with the carrying space of any appropriate element of $\widehat{L}_r$ and the corresponding principal series representation (see [3, Corollary 3.5]).

The next result allows us to consider Schwartz functions on X as a subspace of $\mathcal{E}(G/N)$. It was suggested to me by Vincent Lafforgue.

Proposition 2.2 *The inclusion map $\iota : C_c^\infty(M_{n+1,n}^\times(F)) \rightarrow \mathcal{E}(G/N)$ extends continuously to $\mathcal{S}(X)$.*

Proof According to [9, Chap.VII §3], we can use the KAK decomposition to write any element of $\mathrm{GL}(n, F)$ as a product $k_1 a k_2$, where k_1 and k_2 are elements in a maximal compact subgroup of $\mathrm{GL}(n, F)$ and a is a diagonal matrix with positive entries $a_1, \dots, a_n$. Then for f and h in $\mathcal{S}(X)$, we prove that there exists a constant C such that:

$$|\langle f, h \rangle_X(k_1 a k_2)| \leq C \prod_{i=1}^n |\min(a_i, a_i^{-1})|^{\frac{n+1}{2}}. \quad (2.5)$$

By polarization, we may assume that $f = h$. Moreover, since K is compact, it is enough to treat the case $k_1 = k_2 = 1$. Identifying X to $(F^{n+1})^n$ via $x = (x_1, \dots, x_n)$, the assumption that f is rapidly decreasing implies the existence of positive bounded integrable functions f_i on F^{n+1} such that $|f(x)| \leq f_1(x_1) \cdots f_n(x_n)$. Observe that a acts on x by multiplying each x_i by a_i , so that it is enough to consider the case of one f_i . One has, for f positive bounded and integrable on F^{n+1} and $a > 0$,

$$\begin{aligned} |a|^{\frac{n+1}{2}} \int_{F^{n+1}} f(x) f(ax) dx &\leq |a|^{\frac{n+1}{2}} \|f\|_\infty \int_{F^{n+1}} f(ax) dx \\ &\leq \|f\|_\infty \|f\|_1 |a|^{-\frac{n+1}{2}}. \end{aligned}$$

Since

$$|a|^{\frac{n+1}{2}} \int_{F^{n+1}} f(x) f(ax) dx = |a|^{-\frac{n+1}{2}} \int_{F^{n+1}} f(a^{-1}x) f(x) dx,$$

the same computation yields

$$|a|^{\frac{n+1}{2}} \int_{F^{n+1}} f(x) f(ax) dx \leq \|f\|_\infty \|f\|_1 |a|^{\frac{n+1}{2}},$$

so that

$$|a|^{\frac{n+1}{2}} \int_{F^{n+1}} f(x) f(ax) dx \leq \|f\|_\infty \|f\|_1 |\min(a, a^{-1})|^{\frac{n+1}{2}},$$

which implies (2.5). As a consequence, we will prove that $\langle f, h \rangle_X$ belongs to the reduced C*-algebra of $\mathrm{GL}(n, F)$.

Following [14, §4.4], we consider the generalized Harish-Chandra Schwartz space associated to a real reductive group L consisting of functions F on L such that for any p , there exists a

constant C_p such that the inequality

$$|F(k_1 a k_2)| \leq C_p \frac{e^{-\rho \log a}}{(1 + \|\log a\|)^p} \quad (2.6)$$

is satisfied for all $k_1 a k_2$ in the KAK decomposition of L , where ρ denotes the half sum of positive roots for $(\mathfrak{l} : \mathfrak{a})$. We will use the fact, whose proof is recalled in [14], that for p large enough, there exists C_p such that (2.6) defines a norm that yields a Banach algebra which is a subalgebra of $C_r^*(G)$. In the case at hand, denoting $d_F = \dim_{\mathbb{R}}(F)$, one has

$$\begin{aligned} 2\rho \log a &= \sum_{i=1}^n (n+1-2i) d_F \log a_i \\ &= (n-1) d_F \log a_1 + (n-3) d_F \log a_2 + \cdots + (n-1) d_F \log a_n, \end{aligned}$$

so that

$$\begin{aligned} e^{-\rho \log a} &= \prod_{i=1}^n |a_i|^{i - \frac{n+1}{2}} = (|a_1|^{-1})^{\frac{n-1}{2}} (|a_2|^{-1})^{\frac{n-3}{2}} \cdots |a_{n-1}|^{\frac{n-3}{2}} |a_n|^{\frac{n-1}{2}} \\ &\geq \prod_{i=1}^n |\min(a_i, a_i^{-1})|^{\frac{n-1}{2}} \geq \prod_{i=1}^n |\min(a_i, a_i^{-1})|^{\frac{n+1}{2}}, \end{aligned} \quad (2.7)$$

since $\frac{n+1}{2} > \frac{n-1}{2}$ and $\min(a_i, a_i^{-1}) \leq 1$ for all i . Therefore, (2.5) implies that $\langle f, h \rangle_X$ satisfies (2.6) for any p , and hence defines an element of $C_r^*(\mathrm{GL}(n, F))$.

Next we prove that functions in $\mathcal{S}(X)$ can be approximated by elements of $C_c^\infty(G/N)$ for the $C_r^*(\mathrm{GL}(n, F))$ -valued norm used to build the Hilbert module $\mathcal{E}(G/N)$. Let μ denote the Lebesgue measure of X . Since $\mu(X \setminus M_{n+1, n}^\times(F)) = 0$, there exists a sequence $\{T_m\}_{m \geq 1}$ of open subsets of X containing $X \setminus M_{n+1, n}^\times(F)$, satisfying $\overline{T_{m+1}} \subset T_m$ and such that $\lim_{m \rightarrow \infty} \mu(T_m) = 0$. Let $\{B_m\}_{m \geq 1}$ be the family of closed balls centered at the origin in X with radius m for the usual norm and consider a sequence $\{\chi_m\}_{m \geq 1}$ of compactly supported non-negative smooth functions on X such that

$$\begin{aligned} \chi_m &\equiv 1 \quad \text{on } B_m \cap (X \setminus T_m) \\ &\equiv 0 \quad \text{on } T_{m+1}. \end{aligned}$$

Let f be in $\mathcal{S}(X)$. For all m , the pointwise product $f_m = f \chi_m$ is compactly supported away from singular matrices, and hence defines a function in $C_c^\infty(G/N)$. We will prove that the $C_r^*(\mathrm{GL}(n, F))$ -valued norm of $f - f_m$ converges to 0, so that f can be viewed as the limit in $\mathcal{E}(G/N)$ of the sequence $\{f_m\}_{m \geq 1}$. To do so, we denote $\varphi_m = \langle f - f_m, f - f_m \rangle_X$. By the same argument used above, (2.5) implies that φ_m belongs to the generalized Harish-Chandra Schwartz space of $\mathrm{GL}(n, F)$. Furthermore, (2.5) and (2.7) imply that

$$\left| \varphi_m(k_1 a k_2) \frac{(1 + \|\log a\|)^p}{e^{-\rho \log a}} \right| \leq C (1 + \|\log a\|)^p \prod_{i=1}^n |\min(a_i, a_i^{-1})|$$

for some constant C that can be chosen independently of m by the construction of f_m . It follows that for any $\varepsilon > 0$, there exists a compact Z in $\mathrm{GL}(n, F)$ such that

$$\sup_{k_1 a k_2 \notin Z} \left| \varphi_m(k_1 a k_2) \frac{(1 + \|\log a\|)^p}{e^{-\rho \log a}} \right| \leq \varepsilon.$$

By the definition of f_m , the sequence $\{\varphi_m\}_{m \geq 1}$ converges to 0 uniformly on any compact subset of $\mathrm{GL}(n, F)$, such as Z . Therefore,

$$\sup_{k_1 a k_2 \in \mathrm{GL}(n, F)} \left| \varphi_m(k_1 a k_2) \frac{(1 + \|\log a\|)^p}{e^{-\rho \log a}} \right| \leq \varepsilon$$

for m large enough, so that φ_m converges to 0 in the generalized Harish-Chandra Schwartz space of $\mathrm{GL}(n, F)$, and hence in $C_r^*(\mathrm{GL}(n, F))$, which concludes the proof.

Remark 2.1 The advantage of $\mathcal{S}(X)$ compared with $C_c^\infty(G/N)$ as a submodule of $\mathcal{E}(G/N)$ is the fact that, unlike compactness of the support, the properties of asymptotic decay of elements in $\mathcal{S}(X)$ will be preserved by the Fourier transform used in the normalization process of Section 3.3, thus allowing to properly define an operator between Hilbert modules.

Remark 2.2 Starting from the pre-Hilbert module structure on $C_c^\infty(G/N)$, it is possible to consider various completions, for instance, with respect to $L^1(L)$ or the full C*-algebra $C^*(L)$. However, the estimate (2.5) on which the proof of Proposition 2.2 relies does not allow to exhibit $\mathcal{S}(X)$ as a submodule for these norms.

The universal principal series $\mathcal{E}(G/\overline{N})$ associated to the pair $(G, \overline{P})$ can be described in the same way. More precisely, all the arguments of the above discussion hold, working on $\overline{X}$ and using the following formulas for the actions and the inner product:

$$g.f(y) = f(yg), \quad (2.2)$$

$$f.a(y) = |\det(a)|^{\frac{n+1}{2}} f(ay) = |\det(a)|^{\frac{n+1}{2}} {}^a f(y), \quad (2.3)$$

$$f.\varphi(y) = \int_{\mathrm{GL}(n, F)} f(ay) |\det(a)|^{\frac{n+1}{2}} d^\times a, \quad (2.3')$$

$$\langle f, h \rangle_{\overline{X}}(a) = |\det(a)|^{-\frac{n+1}{2}} \int_{\overline{X}} \overline{f(y)} h(a^{-1}y) dy = |\det(a)|^{-\frac{n+1}{2}} \langle f, {}^{a^{-1}} h \rangle_{L^2}, \quad (2.4)$$

for f and h functions on $\overline{X}$, g in G , a in L and φ in $C_c(L)$. As in Proposition 2.2, the inclusion map $\bar{\iota} : C_c^\infty(M_{n, n+1}^\times(F)) \rightarrow \mathcal{E}(G/\overline{N})$ extends to $\mathcal{S}(\overline{X})$.

3 Intertwining Operators

The purpose of this section is to study and normalize standard intertwining integrals at the level of the Hilbert modules introduced above.

3.1 Standard intertwining integrals

The study of intertwiners between principal series induced from P and $\overline{P}$ respectively relies on the construction of operators transforming N -invariant functions into $\overline{N}$ -invariant ones (see [8, 10–11]). Working formally, that is outside of convergence considerations, one is naturally led to consider integrals of the form

$$\mathcal{I}(f)(g\overline{N}) = \int_{\overline{N}} f(g\overline{n}) d\overline{n}. \quad (3.1)$$

It was observed (see [12]) that such operators can be interpreted as Radon transforms and understood in the context of double fibrations as in [7]. The present situation is described by

the diagram

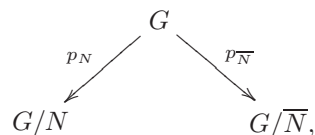


Figure 1

where p_N and $p_{\overline{N}}$ denote the natural projections. Lemma 6.2 in [3] implies that (3.1) defines a map $\mathcal{I} : C_c(G/N) \rightarrow C(G/\overline{N})$ which does not extend to an operator between Hilbert modules, the latter fact being directly related to reducibility phenomena occurring in the principal series (see [1, Appendix]).

In the notations of Figure 1, the standard intertwining integral can be written as

$$\mathcal{I}(f)(y) = \int_{p_N(p_{\overline{N}}^{-1}(y))} f(x) dx.$$

In order to study this integral at the level of Hilbert modules, it will be convenient to work in X and $\overline{X}$ rather than G/N and $G/\overline{N}$, so Figure 1 will be replaced by

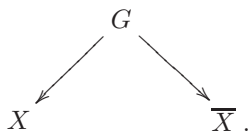


Figure 2

First, we observe that two elements x and y in X and $\overline{X}$ come from the same g in G , if and only if $x = gx_0$ and $y = y_0g^{-1}$, with the notations used in the proof of Proposition 2.1. Denoting by δ the Dirac distribution on $M_n(F)$, it follows that the integral operator $\mathcal{I}$ is given by the distributional kernel $k_{\mathcal{I}}$ defined on $X \times \overline{X}$ by

$$k_{\mathcal{I}}(x, y) = \delta(yx - I_n),$$

in the sense that

$$\mathcal{I}(f)(y) = \int_X k_{\mathcal{I}}(x, y) f(x) dx$$

for any suitable function f , that is

$$\mathcal{I}(f)(y) = \int_X \delta(yx - I_n) f(x) dx. \quad (3.2)$$

The following properties of $k_{\mathcal{I}}$ will be useful in the normalization process studied in the next section. Formulas (2.3) and (2.3) imply that $k_{\mathcal{I}}$ is invariant under the diagonal L -action:

$$k_{\mathcal{I}}(x.a, y.a) = k_{\mathcal{I}}(x, y) \quad (3.3)$$

for a in $\mathrm{GL}_n(F)$. Moreover, the homogeneity property of the Dirac distribution implies that

$$k_{\mathcal{I}}(x.a, y) = |\det(a)|^{-(n+1)} \delta(yx - a). \quad (3.4)$$

Our main purpose will be to construct a G -equivariant unitary operator between Hilbert modules that differs from $\mathcal{I}$ only by a convolution over L . In order to state a more precise definition, let us introduce one more notation.

Definition 3.1 *If T is a function on L , the operator C_T of convolution by T acts on a function f on G/N by*

$$C_T(f)(x) = f *_L T(x) = \int_L f \cdot a(x) T(a) da.$$

In the present situation, (2.3') implies that

$$C_T(f)(x) = \int_{\mathrm{GL}(n, F)} f(xa^{-1}) |\det(a)|^{-\frac{n+1}{2}} T(a) d^\times a. \quad (3.5)$$

Definition 3.2 *A bounded operator $\mathcal{U}: \mathcal{E}(G/N) \rightarrow \mathcal{E}(G/\overline{N})$ is said to normalize the standard intertwining integral $\mathcal{I}$ defined by (3.2) if*

- (i) *it is unitary and G -equivariant;*
- (ii) *there exists a function γ on L such that $(\mathcal{I} \circ C_\gamma)(f) = \mathcal{U}(f)$ for f in a dense subspace of $\mathcal{E}(G/N)$.*

Remark 3.1 Definition 3.2 above should be compared with Definition 3.1 in [1], where normalization is defined for standard intertwining integrals associated to Weyl elements.

3.2 Fourier transform on X

The use of the Fourier transform on matrix spaces to study principal series representations goes back to the work of Stein [18]. Let us briefly collect here the main properties of the integral operator that will give rise to the C*-algebraic intertwiner studied in the next section.

The transform defined by

$$\mathcal{F}f(y) = \int_X f(x) e^{-2i\pi \operatorname{Re}(\operatorname{Tr}(yx))} dx \quad (3.6)$$

maps $\mathcal{S}(X)$ to $\mathcal{S}(\overline{X})$. It satisfies the relation

$$\mathcal{F}(f^{a^{-1}}) = |\det(a)|^{-(n+1)} {}^a(\mathcal{F}(f)). \quad (3.7)$$

3.3 Normalization

We can now establish the main result of the article, relating the Fourier transform of the previous paragraph to the standard intertwining operator at the level of the C*-algebraic universal principal series.

Theorem 3.1 *The transform $\mathcal{F}$ between $\mathcal{S}(X)$ and $\mathcal{S}(\overline{X})$ extends to a unitary operator of Hilbert modules*

$$\mathcal{U}: \mathcal{E}(G/N) \rightarrow \mathcal{E}(G/\overline{N})$$

that normalizes $\mathcal{I}$ in the sense of Definition 3.2. The corresponding normalizing distribution is defined by

$$\gamma_n(a) = |\det(a)|^{\frac{1-n}{2}} e^{-2i\pi \operatorname{Re}(\operatorname{Tr}(a^{-1}))}$$

for a in L .

Proof The G -equivariance is a direct consequence of the G -invariance of the measure. To establish that $\mathcal{F}$ is $C_c(L)$ -linear, it is enough to show that it is L -equivariant.

In view of Proposition 2.2 and Remark 2.1, we work at the level of Schwartz functions on X . For f in $\mathcal{S}(X)$, one has, according to (2.3),

$$\begin{aligned}\mathcal{F}(f.a) &= \mathcal{F}(|\det(a)|^{-\frac{n+1}{2}} f^{a^{-1}}) \\ &= |\det(a)|^{\frac{n+1}{2}} .^a \mathcal{F}(f) \text{ (by (3.7))} \\ &= \mathcal{F}(f).a \text{ (by (2.3)).}\end{aligned}$$

The fact that $\mathcal{F}$ preserves the $C_r^*(L)$ -valued inner products also follows from a straightforward computation relying on its equivariance properties, namely

$$\begin{aligned}\langle \mathcal{F}(f), \mathcal{F}(f) \rangle_{\overline{X}}(a) &= |\det(a)|^{-\frac{n+1}{2}} \langle \mathcal{F}(f), {}^{a^{-1}} \mathcal{F}(f) \rangle_{L^2} \\ &= |\det(a)|^{-\frac{n+1}{2}} \langle \mathcal{F}(f), |\det(a)|^{(n+1)} \mathcal{F}(f^a) \rangle_{L^2} \\ &= |\det(a)|^{\frac{n+1}{2}} \langle \mathcal{F}(f), \mathcal{F}(f^a) \rangle_{L^2} \\ &= |\det(a)|^{\frac{n+1}{2}} \langle f, f^a \rangle_{L^2} = \langle f, f \rangle_X(a),\end{aligned}$$

the last line relying on the Plancherel equality for square integrable functions.

Let now γ be a function on L . According to (3.5), the composition $\mathcal{I} \circ C_\gamma$ acts on a function f by

$$\begin{aligned}(\mathcal{I} \circ C_\gamma)(f)(y) &= \int_X \int_{\mathrm{GL}(n, F)} k_{\mathcal{I}}(x, y) f(xa^{-1}) |\det(a)|^{-\frac{n+1}{2}} \gamma(a) d^\times a dx \\ &\stackrel{x \mapsto xa}{=} \int_X \int_{\mathrm{GL}(n, F)} k_{\mathcal{I}}(xa, y) f(x) |\det(a)|^{\frac{n+1}{2}} \gamma(a) d^\times a dx.\end{aligned}$$

Therefore, the kernel $k_\circ$ defining the composition can be written as

$$\begin{aligned}k_\circ(x, y) &= \int_{\mathrm{GL}(n, F)} k_{\mathcal{I}}(xa, y) |\det(a)|^{\frac{n+1}{2}} \gamma(a) d^\times a \\ &\stackrel{a \mapsto a^{-1}}{=} \int_{\mathrm{GL}(n, F)} k_{\mathcal{I}}(xa^{-1}, y) |\det(a)|^{-\frac{n+1}{2}} \gamma(a^{-1}) d^\times a \\ &= \int_{\mathrm{GL}(n, F)} \delta(yx - a) |\det(a)|^{\frac{n+1}{2}} \gamma(a^{-1}) d^\times a \text{ (by (3.4))} \\ &= \int_{\mathrm{GL}(n, F)} \delta(yx - a) |\det(a)|^{\frac{1-n}{2}} \gamma(a^{-1}) da.\end{aligned}$$

Comparing this last expression with the kernel of the Fourier transform of the previous paragraph

$$k_{\mathcal{F}}(x, y) = \int_{M_n(F)} \delta(yx - a) e^{-2i\pi \operatorname{Re}(\operatorname{Tr}(a))} da,$$

it is easily seen that the result follows from choosing

$$\gamma_n(a) = |\det(a)|^{\frac{1-n}{2}} e^{-2i\pi \operatorname{Re}(\operatorname{Tr}(a^{-1}))}.$$

Pursuing the comparison with the case of $\mathrm{SL}(2)$ suggested in Remark 3.1, we observe that this result is consistent with the one obtained in the case $n = 1$ in [1, Theorem 3.2], where the composition $\mathcal{I}_w \circ \mathcal{F}_w$ was proved to be equal to a convolution operator by a distribution analogous to γ_1 .

4 The Non-archimedean Case

The purpose of this last paragraph is to describe how the construction and the main result presented above extend to the non-archimedean case.

Let F be a non-archimedean local field, that is, a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$. We fix a non-trivial continuous character $\chi : F \rightarrow \mathrm{U}(1)$ and a Haar measure dx on F so that the Fourier transform $\mathcal{F}_\chi$ defined by

$$\mathcal{F}_\chi(f)(y) = \int_F f(x)\chi(xy)dx$$

is an isometry of $L^2(F)$.

A norm $|\cdot|_F$ is fixed on F that coincides with the modular function related to the Haar measure (see [20]). It is specified by the relation $|\pi|_F = q^{-1}$, where π is a uniformizer and q is the cardinality of the residual field.

The structure theory of the group G_F remains the same in this context. The definitions of functional spaces in the case of a totally discontinuous space T need to be adapted as follows: $C_c(T)$ still denotes a continuous function with compact support on T , while $C_c^\infty(T)$ (and $\mathcal{S}(T)$) denotes a locally constant compactly supported function on T . Up to these modifications and replacing absolute values by $|\cdot|_F$ everywhere, the Hilbert module construction of Section 2.3 providing the reduced C*-algebraic universal principal series can be carried out in the same way.

Finally, the proof of the analogue of Proposition 2.2 is straightforward and the normalization process can be achieved by defining the Fourier transform $\mathcal{F}$ between $\mathcal{S}(X)$ and $\mathcal{S}(\overline{X})$ by

$$\mathcal{F}f(y) = \int_X f(x)\chi(\mathrm{Tr}(yx))dx, \quad (4.1)$$

then extending it as an operator of Hilbert modules $\mathcal{E}(G/N) \rightarrow \mathcal{E}(G/\overline{N})$ as in Theorem 3.1. The normalising function γ_n is then given by

$$\gamma_n(a) = |\det(a)|_F^{\frac{1-n}{2}} \chi(\mathrm{Tr}(a^{-1})) \quad (4.2)$$

for any a in L .

Concluding remark The point of view on principal series and intertwining operators presented here relates to other works on the subject. More precisely, the distribution γ_n in (4.2) allows to recover local γ factors introduced by Godement and Jacquet in [6] and the Hilbert module operator obtained from (4.1) by Theorem 3.1 is a way of considering simultaneously a whole family of the normalized intertwiners studied by Shahidi in [17]. The connection with these results will be studied in detail in future works.

References

- [1] Clare, P., C^* -algebraic intertwiners for principal series: Case of $SL(2)$, *J. Noncommut. Geom.*, 2011, to appear.
- [2] Clare, P., On the degenerate principal series of complex symplectic groups, *J. Funct. Anal.*, **262**(9), 2012, 4160–4180.
- [3] Clare, P., Hilbert modules associated to parabolically induced representations of semisimple Lie groups, *J. Operator Theory*, **69**(2), 2013, 483–509.
- [4] Clerc, J. L., Intertwining operators for the generalized principal series on symmetric R -spaces, 2012, arXiv: 1209.0691.
- [5] Dixmier, J., *Les C^* -algèbres et leurs représentations*, Gauthier-Villars and Cie, Editeur-Imprimeur, Paris, 1964.
- [6] Godement, R. and Jacquet, H., *Zeta Functions of Simple Algebras*, Lecture Notes in Mathematics, **260**, Springer-Verlag, Berlin, New York, 1972.
- [7] Helgason, S., *The Radon Transform*, Second Edition, Progress in Mathematics, **5**, Birkhäuser, 1999.
- [8] Knapp, A. W., *Representation Theory of Semisimple Groups*, Princeton Mathematical Ser., 36, Princeton University Press, Princeton, NT, 1986.
- [9] Knapp, A. W., *Lie Groups, Beyond an Introduction*, Second Edition, Progress in Mathematics, **140**, Birkhäuser Boston, Inc., Boston, MA, 2002.
- [10] Knapp, A. W. and Stein, E. M., Intertwining operators for semisimple groups, *Ann. of Math.*, **93**, 1971, 489–578.
- [11] Knapp, A. W. and Stein, E. M., Intertwining operators for semisimple groups II, *Invent. Math.*, **60**(1), 1980, 9–84.
- [12] Kobayashi, T. and Mano, G., The Schrödinger model for the minimal representation of the indefinite orthogonal group $O(p, q)$, *Mem. Amer. Math. Soc.*, **213**(1000), 2011, 132 pages.
- [13] Kobayashi, T., Ørsted, B. and Pevzner, M., Geometric analysis on small unitary representations of $GL(N, \mathbb{R})$, *J. Funct. Anal.*, **260**(6), 2011, 1682–1720.
- [14] Lafforgue, V., K -Théorie bivariante pour les algèbres de banach et conjecture de baum-connes, *Invent. Math.*, **149**(1), 2002, 1–95.
- [15] Ólafsson, G. and Pasquale, A., The $\cos^\lambda$ and $\sin^\lambda$ transforms as intertwining operators between generalized principal series representations of $SL(n + 1, K)$, *Advances in Math.*, **229**(1), 2012, 267–293.
- [16] Rieffel, Marc A., Induced representations of C^* -algebras, *Advances in Math.*, **13**, 1974, 176–257.
- [17] Shahidi, F., Fourier transforms of intertwining operators and Plancherel measures for $GL(n)$, *Amer. J. Math.*, **106**(1), 1984, 67–111.
- [18] Stein, E. M., Analysis in matrix spaces and some new representations of $SL(N, C)$, *Ann. of Math.*, **86**, 1967, 461–490.
- [19] Vogan, D. and Wallach, N., Intertwining operators for real reductive groups, *Advances in Math.*, **82**(2), 1990, 203–243.
- [20] Weil, A., *Basic Number Theory*, Classics in Mathematics, Springer-Verlag, Berlin, 1995.