Chin. Ann. Math.
35B(5), 2014, 703–732
DOI: 10.1007/s11401-014-0856-6

Chinese Annals of Mathematics, Series B

K-Theory and the Quantization Commutes with Reduction Problem

Nigel HIGSON[1] Yanli SONG[2]

Abstract The authors examine the quantization commutes with reduction phenomenon for Hamiltonian actions of compact Lie groups on closed symplectic manifolds from the point of view of topological K-theory and K-homology. They develop the machinery of K-theory wrong-way maps in the context of orbifolds and use it to relate the quantization commutes with reduction phenomenon to Bott periodicity and the K-theory formulation of the Weyl character formula.

Keywords Symplectic reduction, Quantization, K-Theory, K-Homology, Compact Lie groups
2000 MR Subject Classification 53D20, 19L47, 53D50

1 Introduction

The purpose of this paper is to examine the quantization commutes with reduction phenomenon from the point of view of topological K-theory and K-homology. The phenomenon, which concerns the quantization of Hamiltonian actions of compact groups on compact symplectic manifolds, was discovered by Guillemin and Sternberg (see [16]) and by now it has been extensively studied; see for example [24] or [14, Section 8.10] for summaries. The usual contexts are symplectic geometry or Kahler geometry or geometric invariant theory, but even the topological point of view that is the concern of this paper has received a good deal of attention, and much of what appears below can be found in one form or another in earlier works. Our new contribution is an explanation, in the later sections of the paper, of the role Bott periodicity and the Weyl character formula in the transition from the commutative to the noncommutative cases of the quantization commutes with reduction problem. In addition, perhaps the paper will help to introduce the quantization commutes with reduction to a new audience in noncommutative geometry.

Here, in brief, is an overview of the phenomenon as seen from a K-theoretical perspective. Denote by D the Dolbeault operator on a closed complex (or almost-complex, or stable complex) manifold M. If E is any line bundle (or indeed any vector bundle at all) on M, then we can couple D to E so as to form a new elliptic operator D_E. The Fredholm index of D_E is an integer, of course, but if a compact connected Lie group G acts on M and E, then this integer

Manuscript received February 3, 2014.
[1]Department of Mathematics, Pennsylvania State University, University Park, PA16802, U. S.
E-mail: higson@math.psu.edu
[2]Department of Mathematics, University of Toronto, Room 6290, 40 St. George Street, Toronto, Ontario, Canada M5S2E4. E-mail: songyanl@math.utoronto.ca

is the virtual dimension of a more refined index, which is a virtual representation

$$\operatorname{Index}(D_E) \in R(G).$$

The trace of this virtual representation can be calculated using the fixed-point methods of Atiyah, Bott and Segal. For example, if M is the Riemann sphere $\mathbb{C}_\infty$, equipped with the natural rotation action of $U(1)$, then the index is given by the formula

$$\frac{z^{w_0}}{1-z^{-1}} + \frac{z^{w_\infty}}{1-z}. \tag{1.1}$$

The two terms represent contributions from the two fixed-points, and the integers w_0 and w_∞ are the weights of the representations of $U(1)$ on the fibers of E over these points.

In favorable circumstances there is however an entirely different formula that gives not the character but the multiplicities of individual irreducible representations of G in the index. The new formula is not organized around fixed-point data at all.

We will concentrate on the multiplicity of the trivial representation. Suppose that ∇ is a G-invariant connection on E that is compatible with some Hermitian structure. The Kostant moment map

$$\mu\colon\ M \to \mathfrak{g}^*$$

associated to ∇ is defined by

$$\mu_X = \frac{\sqrt{-1}}{2\pi}(\nabla_X - L_X), \tag{1.2}$$

where $X \in \mathfrak{g}$, L_X denotes the infinitesimal action of X on sections of E, and ∇_X is covariant differentiation in the direction of the Killing vector field associated to X (the difference of these two differential operators is multiplication by a smooth function). Assume that $0 \in \mathfrak{g}^*$ is a regular value of the Kostant moment map, and that G acts (locally) freely on the submanifold

$$\mu^{-1}[0] \subseteq M.$$

In the Kahler or symplectic contexts, these two conditions are equivalent, but in general we can simply assume that both conditions hold, and then we can form the reduced manifold (or orbifold if the action is only locally free)

$$M_0 = \mu^{-1}[0]/G.$$

It is also a (stable) complex manifold, and the line bundle E descends a line bundle E_0 on M_0. The quantization commutes with reduction formula is the rather remarkable identity

$$\operatorname{Index}(D_{M,E})_0 = \operatorname{Index}(D_{M_0,E_0}). \tag{1.3}$$

The left side is the multiplicity of the trivial representation of G in $\operatorname{Index}(D_{M,E})$ and the right side is the integer index of the associated Dolbeault operator on the reduced manifold. The formula is striking in that it is nearly as far from a formula of fixed-point type as it could possibly be.

For example, on the Riemann sphere, if $w_0 > 0 > w_\infty$, then the reduced manifold M_0 is a point and $\operatorname{Index}(D_{M_0,E_0}) = 1$. So a quick check shows that the formulas (1.1) and (1.3) are consistent with one another. If the signs are reversed, then M_0 is a point with the opposite stable complex structure and $\mathrm{Index}(D_{M_0,E_0}) = -1$, so the formulas are still consistent.

Actually, the formula (1.3) is not quite correct in the general context that we are considering, which reaches beyond symplectic manifolds. We shall begin by considering abelian group actions on stable complex manifolds (or orbifolds) and review some fairly well-known arguments that lead to a proof that (1.3) is correct for these. But in the nonabelian case the best available result involves tensor powers of the line bundle E, and asserts that

$$\mathrm{Index}(D_{M,E^k})_0 = \mathrm{Index}(D_{M_0,E_0^k}), \quad \forall k \gg 0, \tag{1.4}$$

refer to [21] or [15], for example. This is our objective, and as we stated at the beginning the main goal of this paper is to explain how the nonabelian result follows rather easily from a slight K-theoretic strengthening of the abelian result (applied to the maximal torus), thanks to Bott periodicity and the Weyl character formula.

Necessary preliminaries on K-theory and stable complex manifolds are set out in Sections 2–4. In Section 8 we review (and generalize very slightly) the proof of the formula (1.3) for circle actions. To go further we need to review some facts about orbifolds, and this is done in Sections 10–11. This allows us to pass from circles to tori in Section 12, and then to general groups in Section 13. Finally, many index-theoretic arguments become a bit more conceptual when set within the context of K-homology, and we shall do this for the quantization commutes with reduction phenomenon in the more or less independent Section 9.

2 Complex Manifolds and *K*-Theory

If X is a compact Hausdorff space, then as usual we shall denote by $K(X)$ the Atiyah-Hirzebruch K-theory group of X, or in other words the Grothendieck group generated from the semigroup of isomorphism classes of complex vector bundles on X. If X is a locally compact Hausdorff space, then we shall denote by $K(X)$ the Atiyah-Hirzebruch K-theory group "with compact supports". This is the reduced K-theory of the one-point compactification of X.

Individual vector bundles do not determine elements in K-theory when X is not compact, but bounded complexes of vector bundles

$$E_0 \leftarrow E_1 \leftarrow \cdots \leftarrow E_n$$

do determine K-theory elements whenever the support of the complex (the support is the smallest closed subset of X outside of which the complex is fiberwise exact) is compact. See for example [6] for this construction, which in fact accounts for all K-theory elements.

The most important instance of the construction of K-theory elements from complexes is that of the Bott element of an n-dimensional complex vector space V, which is the element in $K(V)$ determined by the complex

$$\Lambda^0 V^* \leftarrow \Lambda^1 V^* \leftarrow \cdots \leftarrow \Lambda^n V^*$$

of vector bundles with the indicated constant fibers, for which the differentials at $v \in V$ are given by contraction with v. The support of this complex is $\{0\} \subseteq V$.

The Bott element can be generalized, as follows. If V is a complex vector bundle over a locally compact space X, then by applying the Bott element construction to the fibers of the V we obtain a complex of vector bundles over the locally compact space V whose support is the zero section. If the original base space V is compact, then we obtain an element of $K(V)$, called the Thom element. If X is noncompact, then we obtain a Thom homomorphism

$$K(X) \to K(V).$$

It maps the element of $K(X)$ associated to a compactly supported complex E_* over X to the element of $K(V)$ associated to the tensor product of the pullback of E_* to V with the "Thom" complex just described (the tensor product complex is compactly supported). The Thom homomorphism is an isomorphism.

For our purposes, the most important feature of K-theory is the existence of wrong-way maps

$$f_*\colon\ K(M_1) \to K(M_2)$$

associated to continuous maps $f\colon M_1 \to M_2$ between, for now, almost-complex smooth manifolds (we shall consider broader classes of spaces in due course). The construction, which was carried out by Atiyah and Hirzebruch in their earliest works on K-theory (see [3], but see for example [18] for a complete treatment), is functorial and homotopy-invariant. As a result, the construction for smooth embeddings determines the construction in general, since we can factor any map, up to homotopy as a composition

$$M_1 \to M_2 \times V \to M_2,$$

where V is a finite-dimensional complex vector space, the first map is a smooth embedding and the second is the projection, which is homotopy inverse to a smooth embedding. As for embeddings, the map f_* is defined as follows in the special case where the normal bundle NM_1 admits a complex vector bundle structure for which there is an isomorphism

$$TM_1 \oplus NM_1 \cong f^*TM_2$$

of complex vector bundles over M_1 (this is the only case that shall concern us, and in any case the general case quickly reduces to this one). The wrong-way map is defined to be the composition

$$K(M_1) \to K(NM_1) \to K(M_2)$$

of the Thom homomorphism with the natural map on K-theory groups associated to the inclusion of the normal bundle NM_1 as an open tubular neighborhood in M_2.

3 Stable Complex Structures

Recall that a stable complex structure on a real vector bundle V is a complex vector bundle structure on some direct sum $\mathbb{R}^k \oplus V$ (the first summand is the trivial bundle of rank k). Two stable complex structures on V, given by complex structures on $\mathbb{R}^{k_1} \oplus V$ and $\mathbb{R}^{k_2} \oplus V$, are equivalent if there exist trivial complex vector bundles $\mathbb{C}^{n_1}$ and $\mathbb{C}^{n_2}$ and an isomorphism of complex vector bundles

$$\mathbb{R}^{k_1} \oplus V \oplus \mathbb{C}^{n_1} \cong \mathbb{R}^{k_2} \oplus V \oplus \mathbb{C}^{n_2}.$$

The definition of stable complex structure may be applied to the tangent bundle of a smooth manifold: a stable complex structure on a smooth manifold is a stable complex structure on TM, and a stable complex manifold is a smooth manifold equipped with an equivalence class of stable complex structure.

The theory of wrong-way maps sketched in the last section extends to stable complex manifolds: Any smooth, or indeed continuous map between even-dimensional stable complex manifolds induces a wrong-way map

$$f_*\colon\ K(M_1) \to K(M_2)$$

with the same homotopy invariance and functoriality properties as before.[1] One way to see this is to note that a stable complex structure on an even-dimensional M is the same as an almost-complex structure on some $\mathbb{R}^{2k} \times M$. Now apply Bott periodicity.[2]

4 Compact Groups

Let G be a compact Lie group. Everything that we have just discussed generalizes to equivariant K-theory (see [23]), defined by using equivariant complex vector bundles over compact or locally compact G-spaces (and indeed it is not necessary for these things to assume that the compact group G is a Lie group).

There is one detail worth spelling out: In this paper we shall assume that the trivial bundles appearing in the definition of stable complex structure are trivial as equivariant bundles: The action of G on the constant fiber is the trivial action (for other purposes it would suffice to assume that equivariant Bott periodicity holds for these trivial bundles). This has the following useful consequence.

Lemma 4.1 *Let M be a G-equivariant smooth manifold and let*

$$J\colon\ \mathbb{R}^k \oplus TM \to \mathbb{R}^k \oplus TM$$

be a G-equivariant complex structure, giving M to be the structure of a G-equivariant stable complex manifold. Let g be an element of G and let $F \subseteq M$ be the fixed manifold of g.

(a) *The complex structure J over F restricts to a complex structure*

$$J|_F\colon\ \mathbb{R}^k \oplus TF \to \mathbb{R}^k \oplus TF,$$

giving F the structure of a stable complex manifold.

(b) *The complex structure J induces a complex structure on the normal bundle*

$$NF = TM|_F/TF \cong (\mathbb{R}^k \oplus TM)|_F/\mathbb{R}^k \oplus TF.$$

Proof The subbundle $\mathbb{R}^k \oplus TF$ of $(\mathbb{R}^k \oplus TM)|_F$ is precisely the subbundle on which G acts trivially in each fiber. By equivariance it is invariant under J, and so (a) is proved. Item (b) is an immediate consequence of (a).

Wrong-way maps in equivariant K-theory are defined for continuous, equivariant maps between smooth, equivariant, stable complex manifolds. They are compatible with wrong-way maps in the non-equivariant theory in a number of ways. For instance if $f\colon M \to N$ is a map

[1] Moreover, the restriction to even-dimensional manifolds can be dropped if one introduces the odd K-theory group as well.

[2] This requires that we identify $\mathbb{R}^{2k}$ with $\mathbb{C}^k$, which we do by mapping $(x_1, \cdots, x_{2k})$ to $(x_1 + \mathrm{i}x_2, \cdots, x_{2k-1} + \mathrm{i}x_{2k})$.

between stable complex manifolds on which G acts trivially, then there is a commuting diagram

$$\begin{tikzcd} K_G(M) \arrow[r, "f_*"] & K_G(N) \\ K(M)\otimes R(G) \arrow[u] \arrow[r, "f_*\otimes 1"'] & K(M)\otimes R(G) \arrow[u] \end{tikzcd}$$

Figure 1

with the vertical maps given by tensor product of bundles (viewed upstairs as bundles with trivial G-action) with representations (viewed as equivariant bundles pulled back from a one-point space). Incidentally, the vertical maps are isomorphisms (see [23, Proposition 2.2]).

5 Quantization

Of special interest in the theory of wrong-way maps is the collapse of a complex or even-dimensional stable complex manifold to a point,

$$p\colon\ M \to \mathrm{pt}.$$

According to the Atiyah-Singer index theorem, if we identify $K_G(\mathrm{pt})$ with the representation ring $R(G)$, and if E is a smooth, G-equivariant complex vector bundle over a closed almost-complex G-manifold M, then the wrong-way map

$$p_*\colon\ K(M) \to K_G(\mathrm{pt})$$

takes the class[3] $[E] \in K_G(M)$ to the Fredholm index of the Dolbeault operator on M coupled to E:

$$p_*(E) = \mathrm{Index}(D_{M,E}) \in R(G). \tag{5.1}$$

In recognition of the special role played by the wrong-way map, we shall introduce the following notations and terminologies.

Definition 5.1 *Let M be a smooth, even-dimensional, G-equivariant almost complex manifold and let $\alpha \in K_G(M)$ be any K-theory class. Let $p\colon M \to \mathrm{pt}$ be the map that collapses M to a point. We shall call the class*

$$p_*(\alpha) \in K_G(\mathrm{pt})$$

the quantization of the pair (M, α), and denote it by

$$Q_G(M, \alpha) \in R(G).$$

Let us now formulate the simplest instances of the theorems that we wish to prove in this paper.

Definition 5.2 *If $\alpha \in K_G(M)$ is any K-theory class, then denote by*

$$Q_G(M, \alpha)_0 \in \mathbb{Z}$$

the multiplicity of the trivial representation within the virtual representation $Q_G(M, \alpha) \in R(G)$.

[3] We will abuse the notation slightly and write E for both the bundle and its K-theory class.

Definition 5.3 *Assume that $0 \in \mathfrak{g}^*$ is a regular value of μ and that G acts freely on the submanifold $\mu^{-1}[0] \subseteq M$. Let*

$$M_0 = \mu^{-1}[0]/G.$$

Definition 5.4 *Let $\alpha \in K_G(M)$ be any K-theory class. Assume that $0 \in \mathfrak{g}^*$ is regular value of μ and assume that G acts freely on the submanifold $\mu^{-1}[0] \subseteq M$. Denote by $\alpha_0 \in K(M_0)$ the class that maps to the restriction of α to $\mu^{-1}[0]$ under the pullback isomorphism*

$$K(\mu^{-1}[0]/G) \xrightarrow{\cong} K_G(\mu^{-1}[0]).$$

Now assume that M is an even-dimensional stable complex G-manifold, that E is a G-equivariant Hermitian line bundle on M, equipped with a connection using which we define a moment map μ, that $0 \in \mathbb{R}$ is a regular value and that G acts freely on $\mu^{-1}[0]$. We will note later that the reduced manifold M_0 carries a natural stable complex structure.

Theorem 5.1 *If M is closed and $G = U(1)$, then*

$$Q_G(M, E)_0 = Q(M_0, E_0)$$

for all k.

Theorem 5.2 *Whether or not M is closed and whether or not $G = U(1)$, if $\alpha \in K_G(M)$ is any K-theory class, then*

$$Q_G(M, E^k \otimes \alpha)_0 = Q(M_0, E_0^k \otimes \alpha_0)$$

for all sufficiently large $k > 0$.

6 A Bordism Construction

The definition of stable complex manifold applies equally well to manifolds with boundary, and every stable complex structure on a smooth manifold W induces one on ∂W thanks to the isomorphism

$$\mathbb{R}^{k+1} \oplus T\partial W \cong \mathbb{R}^k \oplus \mathbb{R} \oplus T\partial W \cong \mathbb{R}^k \oplus TW|_{\partial W}$$

determined by an outward-pointing normal vector field on the boundary.

An important property of the wrong-way map construction in the context of stable complex manifolds, and hence of quantization, is its bordism invariance: If M is the boundary of W, and if $f\colon M \to N$ extends to W, then the composition

$$K(W) \xrightarrow{\iota^*} K(M) \xrightarrow{f_*} K(N),$$

in which the first map is restriction to the boundary, is the zero map. In particular, we have the following proposition.

Proposition 6.1 *If $M_1 \sqcup -M_2$ is the boundary of a compact stable complex manifold W, and α_1 and α_2 are the restrictions M_1 and M_2 of a K-theory class on W, then $Q(M_1, \alpha_1) = Q(M_2, \alpha_2)$.*

This may be applied in the following context. For the rest of this section $G = U(1)$, let M be smooth, closed G-equivariant stable complex manifold, and let E be a G-equivariant smooth Hermitian line bundle over M. Let

$$\mu = \mu_X : M \to \mathbb{R} \tag{6.1}$$

be the Kostant moment map associated to some Hermitian connection on E, as in the introduction, where the Lie algebra generator

$$X \in \mathrm{g} = \mathfrak{u}(1) \tag{6.2}$$

is chosen so that

$$\exp(tX) = \mathrm{e}^{2\pi \mathrm{i} t} \in G = U(1)$$

for all $t \in \mathbb{R}$. It follows from (1.2) that if $m \in M$ is fixed by G, then the action of G on the fiber E_m is given by the formula

$$\exp(tX) \cdot s = \mathrm{e}^{2\pi \mathrm{i} t \mu(m)} s, \quad \forall s \in E_m,\ \forall t \in \mathbb{R}. \tag{6.3}$$

So the value $\mu(m)$ is an integer, namely the weight of the representation of $U(1)$ on the line E_m. Conversely, if $m \in M$ and $\mu(m)$ is non-integral, then m is not fixed by G, so the action is locally free near the G-orbit of m. Thus, we have the following lemma.

Lemma 6.1 *The G-action on M is locally free everywhere except on the integral level sets of μ.*

Choose real numbers such that

$$0 < a < b < 1 \tag{6.4}$$

and such that a and b are regular values of μ (the exact values are not important and make no essential difference in the construction that follows). By the lemma, the group G acts locally freely on $\mu^{-1}[a]$ and $\mu^{-1}[b]$. In fact we shall assume in this section that the action is free on these level sets and the others in between (the locally free case will be considered later).

We are going to construct a bordism of stable complex G-manifolds from M to a disjoint union of two manifolds $M_{\leq a}$ and ${}_{b\leq}M$. The two manifolds are easy to describe topologically:

$M_{\leq a}$ is obtained from the region of M where $\mu \leq a$ by collapsing each G -orbit where $\mu = a$ to a single point. (6.5)

${}_{b\leq}M$ is obtained from the region of M where $b \leq \mu$ by collapsing each G-orbit where $\mu = b$ to a single point. (6.6)

The G-actions on these manifolds are the obvious ones inherited from M. Moreover on the complements of the collapse regions the manifolds agree with the corresponding open subsets of M as stable complex G-manifolds. The complementary closed sets

$$M_a = \mu^{-1}[a]/G \quad \text{and} \quad M_b = \mu^{-1}[b]/G$$

are fixed-set components in $M_{\leq a}$ and ${}_{b\leq}M$, respectively. So according to Lemma 4.1, they acquire from $M_{\leq a}$ and ${}_{b\leq}M$ the structure of stable complex manifolds in their own right, as well as submanifolds with complex normal bundles. The normal bundles may be described as follows.

$$\begin{array}{l}\text{The group } G = U(1) \text{ acts on the fibers of the either normal bundle with weight} \\ \text{1. In fact the normal bundle of } M_a \text{ is isomorphic to the complex line bundle} \\ (\mu^{-1}[a] \times \mathbb{C})/U(1),\end{array} \tag{6.7}$$

where $U(1)$ acts on $\mu^{-1}[a]$ through the G-action, and on C through the weight -1 action, while G acts only on the left hand factor of the product. The same holds for M_b.

As for the fixed-set stable complex structures on M_a and M_b, they may be described as follows. If c is any regular value of μ, and if G acts freely on the level set $\mu^{-1}[c]$, then we can form the "reduced" manifold $M_c = \mu^{-1}[c]/G$ and equip it with a stable complex structure through the identification

$$\mathbb{R} \oplus T(\mu^{-1}[c]) \cong (TM)|_{\mu^{-1}[c]}$$

that maps $1 \in \mathbb{R}$ to any G-invariant vector field Y defined on M near $\mu^{-1}[c]$ with $Y(\mu) > 0$ (this gives $\mu^{-1}[c]$ a G-invariant stable complex structure), and the identification

$$\mathbb{R} \oplus TM_c \cong T\mu^{-1}[c]/G$$

that maps $1 \in \mathbb{R}$ to the G-invariant vector field on $\mu^{-1}[c]$ determined by the Lie algebra generator $X \in \mathfrak{g}$ in (6.2).

$$\begin{array}{l}\text{The stable complex structure that } M_a \text{ acquires as a fixed manifold in } M_{\leq a} \\ \text{agrees with the stable complex structure that acquires as a reduction of } M. \\ \text{The stable complex structure that } M_b \text{ acquires as a fixed manifold in } {}_{b\leq}M \\ \text{is opposite to the stable complex structure that acquires as a reduction of } M.\end{array} \tag{6.8}$$

Let us now construct the bordism. Define $P \subseteq M \times \mathbb{C}$ by

$$P = \{(m, z) \in M \times \mathbb{C} : |z|^2 \geq (\mu(m) - a)(\mu(m) - b) \geq |z|^2 + k\}. \tag{6.9}$$

Here k is any value that is less than the minimum value of the function $(\mu - a)(\mu - b)$. With this choice of k, the function

$$(m, z) \mapsto (\mu(m) - a)(\mu(m) - b) - |z|^2$$

is regular on $M \times \mathbb{C}$ at the values 0 and k, so P is indeed a smooth submanifold with boundary of $M \times \mathbb{C}$.

Equip $\mathbb{C}$ with the weight -1 action of $U(1)$.[4] The diagonal action of $U(1)$ on $M \times \mathbb{C}$ is free on P (this uses our hypothesis that the action of G on M is free between $\mu = a$ and $\mu = b$), so the quotient

$$W = P/U(1)$$

is a smooth manifold with boundary. The action of G on the first factor of $M \times \mathbb{C}$ alone descends to an action of G on W. The manifold P inherits a G- and $U(1)$-invariant stable

[4]Other choices of action are possible, and give useful variations on the construction (see Section 9).

complex structure from $M \times \mathbb{C}$. The quotient bundle TP/G over W inherits a G-invariant stable complex structure from TP. Now use the identification

$$\mathbb{R} \oplus TW \cong TP/U(1)$$

that maps the trivial summand $\mathbb{R}$ to the subbundle of $TP/U(1)$ spanned by the vector field X generating the diagonal $U(1)$-action to equip W with a stable complex structure. The stable complex G-manifold W so-obtained will be our bordism.

Consider the boundary component of P with

$$(\mu(m) - a)(\mu(m) - b) = |z|^2 + k.$$

By the definition of the constant k that appears here, if (m, z) is a point in this boundary component, then $z \neq 0$. The map

$$(m, z) \mapsto \text{phase}(z) \cdot m, \tag{6.10}$$

where the dot denotes the action of $G = U(1)$ on M, identifies the corresponding boundary component of the quotient W with the stable complex G-manifold $-M$.

Consider next the boundary component of P with

$$(\mu(m) - a)(\mu(m) - b) = |z|^2.$$

It separates into two parts: (1) $\mu \leq a$, (2) $b \leq \mu$. Write the corresponding boundary component of $W = P/U(1)$ as

$$M_{\leq a} \sqcup {}_{b\leq}M$$

accordingly.

The open subset of $M_{\leq a}$ with $\mu < a$ identifies as a G-stable complex manifold with the similar part of M via the map (6.10), and the complement, the closed submanifold with $\mu = a$, identifies with the smooth manifold

$$M_a = \mu^{-1}[a]/G$$

via the projection from $M \times \mathbb{C}$ to M. Near $\mu^{-1}[a] \subseteq M$, the manifold M has a product form

$$M \cong \mu^{-1}[a] \times \mathbb{R},$$

in such a way that μ is the projection onto the $\mathbb{R}$-factor. We find from this that

$$M_{\leq a} \cong (\mu^{-1}[a] \times \mathbb{C})/U(1)$$

near $\mu = a$ via the projection

$$\mu^{-1}[a] \times \mathbb{R} \times \mathbb{C} \to \mu^{-1}[a] \times \mathbb{C}.$$

The structure of the normal bundle can be determined from this. Similar remarks apply to ${}_{b\leq}M$.

Now suppose that $\alpha \in K_G(M)$. We can think of α as a class

$$\alpha \in K_{G\times U(1)}(M)$$

by making $U(1)$ act through its identification with G. Then we can use the projection map from the manifold P in (6.9) to M (which is a proper map) to pull back α to a class in $K_{G\times U(1)}(P)$. Finally, since the action of $U(1)$ on P is free, we can use the pullback isomorphism

$$K_G(W) \xrightarrow{\cong} K_{G\times U(1)}(P) \tag{6.11}$$

to obtain a class in $K_G(W)$ from α. Its restriction to the boundary component M is the original class α. We will use the same notation α for the restrictions to $M_{\leq a}$ and ${}_{b\leq}M$, and we arrive at the following formula from the bordism invariance of quantization:

$$Q(M,\alpha) = Q(M_{\leq a},\alpha) + Q({}_{b\leq}M,\alpha). \tag{6.12}$$

As was pointed out by Duistermaat et al [13], this may be used to give very simple proofs of Theorems 5.1–5.2 in the case $G = U(1)$.[5] The argument uses the well-known fixed-point formulas in K-theory that we shall review in the next section.

7 Fixed-Point Formulas

We shall assume that G is a torus. We review the basic fixed-point formulas in equivariant K-theory with a view to extend them a little in subsequent sections.

Definition 7.1 *Denote by $R(G)_g$ the localization of the character ring $R(G)$ at $g\in G$, or in other words, the ring of fractions obtained by inverting all virtual representations whose character at g is non-vanishing. Similarly, denote by $K_G(X)_g = K_G(X)\otimes_{R(G)} R(G)_g$ the localization of the $R(G)$-module $K_G(X)$.*

Denote by $F\subseteq M$ the fixed set of g (which is the same as the G-fixed set since $g\in G$ is a topological generator). As explained in Lemma 4.1, *the fixed-set is a submanifold of M and it carries a canonical stable complex structure. Denote by $\iota\colon F\to M$ the inclusion.*

Theorem 7.1 (see [23, Proposition 4.1] or [5, Theorem 1.1]) *The restriction map*

$$\iota^* :\ K_G(M)_g \to K_G(F)_g$$

in localized equivariant K-theory is an isomorphism.

Definition 7.2 *If H is any G-equivariant complex vector bundle over a compact G-space X, then write*

$$\lambda(H) = \Sigma(-1)^p[\Lambda^p(H^*)] \in K_G(X).$$

The relevance of this class to the present discussion is that the composition

$$K_G(F) \xrightarrow{\iota_*} K_G(M) \xrightarrow{\iota^*} K_G(F)$$

is given by multiplication by $\lambda(NF)$, where NF is the normal bundle of F in M. This is evident from the definition of wrong-way maps in Section 2.

Theorem 7.2 (see [5, Lemma 2.7]) *If NF is the normal bundle of F in M (with the complex structure given in Lemma* 4.1), *then the element $\lambda(NF)$ is invertible in the localized equivariant K-theory ring $K_G(F)_g$.*

---

[5]The arguments also have a lot in common with the works of Lusztig and Kosniowski (see [20]) and Atiyah and Hirzebruch (see [4]). These authors worked in the holomorphic context, and Duistermaat et al worked in the symplectic context, but the arguments for stable complex manifolds are the same.

Remark 7.1 If F is noncompact, then the statement "$\lambda(NF)$ is invertible" should be taken to mean "multiplication by $\lambda(NF)$ is an invertible map from $K_G(F)_g$ to itself", where "multiplication by $\lambda(NF)$" means "the alternating sum of tensor products with the bundles $\Lambda^p(NF^*)$".

The definition of quantization as a wrong-way map and functoriality of wrong-way maps now immediately give the following Atiyah-Bott-Segal formula, in which we have altered slightly our usage of the symbol F.

Theorem 7.3 (Compare [5, Proposition 2.8])

$$Q_G(M,\alpha) = \sum_F Q_G(F, \lambda(NF)^{-1} \cdot \iota^*\alpha) \in R(G)_g$$

for every $\alpha \in K_G(M)$. Here the sum is over the connected components of the fixed set in M.

Remark 7.2 We could of course consider the fixed set as a single manifold, as we have done up to now, rather than as the union of its components. But calculations are usually done component-wise, so the above formulation of the fixed-point theorem is usually the most convenient.

8 Quantization Formula for Circle Actions

Our goal in this section is to prove Theorems 5.1–5.2 for $G{=}U(1)$ following the method of Duistermaat et al (see [13]). The method has been repeated, with variations, several times in the literature (see for example [12, 22]). Our purpose in repeating it one more is to record enough details to determine how the method generalizes to the contexts that we shall consider in the coming sections.

To begin with, we shall calculate in the case of a stable complex G-manifold M and equivariant line bundle E for which over each component of the G-fixed set in M the group $G = U(1)$ acts on line bundle E with non-negative weight. Obviously the formula (6.12) makes this assumption relevant to our ultimate goal, and we shall come back to the relevance of (6.12) after handling the special case.

Consider the ring $\mathbb{Q}[z^{-1}, z]\!]$ of rational Laurent series with finite singular parts. Because $\mathbb{Q}[z^{-1}, z]\!]$ is in fact a field, the inclusion

$$R(G) \cong \mathbb{Z}[z^{-1}, z] \to \mathbb{Q}[z^{-1}, z]\!]$$

extends to a homomorphism

$$R(G)_g \to \mathbb{Q}[z^{-1}, z]\!],$$

and it is in the Laurent series ring that we shall do our calculations. The advantage of doing so is that we can speak of the "coefficient of z^0" in the contribution of each component of the fixed set (which is an element of $R(G)_g$) to the overall quantization $Q(M, E^k \cdot \alpha)$ (which is an element of $R(G)$).

Lemma 8.1 *If F is a component of the fixed set in M, and if G acts with positive weight on the fibers of the line bundle E over F, then the coefficient of z^0 in*

$$Q(F, \lambda(NF)^{-1} \cdot E^k \cdot \alpha) \in \mathbb{Q}[z^{-1}, z]\!]$$

is zero for all $k \gg 0$. When $\alpha = 1$ the coefficient is zero for all $k > 0$.

Proof To keep the notation simple we shall consider the case $\alpha = 1$ and $k = 1$. The other cases are easy variations.

Since G acts trivially on F the natural map

$$K(F) \otimes_{\mathbb{Z}} R(G) \to K_G(F)$$

is an isomorphism. So for each $s \in \mathbb{Z}$ we can speak of the coefficient of z^s in any element $\gamma \in K_G(F)$, and define the valuation of γ to be the least s for which the coefficient is nonzero. With this terminology, our aim is to show that

$$\mathrm{val}(Q_G(F, \lambda(NF)^{-1} \cdot E)) > 0.$$

The key point is that the valuation of $\lambda(NF)$ is nonpositive:

$$\mathrm{val}(\lambda(NF)) \leq 0, \tag{8.1}$$

and moreover the corresponding leading coefficient is a unit in the ring $K(F)$.

According to Theorem 7.2 there is some $\beta \in K_G(F)$ and an element $\gamma \in R(G)$ (whose character is non-vanishing at $g \in G$) such that

$$\lambda(NF) \cdot \beta = \gamma \in K_G(F). \tag{8.2}$$

Since the leading coefficients of $\lambda(NF)$ and γ are units, we have that

$$\mathrm{val}(\lambda(NF)) + \mathrm{val}(\beta) = \mathrm{val}(\gamma),$$

so

$$\mathrm{val}(\beta) - \mathrm{val}(\gamma) \geq 0.$$

Now

$$Q_G(F, \lambda(NF)^{-1} \cdot E) = Q_G(F, \beta \cdot E) \cdot \gamma^{-1}$$

and

$$\begin{aligned}
\mathrm{val}(Q_G(F, \beta \cdot E) \cdot \gamma^{-1}) &= \mathrm{val}(Q_G(F, \beta \cdot E)) + \mathrm{val}(\gamma^{-1}) \\
&\geq \mathrm{val}(\beta \cdot E) + \mathrm{val}(\gamma^{-1}) \\
&= \mathrm{val}(\beta) + \mathrm{val}(E) - \mathrm{val}(\gamma) \\
&\geq \mathrm{val}(E),
\end{aligned}$$

where the first inequality is a result of Figure 1. Our assumption on E completes the proof.

Now suppose that G acts with weight zero on the fibers of E over some component F of the fixed set. Lemma 8.1 does not apply, but a different hypothesis allows us to reach the same conclusion. Namely assume that the normal bundle is a line bundle of weight 1.[6] Then since

$$\lambda(NF) = 1 - NF^{-1},$$

we see that $\mathrm{val}(\lambda(NF)) = -1$, which is a bit better than (8.1). The argument in the lemma shows that

$$\mathrm{val}(Q_G(F, \lambda(NF)^{-1} \cdot E)) > 0$$

[6] Actually it would be enough to have such a summand in the normal bundle.

once again.

We now return to the bordism constructed in Section 6 and the formula (6.12). The above arguments show that

$$Q_G({}_{b\leq}M, E)_0 = 0$$

and also

$$Q_G({}_{b\leq}M, E^k \cdot \alpha)_0 = 0,$$

when $k \gg 0$.

We can analyze $M_{\leq a}$ in a similar way, but using the field $\mathbb{Q}\llbracket z^{-1}, z]$ of Laurent series with only finitely many positive order terms, and the valuation that indicates the highest nonzero coefficient instead of the lowest. We find that if F is a component of the fixed set where G acts on E with a negative weight, then F does not contribute to $Q_G(M_{\leq a}, E^k \cdot \alpha)_0$.

Let us assume now that (as in the statements of Theorems 5.1–5.2) G acts locally freely on $\mu^{-1}[0]$. Then the only component of the G-fixed set in $M_{\leq a}$ remaining to be analyzed is $F = M_a$. This does contribute to $Q_G(M_{\leq a}, E^k \cdot \alpha)_0$. To calculate what it contributes, use the Neumann series with remainder

$$\begin{aligned}\lambda(NF)^{-1} &= (1 - NF^{-1})^{-1}\\ &= 1 + NF^{-1} + NF^{-2} + \cdots + NF^{-n}\lambda(NF)^{-1}.\end{aligned}$$

The previous arguments show that when we substitute this into the quantity

$$Q(F, \lambda(NF)^{-1} \cdot E) \in \mathbb{Q}\llbracket z^{-1}, z],$$

only the constant term 1 contributes to the z^0 term (for the last term, use NF^{-n} in place of E in the argument used in the proof of Lemma 8.1). As a result

$$Q_G(M_{\leq a}, E)_0 = Q(M_a, E),$$

and similarly

$$Q_G(M_{\leq a}, E^k \cdot \alpha)_0 = Q(M_a, E^k \cdot \alpha)$$

for $k \gg 0$.

These calculations complete the proofs of Theorem 5.1 and the G=$U(1)$ case of Theorem 5.2. If 0 is a regular value of μ, and if G acts freely on $\mu^{-1}[0]$, then for sufficiently small a, M_a is bordant to M_0, indeed isomorphic to it. So we can substitute M_0 for M_a in the formulas above.

Remark 8.1 Denote by $\mathbb{Q}\llbracket z^{-1}, z\rrbracket$ the vector space of doubly infinite Laurent series. This is no longer a ring, of course, but it contains both of the rings $\mathbb{Q}[z^{-1}, z\rrbracket$ and $\mathbb{Q}\llbracket z^{-1}, z]$ that we used above. In the situation where 0 is a regular value of μ and G acts freely on $\mu^{-1}[0]$, we may use this fact to define an alternative quantization

$$Q_G^{\text{local}}(M, E) \in \mathbb{Q}\llbracket z^{-1}, z\rrbracket \tag{8.3}$$

as follows: For each component F of the fixed-point set, expand its contribution to $Q_G(M, E)$ (via the fixed-point formula) in either $\mathbb{Q}[z^{-1}, z\rrbracket$ or $\mathbb{Q}\llbracket z^{-1}, z]$ according as G acts on the fibers of E over F by a positive or negative weight. Then add up these contributions in $\mathbb{Q}\llbracket z^{-1}, z\rrbracket$ to obtain (8.3).

The quantization is “local” in the sense that it is explicitly localized near the fixed set of the G-action. Using Lemma 8.1, one can see that no fixed set contributes to the coefficient of z^0 at all! On the other hand, the calculations above relate the local and original quantizations through the following attractive formula:

$$Q_G(M,E) = Q_G^{\text{local}}(M,E) + \sum_{n\in\mathbb{Z}} Q(M_0, E_0 \otimes {L_0}^n) z^n. \tag{8.4}$$

Here L_0 is the line bundle $(\mu^{-1}[0]\times\mathbb{C})/G$ over M_0 associated to the weight 1 action of G on $\mathbb{C}$ (viewed as an ordinary line bundle, with no group action).

Of course, the quantization commutes with reduction formula can be immediately deduced from (8.4), and to some extent the formula “explains” the relevance of reduced manifolds like M_0 to index theory and fixed-point theory. Therefore, it is an interesting problem to derive counterparts of (8.4) for other groups.

9 Some Remarks on *K*-Homology

In this section we shall briefly examine the quantization commutes with reduction problem from the perspective of K-homology.

K-homology is the dual theory to (equivariant) K-theory, and it is related to K-theory by functorial pairings

$$K^G(X) \otimes K_G(X\times Y) \to K_G(Y)$$

(here K_G denotes equivariant K-theory, as in the rest of this paper, and K^G is equivariant K-homology). In fact K-homology is characterized by these pairings in the sense that for a fixed X every transformation

$$K_G(X\times Y) \to K_G(Y)$$

that is natural in Y and a $K_G(Y)$-module homomorphism determines a class in the K-homology group $K^G(X)$, and conversely.[7] See [10] for more on this point of view on K-homology.

Along with the above abstract definition there are concrete constructions of K-homology. A functional-analytic definition was proposed by Atiyah [1] and worked out in detail by Kasparov (see [19]). An element of the analytic group $K^G_{\text{anal.}}(X)$ is determined by an equivariant Fredholm operator on a G-Hilbert space. The Hilbert space is required to carry an action of the algebra of continuous functions on the compact space X, and the operator should be compatible with this action in a certain sense. See [17] for a complete treatment of Kasparov's theory, at least in the non-equivariant case.

There are also geometric definitions. In the approach of Baum (see [8]), an element of the geometric group $K^G_{\text{geom.}}(X)$ is determined by a closed, even-dimensional stable complex (or, more generally spin^c) G-manifold M, together with an equivariant complex vector bundle E on M and a continuous map f from M to X. A natural transformation

$$Q^G \colon K^G_{\text{geom.}}(X) \to K^G_{\text{anal.}}(X)$$

is defined by associating to (M,E,f) a Fredholm operator obtained from the $\overline{\partial}_E$-operator (the algebra X acts on the associated Hilbert space of L^2-forms on M through f, which determines a homomorphism from $C(X)$ to $C(M)$).

[7] This holds for reasonable spaces X at least, for instance, the smooth closed manifolds that we shall be considering.

For a full treatment of the natural transformation Q in the non-equivariant case (see [9]). The further issues in the equivariant case are considered in [7].

As our notation suggests, we want to view the natural transformation as a quantization map, and indeed when X is a point, it may be identified as such, thanks to the index theorem. But what about reduction?

View the quotient map $X \to X/G$ as a G-equivariant map to a space with a trivial G-action. It induces a map in equivariant homology

$$K^G_{\text{anal./geom.}}(X) \to K^G_{\text{anal./geom.}}(X/G),$$

(either analytic or geometric), and because the G-action is trivial on the right-hand side there is a standard isomorphism

$$K_{\text{anal./geom.}}(X) \otimes_{\mathbb{Z}} R(G) \xrightarrow{\cong} K^G_{\text{anal./geom.}}(X), \tag{9.1}$$

which is analogous to the similar isomorphism in K-theory. Combining the two displays we obtain reduction homomorphisms in geometric and analytic K-homology

$$R\colon K^G_{\text{anal./geom.}}(X) \to K_{\text{anal./geom.}}(X) \otimes_{\mathbb{Z}} R(G).$$

Simply by naturality of the transformation Q, these fit into a "quantization commutes with reduction diagram"

$$\begin{tikzcd}
K^G_{\text{geom.}}(X) \arrow[r, "Q^G"] \arrow[d, "R_{\text{geom.}}"'] & K^G_{\text{anal.}}(X) \arrow[d, "R_{\text{anal.}}"] \\
K_{\text{geom.}}(X/G) \otimes R(G) \arrow[r, "Q"'] & K_{\text{anal.}}(X/G) \otimes_{\mathbb{Z}} R(G)
\end{tikzcd}$$

At this stage there is no "quantization commutes with reduction problem" since the diagram automatically commutes! The problem arises when one asks for explicit formulas for the reduction maps.

In fact there is a simple, concrete formula for the analytic reduction map: Take an equivariant Fredholm operator on a G-Hilbert space, and decompose it and the Hilbert space into their G-isotypical parts:

$$R_{\text{anal.}}(F, H) = \sum_{\sigma \in \widehat{G}} (F|_{H_\sigma}, H_\sigma) \otimes \sigma \in K_{\text{anal.}}(X/G) \otimes_{\mathbb{Z}} R(G).$$

Note that each isotypical component H_σ, although not a $C(X)$-module, is a $C(X/G)$-module.

If X is a point, then there is an isomorphism $K(X/G) \cong \mathbb{Z}$ that maps a Fredholm operator to its Fredholm index. The composition of quantization followed by reduction in the diagram is computed from a geometric cycle, and the multiplicities with which the irreducible representations of G occur in the index of the associated elliptic operator. This is half of the quantization commutes with reduction formula.

Unfortunately, there seems to be no general formula for the geometric reduction map: The problem is that the easily defined isomorphism (9.1) needs to be inverted, and there is in general no known way to do this in the geometric theory.

However, the methods of the previous section may be extended very slightly to handle some cases when $G = U(1)$.

Assume that M is a closed, even-dimensional stable complex G-manifold and that E is a smooth equivariant Hermitian line bundle on M. Define, as usual, a moment map by applying Kostant's formula to a connection on E. Assume that there are regular values $\pm a$ near $0 \in \mathbb{R}$ so that the action of G is free on the level sets $\mu^{-1}[\pm a]$.

Construct a manifold M^0 by starting from the region of M with $-a \leq \mu \leq a$ and collapsing each of the orbits with $\mu = \pm a$ to points. We might also write

$$M^0 = {}_{-a\leq}M_{\leq a}$$

in the notation of Section 6.

The detailed construction of M^0 can be carried out in two stages. The first one is exactly described in Section 6 and results in $M_{\leq a}$. The second one is similar, using $M_{\leq a}$ as input, except that we cut at $-a$, of course. But in addition, during the construction described in Section 6 we equip the second factor in $M_{\leq a} \times \mathbb{C}$ with the weight $+1$ action of $U(1)$ rather than the weight -1 action.

The effect of this is that the normal bundle of $M_{-a} \subseteq M^0$ carries a weight $+1$ action of G, rather than a weight -1 action as before. As for the fixed-set stable complex structure on M_{-a}, it is the opposite of the stable complex structure M_{-a} received as a reduced manifold of M.

We have the following theorem.

Theorem 9.1 *Given a G-equivariant geometric cycle (M, E, f), the multiplicity of the trivial representation in the class*

$$R_{\text{geom.}}(M, E, f) \in K_{\text{geom.}}(X/G) \otimes_{\mathbb{Z}} R(G)$$

is the K-homology class of the triple (M^0, E^0, f^0), where M^0 is as above, E^0 is the line bundle on M^0 obtained from E, and $f^0\colon M^0 \to X/G$ is the continuous map induced from f.

To make the theorem plausible, it should be noted that geometric cycles related by a bordism (see [8–9]) determine the same geometric K-homology class. The theorem is proved by using this and essentially the same fixed-point techniques used in the previous section (so there is nothing new in the approach).

Theorem 9.1 is related to the usual statement of quantization commutes with reduction in the following way. If the action of G is free throughout the region $-a \leq \mu \leq a$, then M^0 can be constructed more simply as a fiber bundle

$$M^0 = (\mu^{-1}[0] \times \mathbb{CP}^1)/G$$

over the reduced manifold M_0, with complex projective fibers. Now it is built into Baum's geometric theory that if one cycle is a bundle over the other with $\mathbb{CP}^1$ fibers (with a compact structure group preserving the complex structure on the projective space), then the two cycles determine the same cohomology class (see [8–9]). So in this instance we can replace (M^0, E^0, f^0) in the above theorem with (M_0, E_0, f_0), and thus recover the usual quantization commutes with reduction result.

A final remark: If we make further assumptions about the freeness of the G-action on sufficiently many level sets (or pass to orbifolds), then we can calculate all the multiplicities in

$R_{\mathrm{geom.}}(M,E,f)$ geometrically by shifting the construction above to other $n \in \mathbb{Z}$. We obtain a sort of "Fourier series"

$$R_{\mathrm{geom.}}(M,E,f) = \sum_{n\in\mathbb{Z}} (M^n, E^n, f^n) \otimes z^n$$

in the group $K_{\mathrm{geom.}}(X/G) \otimes_{\mathbb{Z}} R(G)$.

10 Orbifolds

As we move from $G = U(1)$ to other groups it will become helpful to drop the hypothesis that G acts freely on $\mu^{-1}[0]$, and assume only that the action is locally free. The reason is that we shall work inductively on the rank of G, and it is much easier to keep track of local freeness during the inductive stages than to keep track of freeness.

To accommodate the change we shall work with orbifolds. The following simplified version of the general definition is the most convenient for our purposes (compare [25]).

Definition 10.1 *An orbifold structure on a locally compact Hausdorff space M is a triple*

$$\widehat{M} = (H, P, q)$$

consisting of a compact Lie group H, a smooth manifold P that is equipped with a locally free action of H, and a homeomorphism $q\colon P/H \to M$.

We shall also use the term orbifold, meaning a space with an orbifold structure. The dimension of an orbifold $\widehat{M}$ is the dimension of P minus the dimension of H.

Definition 10.2 *Two orbifold structures (H_1, P_1, q_1) and (H_2, P_2, q_2) on the same space M are equivalent if there is a third orbifold structure of the form $(H_1 \times H_2, P, q)$ and a commuting diagram*

$$\begin{tikzcd}
& P \arrow[dl, "p_1"'] \arrow[dd, "q"] \arrow[dr, "p_2"] & \\
P_1 \arrow[dr, "q_1"'] & & P_2 \arrow[dl, "q_2"] \\
& M &
\end{tikzcd}$$

Figure 2

in which

(a) *the individual actions of H_1 and H_2 on the manifold P are free, not just locally free,*
(b) *the maps p_1 and p_2 are H_1-equivariant and H_2-equivariant, respectively, and*
(c) *these maps identify P_1 and P_2 with the quotient manifolds P/H_2 and P/H_1, respectively.*

Example 10.1 If $\widehat{M} = (H, P, q)$ is an orbifold structure on M, and the action of H on P is free, not just locally free, then the map $q\colon P \to M$ identifies M with the smooth manifold H/P, and $\widehat{M}$ is equivalent to the trivial orbifold structure $(\{e\}, M, \mathrm{id})$ on M.

Definition 10.3 *The orbifold K-theory of an orbifold structure $\widehat{M} = (H, P, q)$ is the equivariant K-theory group*

$$\widehat{K}(\widehat{M}) = K_H(P).$$

Equivalences between orbifold structures determine isomorphisms between orbifold K-theory groups via pullbacks, as in the following diagram:

$$\begin{tikzcd} & K_{H_1\times H_2}(P) & \\ K_{H_1}(P_1) \arrow[ur, "p_1^*", "\cong"'] & & K_{H_2}(P_2) \arrow[ul, "p_2^*"', "\cong"] \end{tikzcd}$$

We define stable complex orbifolds as follows.

Definition 10.4 *A stable complex structure on an orbifold $\widehat{M} = (H, P, q)$ is an H-equivariant stable complex structure on the quotient bundle $TP/\mathfrak{h}$, where $\mathfrak{h}$ denotes the subbundle of TP tangent to the action of H.*

Next, it is easy to generalize all these definitions to the equivariant context, in which a compact group G acts on M. For instance in a G-equivariant orbifold structure $\widehat{M} = (H, P, q)$ on a G-space M, the manifold P is required to carry a smooth G-action commuting with the H-action and the map q is required to be G-equivariant. In addition, then G-equivariant K-theory of $\widehat{M}$ is the $G \times H$-equivariant K-theory of P, and a G-equivariant stable complex structure is a $G \times H$-equivariant stable complex structure on $TP/\mathfrak{h}$.

We shall also use concepts such as bordism (of orbifold structures involving the same group H) and smooth maps between orbifolds (again, in the simplest situations involving the same group H) without further explanation.

We shall need suitable quantization maps

$$Q\colon\ \widehat{K}_G(\widehat{M}) \to R(G) \tag{10.1}$$

in the orbifold context. A geometric definition (in terms of Bott periodicity, wrong-way maps, and so on) does not appear to be known. However a quantization map (10.1) is easy to define using transversally elliptic operators (see [2]): The stable complex structure on $\widehat{M} = (H, P, q)$ determines a $G \times H$ equivariant and H-transversally elliptic Dolbeault operator on P. Restricting to the H-invariant sections we obtain a Fredholm operator with index in $R(G)$. More generally, coupling with classes in the group $\widehat{K}(\widehat{M}) = K_{G\times H}(P)$ gives us the quantization map that we need.

In any case we shall assume given quantization maps (10.1) with the following properties (that are routines to verify from the analytic definition):

(a) $R(G)$-linearity and independence of presentation. The quantization map is $R(G)$-linear and compatible with the isomorphisms in Figure 2.

(b) Compatibility with the wrong-way maps. If $\iota\colon \widehat{M}_1 \to \widehat{M}_2$ is a G-equivariant embedding, then the following diagram is commutative:

$$\begin{tikzcd} \widehat{K}_G(\widehat{M}_1) \arrow[r, "Q_G"] & R(G) \\ \widehat{K}_G(\widehat{M}_2) \arrow[u, "\iota_*"] \arrow[r, "Q_G"'] & R(G) \arrow[u, "="'] \end{tikzcd}$$

(c) Bordism-invariance. If $(\widehat{M}, \alpha)$ is bordant to $(\widehat{M}', \alpha')$, then $Q(\widehat{M}, \alpha) = Q(\widehat{M}', \alpha')$.

(d) Trivial action. If $G = G_2 \times G_1$, and G_1 fixes the manifold P in an orbifold structure (P, H, q) pointwise, then the diagram

$$\begin{tikzcd} \widehat{K}_{G_2\times G_1}(\widehat{M}) \arrow[r, "Q_{G_2\times G_1}"] & R(G_2 \times G_1) \\ \widehat{K}_{G_2}(\widehat{M}) \otimes R(G_1) \arrow[u] \arrow[r, "Q_{G_2}\otimes 1"'] & R(G_2) \otimes_{\mathbb{Z}} R(G_1) \arrow[u] \end{tikzcd}$$

is commutative (compare Figure 1).

At the very end of the paper we shall use one additional property, borrowed from the theory of the analytic index. We shall only formulate it in the context we need, though it holds more generally.

(e) Multiplicative axiom. Let F be a stable complex G-manifold, let $\beta \in K_G(F)$, and assume that

$$Q(F, \beta) = 1 \in R(G).$$

Let $\widehat{M} = (G, P, q)$ be any stable complex orbifold and let $\alpha \in \widehat{K}(\widehat{M})$. If we form the locally compact space

$$W = P \times_G F$$

and the orbifold structure

$$\widehat{W} = (G, P \times F, q),$$

then

$$Q(\widehat{W}, \alpha \cdot \beta) = Q(\widehat{M}, \alpha).$$

This is a mild generalization (to orbifolds) of the multiplicative property of the index used by Atiyah and Singer (compare [6, Axiom B3], and compare [9] for a more modern proof which adapts easily to the present circumstances).

11 Fixed-Point Formulas for Orbifolds

Unless otherwise indicated, we shall assume that the Lie groups (G, H, etc.) occurring in this section are compact, connected and abelian (that is, they are tori). Our aim is to formulate and prove fixed-point formula in the orbifold context, and then use it to generalize Theorems 5.1–5.2.

The orbifold fixed-point formula is not a completely straightforward generalization of the manifold case, mostly because if $\widehat{M} = (H, P, q)$ is a G-equivariant orbifold structure on M, then there is a difference between being fixed by G in M and being fixed by G in P: G-fixed-points in M do not necessarily lift to G-fixed-points in P. In what follows we shall indicate how to handle this point. At the same time we shall work in a slightly more general context than that of Section 7 by studying the fixed set of a subtorus $T \subseteq G$ rather than the fixed set of G itself.

Lemma 11.1 *Let $\widehat{M} = (H, P, q)$ be a G-equivariant orbifold structure, and let $T \subseteq G$ be a subtorus. If F is the fixed set of T in M, then the set $q^{-1}[F] \subseteq P$ is a disjoint union of a locally finite collection of smooth, closed submanifolds of P, each of which is the fixed-point manifold of a torus $L < G \times H$ that projects onto T with finite kernel.*

Proof Let $g \in T$ be a topological generator. For each point $p \in q^{-1}[F]$ there exists $h \in H$ such that the element $(g, h) \in G \times H$ fixes p. The connected component of the identity in the subgroup of $G \times H$ generated by (g, h) is a torus that projects onto $T \subseteq G$ with finite kernel. We call this a good torus. The fixed-point set of any good torus in $G \times H$ is a smooth closed submanifold of P that is contained in $q^{-1}[F]$, and the union of all such fixed-point subsets is $q^{-1}[F]$. The fixed-point sets for different good tori in $G \times H$ are disjoint, because if a point was fixed by two different good tori it would be fixed by the torus generated by the pair, and this would contain a circle subgroup of H, contrary to the assumption that the action of H on P is locally free. Finally, there are only finitely many distinct good tori with nonempty fixed-point sets that meet any given compact set in P. So $q^{-1}[F]$ is a locally finite disjoint union of smooth, closed submanifolds of P, as required.

Corollary 11.1 *The datum $\widehat{F} = (H, q^{-1}[F], q)$ is a G-equivariant orbifold structure on F.*

Definition 11.1 *Let $\widehat{M} = (H, P, q)$ be a G-equivariant orbifold structure on a G-space M, and let $g \in G$. The localized K-theory group $\widehat{K}(\widehat{M})_g$ is defined by*

$$\widehat{K}_G(\widehat{M})_g = \widehat{K}_G(\widehat{M}) \otimes_{R(G)} R(G)_g.$$

Theorem 11.1 *Assume that $g \in G$ generates a torus $T \subseteq G$, and we denote by $F \subseteq M$ the T-fixed set in M. The restriction map*

$$\widehat{K}_G(\widehat{M})_g \to \widehat{K}_G(\widehat{F})_g$$

in localized K-theory is an isomorphism.

Proof The standard argument, which uses the cohomological properties of the K-theory functor, can be adapted to the current situation.

It suffices to show that the localized group $\widehat{K}_G(\widehat{M \setminus F})_g$ is zero. Cover $M \setminus F$ by G-invariant open sets U such that $q^{-1}[U]$ has the form

$$q^{-1}[U] \cong (G \times H) \times_\Gamma V,$$

where Γ is the isotropy group of a point in $q^{-1}[U]$ and V is some Γ-space (here we are making an exception to our general rule: Γ need not be connected). There is an induction isomorphism

$$\widehat{K}_G(\widehat{U}) \cong K_\Gamma(V).$$

The ring $R(G)$ acts on the right-hand side through the projection map from Γ to G and the associated homomorphism from $R(G)$ to $R(\Gamma)$.

The image of Γ in G can not contain the generator $g \in T$, so there is a character of G that is trivial on the image of Γ but nontrivial on g. From this it follows that

$$R(\Gamma) \otimes_{R(G)} R(G)_g = 0$$

and of course it follows in turn that

$$\widehat{K}_G(\widehat{U})_g = K_\Gamma(V)_g = 0.$$

So $\widehat{K}_G(\widehat{M \setminus F})_g$ is zero by a Mayer-Vietoris argument.

Assume now that $\widehat{M} = (H, P, q)$ is equipped with a G-equivariant, stable complex structure. Let us continue to denote by g a generator of a torus $T \subseteq G$, and by $F \subseteq M$ the fixed set of g. Denote by $\widehat{NF}$ the orbifold normal bundle, that is, the normal bundle to $q^{-1}[F]$ in P. The proof of Lemma 4.1 shows that the normal bundle has a $G{\times}H$-invariant complex structure, and we denote by

$$\lambda(\widehat{NF}) \in \widehat{K}_G(\widehat{F})$$

the class from Definition 7.2.

Theorem 11.2 *Let $\widehat{M} = (H, P, q)$ be a G-equivariant, stable complex orbifold. Assume that $g \in G$ generates a torus $T \subseteq G$, and denote by $F \subseteq M$ the T-fixed set in M. The class*

$$\lambda(\widehat{NF}) \in \widehat{K}_G(\widehat{F})$$

becomes invertible when viewed as an element of the localized ring $K_G(\widehat{F})_g$.

We need a small algebraic computation.

Lemma 11.2 *Let $f\colon \Gamma \to G$ be a homomorphism between compact abelian Lie groups (Γ need not be connected), and assume that f has finite kernel. Let L be the connected component of the identity in Γ and let $g \in G$ be a topological generator of the image of L. If $\varphi\colon \Gamma \to U(1)$ is any character that is nontrivial on L, then the element $(1 - \varphi) \in R(\Gamma)$ is invertible in the localized ring $R(\Gamma) \otimes_{R(G)} R(G)_g$.*

Proof There exists some $N > 0$ such that φ^N vanishes on the kernel of f. There exists then a character $\psi\colon G \to U(1)$ such that $\psi \circ f = \phi^N$. Indeed, by Frobenius reciprocity, any constituent of the induced representation $\mathrm{Ind}^G_{\Gamma/\ker(f)} \varphi^N$ will do. Now we can write

$$((1 - \varphi) \otimes 1)^{-1} = (1 + \varphi + \cdots + \varphi^{N-1}) \otimes (1 - \psi)^{-1},$$

as required (the element $(1 - \psi)$ can not vanish on g because ϕ, and hence ϕ^N, is nontrivial on L).

Proof of Theorem 11.2 It suffices to prove that the restriction of $\lambda(\widehat{NF})$ to any $G \times H$ orbit $\mathcal{O} \subseteq q^{-1}[F]$ is invertible in the localized equivariant K-theory of that orbit. If Γ is the isotropy group of the orbit, then

$$K_{G\times H}(\mathcal{O}) \cong R(\Gamma)$$

and the restriction of $\lambda(\widehat{NF})$ takes the form

$$e(\widehat{NF})\big|_{\mathcal{O}} = \prod (1 - \varphi_j),$$

where the characters $\varphi_j\colon \Gamma \to U(1)$ are the Γ-weights of the restriction of the normal bundle to $\mathcal{O}$. If L is the connected component of the identity in Γ, then L is a good torus in the sense of the proof of Lemma 11.1, and the characters ϕ_j are nontrivial even when restricted to L. The invertibility of this product follows from Lemma 11.2.

With these preliminaries in place, we obtain the following fixed-point formula to orbifolds, exactly as in the manifold case (and as in the manifold case, we shall adjust our notation a bit in the formulation of the theorem below).

Theorem 11.3 *Let $\widehat{M} = (H, P, q)$ be a G-equivariant, even-dimensional stable complex orbifold, and let $g \in G$ be a topological generator of a torus $T \subseteq G$. If $\alpha \in \widehat{K}_G(\widehat{M})$, then*

$$Q(\widehat{M}, \alpha) = \sum Q(\widehat{F}, \lambda(\widehat{NF})^{-1} \cdot \iota^* \alpha) \in R(G)_g$$

for every $\alpha \in \widehat{K}_G(\widehat{M})$, where the sum is over the components of the T-fixed set in M.

Now we can formulate and prove our orbifold extensions of Theorems 5.1–5.2. Let

$$\widehat{M} = (H, P, q)$$

be a G-equivariant, even-dimensional, stable complex orbifold. Assume that G decomposes as a product

$$G = G_1 \times T$$

with $T \cong U(1)$. Let $\widehat{E}$ be a $G \times H$ equivariant smooth Hermitian line bundle on P, with connection, and let

$$\mu\colon\ P \to \mathbb{R}$$

be the moment map associated to the T-action by Kostant's formula. It is of course a smooth $(G \times H)$-invariant function. We want to examine the quantization

$$Q_G(\widehat{M}, \widehat{E}) \in R(G),$$

and in particular the multiplicity

$$Q_G(\widehat{M}, \widehat{E})_0 \in R(G_2)$$

of the trivial representation of T. To this end, we construct the bordism of stable complex G-orbifolds

$$\widehat{M} \sim \widehat{M}_{\leq a} \sqcup {}_{b\leq}\widehat{M} \tag{11.1}$$

as in Section 6.[8]

Lemma 11.3 $Q_G({}_{b\leq}\widehat{M}, \widehat{E})_0 = 0$.

This may be proved by using the method described in Section 8, but first we need to make a note of one further property of quantization in the orbifold case, to be added to the list at the end of Section 10:

(f) Change of groups. If $\varphi\colon J \to G \times H$ is a homomorphism of tori, then the homomorphism from $J \times H$ to $G \times H$ determined by ϕ and multiplication in H induces a K-theory map from $\widehat{K}_G(\widehat{M})$ to $\widehat{K}_J(\widehat{M})$. In addition ϕ and the projection from $G \times H$ to G induce a map from $R(G)$ to $R(J)$. The following diagram commutes:

$$\begin{array}{ccc} \widehat{K}_G(\widehat{M}) & \xrightarrow{Q_G} & R(G) \\ \downarrow & & \downarrow \\ \widehat{K}_J(\widehat{M}) & \xrightarrow[Q_J]{} & R(J) \end{array}$$

[8] Note that even if M itself is a manifold, unless we explicitly hypothesize that the action of T is free between $\mu = a$ and $\mu = b$ (rather than locally free, which is guaranteed) the bordism will only be an orbifold bordism.

As with the other properties in our list, this is straightforward to verify if quantization is defined by using the index of transversally elliptic operators.

Proof of Lemma 11.3 We shall use the fixed-point formula and calculate the contribution of each component F of the T-fixed set in M. Let $L \subseteq G \times H$ be the corresponding good torus (see the proof of Lemma 11.1) that fixes the inverse image of F in P pointwise, and consider the multiplication map

$$J = G_1 \times L \to G \times H.$$

Because the associated homomorphism $R(G) \to R(L)$ is injective, it suffices to calculate the quantization map

$$Q_J \colon \; \widehat{K}_J(\widehat{F})_\ell \to R(J)_\ell,$$

where $\ell \in L$ is any lift of the topological generator $g \in T$. But if $S \subseteq P$ is the inverse image of F, then since L acts trivially on S there exists an isomorphism

$$K_J(\widehat{F}) = K_{G_1 \times H}(S) \otimes R(L)$$

and we can argue exactly as in the proof of Lemma 8.1 or the discussion following it that

$$Q(\widehat{F}, \lambda(\widehat{NF})^{-1} \cdot \widehat{E})_0 = 0,$$

as required.

Let us now turn to the second component in the bordism (11.1), namely the orbifold $\widehat{M}_{\leq a}$. At this point we need to make the further assumption concerning $\mu^{-1}[0] \subseteq M$, as we did in Section 8, that it is disjoint from the T-fixed set. Then we can proceed as above and as in Section 8 to compute $Q(\widehat{M}_{\leq a}, \widehat{E})_0$.

To formulate the final result denote by

$$\mu_M \colon \; M \to \mathbb{R}$$

the map induced from $\mu\colon P \to \mathbb{R}$. Then denote by M_a the quotient topological space $\mu_M^{-1}[a] \subseteq M$ by T. Then form the G_1-equivariant stable complex orbifold structure

$$\widehat{M}_a = (T \times H, \mu^{-1}[a], q)$$

constructed from $\mu^{-1}[a] \subseteq P$.

Theorem 11.4 *If $a \in \mathbb{R}$ is any regular value of μ between 0 and 1, and T acts locally freely on $\mu_M^{-1}[0]$, then*

$$Q(\widehat{M}, \widehat{E})_0 = Q(\widehat{M}_a, \widehat{E}_a) \in R(G_1).$$

Similarly, we have the following theorem.

Theorem 11.5 *If $a \in \mathbb{R}$ is any regular value of μ between 0 and 1, T acts locally freely on $\mu_M^{-1}[0]$, and $\widehat{\alpha} \in \widehat{K}_G(\widehat{M})$, then*

$$Q(\widehat{M}, \widehat{E}^k \cdot \widehat{\alpha})_0 = Q(\widehat{M}_a, \widehat{E}_a{}^k \cdot \widehat{\alpha}_a) \in R(G_1)$$

for all $k \gg 0$.

In both theorems, we can replace $a \in \mathbb{R}$ by $0 \in \mathbb{R}$, if 0 is a regular value (but for our purposes the formulations above are better).

12 Quantization Formula for Torus Actions

Let G be a torus. Let $\widehat{M} = (H, P, q)$ be a smooth, G-equivariant, even-dimensional, stable complex orbifold structure on a G-space M, with H a torus. Let $\widehat{E}$ be a G-equivariant, smooth Hermitian line bundle on $\widehat{M}$ (that is, a $G \times H$-equivariant Hermitian line bundle E on P). Let

$$\mu\colon\ P \to \mathfrak{g}^* \tag{12.1}$$

be a Kostant moment map (associated to a $G \times H$-equivariant connection on E). Finally, assume that G acts without fixed-points on $\mu^{-1}[0]$.

It is convenient not to assume that $0 \in \mathfrak{g}^*$ is a regular value of μ, but in any case every neighborhood of $0 \in \mathfrak{g}^*$ contains a regular value a, and we may form the reduced orbifold structure

$$\widehat{M_a} = (G \times H, \mu^{-1}[a], q)$$

on the locally compact space $M_a = \mu_M^{-1}[a]/G$, where, as in the previous section, the map

$$\mu_M\colon\ M \to \mathfrak{g}^*$$

is obtained by factoring (12.1) through the quotient map $q\colon P \to M$.

We may also form the reduced orbifold line bundle $\widehat{E_a} = E|_{\mu^{-1}[a]}$ over $\widehat{M_a}$. More generally, if $\widehat{\alpha}$ is any class in $\widehat{K}_G(\widehat{M})$, then restricting to $\mu^{-1}[a]$ we obtain from it a class

$$\widehat{\alpha_a} \in \widehat{K}(\widehat{M_a}).$$

We want to prove the following versions of the quantization commutes with reduction theorem, in which $Q_G(\widehat{M}, \widehat{E})_0$ and $Q_G(\widehat{M}, E^k \otimes \alpha)_0$ refer to the multiplicities of the trivial representation of the full torus G in the given quantizations (up to now we have been dealing with the trivial representation of some circle subgroup).

Theorem 12.1 *If M is closed, then*

$$Q_G(\widehat{M}, \widehat{E})_0 = Q(\widehat{M_a}, \widehat{E}_a)$$

for all regular values $a \in \mathfrak{g}^*$ sufficiently close to 0.

Theorem 12.2 *If α is any class in $\widehat{K}_G(\widehat{M})$, then*

$$Q_G(\widehat{M}, \widehat{E}^k \otimes \widehat{\alpha})_0 = Q(\widehat{M_a}, \widehat{E}_a^k \otimes \widehat{\alpha}_a)$$

for all regular values $a \in \mathfrak{g}^*$ sufficiently close to 0 and all sufficiently large $k > 0$.

The lower bound on k is independent of the particular choice of a. Of course, if $0 \in \mathfrak{g}^*$ *is* a regular value then we can simply take $a = 0$.

Proof of Theorem 12.1 If G has dimension one, then the result is covered by Theorem 11.4. We shall prove the theorem in general by induction on the dimension of G, and the only real difficulty comes from having to carry along the hypothesis that G acts without fixed-points on $\mu^{-1}[0]$.

In order to handle this point, we use the change of groups property of quantization (see item (e) following Lemma 11.3) and argue as follows. If G has dimension more than one, then we can find a finite covering

$$G_1 \times G_2 \to G$$

by a product of tori such that if a point in M is fixed by either G_1 or G_2 then in fact it is fixed by G (just choose G_1 and G_2 so that they are not contained in any of the finitely many proper isotropy subgroups in G).

Now the fixed-points of G only occur where μ assumes integral values in $\mathfrak{g}^*$, and it is a simple matter to further adjust G_1 and G_2 so that in the decomposition

$$\mathfrak{g}^* \cong \mathfrak{g}_1^* \oplus \mathfrak{g}_2^*,$$

neither of the axes $\mathfrak{g}_1^*$ and $\mathfrak{g}_2^*$ contains any of the finitely many $\mathfrak{g}^*$-integral points in the range of μ (the number is finite by compactness).

By Theorem 11.4 again,

$$Q_G(\widehat{M}, \widehat{E}^k \otimes \alpha)_{0\in \widehat{G}_1} = Q_{G_2}(\widehat{M}_{a_1}, \widehat{E}^k_{a_1} \otimes \widehat{\alpha}_{a_1}) \in R(G_2)$$

for all regular $a_1 \in \mathfrak{g}_1^*$ sufficiently close to zero, where the subscript on the left-hand side indicates the multiplicity of the trivial representation of G_1. Now apply the induction hypothesis to conclude the theorem, but for all $a = (a_1, a_2)$ sufficiently close to zero for which a_1 and a_2 are separately regular values. However if a is any regular value sufficiently close to zero, then we can find a neighborhood around a consisting entirely of regular values, and within that neighborhood an element (a_1, a_2) to which our argument applies. By bordism invariance

$$Q(\widehat{M}_a, \widehat{E}^k_a \otimes \widehat{\alpha}_a) = Q(\widehat{M}_{(a_1,a_2)}, \widehat{E}^k_{(a_1,a_2)} \otimes \widehat{\alpha}_{(a_1,a_2)}),$$

so the theorem is proved.

The proof of Theorem 12.2 is similar. The only additional point is that for any given class $\widehat{\alpha}$, we can replace M by an open and relatively compact G-invariant subset that contains the support of $\widehat{\alpha}$. The compactness allows us to use the same argument as just given.

13 Non-Abelian Case

In this final section we shall consider arbitrary connected and compact groups. We begin by discussing reduction in the context of non-abelian groups and stable complex manifolds, since it is slightly more complicated than in the abelian case (which in effect we handle by writing a torus as a product of circles).

Let G be a compact connected Lie group. Let M be a G-equivariant stable complex manifold (or orbifold), assume that $0 \in \mathfrak{g}^*$ is a regular value of a moment map

$$\mu\colon\ M \to \mathfrak{g}^*$$

associated to a connection on an equivariant Hermitian line bundle, as in the introduction, and assume that G acts locally freely on the inverse image of 0. There exists an exact sequence of vector bundles

$$0 \to T\mu^{-1}[0] \to TM|_{\mu^{-1}[0]} \to \mathfrak{g}^* \to 0,$$

where the quotient map sends a tangent vector X to the form $Y \mapsto X(\mu_Y)$, and another exact sequence

$$0 \to \mathfrak{g} \to T\mu^{-1}[0] \to \widehat{TM}_0 \to 0,$$

where $M_0 = \mu^{-1}[0]/G$. Choosing equivariant splittings of both we obtain an isomorphism

$$\widehat{TM}_0 \oplus \mathfrak{g} \oplus \mathfrak{g}^* \cong TM|_{\mu^{-1}[0]}.$$

Equip the equivariant bundle $\mathfrak{g} \oplus \mathfrak{g}^*$ with an equivariant complex structure J that exchanges to the two summands, with $\langle X, JX \rangle > 0$ for all $X \in \mathfrak{g}$ (the angle brackets denote the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$). The above isomorphism then endows $\widehat{M}_0$ with a stable complex structure, assuming M has been given one.

We shall prove the following theorem, which is essentially due to Meinrenken (see [21]).

Theorem 13.1 *Assume that $0 \in \mathbb{R}$ is a regular value of μ and that G acts locally freely on $\mu^{-1}[0]$. If $\widehat{M}_0$ and $\widehat{E}_0$ are the reduced stable complex orbifold and orbifold line bundle, respectively, then*

$$Q_G(M, E^k)_0 = Q(\widehat{M}_0, \widehat{E}_0^k)$$

for all sufficiently large $k > 0$.

We shall reduce the theorem to the abelian case, and the first step is Lemma 13.1 below. See for example [11] for this approach (where it is credited to Michele Vergne).

Fix a maximal torus T in G. Recall that the Weyl character formula describes the restriction map

$$R(G) \to R(T) \tag{13.1}$$

as follows. A choice of positive roots determines a complex structure on the vector space $\frac{\mathfrak{g}}{\mathfrak{t}}$ that is invariant under the adjoint action of T, and for which the weights of the associated complex T-representation are precisely the negative roots (this convention suits well our purposes). The characters of the irreducible representations of G, when restricted to T, take the form

$$\chi|_T = \frac{\sum (-1)^{\det(w)} \exp(w(\phi + \rho) - \rho)}{\Delta}. \tag{13.2}$$

Most of the detailed aspects of this formula are not relevant to us, but to fix the notation:

(a) $\phi \colon \mathfrak{t} \to \mathfrak{u}(1)$ is the highest (infinitesimal) weight of the representation.

(b) $\rho \colon \mathfrak{t} \to \mathfrak{u}(1)$ is the half-sum of the positive roots.

(c) The exponentials are characters of T defined by

$$\exp(\psi)(\exp(X)) := \exp(\psi(X)).$$

(d) The sum is over the Weyl group, and the signs are given by the sign representation of the Weyl group.

(e) The denominator is the Euler class in $K_G(\mathrm{pt}) = R(T)$ of $(\mathfrak{g}/\mathfrak{t})^*$, thought of as a T-equivariant vector bundle over a point. Thus

$$\Delta = \lambda((\mathfrak{g}/\mathfrak{t})^*) = \wedge^*(\mathfrak{g}/\mathfrak{t}) \in R(T). \tag{13.3}$$

All we need to extract from the character formula (13.2) is that since the restriction map (13.1) is injective, the multiplicity of the trivial representation in any G-representation V is equal to the multiplicity of the trivial representation of T in the virtual representation

$$V|_T \cdot \Delta \in R(T).$$

Hence we have the following lemma.

Lemma 13.1 *The multiplicity of the trivial representation of G in the quantization $Q_G(M,E) \in R(G)$ is equal to the multiplicity $Q_T(M,E\cdot\Delta)_0$ of the trivial weight of T in the quantization $Q_T(M,E\cdot\Delta) \in R(T)$.*

Our goal now is to compute the T-multiplicity $Q_T(M,E\cdot\Delta)_0$ by using the results of the previous section. The first step is to decompose $\mathfrak{g}^*$ as

$$\mathfrak{g}^* = \mathfrak{t}^* \oplus (\mathfrak{g}/\mathfrak{t})^* \tag{13.4}$$

(this is possible since $\mathfrak{t}^*$ is not only a quotient of $\mathfrak{g}^*$, but also embeds in $\mathfrak{g}^*$ as the linear functionals on $\mathfrak{g}$ that vanish on all the root spaces). Denote by

$$\mu_T\colon\ M \to \mathfrak{t}^* \quad \text{and} \quad \overline{\mu}\colon\ M \to (\mathfrak{g}/\mathfrak{t})^*$$

the two components of the moment map under the decomposition (13.4). The first one is the moment map for the T-action. We will use the second map to pull back the Bott generator β of the complex vector space $(\mathfrak{g}/\mathfrak{t})^*$ to M. Since the map $\overline{\mu}$ is obviously homotopic to the zero map, and the pullback of the Bott generator to M along this map is $\Delta \in K_T(M)$ (that is, it is the trivial virtual vector bundle with fiber Δ as described in (13.3) above), we find that

$$\Delta = \overline{\mu}^*(\beta) \in K_T(M),$$

and as a result

$$Q_T(M,E\cdot\Delta) = Q_T(M,E\cdot\overline{\mu}^*(\beta)).$$

Next, since the Bott element is supported on $\{0\} \subseteq (\mathfrak{g}/\mathfrak{t})^*$, the pullback class $\overline{\mu}^*(\beta)$ is supported on $\overline{\mu}^{-1}[0] \subseteq M$. So if $B \subseteq (\mathfrak{g}/\mathfrak{t})^*$ is any (T-invariant) open ball around $0 \in (\mathfrak{g}/\mathfrak{t})^*$, then the pullback of the Bott element defines a class in $K_T(\overline{\mu}^{-1}[B])$.

Let us write

$$U = \overline{\mu}^{-1}[B] \subseteq M$$

(which is of course an open subset of M).

Lemma 13.2 *If k is any integer, then*

$$Q_T(M,E^k\cdot\overline{\mu}^*(\beta)) = Q_T(U,E^k\cdot\overline{\mu}^*(\beta)).$$

Proof This is just an instance of the excision property

$$Q_T(M,\iota_*(\alpha)) = Q_T(U,\alpha),$$

where ι is the inclusion of U into M and α is any class in $K(U)$. See property (b) of the quantization map in the list at the end of Section 10.

Lemma 13.3 *If the open ball $B \subseteq (\mathfrak{g}/\mathfrak{t})^*$ is chosen to be sufficiently small, then 0 is a regular value of the T-moment map*

$$\mu_T\colon U \to \mathfrak{t}^*$$

and T acts locally freely on the inverse image of zero.

Proof This follows from the fact that $\mu_T^{-1}[0] \cap \overline{\mu}^{-1}[0] = \mu_G^{-1}[0]$ and our assumptions concerning regularity and locally free actions for G.

Now form the locally compact space

$$U_0 = (U \cap \mu_T^{-1}[0])/T$$

and the orbifold

$$\widehat{U}_0 = (T, U \cap \mu_T^{-1}[0], q),$$

where q is the obvious projection mapping.

Lemma 13.4 *If k is sufficiently large, then*

$$Q(U, E^k \cdot \overline{\mu}^*(\beta))_0 = Q(\widehat{U}_0, E_0^k \cdot \overline{\mu}^*(\beta)_0).$$

Proof This follows from Theorem 12.2.

It now remains to identify the quantization of the orbifold $\widehat{U}_0$ that appears in Lemma 13.4 to the quantization of the orbifold $\widehat{M}_0 = (G, \mu_G^{-1}[0], q)$ that appears in Theorem 13.1.

The orbifold $\widehat{U}_0$ is a fiber bundle (with structure group G) over $\widehat{M}_0$ with fiber $G \times_T B$. The fiber is in turn a bundle over the complex flag variety $\frac{G}{T}$ with fiber the open ball B.

The K-theory class $E^k \cdot \overline{\mu}^*(\beta)$ on $\widehat{U}_0$ is the fiber product of E_0^k on $\widehat{M}_0$ with the Thom class on $G \times_T B$. The latter class has G-equivariant quantization (index) $1 \in R(G)$, so using the multiplicative property (f) of quantization in the list at the end of Section 10 we find that

$$Q_T(\widehat{U}_0, E_0^k \cdot \overline{\mu}^*(\beta)_0) = Q(\widehat{M}_0, \widehat{E}_0^k).$$

This is precisely the relation we require, and Theorem 13.1 is proved.

References

[1] Atiyah, M. F., Global Theory of Elliptic Operators, In Proc. Internat. Conf. on Functional Analysis and Related Topics (Tokyo, 1969), Univ. of Tokyo Press, Tokyo, 1970, 21–30.

[2] Atiyah, M. F., Elliptic Operators and Compact Groups, Lecture Notes in Mathematics, Vol. 401, Springer-Verlag, Berlin, 1974.

[3] Atiyah, M. F. and Hirzebruch, F., Riemann-Roch theorems for differentiable manifolds, *Bull. Amer. Math. Soc.*, **65**, 1959, 276–281.

[4] Atiyah, M. F. and Hirzebruch, F., Spin-manifolds and Group Actions, in Essays on Topology and Related Topics (Mémoires Dédiés à Georges de Rham), Springer-Verlag, New York, 1970, 18–28.

[5] Atiyah, M. F. and Segal, G. B., The index of elliptic operators. II, *Ann. of Math.* (2), **87**, 1968, 531–545.

[6] Atiyah, M. F. and Singer, I. M., The index of elliptic operators. I, *Ann. of Math.* (2), **87**, 1968, 484–530.

[7] Baum, P., Oyono-Oyono, H., Schick, T., et al., Equivariant geometric K-homology for compact Lie group actions, *Abh. Math. Semin. Univ. Hambg.*, **80**(2), 2010, 149–173.

[8] Baum, P. F. and Douglas, R. G., K-Homology and Index Theory, in Operator Algebras and Applications, Part I (Kingston, Ont., 1980), Vol. 38, Amer. Math. Soc., Providence, RI, 1982, 117–173.

[9] Baum, P. F., Higson, N. and Schick, T., On the equivalence of geometric and analytic K-homology, *Pure Appl. Math. Q.*, **3**, 2007, 1–24.

[10] Block, J. and Higson, N., Weyl Character Formula in K-Theory, in Proceedings of the RIMS Thematic Year 2010 on Perspectives in Deformation Quantization and Noncommutative Geometry, Kyoto University, World Scientific, 2013, 299–334.

[11] Cannas da Silva, A. and Guillemin, V., Quantization of Symplectic Orbifolds and Group Actions, in Northern California Symplectic Geometry Seminar, Vol. 196, Amer. Math. Soc. Transl. Ser. 2, Amer. Math. Soc., Providence, RI, 1999, 1–12.

[12] Cannas da Silva, A., Karshon, Y. and Tolman, S., Quantization of presymplectic manifolds and circle actions, *Trans. Amer. Math. Soc.*, **352**(2), 2000, 525–552.

[13] Duistermaat, H., Guillemin, V., Meinrenken, E. and Wu, S., Symplectic reduction and Riemann-Roch for circle actions, *Math. Res. Lett.*, **2**(3), 1995, 259–266.

[14] Guillemin, V., Ginzburg, V. L. and Karshon, Y., Moment Maps, Cobordisms, and Hamiltonian Group Actions, Vol. 98, Mathematical Surveys and Monographs, Amer. Math. Soc., Providence, RI, 2002.

[15] Guillemin, V. and Sternberg, S., Homogeneous quantization and multiplicities of group representations, *J. Funct. Anal.*, **47**(3), 1982, 344–380.

[16] Guillemin, V. and Sternberg, S., Geometric quantization and multiplicities of group representations, *Invent. Math.*, **67**(3), 1982, 515–538.

[17] Higson, N. and Roe, J., Analytic K-Homology, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[18] Karoubi, M., K-Theory, Springer-Verlag, Berlin, 1978.

[19] Kasparov, G. G., Topological invariants of elliptic operators, I, K-homology, *Izv. Akad. Nauk SSSR Ser. Mat.*, **39**(4), 1975, 796–838.

[20] Kosniowski, C., Applications of the holomorphic Lefschetz formula, *Bull. London Math. Soc.*, **2**, 1970, 43–48.

[21] Meinrenken, E., On Riemann-Roch formulas for multiplicities, *J. Amer. Math. Soc.*, **9**(2), 1996, 373–389.

[22] Metzler, D. S., A K-theoretic note on geometric quantization, *Manuscripta Math.*, **100**(3), 1999, 277–289.

[23] Segal, G. B., Equivariant K-theory, *Inst. Hautes Études Sci. Publ. Math.*, **34**, 1968, 129–151.

[24] Sjamaar, R., Symplectic reduction and Riemann-Roch formulas for multiplicities, *Bull. Amer. Math. Soc.* (*N. S.*), **33**(3), 1996, 327–338.

[25] Vergne, M., Equivariant index formulas for orbifolds, *Duke Math. J.*, **82**(3), 1996, 637–652.