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Coarse Embedding into Uniformly Convex Banach Spaces*

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Abstract In this paper, the author studies the coarse embedding into uniformly convex Banach spaces. The author proves that the property of coarse embedding into Banach spaces can be preserved under taking the union of the metric spaces under certain conditions. As an application, for a group G strongly relatively hyperbolic to a subgroup H, the author proves that $B(n) = \{g \in G \mid |g|_{S\cup\mathscr{H}} \leq n\}$ admits a coarse embedding into a uniformly convex Banach space if H does.

Keywords Coarse embedding, Uniformly convex Banach spaces, Relative hyperbolic groups
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1 Introduction

After Gromov pointed out that the coarse embedding (also referred to as the uniform embedding) should be relevant to Novikov conjecture (see [7, 9]), Yu introduced the concept of property A for discrete metric spaces (see [16]). A metric space with property A admits a coarse embedding into a Hilbert space. And Yu proved that the coarse Baum-Connes conjecture holds for the metric spaces with bounded geometry, which admits a coarse embedding into a Hilbert space (see [16]). Subsequently, Kasparov and Yu proved that the coarse geometric Novikov conjecture holds for discrete metric spaces with bounded geometry, which admits a coarse embedding into a uniformly convex Banach space (see [12]). Recently, Chen, Wang and Yu showed that the maximal coarse Baum-Connes conjecture holds for metric spaces with bounded geometry, which admits a fibred coarse embedding into Hilbert space (see [3]). Coarse embedding into a Hilbert space has been studied deeply these years (see [4]). But there are less results on the coarse embedding into a uniformly convex Banach space. Also, Lafforgue constructed an example which can not be coarsely embedded into uniformly convex Banach spaces (see [13]). We should mention that Brown and Guentner proved that every metric space with bounded geometry admits a coarse embedding into a strictly convex and reflexive Banach space (see [2]).

In this paper, we study the coarse embedding into a uniformly convex Banach space. Our first theorem is as follows.

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Theorem 1.1 *Let X be a metric space and E be a Banach space, and $1 \leq p < +\infty$. If there exists some $\delta > 0$, such that for every $R > 0$, $\varepsilon > 0$, there exists a map $\varphi : X \to E$ satisfying:*

(1) $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X,\ d(x, y) \leq R\} \leq \varepsilon$;

(2) *for every* $m \in \mathbb{N}$, $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X,\ d(x, y) \leq m\} < +\infty$;

(3) $\lim\limits_{s\to+\infty} \inf\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \geq s\} \geq \delta$,

then X admits a coarse embedding into E^p.

The conditions above are generalized from the conditions for coarse embedding into a Hilbert space (see [4]). We then study the coarse embedding under the gluing property. It is easy to prove in the case of Hilbert space, but more complicated in the case of Banach space. We only obtained some partial results.

Proposition 1.1 *Let X be a metric space, and $X = X_1 \cup X_2$, such that X_1, X_2 admit a coarse embedding into a Banach space E and $1 \leq p < +\infty$. If for any $s > 0$, there exists a bounded set C_s such that the sets $X_1\backslash C_s$ and $X_2\backslash C_s$ are s-separated, then X admits a coarse embedding into E^p.*

Recall that two subsets X_1, X_2 of a metric space X are s-separated if

$$d(X_1, X_2) = \inf\{d(x, y), x \in X_1, y \in X_2\} \geq s.$$

Gromov introduces the following property for metric space (see [9]).

Definition 1.1 *A metric space X is called long-range disconnected at infinity if for every $n \in \mathbb{N}$, there exist two subsets X_1^n and X_2^n in X such that:*

(1) $d(X_1^n, X_2^n) = \inf\{d(x_1, x_2) \mid x_1 \in X_1^n, x_2 \in X_2^n\} \geq d$;

(2) *X_1^n and X_2^n cover almost all X, i.e., $X\backslash(X_1^n \cup X_2^n)$ is bounded.*

Proposition 1.2 *If X is long-range disconnected at infinity and all $\{X_i^n\}$ are equivalently coarsely embedded into a Banach space E by coarse maps $\{\varphi_i^n\}$, then X admits a coarse embedding into E^p.*

In the case of infinite union, we have the following result.

Proposition 1.3 *Let X be a metric space with $X = \bigcup\limits_{i\in I} X_i$, and for any $s > 0$, there exists a bounded set C_s such that $X_i \cap C_s \neq \emptyset$ for every $i \in I$ and $\{X_i\backslash C_s\}_{i\in I}$ are pairwise s-separated. If $\{X_i\}$ can be equivalently coarsely embedded into E, then X can be coarsely embedded into E^p.*

As an application, we study the coarse embeddability of the relative hyperbolic group and prove the following theorem.

Theorem 1.2 *If a group G is strongly relatively hyperbolic to a subgroup H and H admits a coarse embedding into a uniformly convex Banach space E, let $B(n) = \{g \in G \mid |g|_{S\cup\mathscr{H}} \leq n\}$, and then $B(n)$ admits a coarse embedding into E^p.*

2 Coarse Geometry and Convex Banach Space

We first recall some definitions in coarse geometry (see [15]).

Definition 2.1 *Let X, Y be metric spaces, and f be a map from X to Y:*

(1) *The map f is proper, if the inverse image, under f, of any bounded subset of Y, is a bounded subset of X.*

(2) *The map is bornologous, if for every $R > 0$, there exists an $S > 0$, such that $d(x, y) \leq R$ implies $d(f(x), f(y)) \leq S$.*

(3) *f is coarse if it is proper and bornologous.*

We say that X admits a coarse embedding into Y if there exists a coarse map $f : X \to Y$. We usually consider the case where Y is a Banach space.

Definition 2.2 *A family of metric spaces $\{X_i\}_{i\in I}$ is called equivalently coarsely embedded into a metric space Y if there exists a family of maps $\{f_i : X_i \to Y\}_{i\in I}$ satisfying:*

(1) *For each $s \geq 0$, there exists some $S \geq 0$, such that $d(x_i, x_i') \leq s$ implies $d(f_i(x_i), f_i(x_i')) \leq S$ for all $i \in I$.*

(2) *For each $r \geq 0$, there exists some $R \geq 0$, such that $d(f_i(x_i), f_i(x_i')) \leq r$ implies $d(x_i, x_i') \leq R$ for all $i \in I$.*

Uniformly the convex Banach space is an important object to study in classical Banach space theory (see [10]).

Definition 2.3 *A Banach space E is called uniformly convex if for any $\epsilon > 0$, there exists a $\delta > 0$, for any $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$, then $\left\|\frac{x+y}{2}\right\| \leq 1 - \delta$.*

We know that ℓ^p $(1 < p < +\infty)$ is a uniformly convex Banach space. If E is a uniformly convex Banach space, let

$$E^p = \Big\{x = (x_i)_{i\in\mathbb{N}} \mid x_i \in E, \sum_{n\in\mathbb{N}} \|x_i\|^p < +\infty\Big\}$$

with the norm $\|x\| = \big(\sum\limits_{n\in\mathbb{N}} \|x_i\|^p\big)^{\frac{1}{p}}$. If $1 < p < +\infty$, then E^p is also a uniformly convex Banach space.

3 Coarse Embedding into Uniformly Convex Banach Space

We first rewrite the condition for coarse embedding into a uniformly convex Banach space.

Theorem 3.1 *Let X be a metric space and E be a Banach space, and $1 \leq p < +\infty$. If there exists some $\delta > 0$ such that for every $R > 0$, $\varepsilon > 0$, there exists a map $\varphi : X \to E$ satisfying:*

(1) $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X,\ d(x, y) \leq R\} \leq \varepsilon$;

(2) *for each $m \in \mathbb{N}$, $\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X,\ d(x, y) \leq m\} < +\infty$;*

(3) $\lim\limits_{s\to+\infty} \inf\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \geq s\} \geq \delta$,

then X admits a coarse embedding into E^p.

Proof For $n \in \mathbb{N}$, let $R_n = n$, $\varepsilon_n = \frac{1}{2^n}$, and there exists a $\varphi_n : X \to E$ satisfying the above conditions. And we can find an s_n such that $\|\varphi_n(x) - \varphi_n(y)\| > \frac{\delta}{2}$ if $d(x, y) \geq s_n$, and we can choose $\{s_n\}$ to be an increasing sequence. Fix a point $x_0 \in X$, define

$$\begin{aligned}\varphi: \quad & X \to E^p,\\ & x \mapsto \bigoplus_{n=1}^{\infty}(\varphi_n(x) - \varphi_n(x_0)).\end{aligned}$$

It is easy to see that $\|\varphi(x)\| < +\infty$ for each $x \in X$. We claim that φ is coarse.

(1) For any $x, y \in X$, assume $k-1 < d(x,y) \le k$, then

$$\begin{aligned}\|\varphi(x) - \varphi(y)\|^p &= \sum_{n=1}^{+\infty} \|\varphi_n(x) - \varphi_n(y)\|^p\\ &= \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p + \sum_{n=k}^{+\infty} \|\varphi_n(x) - \varphi_n(y)\|^p\\ &\le \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p + \sum_{n=k}^{+\infty} \frac{1}{2^{np}}.\end{aligned}$$

Let $C_n^k = \sup\{\|\varphi_n(x) - \varphi_n(y)\|, d(x,y) \le k\}$, then

$$\|\varphi(x) - \varphi(y)\|^p \le \sum_{n=1}^{k-1} (C_n^k)^p + 1.$$

(2) For any $x, y \in X$, assume $s_{k-1} \le d(x,y) \le s_k$, then

$$\|\varphi(x) - \varphi(y)\|^p = \sum_{n=1}^{\infty} \|\varphi_n(x) - \varphi_n(y)\|^p \ge \sum_{n=1}^{k-1} \|\varphi_n(x) - \varphi_n(y)\|^p > (k-1)\Big(\frac{\delta}{2}\Big)^p.$$

$d(x,y) \to +\infty$ implies $k \to +\infty$, so $(k-1)\big(\frac{\delta}{2}\big)^p \to +\infty$.

Example 3.1 ℓ^p $(1 \le p < +\infty)$ satisfies the above conditions for $E = \ell^p$.

Proof Let $\delta = 1$, $\forall R > 0$ and $\varepsilon > 0$, choose a natural number h such that $\frac{R}{h} < \varepsilon$ and define $\varphi : \ell^p \to \ell^p$ by $\varphi(x) = \frac{x}{h}$. Then

(1) $\sup\{\|\varphi(x) - \varphi(y)\|, d(x,y) \le R\} = \sup\{\frac{1}{h}\|x - y\|,\ d(x,y) \le R\} < \varepsilon$.

(2) For every $m \in \mathbb{N}$, $\sup\{\|\varphi(x) - \varphi(y)\|, d(x,y) \le m\} = \frac{m}{h} < +\infty$.

(3) $\inf\{\|\varphi(x) - \varphi(y)\|, d(x,y) \ge s\} = \frac{s}{h}$, $\lim\limits_{s\to+\infty} \frac{s}{h} = +\infty$.

Johnson and Randrianarivony proved that ℓ^p $(p > 2)$ does not admit a coarse embedding into a Hilbert space (see [11]). So the conditions for a coarse embedding into a uniformly convex Banach space E in the above theorem is very different from the coarse embedding into a Hilbert space (see [4]).

Example 3.2 If X admits a coarse embedding into ℓ^p, then X satisfies the above conditions.

Proof In fact, X admits a coarse embedding into Y is equivalent to that there exists a $\psi : X \to Y$ and nondecreasing functions $\rho_\pm : [0,\infty) \to [0,\infty)$, satisfying $\rho_-(d(x,y)) \le d(\psi(x) - \psi(y)) \le \rho_+(d(x,y))$ and $\lim\limits_{r\to+\infty} \rho_-(r) = +\infty$.

Fix such a map ψ and for every $R > 0$, $\varepsilon > 0$, choose a nature number h such that $\frac{\rho_+(R)}{h} < \varepsilon$, and define $\varphi : X \to \ell^p$ by $\varphi(x) = \frac{\psi(x)}{h}$. Then it is easy to verify the conditions in Theorem 3.1 for φ.

Remark 3.1 (a) We can see from the proof that this δ is not important. We can take it to be infinity in general, i.e., to replace the third condition with

$$\lim_{s\to+\infty} \inf\{\|\varphi(x) - \varphi(y)\|_p : x, y \in X, d(x, y) \geq s\} = +\infty.$$

(b) If we take $E = \ell^p$ $(1 < p < +\infty)$ and change the condition (2) of Theorem 3.1 with

$$\sup\{\|\varphi(x) - \varphi(y)\| : x, y \in X, d(x, y) \leq m\} < L_0, \quad \forall m \in \mathbb{N}$$

for some fixed L_0, by the Mazur map mentioned in [1], then it can be embedded into ℓ^2, a Hilbert space.

4 On the Union of Metric Spaces

In this section, we study the coarse embeddability under taking the union of metric spaces.

Proposition 4.1 *Let X be a metric space, and $X = X_1 \cup X_2$, such that X_1, X_2 admit a coarse embedding into a Banach space E and $1 \leq p < +\infty$. If for any $s > 0$, there exists a bounded set C_s such that $X_1\backslash C_s$ and $X_2\backslash C_s$ are s-separated, then X admits a coarse embedding into E^p.*

Proof We first assume that $X_1 \cap X_2 \neq \emptyset$. Take an $x_0 \in X_1 \cap X_2$, and replace C_s with $C_s \cup \{x_0\}$ if necessary, so then we can assume that $x_0 \in C_s$ for any s.

For any $R \geq 0$, $\varepsilon \geq 0$, there exists a bounded set C_{2R} such that $X_i\backslash C_{2R}$ is $2R$-separated. Suppose $C_{2R} \subset B(x_0, k)$ for some k. As X_i admits a coarse embedding into E, we can find a number $r > 2R + k$ and a map φ_r^i, such that

(1) $\sup\{\|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \leq r\} \leq \varepsilon$,

(2) $\forall m \in \mathbb{N}$, $\sup\{\|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \leq m\} < +\infty$,

(3) $\lim\limits_{s\to+\infty} \inf\{\|\varphi_r^i(x) - \varphi_r^i(y)\| : x, y \in M_i, d(x, y) \geq S\} = +\infty$.

We define

$$\begin{aligned} \varphi : \quad & X \to (E \oplus E)_p, \\ & x \mapsto (\varphi_r^1(a) - \varphi_r^1(x_0), 0), \quad \text{if } x \in M_1\backslash C_{2R}, \\ & y \mapsto (0, \varphi_r^2(b) - \varphi_r^2(x_0)), \quad \text{if } y \in M_2\backslash C_{2R}, \\ & z \mapsto (0, 0), \quad \text{if } z \in C_{2R}. \end{aligned}$$

We need to verify the conditions in Theorem 3.1.

(i) For $d(x, y) \leq R$, if $x \in M_1\backslash C_{2R}$, $y \in C_{2R}$, then $d(x_0, y) \leq k$, and we have $d(x, x_0) \leq d(x, y) + d(y, x_0) \leq k + R$, so

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^1(x) - \varphi_r^1(x_0)\| < \epsilon.$$

If $x \in M_2\backslash C_{2R}$, $y \in C_{2R}$ or $x, y \in C_{2R}$ or $x, y \in M_i\backslash C_{2R}$, it is similar to prove $\|\varphi(x) - \varphi(y)\| < \epsilon$.

(ii) For $\forall m > 0$, there exists a bounded set C_m, such that $X_i \backslash C_m$ is m-separated, and we can find a number h such that $C_m \subset B(C_{2R}, h)$. For $d(x,y) < m$, if there exists i, such that $x \in X_i \backslash C_{2R},\ y \in X_i \backslash C_{2R}$, then

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(y)\|.$$

If $x \in X_i \backslash C_{2R}$, $y \in C_{2R}$, then $d(x,x_0) \leq d(x,y) + d(y,x_0) \leq m + k$. Then

$$\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(x_0)\|.$$

If $x \in X_1 \backslash C_{2R}$, $y \in X_2 \backslash C_{2R}$ for $d(x,y) \leq m$, then either $x \in C_m$ or $y \in C_m$. Suppose $x \in C_m$, so $d(x,x_0) \leq h + k; d(y,x_0) \leq h + k + m$. Then

$$\|\varphi(x) - \varphi(y)\| = (\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}}.$$

Let $t = h + m + k$. We get

$$\begin{aligned}
&\sup\{\|\varphi(x) - \varphi(y)\|, d(x,y) \leq m\} \\
&\leq \max\Big\{ \sup_{d(x,y)\leq t} \|\varphi_r^i(x) - \varphi_r^i(y)\|, \\
&\quad \sup_{\substack{d(x,x_0)\leq t \\ d(y,x_0)\leq t}} \{(\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}}\}\Big\} \\
&< +\infty.
\end{aligned}$$

(iii) Let $s = d(x,y)$, and when s tends to infinity, if $x, y \in X_i \backslash C_{2R}$, then$\|\varphi_r^i(x) - \varphi_r^i(y)\| \to +\infty$ by the property of φ_r^i. If $x \in X_i \backslash C_{2R}$, $y \in C_{2R}$, then for $d(y,x_0) < k$, $d(x,y) \to +\infty$ implies $d(x,x_0) \to +\infty$, so $\|\varphi(x) - \varphi(y)\| = \|\varphi_r^i(x) - \varphi_r^i(x_0)\| \to +\infty$. If $x \in X_1 \backslash C_{2R}$, $y \in X_2 \backslash C_{2R}$, $d(x,y) \to +\infty$ implies either $d(x,x_0) \to +\infty$ or $d(y,x_0) \to +\infty$, thus

$$\|\varphi(x) - \varphi(y)\|^p = (\|\varphi_r^1(x) - \varphi_r^1(x_0)\|^p + \|\varphi_r^2(y) - \varphi_r^2(x_0)\|^p)^{\frac{1}{p}} \to +\infty.$$

Applying Theorem 3.1, we get the desired result.

In the case of $X_1 \cap X_2 = \emptyset$, we can assume that $X_i \cap C_s \neq \emptyset$ $(\forall s > 0)$. Take $x_0 \in X_1 \cap C_{2R}$, $y_0 \in X_2 \cap C_{2R}$, and define

$$\begin{aligned}
\varphi:\quad & X \to E \oplus E, \\
& x \mapsto (\varphi_r^1(a) - \varphi_r^1(x_0), 0), \quad \text{if } x \in X_1 \backslash C_{2R}, \\
& y \mapsto (0, \varphi_r^2(b) - \varphi_r^2(y_0)), \quad \text{if } y \in X_2 \backslash C_{2R}, \\
& z \mapsto (0,0), \quad \text{if } z \in C_{2R}.
\end{aligned}$$

The proof follows.

Proposition 4.2 *If X is long-range disconnected at infinity and all $\{X_i^n\}$ are equivalently coarse embedded into a Banach space E by coarse maps $\{\varphi_i^n\}$, then X admits a coarse embedding into E^p.*

Proof For any $R \geq 0$, $\epsilon \geq 0$, choose a number $n \in \mathbb{N}$ such that $n > R$. Choose a point $x_n \in X \backslash (X_1^n \cup X_2^n)$, and define

$$\begin{aligned} \varphi : \quad & X \to (E \oplus E)_p, \\ & x \mapsto (\varphi_1^n(a) - \varphi_1^n(x_n), 0), \quad \text{if } x \in X_1^n, \\ & y \mapsto (0, \varphi_2^n(b) - \varphi_2^n(x_n)), \quad \text{if } y \in X_2^n, \\ & z \mapsto (0, 0), \quad \text{otherwise.} \end{aligned}$$

Using the similar argument as in Proposition 4.1, it can be shown that φ satisfies the condition of Theorem 3.1.

Proposition 4.3 *Let $X = \bigcup_{i \in I} X_i$ be a metric space. If for any $s > 0$, there exists a bounded set C_s with $X_i \cap C_s \neq \emptyset$ for every $i \in I$ and $\{X_i \backslash C_s\}$ are pairwise s-separated. If X_i can be equivalently coarsely embedded into a Banach space E, then X can be coarsely embedded into E^p.*

Proof $\forall R > 0$, $\epsilon > 0$, there exists a bounded set C_R such that $X_i \backslash C_R$ is R-separated. Suppose that $C_R \subset B(x_0, k)$ for some k, and take an $r > k + 2R$. Since $\{X_i\}$ are equivalently coarsely embedded into E, we can find $\varphi_r^i : X_i \to E$, such that

(1) $\sup_i \sup\{\| \varphi_r^i(x) - \varphi_r^i(y) \| < \epsilon, x, y \in X_i, d(x,y) < r\} < \epsilon$,

(2) $\sup_i \sup\{\| \varphi_r^i(x) - \varphi_r^i(y) \|, x, y \in X_i, d(x,y) < m\} < \infty, \forall m \in \mathbb{N}$,

(3) $\lim_{s \to \infty} \inf_i \inf\{\| \varphi_r^i(x) - \varphi_r^i(y) \|, x, y \in X_i, d(x,y) > s\} = \infty$.

For each i, fix an $x_i \in X_i \cap C_R$. Define

$$\begin{aligned} \varphi : \quad & X \to E^p, \\ & a \mapsto (0, \cdots, \underbrace{\varphi_r^i(x) - \varphi_r^i(x_i)}_{i\text{th item}}, 0 \cdots), \quad \text{if } a \in X_i \backslash C_R, \\ & b \mapsto (0, \cdots, 0), \quad \text{if } b \in C_R. \end{aligned}$$

The proof follows using the similar argument as in Proposition 4.1.

5 Relative Hyperbolic Group

Let G be a finitely generated group with generating set S (closed under taking the inverse), and then G is a proper metric space with word-length metric induced by the generating set S. Let H be a finitely generated subgroup of G. We denoted $H \backslash \{e\}$ by $\mathscr{H}$. Then the Cayley graphs (G, S) and $(G, S \cup \mathscr{H})$ are both metric spaces with word-length metrics d_S and $d_{S \cup \mathscr{H}}$, respectively.

Definition 5.1 *Let p be a path in $(G, S \cup \mathscr{H})$. An $\mathscr{H}$-component of p is a maximal sub-path of p contained in the same left coset gH. The path is said to be without backtracking, if it does not have two distinct $\mathscr{H}$-components in the same coset gH_i.*

Definition 5.2 *A path-metric space X is hyperbolic if there exists some $\delta > 0$ such that the δ-neigborhood of any two sides of a geodesic triangle contains the third side. The group G is said to be weakly hyperbolic relative to H if the Cayley graph $(G, S \cup \mathscr{H})$ is hyperbolic.*

Definition 5.3 (see [6]) *We say the pair (G,H) satisfies the bounded coset penetration property (BCP for short), if for every $R \geq 0$, there exists $a = a(R)$ such that p, q are two geodesics in $(G, S \cup \mathscr{H})$ with $p_- = q_-$ and $d_S(p_+, q_+) \leq R$. So then*

(1) *suppose that p has an $\mathscr{H}$-component s with $d_S(s_-, s_+) \geq a(R)$, and then q has an $\mathscr{H}$-component contained in the same left coset of s.*

(2) *Suppose s, t that are two $\mathscr{H}$-components of p, q respectively, contained in the same left coset, and then $d_S(s_-, t_-) \leq a(R)$, $d_S(s_+, t_+) \leq a(R)$.*

Definition 5.4 *The group G is strongly relatively hyperbolic to H, if it is weakly hyperbolic to H and satisfies BCP.*

Denote $B(n) = \{g \in G \mid |g|_{S\cup\mathscr{H}} \leq n\}$. Osin proved in [14] that $B(n)$ has an asymptotic dimension at most d if the subgroup H has an asymptotic dimension at most d. Dadarlat and Guentner proved that G admits a coarse embedding into a Hilbert space if H admits a coarse embedding into a Hilbert space (see [5]). Fukaya and Oguni proved the coarse Baum-Connes conjecture holds for G if H does (see [8]). We prove the following theorem.

Theorem 5.1 *If H admits a coarse embedding into E, then $B(n)$ admits a coarse embedding into E^p for each $n \in \mathbb{N}$.*

Proof We proceed by induction on n. First $B(0) = \{e\}$ is trivial. $B(1) = H \cup S$ is just in the 1-neighborhood of H, so it can be coarsely embedded. We assume that $B(n-1)$ is coarsely embedded into ℓ^p. We know

$$B(n) = \Big(\bigcup_{x\in S} B(n-1)x\Big) \cup B(n-1)H.$$

Since $B(n-1)x$ is just in the 1-neighborhood of $B(n-1)$ in (G,S), it can be coarsely embedded. We are concerned about $B(n-1)H$. we can find a subset $R(n-1)$ in $B(n-1)$ such that for any $b \in B(n-1)$, $bH = gH$ for a unique $g \in R(n-1)$. Thus

$$B(n-1)H = \bigsqcup_{g\in R(n-1)} gH.$$

$\forall R \geq 0,\ \epsilon \geq 0$, we have an $a(R)$ from the BCP. We can assume that $a(R) \geq R$ and $a(R)$ is increasing. Let $T_R = \{g \in G \mid |g|_S \leq a(R)\}$. Let $Y_R = B(n-1)T_R$, and then Osin proved that $\{gH\backslash Y_R\}_{g\in R(n-1)}$ is R-separated (see [14]). We find maps φ_1 and φ_2 for embedding of Y_R and H, respectively, such that

(1) $\sup\{\|\varphi_i(x) - \varphi_i(y)\|_p : x, y \in X, d_S(x,y) \leq 3a(R)\} \leq \frac{\varepsilon}{2}$,

(2) $C_m = \sup\{\|\varphi_i(x) - \varphi_i(y)\|_p : x, y \in X, d_S(x,y) \leq m\} < +\infty,\ \forall m \in \mathbb{N}$,

(3) $\lim\limits_{t\to+\infty} \inf\{\|\varphi_i(x) - \varphi_i(y)\|_P : x, y \in X, d_S(x,y) \geq t\} \geq \delta$.

We define a map:

$$\varphi:\ B(n-1)H \to E \oplus \Big(\bigoplus_{g_i\in R(n-1)} E\Big)$$

as following.

For $x \in g_iH\backslash Y_R$, fix the shortest word A_i for g_i in $(G, S\cup\mathscr{H})$. Let $A_i = g_i'h_i'$, where h_i' is the $\mathscr{H}$-component in g_iH. Replacing g_i with g_i', we can assume that g_i does not have an $\mathscr{H}$-component in g_iH. Then $x = g_ix_i$ is a geodesic in $(G, S\cup\mathscr{H})$ (see [14]). We define

$\varphi(x) = \varphi_1(g_i) \oplus \varphi_2(x_i)$, where $\varphi_2(x_i)$ is in the g_i item in $\bigoplus_{g_i \in R(n-1)} E$. Let $\varphi(y) = \varphi_1(y)$ for $y \in Y_R$. We need to verify the conditions in Theorem 3.1.

(1) For $d_S(x,y) \leq R$, we have two cases. If $x, y \in Y_R$, then $\|\varphi_1(x) - \varphi_1(y)\| \leq \epsilon$. If $x \in g_iH \backslash Y_R$, we have $y \in g_iH$ by BCP.

If $y \in g_iH \backslash Y_R$, let $y = g_iy_i$, $x = g_ix_i$ be the geodesics in $(G, S \cup \mathscr{H})$. Thus $d_S(x_i, y_i) = d_S(x,y) \leq R$ and $\|\varphi(x) - \varphi(y)\| = \|\varphi_2(x_i) - \varphi_2(y_i)\| < \epsilon$.

Otherwise, $y \in g_iH \cap Y_R$, and let $y = g_i'y_i'$ be a geodesic in $(G, S \cup \mathscr{H})$. For $d_S(x,y) \leq R$, we have $d_S(g_i, g_i') \leq a(R)$ and $|y_i|_s \leq a(R)$. Then

$$\begin{aligned} &d_S(g_i, y) \leq d_S(g_i, g_i') + d_S(g_i', y) \leq 2a(R), \\ &|x_i|_s = d_S(g_i, x) \leq d_S(g_i, y) + d_S(y, x) \leq 3a(R). \end{aligned}$$

We have

$$\|\varphi(x) - \varphi(y)\| = (\|\varphi(g_i) - \varphi(y)\|^p + \|\varphi_2(x_i)\|^p)^{\frac{1}{p}} \leq \epsilon.$$

(2) For $m \in \mathbb{N}$ and $d_S(x,y) \leq m$, let $T_m = \{g \mid |g|_S \leq a(m)\}$ and $Y_m = B(n-1)T_m$. Then $\{g_iH \backslash Y_m\}$ is m-separated. If x, y are both in Y_R or in g_iH for some i, it is easy to see that $\|\varphi(x) - \varphi(y)\|$ is bounded. So we only need to consider $x \in g_iH$, $y \in g_jH$ with $i \neq j$. For $d_S(x,y) \leq m$, either x or y is in Y_m. We assume $y \in Y_m$. For $x \in g_iH \cap (Y_m \backslash Y_R)$, let $x = g_ix_i$ be a geodesic in $(G, S \cup \mathscr{H})$. We have the following two cases.

Case a. If $y \in g_jH \cap (Y_m \backslash Y_R)$, let $y = g_jy_j$ be geodesics in $(G, S \cup \mathscr{H})$. Then

$$\|\varphi(x) - \varphi(y)\|^p = \|\varphi_1(g_i) - \varphi_1(g_j)\|^p + \|\varphi_2(x_i)\|^p + \|\varphi_2(y_j)\|^p.$$

From $d_S(g_i, g_j) \leq a(m)$, $|x|_S \leq a(m)$, and $|y_j|_S \leq a(m)$, we know $\|\varphi(x) - \varphi(y)\| \leq 3C_m$.

Case b. If $y \in g_jH \cap Y_R$, let $y = g_i'y_i'$ be the geodesic in $(G, S \cup \mathscr{H})$ with y_i an $\mathscr{H}$-component and $|y_i'|_S \leq a(R)$. For $d_S(x,y)$, then $d_S(g_i, g_i') \leq a(m)$ and $d_S(g_i, y) \leq 2a(m)$, $|x|_S \leq 3a(m)$. Then

$$\|\varphi(x) - \varphi(y)\|^p = \|\varphi_1(g_i) - \varphi_1(y)\|^p + \|\varphi_2(x_i)\|^p.$$

We have that $\|\varphi(x) - \varphi(y)\| \leq 2C_m$.

(3) We have $d_S(x,y) \leq l(x_g) + d_S(g_i, g_j) + l(y_g)$, and thus that $d_S(x,y)$ tends to infinity implies at least term must tend to infinity. So

$$\lim_{t \to +\infty} \inf\{\|\varphi(x) - \varphi(y)\|, d_S(x,y) \geq t\} = \infty.$$

By Theorem 3.1, $B(n)$ admits a coarse embedding into E^p.

We should mention that the problem whether G admits a coarse embedding into a uniformly Banach space if H dose is still open.

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