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Conjugacy Classes and Characters for Extensions of Finite Groups*

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Abstract Let H be an extension of a finite group Q by a finite group G. Inspired by the results of duality theorems for étale gerbes on orbifolds, the authors describe the number of conjugacy classes of H that map to the same conjugacy class of Q. Furthermore, a generalization of the orthogonality relation between characters of G is proved.

Keywords Group extensions, Conjugacy classes, Orthogonality of characters
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1 Introduction

Extensions of finite groups play an important role in the theory of finite groups. For example, the composition serious of a finite group H consists of a sequence of subgroups H_i

$$1 = H_0 \lhd H_1 \lhd H_2 \lhd \cdots \lhd H_n = H,$$

such that H_i is a strict normal subgroup of H_{i+1} with a simple quotient group H_{i+1}/H_i, for $i = 0, \cdots, n-1$. Therefore, with the classification theorem of finite simple groups, the study of extensions of finite groups would describe and classify all finite groups.

The structure of extensions of finite groups has been studied for a long time (see [7]). In this paper, we look at extensions of finite groups from a geometric point of view. A finite group G is a groupoid with one unit. In the language of stacks (see [1]), such a group(oid) corresponds to the classifying stack BG of principal G-bundles. An extension of a finite group Q by a finite group G

$$1 \to G \to H \to Q \to 1$$

is equivalent to a G-gerbe

$$BH \to BQ,$$

a bundle of BG over BQ (see [5]).

Our study of extensions of finite groups is motivated by a conjecture in mathematical physics (see [4]). Let $\widehat{G}$ be the finite set of isomorphism classes of irreducible unitary representations

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of G. The above extension H of Q by G gives a natural action of Q on $\widehat{G}$. Consider the transformation groupoid $\widehat{G} \rtimes Q \rightrightarrows \widehat{G}$. There is a canonical class c in $H^2(\widehat{G} \rtimes Q, U(1))$ associated to the extension H. The decomposition conjecture in [4] suggests that the geometry of a G-gerbe associated to the extension H is equivalent to the geometry of the orbifold associated to the groupoid $\widehat{G} \rtimes Q$ twisted by c. We studied this conjecture in [8] from the view point of noncommutative geometry. In particular, we proved that the group algebra of H is Morita equivalent to the c-twisted groupoid algebra of $\widehat{G} \rtimes Q$. The details of this are reviewed in Section 2.

In this short note, we present two results from our analysis of the structure of $\mathbb{C}H$. One result concerns the relations between conjugacy classes of H and Q (see Section 3). The other result concerns a generalized orthogonality relation between characters of G (see Section 4).

2 Group Algebras of Finite Group Extensions

Consider an extension of finite groups as in

$$1 \longrightarrow G \xrightarrow{i} H \xrightarrow{j} Q \longrightarrow 1. \tag{2.1}$$

As part of our study of gerbe duality, the structure of the group algebra $\mathbb{C}H$ is analyzed in [8]. We briefly recall the results.

Choose a section $s : Q \to H$ of $j : H \to Q$ above such that $j \circ s = \mathrm{id}$, and $s(1) = 1$. Since G and Q are finite groups, such a section s always exists. For $q_1, q_2 \in Q$, define $\tau(q_1, q_2) := s(q_1)s(q_2)s(q_1q_2)^{-1}$. It is easy to see that $\tau(q_1, q_2) \in \ker(j) = G$, so we obtain

$$\tau : Q \times Q \to G.$$

Clearly τ is trivial (i.e., $\tau(-,-) = 1$) if and only if $s : Q \to H$ is a group homomorphism, which in turn is equivalent to the extension (2.1) being a split extension.

The definition of τ may be written as

$$s(q_1)s(q_2) = \tau(q_1, q_2)s(q_1q_2). \tag{2.2}$$

By associativity, we have $(s(q_1)s(q_2))s(q_3) = s(q_1)(s(q_2)s(q_3))$. It follows that

$$\tau(q_1, q_2)\tau(q_1q_2, q_3) = s(q_1)\tau(q_2, q_3)s(q_1)^{-1}\tau(q_1, q_2q_3). \tag{2.3}$$

Given the section s, we can define a set-theoretic bijection between H and $G \times Q$:

$$\alpha : H \to G \times Q, \quad \alpha(h) := (hs(j(h))^{-1}, j(h)).$$

The inverse of α is

$$G \times Q \to H, \quad (g, q) \mapsto i(g)s(q).$$

The group structure on H induces a new group structure $\cdot$ on $G \times Q$ via α. This group structure is given by

$$(g_1, q_1) \cdot (g_2, q_2) = (g_1 \operatorname{Ad}_{s(q_1)}(g_2)\tau(q_1, q_2), q_1q_2), \tag{2.4}$$

where $\operatorname{Ad}_h(\cdot)$ denotes the conjugation action of an element $h \in H$ on G, which is an automorphism of G because G is normal in H. Denote by

$$G \rtimes_{s,\tau} Q$$

the set $G \times Q$ with the group structure given by (2.4). The definition implies that α is a group isomorphism:

$$\alpha : H \to G \rtimes_{s,\tau} Q.$$

It is easy to check that different choices of the section s yield isomorphic groups $G \rtimes_{s,\tau} Q$.

The group isomorphism α naturally induces an isomorphism of group algebras

$$\alpha : \mathbb{C}H \xrightarrow{\simeq} \mathbb{C}(G \rtimes_{s,\tau} Q).$$

Given s and τ, we let an element $q \in Q$ act on $\mathbb{C}G$ by conjugation by $s(q)$. This does not give an action of Q on $\mathbb{C}G$, and the failure of this to be an action is governed by τ. In other words, this defines a τ-twisted action of Q on $\mathbb{C}G$. Hence the group algebra $\mathbb{C}(G \rtimes_{s,\tau} Q)$ can be written as a twisted crossed product algebra $\mathbb{C}G \rtimes_{s,\tau} Q$.

Let $\widehat{G}$ be the set of isomorphism classes of irreducible complex linear representations of G. Furthermore, for every element $[\rho]$ in $\widehat{G}$, we choose an irreducible representation in the class $[\rho]$ denoted by

$$\rho : G \to \mathrm{End}(V_\rho),$$

where V_ρ is a certain finite dimensional $\mathbb{C}$-vector space. The group algebra $\mathbb{C}G$ is isomorphic to a direct sum of matrix algebras $\bigoplus_{[\rho]\in\widehat{G}} \mathrm{End}(V_\rho)$:

$$\beta : \mathbb{C}G \xrightarrow{\simeq} \bigoplus_{[\rho]\in\widehat{G}} \mathrm{End}(V_\rho), \quad g \mapsto (\rho(g))_{[\rho]\in\widehat{G}}.$$

This is well-known (see e.g. [3, Proposition 3.29]).

Next we define an action of Q on $\widehat{G}$. Let $\rho : G \to \mathrm{End}(V_\rho)$ be a $\mathbb{C}$-linear representation of G. Given $q \in Q$, we obtain another G representation $\widetilde{\rho}$ defined by

$$G \ni g \mapsto \rho(\mathrm{Ad}_{s(q)}(g)).$$

It is easy to see that $\widetilde{\rho}$ is irreducible if and only if ρ is. If $s' : Q \to H$ is another section of j, then we have $\rho \circ \mathrm{Ad}_{s(q)} = \rho \circ \mathrm{Ad}_{s'(q)} \mathrm{Ad}_{s'(q)^{-1}s(q)}$. Since $s'(q)^{-1}s(q) \in G$, $\mathrm{Ad}_{s'(q)^{-1}s(q)}$ is an inner automorphism of G. Hence $\rho \circ \mathrm{Ad}_{s(q)}$ and $\rho \circ \mathrm{Ad}_{s'(q)}$ are isomorphic G-representations. Therefore the assignment $(q, \rho) \mapsto \widetilde{\rho}$ yields a right Q-action on $\widehat{G}$; namely, $q \in Q$ sends the class $[\rho] \in \widehat{G}$ to the class $[\widetilde{\rho}] \in \widehat{G}$. For notational convenience, we write this right action as a left action. We denote the image of the isomorphism class $[\rho] \in \widehat{G}$ under the action by q by $q([\rho])$. By abuse of notation, we denote the chosen irreducible G-representation that represents the class $q([\rho])$ also by $q([\rho]) : G \to \mathrm{End}(V_{q([\rho])})$. Let

$$\widehat{G} \rtimes Q := (\widehat{G} \times Q \rightrightarrows \widehat{G})$$

be the groupoid associated to this Q-action on $\widehat{G}$.

By construction, the representation $q([\rho]) : G \to \mathrm{End}(V_{q([\rho])})$ is equivalent to the representation $\widetilde{\rho} : G \to \mathrm{End}(V_\rho)$ defined by $g \mapsto \rho(\mathrm{Ad}_{s(q)}(g))$. Therefore there exists a $\mathbb{C}$-linear isomorphism

$$T_q^{[\rho]} : V_\rho \to V_{q([\rho])},$$

that intertwines the two representations, namely

$$\rho(\mathrm{Ad}_{s(q)}(g)) = {T_q^{[\rho]}}^{-1} \circ q([\rho])(g) \circ T_q^{[\rho]}.$$

We may choose $T_1^{[\rho]}$ to be the identity map on V_ρ. It can be shown that there are constants $c^{[\rho]}(q_1,q_2)$ such that $T_{q_2}^{q_1([\rho])}\circ T_{q_1}^{[\rho]}\circ\rho(\tau(q_1,q_2))\circ {T_{q_1q_2}^{[\rho]}}^{-1}$ is $c^{[\rho]}(q_1,q_2)$ times the identity map. In other words,

$$T_{q_2}^{q_1([\rho])}\circ T_{q_1}^{[\rho]} = c^{[\rho]}(q_1,q_2)T_{q_1q_2}^{[\rho]}\rho(\tau(q_1,q_2))^{-1}. \tag{2.5}$$

Since the collection $\{\rho\}$ consists of unitary representations, the isomorphisms $T_q^{[\rho]}$ can also be chosen to be unitary. Therefore, $c^{[\rho]}(q_1,q_2)$ actually takes value in $U(1)$. By [8, Proposition 3.1], the function

$$c:\widehat{G}\times Q\times Q\to U(1),\quad ([\rho],q_1,q_2)\mapsto c^{[\rho]}(q_1,q_2)$$

is a 2-cocycle on the groupoid $\widehat{G}\rtimes Q$ such that $c^{[\rho]}(1,q)=c^{[\rho]}(q,1)=1$ for any $[\rho]\in\widehat{G},\ q\in Q$. The cohomology class defined by c is independent of the choices of the section s and the operator $T_q^{[\rho]}$.

Let $C(\widehat{G}\rtimes Q,c)$ be the twisted groupoid algebra associated to the cocycle c on $\widehat{G}\rtimes Q$. We explain the definition of $C(\widehat{G}\rtimes Q,c)$ and refer the readers to [9] for more details. By definition, $C(\widehat{G}\rtimes Q,c)$ is the set of $C(\widehat{G})$-valued functions on Q, i.e., $\mathbb{C}$-valued functions on $\widehat{G}\times Q$. By abuse of notation, for $([\rho],q)\in\widehat{G}\times Q$, we also denote by $([\rho],q)$ the function on $\widehat{G}\times Q$ which takes value 1 at $([\rho],q)$ and 0 elsewhere. The collection $\{([\rho],q)\}$ of functions on $\widehat{G}\times Q$ forms an additive basis of $C(\widehat{G}\rtimes Q,c)$. The set $C(\widehat{G}\rtimes Q,c)$ is endowed with a product structure defined by

$$([\rho],q)\circ([\rho'],q')=\begin{cases}c^{[\rho]}(q,q')([\rho],qq'), & \text{if } [\rho']=q([\rho]),\\ 0, & \text{otherwise.}\end{cases}$$

The cocycle condition of c implies that this product is associative.

Let $\bigoplus_{[\rho]\in\widehat{G}}\mathrm{End}(V_\rho)\otimes\mathbb{C}Q$ be the $\mathbb{C}$-vector space spanned by elements of the form (x_ρ,q), where x_ρ is an element in $\mathrm{End}(V_\rho)$ with $[\rho]\in\widehat{G}$ and $q\in Q$. We equip this space with a product $\circ$ defined as follows:

$$(x_{\rho_1},q_1)\circ(\widetilde{x}_{\rho_2},q_2):=\begin{cases}(x_{\rho_1}{T_{q_1}^{[\rho_1]}}^{-1}\widetilde{x}_{q_1([\rho_1])}T_{q_1}^{[\rho_1]}\rho_1(\tau(q_1,q_2)),q_1q_2), & \text{if } [\rho_2]=q_1([\rho_1]),\\ 0, & \text{otherwise.}\end{cases}$$

Let

$$\bigoplus_{[\rho]}\mathrm{End}(V_\rho)\rtimes_{T,\tau}Q$$

be the space $\bigoplus_{[\rho]\in\widehat{G}}\mathrm{End}(V_\rho)\otimes\mathbb{C}Q$ with the product $\circ$ defined above. We call this the twisted crossed product algebra. This algebra plays an important role in the following structure result on the group algebra $\mathbb{C}H$.

Proposition 2.1 (see [8, Proposition 3.2]) *The map*

$$\kappa: G\times Q\ni(g,q)\mapsto\sum_{[\rho]\in\widehat{G}}(\rho(g),q)$$

defines an algebra isomorphism from the group algebra $\mathbb{C}G\rtimes_{s,\tau}Q$ to the twisted crossed product algebra $\bigoplus_{[\rho]}\mathrm{End}(V_\rho)\rtimes_{T,\tau}Q$. Hence,

$$\kappa\circ\alpha:\mathbb{C}H\to\bigoplus_{[\rho]}\mathrm{End}(V_\rho)\rtimes_{T,\tau}Q$$

is an algebra isomorphism.

Proposition 2.1 is used in [8, Section 3.2] to prove the following structure result of $\mathbb{C}H$.

Theorem 2.1 (see [8, Theorem 3.1]) *The group algebra $\mathbb{C}H$ is Morita equivalent to the twisted groupoid algebra $C(\widehat{G} \rtimes Q, c)$.*

We remark that the proof of Theorem 2.1 is done by explicitly constructing Morita equivalence bimodules between the two algebras.

Since $j : H \to Q$ is a surjective group homomorphism, j induces a surjective homomorphism of algebras from $\mathbb{C}H$ to $\mathbb{C}Q$. It is well-known that the center of $\mathbb{C}Q$ has a canonical additive basis indexed by the conjugacy classes of Q. This decomposition of the center $Z(\mathbb{C}Q)$ and the surjection $\mathbb{C}H \to \mathbb{C}Q$ imply that the center of $\mathbb{C}H$, as a vector space, decomposes into a direct sum of subspaces $Z(\mathbb{C}H)_{\langle q\rangle}$ indexed by conjugacy classes $\langle q\rangle$ of Q,

$$Z(\mathbb{C}H) = \bigoplus_{\langle q\rangle \subset Q} Z(\mathbb{C}H)_{\langle q\rangle}.$$

As shown in [8, Section 3.2], the center $Z(C(\widehat{G} \rtimes Q, c))$ decomposes into a direct sum of subspaces $Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}$ indexed by conjugacy classes of Q,

$$Z(C(\widehat{G} \rtimes Q, c)) = \bigoplus_{\langle q\rangle \subset Q} Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}.$$

The explicit Morita equivalence bimodules in the proof of Theorem 2.1 yield an algebra isomorphism from the center of $\mathbb{C}H$ to the center of $C(\widehat{G} \rtimes Q, c)$, which we denote by I.

Proposition 2.2 (see [8, Proposition 3.4]) *The isomorphism*

$$I : Z(\mathbb{C}H) \to Z(C(\widehat{G} \rtimes Q, c))$$

is compatible with the decompositions into subspaces indexed by conjugacy classes of Q, i.e., I is an isomorphism from $Z(\mathbb{C}H)_{\langle q\rangle}$ to $Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}$.

In the rest of this paper, we discuss some group-theoretic applications of our analysis of the group algebra $\mathbb{C}H$.

3 Counting Conjugacy Classes in Group Extensions

Let $j : H \to Q$ be a surjective homomorphism of finite groups. Let $\langle q\rangle \subset Q$ be a conjugacy class of Q. The pre-image $j^{-1}(\langle q\rangle) \subset H$ may be partitioned into a disjoint union of conjugacy classes of H. It is natural to ask the following question.

Question 3.1 How many conjugacy classes of H are contained in $j^{-1}(\langle q\rangle)$?

In this section, we discuss an answer to this question.

Let G be the kernel of $j : H \to Q$. Then we are in the situation of the exact sequence (2.1). The homomorphism $j : H \to Q$ induces a surjective homomorphism $\mathbf{j} : \mathbb{C}H \to \mathbb{C}Q$ between group algebras. This, in turn, induces a homomorphism $\mathbf{j} : Z(\mathbb{C}H) \to Z(\mathbb{C}Q)$ between centers. The centers $Z(\mathbb{C}H)$ and $Z(\mathbb{C}Q)$, viewed as vector spaces, admit natural bases, $\{1_{\langle h\rangle}\} \subset Z(\mathbb{C}H)$ and $\{1_{\langle q\rangle}\} \subset Z(\mathbb{C}Q)$, indexed by conjugacy classes. These bases satisfy the requirement that if $j(\langle h\rangle) = \langle q\rangle$, then $\mathbf{j}(1_{\langle h\rangle}) \in \mathbb{N}1_{\langle q\rangle}$. As $j(\langle s(q)\rangle) = \langle q\rangle$, the map $j : Z(\mathbb{C}H) \to Z(\mathbb{C}Q)$ is surjective. Let

$$Z(\mathbb{C}H)_{\langle q\rangle} := \bigoplus_{\langle h\rangle \subset j^{-1}(\langle q\rangle)} \mathbb{C}1_{\langle h\rangle}.$$

By construction, the dimension $\dim Z(\mathbb{C}H)_{\langle q\rangle}$ is the number of conjugacy classes of H that are contained in $j^{-1}(\langle q\rangle)$. By Proposition 2.2, the isomorphism $I : Z(\mathbb{C}H) \to Z(C(\widehat{G} \rtimes Q, c))$ restricts to an additive isomorphism

$$Z(\mathbb{C}H)_{\langle q\rangle} \simeq Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}.$$

Clearly, the answer to Question 3.1 is the dimension $\dim Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}$, which we now compute.

Let $\widehat{G}^q \subset \widehat{G}$ be the subset consisting of elements fixed by $q \in Q$. Let $C(q) \subset Q$ be the centralizer subgroup of q. Then, by [6], we have that $Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}$ is additively isomorphic to the c-twisted orbifold cohomology $H^\bullet_{\mathrm{orb}}([\widehat{G}^q/C(q)], c)$. Decompose $\widehat{G}^q$ into a disjoint union of $C(q)$-orbits:

$$\widehat{G}^q = \coprod_i O_i. \tag{3.1}$$

For each $C(q)$-orbit O_i, pick a representative $[\rho_i]$ and denote by $Q_i := \mathrm{Stab}_{C(q)}([\rho_i]) \subset C(q)$ the stabilizer subgroup of $[\rho_i]$. Consider the homomorphism

$$\gamma^{[\rho_i]}_{-,q} : C(q) \to U(1), \quad C(q) \ni q_1 \mapsto \gamma^{[\rho_i]}_{q_1,q} := c^{[\rho_i]}(q_1, q)c^{[\rho_i]}(q, q_1)^{-1}.$$

Here, $c^{[\rho]}(-,-)$ is the cocycle defined in (2.5). It follows from (3.1) that

$$H^\bullet_{\mathrm{orb}}([\widehat{G}^q/C(q)], c) \simeq \bigoplus H^\bullet_{\mathrm{orb}}(BQ_i, c).$$

By [6, Example 6.4], we have that $H^\bullet_{\mathrm{orb}}(BQ_i, c) = \mathbb{C}$ if the following condition holds:

$$\gamma^{[\rho_i]}_{q_1,q} = 1 \quad \text{for all } q_1 \in Q_i. \tag{3.2}$$

Moreover, if (3.2) does not hold, then $H^\bullet_{\mathrm{orb}}(BQ_i, c) = 0$. It follows that $\dim Z(C(\widehat{G} \rtimes Q, c))_{\langle q\rangle}$ is equal to

$$\#\{O_i = C(q)\text{-orbit of } \widehat{G}^q \mid \text{there exists } [\rho_i] \in O_i \text{ s.t. } \gamma^{[\rho_i]}_{q_1,q} = 1 \text{ for all } q_1 \in Q_i = \mathrm{Stab}_{C(q)}([\rho_i])\}.$$

In summary, we have obtained the following theorem as an answer to Question 3.1.

Theorem 3.1 *Let $H = G \rtimes_{s,\tau} Q$ be an extension of Q by G. Consider the canonical quotient map $j : H \to Q$. For $q \in Q$, the number of conjugacy classes of H that are mapped to the conjugacy class $\langle q\rangle$ of Q is equal to*

$$\#\left\{ O_i = C(q)\text{-}\mathit{orbit\ of}\ \widehat{G}^q \;\middle|\; \begin{array}{l} \mathit{there\ exists}\ [\rho_i] \in O_i\ \mathit{s.t.}\ \gamma^{[\rho_i]}_{q_1,q} = 1 \\ \mathit{for\ all}\ q_1 \in Q_i = \mathrm{Stab}_{C(q)}([\rho_i]) \end{array} \right\}. \tag{3.3}$$

In the following, we discuss a few special cases of Theorem 3.1.

Example 3.1 If the group G is abelian, then all irreducible representations of G are 1-dimensional, and all intertwiners in (2.5) can be taken to be the identity. In this case, (3.3) can be simplified into

$$\#\left\{ O_i = C(q)\text{-orbit of } \widehat{G}^q \;\middle|\; \begin{array}{l} \text{there exists } [\rho_i] \in O_i, \text{s.t. } \rho_i(\tau(q_1,q)\tau(q,q_1)^{-1}) = 1 \\ \text{for all } q_1 \in Q_i = \mathrm{Stab}_{C(q)}([\rho_i]) \end{array} \right\}. \tag{3.4}$$

Example 3.2 If the group G is abelian and H is a semi-direct product of G and Q, then the cocycle $\tau(-,-)$ can be taken to be trivial. In this case, (3.3) can be simplified into

$$\#\{C(q)\text{-orbit of } \widehat{G}^q\}. \tag{3.5}$$

Example 3.3 If the Q-action on $\widehat{G}$ is trivial[1], then $\widehat{G}^q = \widehat{G}$, and all intertwiners in (2.5) can be taken to be the identity. In this case, (3.3) can be simplified into

$$\#\left\{ [\rho] = \begin{array}{l} \text{the isomorphism class of} \\ \text{irreducible } G\text{-representations} \end{array} \;\middle|\; \begin{array}{l} \rho(\tau(q_1,q)\tau(q,q_1)^{-1}) = 1 \\ \text{for all } q_1 \in C(q) \end{array} \right\}. \tag{3.6}$$

4 An Orthogonality Relation of Characters

The material in this section is inspired by the proof of the orthogonality relation given in [2, Chapter 2, Section 12]. Using Proposition 2.1, we prove a generalization of the orthogonality relation between characters of G. For $h \in H$, write the centralizer subgroup of h by $C_H(h)$, and the number of elements in $C_H(h)$ by $|C_H(h)|$.

Theorem 4.1 *Let $H = G \rtimes_{s,\tau} Q$ be an extension of Q by G. For $[\rho] \in \widehat{G}$, let χ_ρ^G be the character of the G-representation V_ρ. For $(g_1, g_2) \in G \times G$,*

$$\sum_{[\rho]\in\widehat{G}} \sum_{q\in Q} \chi_\rho^G(g_1^{-1})\chi_{q([\rho])}^G(g_2) = \begin{cases} |C_H(g_1)|, & \textit{if } g_1 \textit{ and } g_2 \textit{ are conjugate in } H, \\ 0, & \textit{otherwise.} \end{cases} \tag{4.1}$$

Proof Consider (2.1) again. The group $H \times H$ acts naturally on the group algebra $\mathbb{C}H$ via $(h_1, h_2) \cdot h = h_1^{-1} h h_2$. In this way, we may view $\mathbb{C}H$ as a representation of $H \times H$. Its character $\chi_{\mathbb{C}H}^{H\times H}$ can be calculated as follows:

$$\begin{aligned} \chi_{\mathbb{C}H}^{H\times H}((h_1,h_2)) &= \#\{h \in H \mid h_1^{-1} h h_2 = h\} = \#\{h \in H \mid h h_2 h^{-1} = h_1\} \\ &= \begin{cases} |C_H(h_1)|, & \text{if } h_1 \text{ and } h_2 \text{ are conjugate in } H, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

We now consider $\mathbb{C}H$ as a representation of the subgroup $G \times G$. The above calculation gives the character of this representation: For $(g_1, g_2) \in G \times G$,

$$\chi_{\mathbb{C}H}^{G\times G}((g_1,g_2)) = \chi_{\mathbb{C}H}^{H\times H}((g_1,g_2)) = \begin{cases} |C_H(g_1)|, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate in } H, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

We calculate the character $\chi_{\mathbb{C}H}^{G\times G}$ by another method. By Proposition 2.1, there is an isomorphism of algebras

$$\mathbb{C}H \simeq \bigoplus_{[\rho]\in\widehat{G}} \mathrm{End}(V_\rho) \rtimes_{T,\tau} Q.$$

[1] Equivalently, this means that the band of the gerbe $BH \to BQ$ is trivial.

Under this isomorphism, the $G \times G$ action on $\mathbb{C}H$ is identified with the following $G \times G$ action on $\bigoplus_{[\rho]\in\widehat{G}} \mathrm{End}(V_\rho) \rtimes_{T,\tau} Q$:

$$\begin{aligned}(g_1, g_2) \cdot (x_\rho, q) &:= \Big(\sum_{\rho_1} \rho_1(g_1^{-1}), 1\Big) \circ (x_\rho, q) \circ \Big(\sum_{\rho_2} \rho_2(g_2), 1\Big) \\ &= (\rho(g_1^{-1}) x_\rho {T_q^{[\rho]}}^{-1} q([\rho])(g_2) T_q^{[\rho]}, q),\end{aligned}$$

where $\circ$ is the algebra structure on $\bigoplus_{[\rho]\in\widehat{G}} \mathrm{End}(V_\rho) \rtimes_{T,\tau} Q$.

For each ρ, fix an isomorphism of $\mathrm{End}(V_\rho)$ with a matrix algebra, and let e^ρ_{st} denote the standard basis of this matrix algebra. We use the symbol $(x_\rho)_{st}$ to denote the s,t-entry of $x_\rho \in \mathrm{End}(V_\rho)$. Then we have $(\rho(g_1^{-1}) e^\rho_{st} {T_q^{[\rho]}}^{-1} q([\rho])(g_2) T_q^{[\rho]})_{st} = (\rho(g_1^{-1}))_{ss} ({T_q^{[\rho]}}^{-1} q([\rho])(g_2) T_q^{[\rho]})_{tt}$. Therefore,

$$\begin{aligned}\mathrm{tr}\,((g_1, g_2)|_{\mathrm{End}(V_\rho)\times\{q\}}) &= \sum_{s,t} (\rho(g_1^{-1}))_{ss} ({T_q^{[\rho]}}^{-1} q([\rho])(g_2) T_q^{[\rho]})_{tt} \\ &= \mathrm{tr}\,(\rho(g_1^{-1}))\mathrm{tr}\,({T_q^{[\rho]}}^{-1} q([\rho])(g_2) T_q^{[\rho]}) \\ &= \chi^G_\rho(g_1^{-1}) \chi^G_{q([\rho])}(g_2),\end{aligned}$$

where χ^G_ρ and $\chi^G_{q([\rho])}$ denote the characters of the G-representations ρ and $q([\rho])$. Summing over $[\rho] \in \widehat{G}$ and $q \in Q$, we find that

$$\chi^{G\times G}_{\mathbb{C}H}((g_1, g_2)) = \sum_{[\rho]\in\widehat{G}} \sum_{q\in Q} \chi_\rho(g_1^{-1}) \chi_{q([\rho])}(g_2). \tag{4.3}$$

Combining the above with (4.2), we obtain the desired identity:

$$\sum_{[\rho]\in\widehat{G}} \sum_{q\in Q} \chi^G_\rho(g_1^{-1}) \chi^G_{q([\rho])}(g_2) = \begin{cases} |C_H(g_1)|, & \text{if } g_1 \text{ and } g_2 \text{ are conjugate in } H, \\ 0, & \text{otherwise.}\end{cases}$$

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