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Permanence of Metric Sparsification Property under Finite Decomposition Complexity*

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Abstract The notions of metric sparsification property and finite decomposition complexity are recently introduced in metric geometry to study the coarse Novikov conjecture and the stable Borel conjecture. In this paper, it is proved that a metric space X has finite decomposition complexity with respect to metric sparsification property if and only if X itself has metric sparsification property. As a consequence, the authors obtain an alternative proof of a very recent result by Guentner, Tessera and Yu that all countable linear groups have the metric sparsification property and hence the operator norm localization property.

Keywords Metric space, Metric sparsification, Asymptotic dimension, Decomposition complexity, Permanence property
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1 Introduction

The metric sparsification property and finite decomposition complexity are notions of metric geometry, which were introduced very recently in studying the coarse Novikov conjecture (see [3–4]) and the stable Borel conjecture (see [9]), respectively. Both properties were motivated by the notion of finite asymptotic dimension of a metric space introduced by Gromov [7].

Recall that a metric space X has finite asymptotic dimension if there is an integer $n \geq 0$, such that for any (large) number $r > 0$, the space X may be written as a union of $n+1$ subspaces X_i, each of which may be further decomposed as an r-disjoint union:

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \bigsqcup_{j=1}^{\infty} X_{ij}, \quad \mathrm{dist}(X_{ij}, X_{ij'}) > r,$$

in which the metric family $\{X_{ij} : i, j\}$ is bounded, i.e., $S := \sup\limits_{i,j} \mathrm{diam}(X_{ij}) < \infty$. In general, we say that a countable family of metric spaces $\mathcal{X} = \{X\}$ is (n, r)-decomposable over another metric family $\mathcal{Y}$, if every $X \in \mathcal{X}$ admits a decomposition as above, where each $X_{ij} \in \mathcal{Y}$.

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Inspired by the feature of finite asymptotic dimension, Guentner, Tessera and Yu [9] introduced the notion of finite decomposition complexity as a measure of computational complexity of metric spaces. Roughly speaking, a metric family $\mathcal{X}$ has finite decomposition complexity, if it can be decomposed, through a finite number of applications of the decomposability relation as above, into a bounded family. Guentner, Tessera and Yu proved that the stable Borel conjecture holds for an aspherical manifold whose fundamental group has finite decomposition complexity (see [9]).

On the other hand, the metric sparsification property was introduced by Chen, Tessera, Wang and Yu [3] to supply a more flexible geometric condition for operator norm localization, which can be applied to the coarse Novikov conjecture in operator K-theory (see [6]). Roughly speaking, a metric space has the metric sparsification property, if there exists a constant $0 < c < 1$, such that, for every positive finite Borel measure μ on X, there exists a Borel subset Ω, which is a union of "well-separated" subsets of uniformly bounded diameters, such that $\mu(\Omega) \geq c\mu(X)$. Chen, Tessera, Wang and Yu proved that any solvable locally compact group equipped with a proper, locally finite left-invariant metric has metric sparsification property. This provides the first example of finitely generated group with infinite asymptotic dimension satisfying operator norm localization property (see [3]).

The motivation for studying the permanence property in coarse geometry is inspired by [5], in which it is proved that the operator norm localization is stable under some operations of coarse metric spaces. In [8], Guentner gave a survey on various permanence properties of coarse metric spaces. In this paper, we shall regard finite decomposition complexity as a type of operation of metric spaces to provide a permanence result for metric sparsification property. To do this, we introduce a notion of finite decomposition complexity with respect to metric sparsification property, and show that if a metric space X has this property, then X itself has the metric sparsification property. That is, the metric sparsification property is stable under large scale decompositions of finite complexity. Combined with a result of Guentner, Tessera and Yu [9] that all countable linear groups have finite decomposition complexity, this also implies another result of Guentner, Tessera and Yu [10] that all countable linear groups have the metric sparsification property, and hence the operator norm localization property. This fact can be used to prove the coarse Novikov conjecture for the box spaces associated to countable linear groups, including many interesting sequences of expander graphs (see [6, 10]).

2 Preliminaries

Definition 2.1 (see [3]) *Let X be a metric space. We say that X has metric sparsification property with constant $0 < c \leq 1$ (we say that X has $MS(c)$ for short), if there exists a (non-decreasing) function $f : \mathrm{R}_+ \to \mathrm{R}_+$, such that for all $m \in \mathrm{R}_+$ and every finite positive Borel measure μ on X, there is a Borel subset $\Omega = \bigsqcup_{i\in I} \Omega_i$, such that*

(i) $\mathrm{dist}(\Omega_i, \Omega_j) \geq m$ *for all* $i \neq j \in I$;

(ii) $\mathrm{diam}(\Omega_i) \leq f(m)$ *for all* $i \in I$;

(iii) $\mu(\Omega) \geq c\mu(X)$.

When we need to be more explicit, we will say that X has $\mathrm{MS}(c)$ with function f. If m, μ are given, and if we want to say that a subset Ω satisfies the Definition 2.1, we will simply write $\Omega = \Omega(\mu, f, m, c)$.

Definition 2.2 (see [3]) *We say that a family of metric spaces has uniform $MS(c)$, if there is a common f that works for all the elements of the family.*

The following property plays an important role in the next section.

Proposition 2.1 (see [3]) *If X has metric sparsification property, then it has the property with constant c for all $0 < c < 1$.*

To recall the notion of finite decomposition complexity (see [9]), we shall use $\mathcal{X}$, $\mathcal{Y}$, etc. to denote the (countable) families of the metric spaces, and use Δ, Γ, etc. to denote the collections of the metric families.

A metric family $\mathcal{X} = \{X_j\}_{j=1}^{\infty}$ is bounded, if there is a uniform bound on the diameter of the individual space X_j, namely, $\sup\{\mathrm{diam} X_j : j\} < \infty$. Let $\mathbb{D}_0$ denote the collection of bounded families:

$$\mathbb{D}_0 = \{\mathcal{X} : \ \mathcal{X} \text{ is bounded}\}.$$

A metric family $\mathcal{X}$ is (n, r)-decomposable over a metric family $\mathcal{Y}$, if every $X \in \mathcal{X}$ admits a decomposition

$$X = \bigcup_{i=0}^{n} X_i, \quad X_i = \bigsqcup_{j=1}^{\infty} X_{ij}, \quad \mathrm{dist}(X_{ij}, X_{ij'}) > r,$$

with each $X_{ij} \in \mathcal{Y}$.

Definition 2.3 (see [9]) *Let $\mathbb{F}$ be a collection of metric families. A family $\mathcal{X}$ is decomposable over $\mathbb{F}$, if there exists an $n \geq 0$, such that for every $r > 0$, there exists a $\mathcal{Y} \in \mathbb{F}$, such that $\mathcal{X}$ is (n, r)-decomposable over $\mathcal{Y}$. The collection $\mathbb{F}$ is closed under decomposability, if every family $\mathcal{X}$ decomposable over $\mathbb{F}$ actually belongs to $\mathbb{F}$.*

Note that a space X, always viewed as a singleton family, is decomposable over the collection $\mathbb{D}_0$ of bounded families precisely when it has finite asymptotic dimension. A family $\mathcal{X} = \{X_i\}$ is decomposable over the collection of bounded families $\mathbb{D}_0$ precisely when the metric spaces X_i comprising it have uniformly finite asymptotic dimension in the sense of Bell and Dranishnikov [1–2].

Definition 2.4 (see [9]) *The collection of metric families $\mathbb{D}$ having finite decomposition complexity is the smallest collection containing the bounded families and closed under decomposability.*

Let F be a Borel map from a metric space X to another metric space Y. Recall that F is said to be a coarse map if

(1) for every $R > 0$, there exists an $S > 0$, such that $d(F(x), F(y)) < S$ for every pair of points $x, y \in X$ with $d(x, y) < R$;

(2) the inverse image $F^{-1}(B)$ for every bounded subset B of Y is bounded.

We say that X is coarsely equivalent to Y if there exist coarse maps $F : X \to Y$ and $G : Y \to X$, such that there exists a constant $C > 0$ satisfying $d(G \circ F(x), x) < C$ for all $x \in X$ and $d(F \circ G(y), y) < C$ for all $y \in Y$. It turns out that both finite decomposition complexity and operator norm localization property are invariant under coarse equivalence (see [3, 9]).

In the following, let $\mathbb{MSP}$ denote the collection of metric families having uniform metric sparsification property. In order to express the idea of this paper, we introduce the following notion.

Definition 2.5 *The collection $\mathbb{FDC}'\mathbb{MSC}$ of metric families, having finite decomposition complexity with respect to metric sparsificaton property, is the smallest metric collection containing $\mathbb{MSP}$, which is closed under decomposability. A metric space X is said to have finite decomposition complexity with respect to metric sparsification property, if the singleton family $\{X\}$ belongs to $\mathbb{FDC}'\mathbb{MSP}$.*

3 Main Result

The main result of this paper is the following permanence property.

Theorem 3.1 *A metric space X has finite decomposition complexity with respect to metric sparsification property if and only if X itself has metric sparsification property.*

It follows that $\mathbb{FDC}'\mathbb{MSP} = \mathbb{MSP}$, or in other words, the collection $\mathbb{MSP}$ of metric families, having uniform metric sparsification property, is closed under decomposability.

Since the converse of the above theorem is obviously true, we only have to show the necessity of Theorem 3.1. To begin with, note that Definition 2.5 can be reformulated as follows.

Proposition 3.1 *A metric space X has finite decomposition complexity with respect to metric sparsification property if and only if, for any sequence $\{r_k\}_{k=1}^{\infty}$ of positive numbers, there exists an integer $m > 0$ and m non-negative integers $\{n_k\}_{k=0}^{m-1}$, where n_0 depends only on X and each n_k with $k > 0$ depends only on $r_1, \cdots, r_{k-1}$ and $n_0, n_1, \cdots, n_{k-1}$, such that we have m levels of decomposition as follows:*

(1) *For X and $r_1 > 0$, we have*

$$X = \bigcup_{i_1=0}^{n_0} X_{i_1}, \quad X_{i_1} = \bigsqcup_{r_1-\text{disjoint}} X_{i_1 j_1}.$$

(2) *For all $X_{i_1 j_1}$ and $r_2 > 0$, we have*

$$X_{i_1 j_1} = \bigcup_{i_2=0}^{n_1} X_{i_1 j_1 i_2}, \quad X_{i_1 j_1 i_2} = \bigsqcup_{r_2-\text{disjoint}} X_{i_1 j_1 i_2 j_2}.$$

$\cdots$

(m) *For all $X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$ and $r_m > 0$, we have that*

$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} = \bigcup_{i_m=0}^{n_{m-1}} X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m},$$
$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m} = \bigsqcup_{r_m-\text{disjoint}} X_{i_1 j_1 \cdots i_m j_m},$$

and the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ has uniform metric sparsification property.

To prove Theorem 3.1, we first prove the following "quantitative version of the finite union theorem" for metric sparsification property of metric spaces.

Let

$$\mathcal{M}(X) = \{\text{all the finite positive Borel measure on } X\}.$$

Lemma 3.1 *Let X be a metric space, expressed as a union of finite, i.e., $n+1$, metric subspaces:*

$$X = X_0 \cup X_1 \cup \cdots \cup X_n.$$

Let $r > 0$ be given, and fix a natural number $k \in \mathrm{N}$. If there exist common constants $S > 0$ and $0 < c \leq 1$ for all $i = 0, 1, \cdots, n$, such that, for all $\nu_i \in \mathcal{M}(X_i)$, there exist $\overline{\Omega}_i = \bigsqcup\limits_j \overline{\Omega}_{ij} \subset X_i$, where j runs through a countable index set, such that

(1) $d(\overline{\Omega}_{i\alpha}, \overline{\Omega}_{i\beta}) \geq \Big(4\Big\lceil \dfrac{1}{1 - \sqrt[n]{1 - \frac{1}{2^{k+1}}}} \Big\rceil - 3\Big)2r$ *for all $\alpha \neq \beta$;*

(2) $\operatorname{diam}(\overline{\Omega}_{i\alpha}) \leq S$ *for each α;*

(3) $\nu_i(\overline{\Omega}_i) \geq c\nu_i(X_i)$,

where $\lceil x \rceil$ denote the least integer greater than or equal to x.

Then, for any finite positive measure μ on X, i.e., $\mu \in \mathcal{M}(X)$, there always exists a subset $\Omega = \bigsqcup\limits_i \Omega_i \subset X$, where i runs through a countable index set, such that

(i) $d(\Omega_i, \Omega_j) \geq r$ *for all $i \neq j$;*

(ii) $\operatorname{diam}(\Omega_j) \leq S + \Big(\Big\lceil \dfrac{1}{1 - \sqrt[n]{1 - \frac{1}{2^{k+1}}}} \Big\rceil - 1\Big)4r$ *for each j;*

(iii) $\mu(\Omega) \geq \Big(1 - \dfrac{1}{2^{k+1}}\Big)c\mu(X)$.

Proof First, consider the case in which X is a union of 2 subspaces, i.e., $n = 1$,

$$X = X_0 \cup X_1.$$

In this case,

$$\Big\lceil \frac{1}{1 - \sqrt[n]{1 - \frac{1}{2^{k+1}}}} \Big\rceil = 2^{k+1}.$$

Let $N = 2^{k+1}$. For each $m = 0, 1, 2, \cdots$, let

$$Z_m = \{x \in X : m(2r) \leq d(x, X_0) < (m+1)(2r)\}.$$

For each $1 \leq i \leq N$, let

$$V_i = \cup\{Z_m : m \equiv i \mod N\}.$$

Then $X = \bigsqcup\limits_{i=1}^{N} V_i$. Since $\mu(X) < \infty$, there is an i_0, such that $\mu(V_{i_0}) \leq \frac{\mu(X)}{N}$. Let $U_0 = \bigcup\limits_{m<i_0} Z_m$ and $U_1 = \bigcup\limits_{m>i_0} Z_m$. Note that U_0 and U_1 are $2r$-disjoint if none of them are empty.

Fix arbitrarily a measure $\mu \in \mathcal{M}(X)$. Let $i : U_1 \to X_1$ be the inclusion map. Take a Borel map $p : U_0 \to X_0$, such that

$$p(x) = x, \quad \text{if } x \in X_0$$

and

$$d(p(x), x) \leq (N-1) \cdot 2r.$$

Let $\mu_0 = p(\mu|_{U_0})$ on X_0 be the push-forward measure of the restriction measure $\mu|_{U_0}$ on U_0. Then for any Borel subset $A \subset X_0$, we have $\mu_0(A) = \mu(p^{-1}(A))$. Similarly, let $\mu_1 = i(\mu|_{U_1})$ be the measure on X_1. It is easy to check that $\mu_0 \in \mathcal{M}(X_0)$ and $\mu_1 \in \mathcal{M}(X_1)$ if none of them are empty.

By the conditions of this lemma, there are subsets

$$\overline{\Omega}_0 = \bigsqcup_j \overline{\Omega}_{0j} \subset X_0, \quad \overline{\Omega}_1 = \bigsqcup_j \overline{\Omega}_{1j} \subset X_1,$$

satisfying

(1) $d(\overline{\Omega}_{i\alpha}, \overline{\Omega}_{i\beta}) \geq (2^{k+3} - 3)r$ for $\alpha \neq \beta$;

(2) $\mathrm{diam}(\overline{\Omega}_{i\alpha}) \leq S$;

(3) $\mu_i(\overline{\Omega}_i) \geq c\mu_i(X_i)$

for $i = 0, 1$.

Let $\widetilde{\Omega}_0 = p^{-1}(\overline{\Omega}_0) = \bigsqcup_j p^{-1}(\overline{\Omega}_{0j}) =: \bigsqcup_j \widetilde{\Omega}_{0j}$ and $\widetilde{\Omega}_1 = i^{-1}(\overline{\Omega}_1) = \bigsqcup_j i^{-1}(\overline{\Omega}_{1j}) =: \bigsqcup_j \widetilde{\Omega}_{1j}$, where j runs through a countable index set.

Then the subset $\widetilde{\Omega}_1 = \bigsqcup_j \widetilde{\Omega}_{1j} \subset U_1$ satisfies $d(\widetilde{\Omega}_{1\alpha}, \widetilde{\Omega}_{1\beta}) \geq (2^{k+3} - 3)r > r$ for $\alpha \neq \beta$, $\mathrm{diam}(\widetilde{\Omega}_{1\alpha}) \leq S$, $\mu(\widetilde{\Omega}_1) \geq c\mu(U_1)$.

The subset $\widetilde{\Omega}_0 = \bigsqcup_j \widetilde{\Omega}_{0j} \subset U_0$ satisfies

$$\begin{aligned} d(\widetilde{\Omega}_{0\alpha}, \widetilde{\Omega}_{0\beta}) &\geq (2^{k+3} - 3)r - 2(N-1) \cdot 2r = (2^{k+3} - 3)r - 4(2^{k+1} - 1)r = r, \\ \mathrm{diam}(\widetilde{\Omega}_{0\alpha}) &\leq S + 2(N-1)2 \cdot r = S + (2^{k+1} - 1) \cdot 4r, \\ \mu(\widetilde{\Omega_0}) &\geq c\mu(U_0). \end{aligned}$$

Taking $\Omega = \widetilde{\Omega}_0 \sqcup \widetilde{\Omega}_1$, we have

$$c\frac{N-1}{N}\mu(X) \leq c\mu(U_0 \sqcup U_1) = c[\mu(U_0) + \mu(U_1)] \leq \mu(\widetilde{\Omega}_0) + \mu(\widetilde{\Omega}_1) = \mu(\Omega).$$

Then we get $c\big(1 - \frac{1}{N}\big)\mu(X) \leq \mu(\Omega)$, i.e., $c\big(1 - \frac{1}{2^{k+1}}\big)\mu(X) \leq \mu(\Omega)$. Now we rearrange the index set of Ω as $\Omega = \bigsqcup_i \Omega_i$. Then, we get

(i) $d(\Omega_i, \Omega_j) \geq r$ for all $i \neq j$;

(ii) $\mathrm{diam}(\Omega_j) \leq S + (2^{k+1} - 1)4r$ for each j;

(iii) $\mu(\Omega) \geq \big(1 - \frac{1}{2^{k+1}}\big)c\mu(X)$.

In the general case, suppose

$$X = X_0 \cup X_1 \cup X_2 \cup \cdots \cup X_n.$$

Let $r > 0$ and $k \in \mathrm{N}$ be given. Let $\delta = 1 - \sqrt[n]{1 - \frac{1}{2^{k+1}}}$, and let $N \in \mathrm{N}$ be the least integer greater than or equal to $\frac{1}{\delta}$, i.e.,

$$N = \left\lceil \frac{1}{1 - \sqrt[n]{1 - \frac{1}{2^{k+1}}}} \right\rceil,$$

where $\lceil x \rceil$ denotes the least integer greater than x. Fix arbitrarily a measure $\mu \in \mathcal{M}(X)$. The strategy of arguments in the above case of 2 subspaces implies that there exist $2r$-disjoint subspaces U_0 and V_1 by cutting off a subset of X of measure at most $\frac{\mu(X)}{N}$ (see Figure 1), such that

(1) U_0 is contained in the $(2N-2)r$-neighborhood of X_0, i.e., $U_0 \subseteq \mathcal{N}_{(2N-2)r}(X_0)$;

(2) V_1 is a subspace of $X_1 \cup X_2 \cup \cdots \cup X_n$, such that

$$(1 - \delta)\mu(X) \leq \mu(U_0) + \mu(V_1).$$

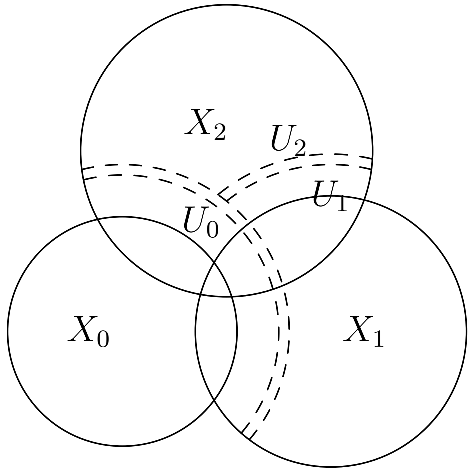

Figure 1 Incursive cutting off

Similarly, there exist $2r$-disjoint subspaces U_1 and V_2 of $X - X_1$ by cutting off a subset of measure at most $\frac{\mu(X-X_1)}{N}$, such that

(1) U_1 is contained in the $(2N-2)r$-neighborhood of X_1, i.e., $U_1 \subseteq \mathcal{N}_{(2N-2)r}(X_1)$;

(2) V_2 is a subspace of $X_2 \cup X_3 \cup \cdots \cup X_n$, such that

$$(1-\delta)\mu(V_1) \leq \mu(U_1) + \mu(V_2).$$

We get

$$\begin{aligned}(1-\delta)^2\mu(X) &\leq (1-\delta)\mu(U_0) + (1-\delta)\mu(V_1)\\ &\leq \mu(U_0) + \mu(U_1) + \mu(V_2)\\ &= \sum_{i=0}^{1}\mu(U_i) + \mu(V_2).\end{aligned}$$

By induction, we assume that at the $(n-1)$-th step, there are $2r$-disjoint subspaces U_{n-2} and V_{n-1}, such that

(1) U_{n-2} is contained in the $(2N-2)r$-neighborhood of X_{n-2};

(2) V_{n-1} is a subspace of X_{n-1}, such that

$$\begin{gathered}(1-\delta)\mu(V_{n-2}) \leq \mu(U_{n-2}) + \mu(V_{n-1}),\\ (1-\delta)^{n-1}\mu(X) \leq \sum_{i=0}^{n-2}\mu(U_i) + \mu(V_{n-1}).\end{gathered}$$

Then at the n-th step, finally there exist $2r$-disjoint subspaces U_{n-1} and $U_n = V_n$, such that

(1) U_{n-1} is contained in the $(2N-2)r$-neighborhood of X_{n-1};

(2) U_n is a subspace of X_n, such that

$$(1-\delta)\mu(V_{n-1}) \leq \mu(U_{n-1}) + \mu(V_n = U_n).$$

Further, we get that

$$\begin{aligned}(1-\delta)^n\mu(X) &= (1-\delta)(1-\delta)^{n-1}\mu(X)\\ &\le (1-\delta)\sum_{i=0}^{n-2}\mu(U_i)+(1-\delta)\mu(V_{n-1})\\ &\le \sum_{i=0}^{n-2}\mu(U_i)+(1-\delta)\mu(V_{n-1})\\ &\le \sum_{i=0}^{n-2}\mu(U_i)+\mu(U_{n-1})+\mu(V_n=U_n)\\ &= \sum_{i=0}^{n-1}\mu(U_i)+\mu(V_n=U_n),\end{aligned}$$

i.e., $(1-\delta)^n\mu(X)\le\sum\limits_{i=0}^{n}\mu(U_i)$.

Note that the family of subsets $\{U_i\}_{j=0}^n$ are disjoint from each other, and each U_i is contained in $(N-1)\cdot 2r$ neighborhood of X_i. Hence, with the same technique as used to deal with the $2r$-neighborhood in the above for the union of 2 subsets, associated to each U_i, $i=0,1,\cdots,n$, there exists a subset $\Omega_i=\bigsqcup\limits_\alpha\Omega_{i\alpha}$, such that

(1) $d(\Omega_{i\alpha},\Omega_{i\beta})\ge r$ if $\alpha\ne\beta$;

(2) $\operatorname{diam}(\Omega_{i\alpha})\le S+2(N-1)\cdot 2r=S+\Big(\Big\lceil\dfrac{1}{1-\sqrt[n]{1-\frac{1}{2^{k+1}}}}\Big\rceil-1\Big)4r$;

(3) $\mu(\Omega_i)\ge c\mu(U_i)$. It follows that

$$(1-\delta)^n c\mu(X)\le\sum_{i=0}^{n}c\mu(U_i)\le\sum_{i=0}^{n}\mu(\Omega_i).$$

Let $\Omega=\sqcup\Omega_i$. Then $\Omega=\sqcup\Omega_i'$, if we rearrange the index of subsets. Hence, we have

(1) $d(\Omega_i',\Omega_j')\ge r$ for $i\ne j$;

(2) $\operatorname{diam}(\Omega_j')\le S+\Big(\Big\lceil\dfrac{1}{1-\sqrt[n]{1-\frac{1}{2^{k+1}}}}\Big\rceil-1\Big)4r$;

(3) $\mu(\Omega)\ge c(1-\delta)^n\mu(X)$. Note that

$$(1-\delta)^n\ge\Big(1-\frac{1}{N}\Big)^n\ge\Big(1-\frac{1}{\frac{1}{1-\sqrt[n]{1-\frac{1}{2^{k+1}}}}}\Big)^n=1-\frac{1}{2^{k+1}}.$$

The proof is complete.

Proof of Theorem 3.1 Suppose that a metric space X has finite decomposition complexity with respect to metric sparsification property. Let $c_0=0.4$. We proceed to find an $f:\mathrm{R}_+\to\mathrm{R}_+$ for X to have the metric sparsification property.

Let

$$N(k,n)=\Big\lceil\frac{1}{1-\sqrt[n]{1-\frac{1}{2^{k+1}}}}\Big\rceil\quad\text{for all } k,n\in\mathrm{N}.$$

For any $r>0$, by Proposition 2.1, there exists an $m>0$ and m non-negative integers $\{n_k\}_{k=0}^{m-1}$ corresponding to the sequence of positive numbers

$$r_1=(4N(1,n_0)-3)r,\quad r_2=(4N(2,n_1)-3)r_1,\ \cdots,\ r_m=(4N(m,n_{m-1})-3)r_{m-1},\ \cdots,$$

such that

(1) for X and r_1, we have

$$X = \bigcup_{i_1}^{n_0} X_{i_1}, \quad X_{i_1} = \bigsqcup_{r_1\text{-disjoint}} X_{i_1 j_1},$$

(2) for all $X_{i_1 j_1}$ and $r_2 > 0$, we have

$$X_{i_1 j_1} = \bigcup_{i_2}^{n_1} X_{i_1 j_1 i_2}, \quad X_{i_1 j_1 i_2} = \bigsqcup_{r_2\text{-disjoint}} X_{i_1 j_1 i_2 j_2},$$

$\cdots$

(m) for all $X_{i_1 j_1 \cdots i_{m-1} j_{m-1}}$ and $r_m > 0$, we have

$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1}} = \bigcup_{i_m=0}^{n_{m-1}} X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m},$$
$$X_{i_1 j_1 \cdots i_{m-1} j_{m-1} i_m} = \bigsqcup_{r_m\text{-disjoint}} X_{i_1 j_1 \cdots i_m j_m},$$

and the family of metric spaces $\{X_{i_1 j_1 \cdots i_m j_m}\}_{i_1, j_1, \cdots, i_m, j_m}$ has uniform metric sparsification property. By proposition 2.1, we can take the common constant $c = 0.8$ and the corresponding function $\widetilde{f} : \mathrm{R}_+ \to \mathrm{R}_+$ as in Definition 2.2.

Let $S_m = \widetilde{f}(r_m)$ for all $m = 0, 1, 2, \cdots$. By the step (m), for any positive finite measure μ on $X_{i_1 j_1 \cdots i_m j_m}$, there exists a subset $\Omega_{i_1 j_1 \cdots i_m j_m} = \bigsqcup_{\alpha} \Omega^{\alpha}_{i_1 j_1 \cdots i_m j_m}$ of $X_{i_1 j_1 \cdots i_m j_m}$, such that

(1) $d(\Omega^{\alpha}_{i_1 j_1 \cdots i_m j_m}, \Omega^{\beta}_{i_1 j_1 \cdots i_m j_m}) \geq r_m$ for all $\alpha \neq \beta$,
(2) $\mathrm{diam}(\Omega^{\alpha}_{i_1 j_1 \cdots i_m j_m}) \leq S_m$ for each α,
(3) $\mu(\Omega_{i_1 j_1 \cdots i_m j_m}) \geq c\mu(X_{i_1 j_1 \cdots i_m j_m})$.

Let $\Omega_{i_1 j_1 \cdots i_m} = \bigsqcup_{j_m} \Omega_{i_1 j_1 \cdots i_m j_m}$. Since $\mathrm{dist}(X_{i_1 j_1 \cdots i_m j_m}, X_{i_1 j_1 \cdots i_m j'_m}) \geq r_m$, by rearranging the index of the subsets $\{\Omega^{\alpha}_{i_1 j_1 \cdots i_m j_m}\}$, we can write

$$\Omega_{i_1 j_1 \cdots i_m} = \bigsqcup_{\alpha} \Omega^{\alpha}_{i_1 j_1 \cdots i_m} \subset X_{i_1 j_1 \cdots i_m}.$$

Then we have

(1)$d(\Omega^{\alpha}_{i_1 j_1 \cdots i_m}, \Omega^{\beta}_{i_1 j_1 \cdots i_m}) \geq r_m$ for all $\alpha \neq \beta$,
(2) $\mathrm{diam}(\Omega^{\alpha}_{i_1 j_1 \cdots i_m}) \leq S_m$ for each α,
(3) $\mu(\Omega_{i_1 j_1 \cdots i_m}) \geq c\mu(X_{i_1 j_1 \cdots i_m})$.

By Lemma 3.1, there exist nonempty subsets of $\Omega = \bigsqcup_{j} \Omega_j$, such that

(1) $d(\Omega_i, \Omega_j) \geq \frac{r_m}{4N(m, n_{m-1}) - 3} = r_{m-1}$ for all $i \neq j$,
(2) $\mathrm{diam}(\Omega_i) \leq S_m + (N(m, n_{m-1}) - 1)4r$ for each i,
(3) $\mu(\Omega) \geq (1 - \frac{1}{2^{m+1}})c\mu(X_{i_1 j_1 \cdots i_{m-1} j_{m-1}})$.

Taking a measure $\mu \in \mathcal{M}(X)$, we have

$$\mu|_{X_{i_1 j_1 \cdots i_m j_m}} \in \mathcal{M}(X_{i_1 j_1 \cdots i_m j_m}).$$

By the above arguments and applying Lemma 3.1 for m times, we have that, for any measure $\mu \in \mathcal{M}(X)$, there exists a subset $\Omega = \bigsqcup_{j} \Omega_j$ of X, such that

(1) $d(\Omega_i, \Omega_j) \geq r_0 = r$ for all $i \neq j$,

(2) $\mathrm{diam}(\Omega_i) \leq S_m + \sum_{j=1}^{m}(N(j, n_{j-1}) - 1)4r$ for each i,

(3) $\mu(\Omega) \geq c \prod_{j=1}^{m} \Big(1 - \frac{1}{2^{j+1}}\Big)\mu(X)$.

Note

$$\prod_{j=1}^{m} \Big(1 - \frac{1}{2^{j+1}}\Big) \geq 1 - \sum_{j=2}^{\infty} \frac{1}{2^j} = \frac{1}{2}.$$

Let

$$f(r) = S_m + \sum_{j=1}^{m}(N(j, n_{j-1}) - 1)4r.$$

We have that X has metric sparsification property relative to f with constant $\frac{c}{2} = 0.4 = c_0$. The proof is complete.

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