Chin. Ann. Math.
35B(5), 2014, 801–816
DOI: 10.1007/s11401-014-0851-y

Chinese Annals of Mathematics, Series B

Distortion of Wreath Products in Thompson's Group F*

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Abstract The authors use geometric techniques to prove that the restricted wreath product $F \wr \mathbb{Z}$ is a quasi-isometrically embedded subgroup of Thompson's group F.

Keywords Thompson's group F, Restricted wreath products, Quasi-isometrically embedded
2000 MR Subject Classification 46L07, 46L80

1 Introduction

The interesting properties of Thompson's group F have made it a favorite object of study among group theorists and topologists. It was discovered by Richard Thompson in 1965, initially used to construct finitely presented groups with unsolvable word problems. This group was invented as a group of certain transformations of terms in the λ-calculus (mathematical logic), and then emerged in such areas as functional analysis, homological algebra, homotopy theory and group theory itself. Many questions about F are still open, and in particular, it is not known whether F is an amenable group. The question is of considerable interest since both the affirmative and negative answers would provide counterexamples to open questions (see [4, 11]).

Questions concerning distortion of subgroups is a topical subject of investigation in geometric group theory (see [3, 8, 14–15]). Bridson asked the question of whether or not a quasi-isometry exists between Thompson's group F and the group $F \times \mathbb{Z}$. Burillo [2] gave a positive answer to this question by finding quasi-isometric embedding in F of subgroups isomorphic to $F \times \mathbb{Z}^n$ ($n \geq 1$). Guba and Sapir [9–10] proved that the subgroups of the form $F^m \times \mathbb{Z}^n$, for integers $m, n \geq 0$, are embedded in F without distortion. Cleary and Taback used shift maps of F, which also shows this result (see [6]). In [5], Clearly showed that $\mathbb{Z} \wr \mathbb{Z}$ is embedded in F without distortion. In this paper, using the important tools of the reduced forest diagrams and the reduced tree diagrams, we prove that the restricted wreath products $F \wr \mathbb{Z}$ and $\mathbb{Z} \wr \mathbb{Z}$ are quasi-isometrically embedded subgroups of Thompson's group F.

Manuscript received October 15, 2011. Revised March 3, 2014.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11226122, 11301224, 11231002) and the Zhejiang Provincial Natural Science Foundation of China (No. LQ12A01015).

2 Preliminaries

2.1 Distortion of subgroups in finitely generated groups

We recall several definitions.

Let S be a finite generating set for a group G, and for any $g \in G$, define $|g|_S$ to be the length of the shortest word representing g in elements of the generating set S. Then we say that $|\cdot|_S$ is a word-length function for G with respect to S.

Let $\phi, \psi : G \to \mathbb{N}$ be functions from G into $\mathbb{N}$, and then we write $\phi \preceq \psi$ if there is a positive integer constant C, such that $\phi(g) \le C\psi(g)$ for all $g \in G$. If $\phi \preceq \psi$ and $\psi \preceq \phi$, then we say that these functions are equivalent, and we denote this fact by $\phi \sim \psi$. It is obvious that $\sim$ is indeed an equivalence relation. If S and T are finite generating sets for the same group G, then an elementary argument shows that the functions $|\cdot|_S$ and $|\cdot|_T$ are equivalent.

Suppose that G and H are finitely generated groups, such that H is a subgroup of G. S and T are finite generation sets for G and H, respectively. The functions $|\cdot|_S$ and $|\cdot|_T$ can be regarded as functions on H. An elementary argument shows that $|\cdot|_S \preceq |\cdot|_T$. If $|\cdot|_T \preceq |\cdot|_S$ also holds, then the subgroup H is said to be embedded in G quasi-isometrically or without distortion. Note that the equivalence of $|\cdot|_S$ and $|\cdot|_T$ does not depend on the choice of finite generating sets S and T. So one can introduce the word-length functions $|\cdot|_G$ and $|\cdot|_H$ for G and H, respectively, which depend only on G and H. The subgroup H is quasi-isometrically embedded in G if and only if $|\cdot|_G \sim |\cdot|_H$.

2.2 Wreath product

Let G and N be finitely generated groups, and let $1_G \in G$ and $1_N \in N$ be their units. The support of a function $f : N \to G$ is the set

$$\mathrm{supp}(f) = \{x \in N \mid f(x) \neq 1_G\}.$$

The direct sum $\bigoplus\limits_N G$ of groups G (or restricted direct product) is the group of functions

$$C_0(N, G) = \{f : N \to G \text{ with finite support}\}.$$

There is a natural action of N on $C_0(N, G)$: For all $a \in N,\ x \in N,\ f \in C_0(N, G)$,

$$a(f)(x) = f(xa^{-1}).$$

The semidirect product $C_0(N, G) \rtimes N$ is called restricted wreath product and is denoted as $G \wr N$. We recall that the product in $G \wr N$ is defined by the formula

$$(f, a)(g, b) = (fa(g), ab).$$

Let S and T be finite generating sets for G and N, respectively. Let $e \in C_0(N, G)$ denote the constant function taking value 1_G, and let $\delta_v^b : N \to G, v \in N, b \in G$ be the δ-function, i.e.,

$$\delta_v^b(v) = b \quad \text{and} \quad \delta_v^b(x) = 1_G \quad \text{for } x \neq v.$$

Note that $a(\delta_v^b) = \delta_{va}^b$, and hence $(\delta_v^b, 1_N) = (e, v)(\delta_{1_N}^b, 1_N)(e, v^{-1})$. Since every function $f \in C_0(N, G)$ can be presented by $\delta_{v_1}^{b_1} \cdots \delta_{v_k}^{b_k}$,

$$(f, 1_N) = (\delta_{v_1}^{b_1}, 1_N) \cdots (\delta_{v_k}^{b_k}, 1_N) \quad \text{and} \quad (f, u) = (f, 1_N)(e, u).$$

The set $\widetilde{S} = \{(\delta_{1_N}^s, 1_N), (e, t) \mid s \in S, t \in T\}$ is a generating set for $G \wr N$. We will use an abbreviations f for $(f, 1_N)$ and t for (e, t) for elements of the group $G \wr N$. So we denote $(f, t) = (f, 1_N)(e, t)$ by ft.

In the case $N = \mathbb{Z}$, let us now state a formula computing the word length of an element of $G \wr \mathbb{Z}$ from [12–13].

Lemma 2.1 (see [12, Theorem 1.2] and [13, Proposition 2.4]) *Let G be a finitely generated group with a finite generating set S, and let $\widetilde{S} = \{(\delta_0^s, 0), (e, 1) \mid s \in S\}$, where e is the identity of $\bigoplus_{\mathbb{Z}} G$. Then $\widetilde{S}$ is a finite generating set for $G \wr \mathbb{Z}$. Let $x = (f, n) \in G \wr \mathbb{Z}$, $m = \min\{k \in \mathbb{Z} \mid f(k) \neq 1_G\}$, and $M = \max\{k \in \mathbb{Z} \mid f(k) \neq 1_G\}$. Then the word-length function with respect to $\widetilde{S}$ of x satisfies*

$$|x|_{\widetilde{S}} = \begin{cases} |n|, & \text{if } f = e, \\ \sum\limits_{i \in \mathbb{Z}} |f(i)|_S + L_{\mathbb{Z}}(x), & \text{otherwise,} \end{cases}$$

where $L_{\mathbb{Z}}(x)$ denotes the length of the shortest path starting from 0, ending at n and passing through m and M in the (*canonical*) *Cayley graph of $\mathbb{Z}$.*

Let Λ be a group, H be a subgroup of Λ and $v : H \to \Lambda$ be an injective homomorphism. The HNN-extension with basis Λ and stable letter t relative to H and v is defined by

$$\mathrm{HNN}(\Lambda, H, v) = \langle \Lambda, t \mid \ t^{-1}ht = v(h), \ \forall h \in H \rangle.$$

One may express $G \wr \mathbb{Z}$ as an HNN-extension in three ways (we denote by s the positive generator of $\mathbb{Z}$ in $G \wr \mathbb{Z}$ and by t_+, t_-, t the stable letters of the HNN-extensions):

(1) Set $\Lambda_+ = \bigoplus_{n \geq 0} G$ and $v_+ : \Lambda_+ \to \Lambda_+$ given by

$$v_+(\lambda)_0 = 1_G, \quad v_+(\lambda)_n = \lambda_{n-1}, \quad \forall n \geq 1.$$

One has $\mathrm{HNN}(\Lambda_+, \Lambda_+, v_+) = G \wr \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda, \ t_+ \mapsto s^{-1}$.

(2) Set $\Lambda_- = \bigoplus_{n \leq 0} G$ and $v_- : \Lambda_- \to \Lambda_-$ given by

$$v_-(\lambda)_0 = 1_G, \quad v_-(\lambda)_n = \lambda_{n+1}, \quad \forall n \leq -1.$$

One has $\mathrm{HNN}(\Lambda_-, \Lambda_-, v_-) = G \wr \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda, \ t_- \mapsto s$.

(3) Set $\Lambda = \bigoplus_{n \in \mathbb{Z}} G$ and $v : \Lambda \to \Lambda$ given by $v(\lambda)_n = \lambda_{n-1}, \ \forall n \in \mathbb{Z}$. One has $\mathrm{HNN}(\Lambda, \Lambda, v) = G \wr \mathbb{Z}$ and the isomorphism is given by $\lambda \mapsto \lambda, \ t_+ \mapsto s^{-1}$.

2.3 Thompson's group *F*

Thompson's group F is a remarkable, finitely generated, finitely presented group which can be understood via a wide range of perspectives. Canon, Floyd and Parry [4] gave an excellent overview of the properties of F. Here we present a brief introduction to Thompson's group F and refer the interested readers to [1, 4] for more detailed and comprehensive descriptions. Thompson's group F has been studied for several decades. We remind the reader of some of its known properties.

(1) It can be described as the group of piecewise-linear homeomorphisms of the unit interval [0,1], all of whose derivatives are integer powers of 2 and have a finite number of break points which are all dyadic rational numbers (i.e., points of the form $\frac{m}{2^n}$, $m, n \in \mathbb{Z}$). The group operation is defined as follows: For every $f, g \in F$,

$$(fg)(t) = g(f(t)), \quad \forall t \in [0,1].$$

(2) It can be described as the group of all functions satisfying the properties indicated above if the segment [0,1] is replaced by $[0, \infty)$ and in addition the derivative is equal to 1 at $+\infty$.

(3) It can also be described as the group with the following infinite presentation:

$$\langle x_0, x_1, \cdots, x_n, \cdots \mid x_n x_k = x_k x_{n+1},\ \forall k < n \rangle.$$

From this presentation, we may see $x_{n+1} = x_0^{-1} x_n x_0$ for $n \geq 1$, and thus F is finitely generated by $\{x_0, x_1\}$.

We define a caret to be a vertex of the tree together with two downward oriented edges, which we refer to as the left and right edges of the caret. Every caret has the form of the rooted tree in Figure 1.

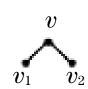

Figure 1 A caret

Elements of F can be viewed as pairs of finite binary rooted trees, each with the same number of carets, called tree diagrams. A binary forest is a sequence $(T_0, T_1, \cdots)$ of finite binary trees. A binary forest is bounded if only finitely many trees T_i are nontrivial. A Forest diagram, which represents an element of F as a pair of bounded binary forests, is another useful diagram representation for F. A forest diagram (or a tree diagram) is reduced if it does not have any opposing pairs of carets.

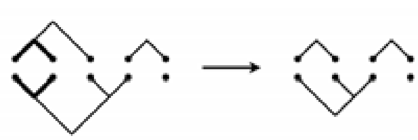

Figure 2 An example of the unreduced and the reduced forest diagrams representing the same element in F

Lemma 2.2 (see [1, Proposition 2.2.4]) *Every element of Thompson's group F has a unique reduced forest diagram* (*or reduced tree diagram*)*.*

It is easy to translate between tree diagrams and forest diagrams (see [1]). Given a tree diagram, we simply remove the right stalk of each tree to get the corresponding forest diagram (see Figure 3).

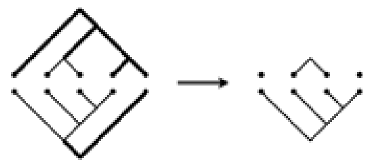

Figure 3 A tree diagram being translated into a forest diagram

The reduced tree diagrams and the reduced forest diagrams for the infinite generating set $\{x_0, x_1, \cdots\}$ are pictured in Figures 4–5.

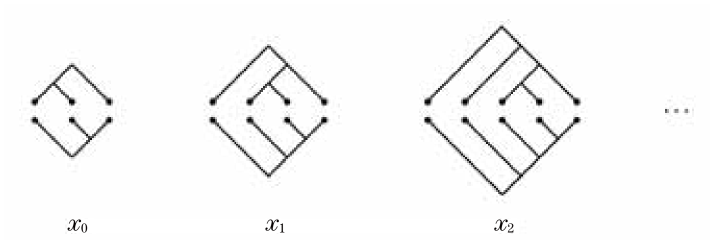

Figure 4 The reduced tree diagrams for $\{x_0, x_1, \cdots\}$

Figure 5 The reduced forest diagrams for $\{x_0, x_1, \cdots\}$

Let T be a tree, and the right side of T is the maximal path of right edges in T which begins at the root of T.

Define the exponents of T as follows. Let $I_0, \cdots, I_n$ be the leaves of T in order. For every integer k with $0 \le k \le n$, let a_k be the length of the maximal path of left edges in T, which begins at I_k and does not reach the right side of T. Then a_k is the kth exponent of T.

Example 2.1 The right side of tree S in Figure 6 is highlighted. Its leaves are labeled 0, 1, 2, 3, 4, 5 in order and the exponents of S in order are 0, 2, 1, 0, 0, 0.

Let f be a non-trivial element of F with the reduced tree diagram (R, S), and once the exponents of the leaves in R and S have been computed, the normal form of f is easily obtained.

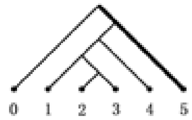

Figure 6 A tree S

Lemma 2.3 (Normal Form (see [4])) *Let f be a non-trivial element of F with the reduced tree diagram (R,S). Let $a_0,\cdots,a_n$ be the exponents of R, and $b_0,\cdots,b_n$ be the exponents of S. Then f can be expressed uniquely in the form: $f=x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}$, such that*

(1) *exactly one of a_n and b_n is nonzero;*

(2) *for every integer i with $0\le i<n$, if $a_i>0$ and $b_i>0$, then either $a_{i+1}>0$ or $b_{i+1}>0$. In this case, we say that $f=x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}$ is the normal form for f.*

Given the normal form of an element f of F, we can easily obtain the reduced tree diagram and the reduced forest diagram of f. Conversely, given either the reduced tree diagram or the reduced forest diagram of f, we can immediately get the normal form of f.

Cleary and Taback [7] estimated the word length $|f|_S$ with respect to $S=\{x_0,x_1\}$ in terms of the number of carets in any tree of the reduced tree diagrams.

Proposition 2.1 (see [7, Theorem 3.1]) *Let f be a non-trivial element of F with the reduced tree diagram (R_f,S_f). The number of carets in R_f (or S_f) is denoted by $N(f)$. Let $S=\{x_0,x_1\}$. Then*

$$N(f)-2\le |f|_S\le 4N(f)-4.$$

In this paper, for every $f\in F$ with the reduced tree diagram (R_f,S_f), let $N(f)$ denote the number of carets in R_f (or S_f).

3 Main Results

We have indicated two presentations of F by piecewise linear functions. Pick the representation by functions on $[0,\infty)$, and for every integer $k\ge 0$, let

$$\Phi_k=\{f\in F\mid f(t)=t,\ \forall t\notin[k+1,k+2]\}.$$

Guba and Sapir [9] proved that x_0 and Φ_0 generate the restricted wreath product $F\wr\mathbb{Z}$.

Lemma 3.1 (see [9]) *The restricted wreath product $F\wr\mathbb{Z}$ is a group isomorphic to the subgroup H generated by x_0 and Φ_0 of F.*

Proof Note that

$$F\wr\mathbb{Z}=\mathrm{HNN}\Big(\bigoplus_{n\ge 0}F,\bigoplus_{n\ge 0}F,v_+\Big)=\Big\langle\bigoplus_{n\ge 0}F,t\,\Big|\,t^{-1}ht=v_+(h),\ \forall h\in\bigoplus_{n\ge 0}F\Big\rangle.$$

For every integer $k\ge 0$, it is easy to see that Φ_k is a group isomorphic to F and

$$\Phi_i\Phi_j=\Phi_j\Phi_i,\quad \forall i\ne j.$$

Therefore, Φ_k $(k \geq 0)$ generate $\bigoplus\limits_{n\geq 0} F$. Besides,

$$x_0^{-1}\Phi_k x_0 = \Phi_{k+1}, \quad \forall k \geq 0.$$

Thus x_0 and Φ_0 generate the restricted wreath product $F \wr \mathbb{Z}$.

Now we give our main result.

Theorem 3.1 *The subgroup H isomorphic to $F \wr \mathbb{Z}$ in F generated by x_0, $x_1^2x_2^{-1}x_1^{-1}$ and $x_1x_2^2x_3^{-1}x_2^{-1}x_1^{-1}$ is quasi-isometrically embedded.*

Proof Note that $x_1^2x_2^{-1}x_1^{-1}$ and $x_1x_2^2x_3^{-1}x_2^{-1}x_1^{-1}$ generate Φ_0.

Indeed, there is a group isomorphism $\sigma : F \to \Phi_0$ defined as follows.

For every $f \in F$, let (R_f, S_f) be the reduced tree diagram of f. Then define $\sigma(f)$ to be the element of Φ_0 with the reduced forest diagram as Figure 7. Since x_0 and x_1 generate F, $\sigma(x_0)$ and $\sigma(x_1)$ generate Φ_0, it is easy to see that $\sigma(x_0) = x_1^2x_2^{-1}x_1^{-1}$ and $\sigma(x_1) = x_1x_2^2x_3^{-1}x_2^{-1}x_1^{-1}$.

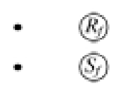

Figure 7 The reduced forest diagram of $\sigma(f)$

By Lemma 2.1 and the fact that $\{x_0, x_1\}$ is a finite generating set for F, $\{(\delta_0^{x_0}, 0), (\delta_0^{x_1}, 0), (e, 1)\}$ is a finite generating set for $F \wr \mathbb{Z}$, where e is the identity element of $\bigoplus\limits_{\mathbb{Z}} F$. There is a group isomorphism $\varphi : F \wr \mathbb{Z} \to H$ which is given by

$$(\delta_0^{x_0}, 0) \mapsto x_1^2x_2^{-1}x_1^{-1}, \quad (\delta_0^{x_1}, 0) \mapsto x_1x_2^2x_3^{-1}x_2^{-1}x_1^{-1}, \quad (e, 1) \mapsto x_0^{-1}.$$

In general, let f be an element of F with the reduced tree diagram (R_f, S_f). Then $\varphi((\delta_0^f, 0))$, which is denoted by $\varphi(\delta_0^f)$, is the element in F with the reduced forest diagram in Figure 7. Moreover, let n be a natural number. Since

$$(\delta_{-n}^f, 0) = (e, -n)(\delta_0^f, 0)(e, n),$$

we have

$$\varphi(\delta_{-n}^f) = x_0^n\varphi(\delta_0^f)x_0^{-n}.$$

Note that if $\varphi(\delta_0^f) = x_0^{a_0}x_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_0^{-b_0}$ is the normal form, then $a_0 = 0$, $b_0 = 0$ and $x_0^nx_1^{a_1}\cdots x_n^{a_n}x_n^{-b_n}\cdots x_1^{-b_1}x_0^{-n}$ is the normal form of $\varphi(\delta_{-n}^f)$. Therefore, we obtain the reduced forest diagram for $\varphi(\delta_{-n}^f)$ in Figure 8.

Similarly, since

$$(\delta_n^f, 0) = (e, n)(\delta_0^f, 0)(e, -n),$$

we have

$$\varphi(\delta_n^f) = x_0^{-n}\varphi(\delta_0^f)x_0^n.$$

We obtain the reduced forest diagram for $\varphi(\delta_n^f)$ in Figure 9.

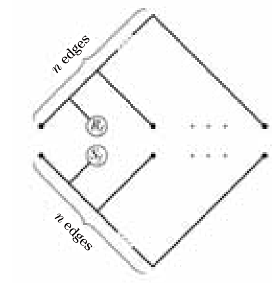

Figure 8 The reduced forest diagram of $\varphi(\delta_{-n}^{f})$

Figure 9 The reduced forest diagram of $\varphi(\delta_{n}^{f})$

For every $f \in F$, let $|f|_F$ denote the word length of f with respect to $\{x_0, x_1\}$. For every $y \in F \wr \mathbb{Z}$, let $|y|_{F\wr\mathbb{Z}}$ denote the word length of y with respect to $\{(\delta_0^{x_0}, 0), (\delta_0^{x_1}, 0), (e, 1)\}$, and let $|\varphi(y)|_H$ denote the word length of of $\varphi(y)$ with respect to $\{x_0, x_1^2x_2^{-1}x_1^{-1}, x_1x_2^2x_3^{-1}x_2^{-1}x_1^{-1}\}$. Note that $|\varphi(y)|_H = |y|_{F\wr\mathbb{Z}}$. We are going to prove that for every $y \in F\wr\mathbb{Z}$, $|\varphi(y)|_H \le 12|\varphi(y)|_F$.

Now let $x = (g, k) \in F \wr \mathbb{Z}$, $m = \min\{j \in \mathbb{Z} \mid g(j) \neq 1_F\}$ and $M = \max\{j \in \mathbb{Z} \mid g(j) \neq 1_F\}$. Let (R_i, S_i) be the reduced tree diagram of $g(i)$, and $N\left(g(i)\right)$ be the number of carets in R_i (or S_i).

If $g = e$, then by Lemma 2.1, $|\varphi(x)|_H = |x|_{F\wr\mathbb{Z}} = |(e, k)|_{F\wr\mathbb{Z}} = |k|$. Besides,

$$|\varphi(x)|_F = |\varphi\left((e, k)\right)|_F = |x_0^{-k}|_F = |k| = |\varphi(x)|_H.$$

Now consider the case when $g \neq e$, and then $\sum_{i\in\mathbb{Z}}|g(i)|_F \ge 1$. Observe that

$$\varphi(x) = \varphi\left((g, k)\right) = \varphi\left((g, 0)(e, k)\right) = \varphi(g)x_0^{-k}$$

and

$$\varphi(g) = \varphi(\delta_m^{g(m)}\delta_{m+1}^{g(m+1)}\cdots\delta_M^{g(M)}) = \varphi(\delta_m^{g(m)})\varphi(\delta_{m+1}^{g(m+1)})\cdots\varphi(\delta_M^{g(M)}).$$

Case 1 $0 \le m \le M$. It is easy to picture the reduced forest diagram for $\varphi(g)$ in Figure 10.

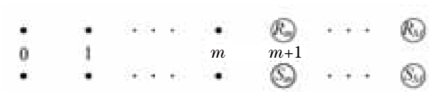

Figure 10 The reduced forest diagram of $\varphi(g)$

(a) If $0 \le k \le M+1$, then by Lemma 2.1,

$$|\varphi(x)|_H = |(g,k)|_{F\wr\mathbb{Z}} = \begin{cases} \Big(\sum\limits_{i\in\mathbb{Z}} |g(i)|_F\Big) + M + 1, & k = M+1, \\ \Big(\sum\limits_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2M - k, & \text{otherwise.} \end{cases}$$

Since $\varphi(x) = \varphi(g)x_0^{-k}$, we can obtain the reduced forest diagram for $\varphi(x)$ in either Figure 11 or Figure 12.

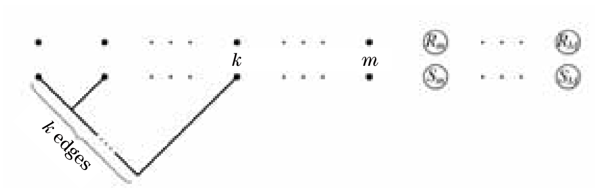

Figure 11 The reduced forest diagram of $\varphi(x)$

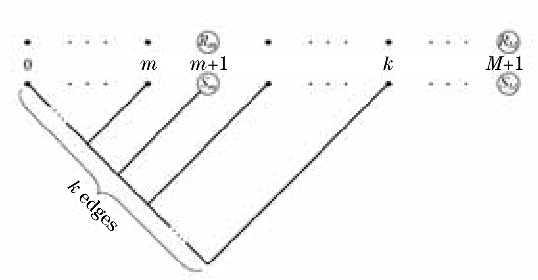

Figure 12 The reduced forest diagram of $\varphi(x)$

By the translation between the reduced forest diagrams and the reduced tree diagrams, we can observe that $N(\varphi(x)) = \big(\sum\limits_{i\in\mathbb{Z}} N(g(i))\big) + M + 2$. By Proposition 2.1,

$$\begin{aligned} |\varphi(x)|_F &\ge N(\varphi(x)) - 2 = \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + M \\ &\ge \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + M \ge \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 2M\Big) \ge \frac{1}{8}|\varphi(x)|_H. \end{aligned}$$

(b) If $k > M+1$, then

$$|\varphi(x)|_H = |(g,k)|_{F\wr\mathbb{Z}} = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + k.$$

And we obtain the reduced forest diagram for $\varphi(x)$ in Figure 13.

Observe that

$$N(\varphi(x)) = \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + k + 1.$$

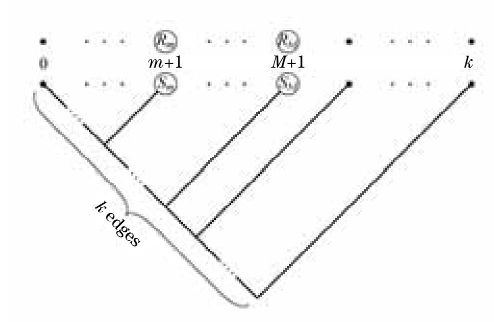

Figure 13 The reduced forest diagram of $\varphi(x)$

Then

$$
\begin{aligned}
|\varphi(x)|_F &\geq N(\varphi(x)) - 2 = \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + k - 1 \\
&\geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + k - 1 = \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 4k - 4\Big) \\
&\geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + k\Big) = \frac{1}{4}|\varphi(x)|_H.
\end{aligned}
$$

(c) If $k < 0$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + |k| + 2M.$$

We obtain the reduced forest diagram for $\varphi(x)$ in Figure 14.

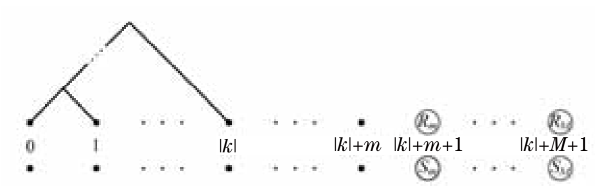

Figure 14 The reduced forest diagram of $\varphi(x)$

Then

$$N(\varphi(x)) = \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |k| + M + 2.$$

So

$$
\begin{aligned}
|\varphi(x)|_F &\geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |k| + M \\
&\geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + |k| + 2M\Big) = \frac{1}{4}|\varphi(x)|_H.
\end{aligned}
$$

Case 2 $m < 0 \leq M$. Then we picture the reduced forest diagram for $\varphi(g)$ in Figure 15.

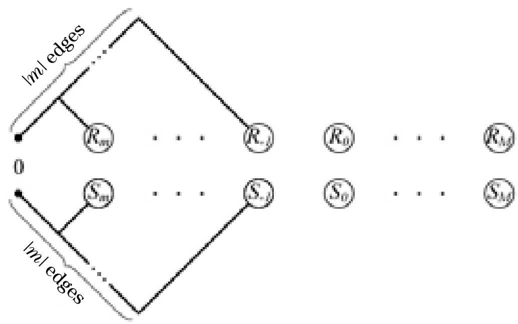

Figure 15 The reduced forest diagram of $\varphi(g)$

(a) If $0 \le k \le M+1$, then

$$|\varphi(x)|_H = \begin{cases} \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2|m| + M + 1, & k = M+1, \\ \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2|m| + 2M - k, & \text{otherwise.} \end{cases}$$

We can obtain the reduced forest diagram for $\varphi(x)$ in Figure 16.

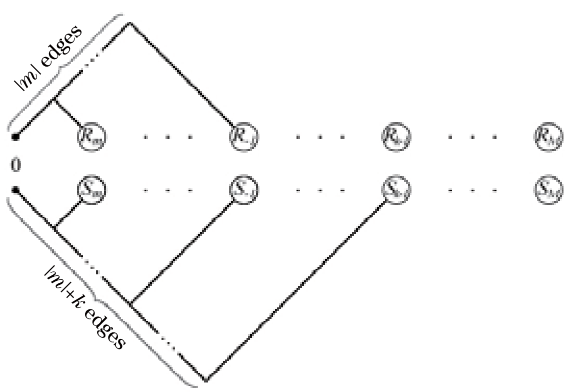

Figure 16 The reduced forest diagram of $\varphi(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |m| + M + 2$ and thus

$$\begin{aligned} |\varphi(x)|_F &\ge \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |m| + M \\ &\ge \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 2|m| + 2M\Big) \ge \frac{1}{8}|\varphi(x)|_H. \end{aligned}$$

(b) If $k > M+1$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + k + 2|m|.$$

We obtain the reduced forest diagram for $\varphi\,(x)$ in Figure 17.

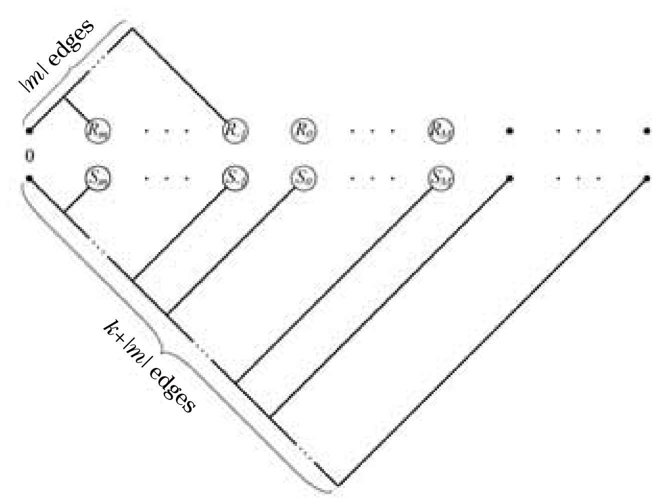

Figure 17 The reduced forest diagram of $\varphi\,(x)$

Then $N(\varphi(x)) = \big(\sum\limits_{i\in\mathbb{Z}} N(g(i))\big) + |m| + k + 1$ and so

$$\begin{aligned}|\varphi(x)|_F &\geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |m| + k - 1 \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 4k + 4|m| - 4\Big)\\ &\geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + k + 2|m|\Big) = \frac{1}{4}|\varphi(x)|_H.\end{aligned}$$

(c) If $m < k < 0$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2M + 2|m| - |k|.$$

We obtain the reduced forest diagram for $\varphi\,(x)$ in Figure 18.

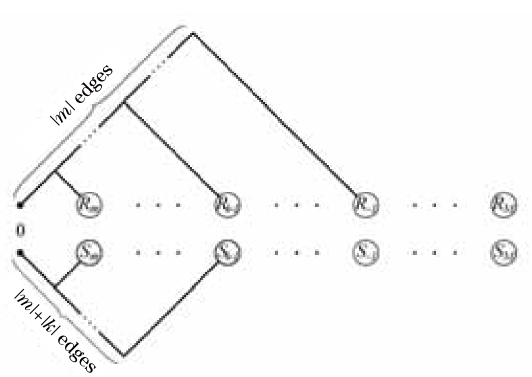

Figure 18 The reduced forest diagram of $\varphi\,(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |m| + M + 2$ and so

$$|\varphi(x)|_F \geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |m| + M \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 2|m| + 2M\Big) \geq \frac{1}{4}|\varphi(x)|_H.$$

(d) If $k \leq m$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2M + |k|.$$

We obtain the reduced forest diagram for $\varphi(x)$ in Figure 19.

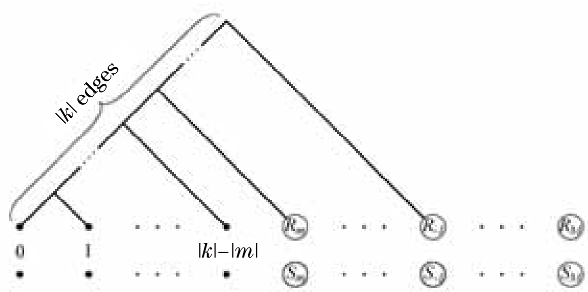

Figure 19 The reduced forest diagram of $\varphi(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |k| + M + 2$ and so

$$|\varphi(x)|_F \geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |k| + M \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + |k| + 2M\Big) = \frac{1}{4}|\varphi(x)|_H.$$

Case 3 $m \leq M < 0$. Then we picture the reduced forest diagram for $\varphi(g)$ in Figure 20.

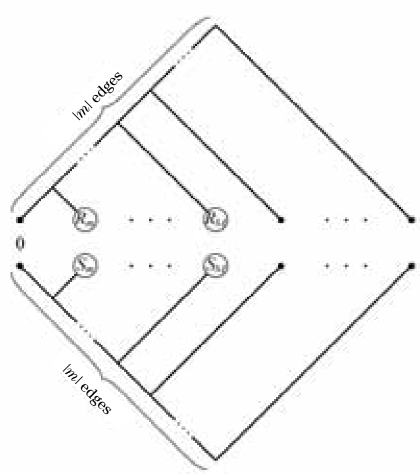

Figure 20 The reduced forest diagram of $\varphi(g)$

(a) If $k \geq 0$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + k + 2|m|.$$

We obtain the reduced forest diagram for $\varphi(x)$ in Figure 21.

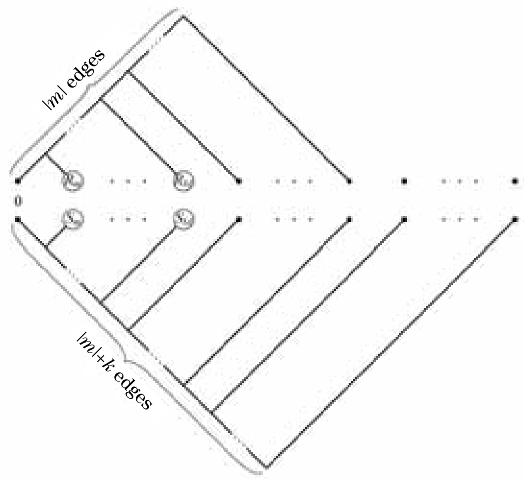

Figure 21 The reduced forest diagram of $\varphi(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |m| + k + 1$ and so

$$|\varphi(x)|_F \geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |m| + k - 1 \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 4|m| + 4k - 4\Big) \geq \frac{1}{12}|\varphi(x)|_H.$$

(b) If $m < k < 0$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + 2|m| - |k|.$$

We obtain the reduced forest diagram for $\varphi(x)$ in Figure 22.

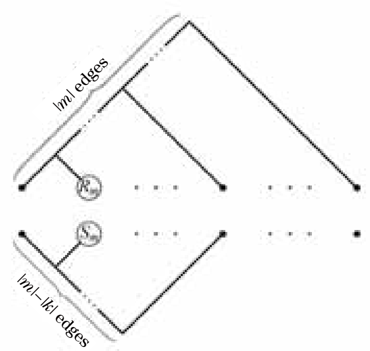

Figure 22 The reduced forest diagram of $\varphi(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |m| + 1$ and so

$$|\varphi(x)|_F \geq \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |m| - 1 \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)| + 4|m| - 4\Big) \geq \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)| + 2|m|\Big) \geq \frac{1}{4}|\varphi(x)|_H.$$

(c) If $k \le m$, then

$$|\varphi(x)|_H = \Big(\sum_{i\in\mathbb{Z}} |g(i)|_F\Big) + |k|.$$

We obtain the reduced forest diagram for $\varphi(x)$ in Figure 23.

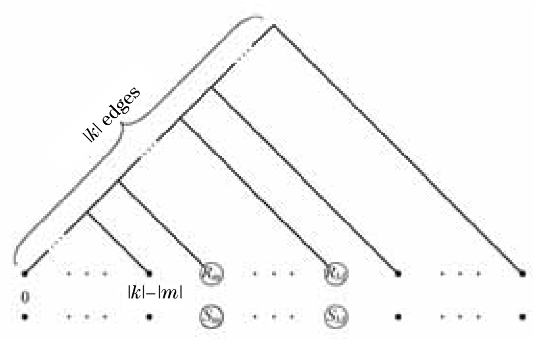

Figure 23 The reduced forest diagram of $\varphi(x)$

Then $N(\varphi(x)) = \big(\sum_{i\in\mathbb{Z}} N(g(i))\big) + |k| + 1$ and so

$$|\varphi(x)|_F \ge \Big(\sum_{i\in\mathbb{Z}} N(g(i))\Big) + |k| - 1 \ge \frac{1}{4}\Big(\sum_{i\in\mathbb{Z}} |g(i)|_F + 4|k| - 4\Big) \ge \frac{1}{8}|\varphi(x)|_H.$$

So far, we have shown that $|\cdot|_H \preceq |\cdot|_F$, i.e., H is quasi-isometrically embedded in F.

In the same way, we can also prove that $\mathbb{Z}\wr\mathbb{Z}$ is quasi-isometrically embedded in F.

Indeed, let H_1 be the subgroup of F generated by x_0 and $x_1^2x_2^{-1}x_1^{-1}$. Then there is a group isomorphism $\psi : \mathbb{Z}\wr\mathbb{Z} \to H_1$ which is given by

$$(\delta_0^1, 0) \mapsto x_1^2x_2^{-1}x_1^{-1}, \quad (e, 1) \mapsto x_0^{-1},$$

where e is the identity element of $\bigoplus_{\mathbb{Z}}\mathbb{Z}$. For every $x = (g, k) \in \mathbb{Z}\wr\mathbb{Z}$, let (R_i, S_i) be the reduced tree diagram of $x_0^{g(i)}$, and $N(g(i))$ be the number of carets in R_i (or S_i). Then one can obtain the reduced forest diagram of $\psi(x)$ by the reduced tree diagram (R_i, S_i) of $x_0^{g(i)}$ and observe that $N(g(i)) = |g(i)| + 1$. Then by the similar proof, we obtain the following theorem.

Theorem 3.2 *The subgroup H_1 isomorphic to $\mathbb{Z}\wr\mathbb{Z}$ in F generated by x_0 and $x_1^2x_2^{-1}x_1^{-1}$ is quasi-isometrically embedded.*

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