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Rankin-Cohen Deformations and Representation Theory*

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Abstract The author uses the unitary representation theory of $SL_2(\mathbb{R})$ to understand the Rankin-Cohen brackets for modular forms. Then this interpretation is used to study the corresponding deformation problems that Paula Cohen, Yuri Manin and Don Zagier initiated. Two uniqueness results are established.

Keywords Modular forms, Rankin-Cohen brackets, Representation theory, Rankin-Cohen deformation
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1 Introduction

Let Γ be a congruence subgroup of $PSL(2,\mathbb{Z})$. For $k\in\mathbb{N}$, a modular form of weight $2k$ is a complex function f on the upper half plane $\mathbb{H}$ which satisfies the following (cf. [16, 25, 32]):

(1) (Holomorphicity) f is holomorphic.

(2) (Modularity) For $\gamma=\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right)\in\Gamma$ and $z\in\mathbb{H}$, $f|_{2k}\gamma=f$, where

$$(f|_{2k}\gamma)(z)=(cz+d)^{-2k}f\Big(\frac{az+b}{cz+d}\Big). \tag{1.1}$$

(3) (Growth Condition at the Boundary) We assume that $|f(z)|$ would be majorized by a polynomial in $\max\{1,\mathrm{Im}(z)^{-1}\}$.

We denote by $\mathcal{M}(\Gamma)=\bigoplus\limits_{k\in\mathbb{N}}\mathcal{M}_{2k}(\Gamma)$ the graded algebra (by the weight) of modular forms with respect to this group.

In the 1950s, Rankin began the study of bi-differential operators over $\mathcal{M}(\Gamma)$ which produce new modular forms (cf. [22]), and twenty years later Henri Cohen gave a complete answer (cf. [4]) by proving that all these operators are linear combinations of the following brackets:

$$[f,g]_n=\sum_{r=0}^{n}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}f^{(r)}g^{(n-r)}\in\mathcal{M}_{2k+2l+2n}(\Gamma), \tag{1.2}$$

where $f\in\mathcal{M}_{2k}$ and $g\in\mathcal{M}_{2l}$ are two modular forms, and $f^{(r)}=\big(\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}\big)^r f$.

These brackets attracted interest of several authors. In [31], Zagier used the Ramanujan derivation $X:\mathcal{M}_{2k}\to\mathcal{M}_{2k+2}$ as follows:

$$Xf=\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}f}{\mathrm{d}z}-\frac{1}{2\pi\mathrm{i}}\frac{\mathrm{d}}{\mathrm{d}z}(\log\eta^4)\cdot kf, \tag{1.3}$$

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and introduced two series of elements by induction as follows:

$$f_{r+1} = Xf_r + r(r+2k-1)\Phi f_{r-1}, \quad g_{s+1} = Xg_s + s(s+2l-1)\Phi g_{s-1}, \tag{1.4}$$

where $\Phi = \frac{1}{144}E_4 \in \mathcal{M}_4$, and E_4 is the Eisenstein series of weight 4. He showed that

$$\sum_{r=0}^{n}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}f_r g_{n-r} = [f,g]_n, \tag{1.5}$$

which made the modularity of $[f,g]_n$ obvious as all the f_r and g_{n-r} are modular.

Moreover, he showed that for all associative $\mathbb{Z}$ (or $\mathbb{N}$)-graded algebra having a derivation which increases the degree by 2, and for all elements Φ of degree 4, the formula (1.5) defines a canonical Rankin-Cohen algebra structure (cf. [31] for the definition and properties).

Remark 1.1 When $\Phi = 0$, the situation is simplified to what Zagier called a standard Rankin-Cohen algebra (cf. [31] for the definition and properties).

Remark 1.2 We remind the readers that in the above definitions only the modularity is used, so we can do the same for nonholomorphic functions.

At about the same time, Paula Cohen, Yuri Manin and Don Zagier established a bijective correspondence between modular forms and invariant formal pseudodifferential operators (cf. [32–33]). They showed that the following formula (plus linear extension) defines an associative product over $\mathcal{M}(\Gamma)[[\hbar]]$: For two modular forms $f \in \mathcal{M}_{2k}$, $g \in \mathcal{M}_{2l}$,

$$\mu^\kappa(f,g) := \sum_{n=0}^{\infty} t_n^\kappa(k,l)[f,g]_n, \tag{1.6}$$

where the coefficients are given by

$$t_n^\kappa(k,l) = \Big(-\frac{1}{4}\Big)^n \sum_{j\geq 0}\binom{n}{2j}\frac{\binom{-\frac{1}{2}}{j}\binom{\kappa-\frac{3}{2}}{j}\binom{\frac{1}{2}-\kappa}{j}}{\binom{-k-\frac{1}{2}}{j}\binom{-l-\frac{1}{2}}{j}\binom{n+k+l-\frac{3}{2}}{j}}. \tag{1.7}$$

A special case is when $\kappa = \frac{1}{2}$ or $\frac{3}{2}$, and the product is reduced to what Eholzer claimed to be an associative product

$$f \star g := \sum_{n=0}^{\infty}[f,g]_n. \tag{1.8}$$

Remark 1.3 In this formulation, only the modularity of f is used, and we do not need either holomorphicity, or the growth condition near the boundary.

In 2003, Connes and Moscovici related the Hopf algebra $\mathcal{H}_1$ introduced in their study of the transversal index theory (cf. [6–10]), which governs the local symmetry in calculating the index of a transversal elliptic operator, to the Rankin-Cohen brackets (cf. [11]). By taking into account the work of Cohen-Manin-Zagier and Eholzer, especially (1.8), Connes and Moscovici proved a theorem stating that for every action of $\mathcal{H}_1$ on an algebra $\mathcal{A}$ with a certain extra structure, there exists a familly of formal deformations of $\mathcal{A}$ where the general terms of the deformed products are defined by some generalized Rankin-Cohen brackets (cf. [12]).

In a joint work with Bieliavsky and Tang [2], we have studied the deformation question from a quite different point of view. We used the deformation quantization theory of Fedosov to construct a realization of Rankin-Cohen deformations. More precisely, we found a specific symplectic connection on the upper half plane (cf. [2] and also [1, 21]):

$$\begin{aligned} \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_1} &= \mu(x_1, x_2)\frac{\partial}{\partial x_2}, & \nabla_{\frac{\partial}{\partial x_1}} \frac{\partial}{\partial x_2} &= \frac{1}{2x_2}\frac{\partial}{\partial x_1}, \\ \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_1} &= \frac{1}{2x_2}\frac{\partial}{\partial x_1}, & \nabla_{\frac{\partial}{\partial x_2}} \frac{\partial}{\partial x_2} &= -\frac{1}{2x_2}\frac{\partial}{\partial x_2}. \end{aligned} \tag{1.9}$$

Here μ is a suitable function. And on the corresponding Weyl algebra we found the same induction relation as that of Connes-Moscovici while calculating the deformed product. Then by an analogous argument, we re-obtained the above theorem of Connes-Moscovici (cf. [24] also).

In this paper, we study the brackets via the unitary representation theory of $SL_2(\mathbb{R})$, and then apply the results thus obtained to the deformation questions.

The rest of this paper is organized as follows: First a (relatively) explicit interpretation of the Rankin-Cohen brackets is given via the representation theory of $SL_2(\mathbb{R})$. The main result is the following theorem.[1]

Theorem 1.1 *Let $f \in \mathcal{M}_{2k},\ g \in \mathcal{M}_{2l}$ be two non-zero modular forms. Let $\pi_f \cong \pi_{\deg f}, \pi_g \cong \pi_{\deg g}$ be the corresponding discrete series representations of $SL_2(\mathbb{R})$. The tensor product of these two representations can be decomposed into a direct sum of discrete series representations, i.e.,*

$$\pi_f \otimes \pi_g = \bigoplus_{n=0} \pi_{\deg f + \deg g + 2n}. \tag{1.10}$$

The Rankin-Cohen bracket $[f, g]_n$ gives (up to scale) the vectors of the minimal K-weight in the representation space of the component $\pi_{\deg f + \deg g + 2n} \cong \pi_{[f,g]_n}$.

These representations are constructed in the following way: Let $f \in \mathcal{M}_{2k}(\Gamma)$ be a modular form. We associate to it a function on $\Gamma\backslash SL_2(\mathbb{R})$ by using the following map: For $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$,

$$(\sigma_{2k} f)(g) = f \mid_k g(\mathrm{i}) = (c\mathrm{i} + d)^{-2k} f\Big(\frac{a\mathrm{i} + b}{c\mathrm{i} + d}\Big). \tag{1.11}$$

This function belongs to

$$C^\infty(\Gamma\backslash SL_2(\mathbb{R}), 2k) = \{F \in C^\infty(\Gamma\backslash SL_2(\mathbb{R})), F(gr_\theta) = \exp(\mathrm{i}2k\theta)F(g)\}.$$

By taking into account the natural right action of $SL_2(\mathbb{R})$ on $C^\infty(\Gamma\backslash SL_2(\mathbb{R}))$:

$$(\pi(h)F)(g) = F(gh), \tag{1.12}$$

[1] The experts certainly had known this long before, as was showed by a remark of Deligne in 1973 (cf. Remark 3.1), but before finishing my Ph.D. thesis (November 2006), I had not found any detailed presentation of this result.

we obtain a representation of $SL_2(\mathbb{R})$ and so of the complexified Lie algebra $\mathfrak{s}l_2(\mathbb{C})$ by taking the smallest invariant subspace which contains the orbit of $\sigma_{2k}f$. We show that this representation is a discrete series representation of weight $2k$. In the end, we pull all the vectors in a basis of the representation space back to a subspace of $C^\infty(\mathbb{H})$ by using the inverse of the $\sigma_{2(k+n)}$'s, $n \geq 0$.

Then we use this representation theoretical interpretation to study certain properties of the deformed products, and mainly we can get the next two results.

Theorem 1.2 *The only formal deformed associative products* $* : \widetilde{\mathcal{M}}[[\hbar]] \times \widetilde{\mathcal{M}}[[\hbar]] \to \widetilde{\mathcal{M}}[[\hbar]]$ *defined by* $\mathbb{C}[[\hbar]]$*-linear extension and the formula*

$$f * g = \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} [f, g]_n \hbar^n, \tag{1.13}$$

where $\widetilde{\mathcal{M}}$ *is the space of functions which satisfy the modularity condition,* $(\alpha)_n := \alpha(\alpha + 1)\cdots(\alpha + n - 1)$, $A_0 = 1$ *and* $A_1(x, y) = xy$, *are those found by Cohen-Manin-Zagier.*

Proposition 1.1 *Let* Γ *be a congruence subgroup of* $SL_2(\mathbb{Z})$ *such that* $\mathcal{M}(\Gamma)$ *admits the unique factorization property (for example* $SL_2(\mathbb{Z})$ *itself), and let* $F_1, F_2, G_1, G_2 \in \mathcal{M}(\Gamma)$, *such that*

$$RC(F_1, G_1) = RC(F_2, G_2) \tag{1.14}$$

is the formal series in $\mathcal{M}(\Gamma)[[\hbar]]$. *Then there exists a constant* C, *such that*

$$F_1 = CF_2, \quad G_2 = CG_1. \tag{1.15}$$

This result implies that to some extent the set of Rankin-Cohen brackets of two modular forms f and g contains all the information of the pair (f, g).

2 From Modular Forms to Discrete Series

In this part, we will describe in detail the way of understanding these Rankin-Cohen brackets from a theoretical point of view on representation. We will partially follow the argument that Jean-Pierre Labesse indicated (cf. [20]):

Let $f \in \mathcal{M}_{2k}(\Gamma)$ be a modular form of weight $2k$ with respect to a congruence subgroup Γ of $SL_2(\mathbb{Z})$. We will associate a Γ-invariant function over $\Gamma\backslash SL_2(\mathbb{R})$ to it.

We define

$$(\sigma_{2k}f)(g) = f|_k g(\mathrm{i}) = (c\mathrm{i} + d)^{-2k} f\Big(\frac{a\mathrm{i} + b}{ci + d}\Big) \tag{2.1}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL_2(\mathbb{R})$. This function is invariant under the left translation of the group Γ: Let $\gamma \in \Gamma$, $f|_k \gamma g = (f|_k \gamma)|_k g = f|_k g$.

We also verify that for

$$r_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in SL_2(\mathbb{R}), \tag{2.2}$$

we have

$$(\sigma_{2k}f)(gr_\theta) = \exp(\mathrm{i}2k\theta)(\sigma_{2k}f)(g). \tag{2.3}$$

As one can observe easily, σ_{2k} gives a bijection between

$$C^\infty(\Gamma\backslash\mathbb{H}, 2k) = \left\{F \in C^\infty(\mathbb{H}),\ f(\gamma.z) = (cz+d)^{2k} f(z),\ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma\right\} \tag{2.4}$$

and

$$C^\infty(\Gamma\backslash SL_2(\mathbb{R}), 2k) = \{F \in C^\infty(\Gamma\backslash SL_2(\mathbb{R})),\ F(gr_\theta) = \exp(\mathrm{i}2k\theta)F(g)\}. \tag{2.5}$$

Taking the space of smooth functions $C^\infty(\Gamma\backslash SL_2(\mathbb{R}))$, we have a natural right action of $SL_2(\mathbb{R})$ on $\Gamma\backslash SL_2(\mathbb{R})$: For $F \in C^\infty(\Gamma\backslash SL_2(\mathbb{R}))$,

$$(\pi(h)F)(g) = F(gh). \tag{2.6}$$

We take the smallest invariant subspace under the action of $SL_2(\mathbb{R})$ which contains the orbit of $\sigma_{2k}f$ for a form $f \in \mathcal{M}_{2k}$, and we are interested in the action of the Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ on this space. We adopt the notation that Lang used in his book [19] (cf. [3, 17–18, 26–28] also). A basis of this Lie algebra is

$$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.7}$$

while a basis for the complexified Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is

$$E_+ = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad E_- = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad W = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{2.8}$$

with

$$\begin{aligned} \exp(tV) &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad \exp(tH) = \begin{pmatrix} \exp t & 0 \\ 0 & \exp(-t) \end{pmatrix}, \\ \exp(tE_+) &= \begin{pmatrix} 1+t & \mathrm{i}t \\ \mathrm{i}t & 1-t \end{pmatrix}, \quad \exp(tE_-) = \begin{pmatrix} 1+t & -\mathrm{i}t \\ -\mathrm{i}t & 1-t \end{pmatrix}, \\ \exp(tW) &= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}. \end{aligned} \tag{2.9}$$

Now we take an arbitrary holomorphic function ξ over the upper half plane $\mathbb{H}$, and for all k, we define

$$(F_k\xi)(g) := (\sigma_{2k}\xi)(g).$$

We calculate first the action of the base vectors described above on $F_k\xi$. We find

$$\begin{aligned} (L_V F_k\xi)(g) &= (-2k)\frac{d\mathrm{i}+c}{c\mathrm{i}+d}(F_k\xi)(g) + 2\Big(F_{k+1}\frac{\mathrm{d}\xi}{\mathrm{d}z}\Big)(g), \\ (L_H F_k\xi)(g) &= (-2k)\frac{c\mathrm{i}-d}{c\mathrm{i}+d}(F_k\xi)(g) + 2i\Big(F_{k+1}\frac{\mathrm{d}\xi}{\mathrm{d}z}\Big)(g), \end{aligned} \tag{2.10}$$

which implies

$$\begin{aligned} L_{E_+}(F_k\xi)(g) &= 2\Big[(-2k)\frac{c\mathrm{i}-d}{c\mathrm{i}+d}(F_k\xi)(g) + 2\mathrm{i}\Big(F_{k+1}\frac{\mathrm{d}\xi}{\mathrm{d}z}\Big)(g)\Big], \\ L_{E_-}(F_k\xi)(g) &= (L_H - \mathrm{i}L_V)(F_k\xi)(g) = 0. \end{aligned} \tag{2.11}$$

As W is the generator of the maximal compact subgroup, we also have

$$(L_W F_k\xi)(g) = 2k\mathrm{i}(\sigma_{2k}\xi)(g) = 2k\mathrm{i}(F_k\xi)(g). \tag{2.12}$$

So by induction, we have the following result.

Lemma 2.1 *For $n \in \mathbb{N}$,*

(1) $(L_{E_+})^n(F_k\xi) = 2^n \sum_{t=0}^{n} (-1)^{n-t} \frac{n!}{t!} \binom{2k+n-1}{n-t} \Big(\frac{\mathrm{ci}-d}{\mathrm{ci}+d}\Big)^{n-t} (2\mathrm{i})^t \Big(F_{k+t} \frac{\mathrm{d}^t\xi}{\mathrm{d}z^t}\Big)(g)$;

(2) $L_W(L_{E_+})^n(F_k\xi)(g) = (2k+2n)\mathrm{i}(L_{E_+})^n(F_k\xi)(g)$;

(3) $L_{E_-}(L_{E_+})^n(F_k\xi)(g) = -4n(2k+n-1)(L_{E_+})^{n-1}(F_k\xi)(g)$.

Next we calculate the action of the Casimir operator defined by

$$\omega = V^2 + H^2 - W^2 = \frac{1}{2}(E_+E_- + E_-E_+) - W^2. \tag{2.13}$$

The above calculation shows that for each vector $(L_{E_+})^n F_k\xi$,

$$\begin{aligned} \omega(L_{E_+})^n F_k\xi &= \frac{1}{2}[-4n(2k+n-1) - 4(n+1)(2k+n)](L_{E_+})^n F_k\xi \\ &\quad + (2k+2n)^2 (L_{E_+})^n F_k\xi \\ &= 4k(k-1)(L_{E_+})^n F_k\xi. \end{aligned} \tag{2.14}$$

Thus the Casimir acts on the space generated by the $(L_{E_+})^n F_k\xi$'s as a constant.

If we start by a modular form f (so a holomorphic function) of weight $2k$ and form a vector space generated by the functions $(L_{E_+})^n F_k f$, the above argument shows then $\mathfrak{sl}_2(\mathbb{C})$ also acts on that space and the Casimir acts as the multiplication by the constant $4k^2 - 4k$. So we have a representation of $\mathfrak{sl}_2(\mathbb{C})$.

Now we prove its irreducibility: For all operators T which commute with the representation, $[T, E_-] = 0$ implies that for the vector of the minimal weight $F_k f$, $TF_k f$ is still a vector of the minimal weight (for it is sent to zero by E_-), so there is a constant λ such that $TF_k f = \lambda F_k f$. By the same argument, by $E_-T(E_+F_k f) = TE_-(E_+F_k f) = T(8kF_k f) = 8k\lambda T_k f$, we have $T(E_+F_k f) = \lambda E_+F_k f$. So by induction we show that T acts by constant, and the representation is therefore irreducible. From the representation theory of $SL_2(\mathbb{R})$ we know the following proposition (cf. [19, 29]).

Proposition 2.1 *What we have constructed is an irreducible representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ which is the infinitesimale version of the discrete series representation of the group $SL_2(\mathbb{R})$ of weight $2k$.*

When we take all these functions of $C^\infty(SL_2(\mathbb{R}))$ back to the space $C^\infty(\mathbb{H})$ by using the bijectivity of the maps σ_{2k+2n}, we get a representation of $\mathfrak{sl}_2(\mathbb{C})$, denoted by π_f. Its representation space consists of some functions defined on the upper half plane. We denote by E_+, E_-, W the operators which correspond to L_{E_+}, L_{E_-}, L_W, respectively.

First, by

$$\begin{aligned} (\sigma_{2k+2}E_+f)(g) &= L_{E_+}(\sigma_{2k}f)(g) \\ &= 2\Big[(-2k)\frac{\mathrm{ci}-d}{\mathrm{ci}+d}(\sigma_{2k}f)(g) + 2\mathrm{i}\Big(\sigma_{2k+2}\frac{\mathrm{d}f}{\mathrm{d}z}\Big)(g)\Big] \\ &= 2\Big[2k\frac{1}{\mathrm{Im}\,\frac{a\mathrm{i}+b}{\mathrm{ci}+d}}\sigma_{2k+2}f + 2\mathrm{i}\sigma_{2k+2}\frac{\mathrm{d}f}{\mathrm{d}z}\Big](g), \end{aligned} \tag{2.15}$$

we can define

$$\widetilde{X}f := -\frac{1}{8\pi}(E_+)f = \frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}f}{\mathrm{d}z} - \frac{2kf}{4\pi \mathrm{Im} z}, \tag{2.16}$$

which is called the Shimura operator by some authors and played an important role in Henri Cohen's paper [4].

In fact, we can verify directly the following result.

Lemma 2.2 *Let f be a differentiable function, such that*

$$f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{2k} f(z).$$

We have

$$\widetilde{X}f\Big(\frac{az+b}{cz+d}\Big) = (cz+d)^{2k+2}\widetilde{X}f(z).$$

Proof It is sufficient to use

$$\mathrm{Im}\left(\frac{az+b}{cz+d}\right) = \mathrm{Im}\left(\frac{az+b}{cz+d}\cdot\frac{c\overline{z}+d}{c\overline{z}+d}\right) = \frac{\mathrm{Im}\, z}{|cz+d|^2}. \tag{2.17}$$

The claim can be obtained by the following calculation:

$$\begin{aligned}
\widetilde{X}f\Big(\frac{az+b}{cz+d}\Big) &= \frac{1}{2\pi \mathrm{i}}\frac{\frac{\partial}{\partial z}\big(f\big(\frac{az+b}{cz+d}\big)\big)}{\frac{\partial}{\partial z}\big(\frac{az+b}{cz+d}\big)} - \frac{2k}{4\pi\,\mathrm{Im}\left(\frac{az+b}{cz+d}\right)} f\Big(\frac{az+b}{cz+d}\Big)\\
&= \frac{1}{2\pi \mathrm{i}}\Big[(cz+d)^{2k}\frac{\mathrm{d}f}{\mathrm{d}z} + 2k(cz+d)^{2k-1}f(z)\Big](cz+d)^2\\
&\quad - \frac{2k}{4\pi}\frac{(cz+d)(c\overline{z}+d)}{\mathrm{Im}\, z}(cz+d)^{2k}f(z)\\
&= (cz+d)^{2k+2}\frac{1}{2\pi \mathrm{i}}\frac{\mathrm{d}f}{\mathrm{d}z} + (cz+d)^{2k+1}\frac{2k}{4\pi\,\mathrm{Im}\, z}(cz+d)f(z)\\
&= (cz+d)^{2k+2}\widetilde{X}f(z).
\end{aligned}$$

By reiterating this operation, we get the following correspondence:

$$\Big(-\frac{1}{8\pi}\Big)^n \frac{1}{2k\cdots(2k+n-1)}(E_+)^n f \leftrightarrow \frac{1}{2k\cdots(2k+n-1)}\Big(\frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial z} - \frac{Y}{2\pi\,\mathrm{Im}\, z}\Big)^n f,$$

where $Yf = kf$ is the Euler operator. Using the representation theory of $SL_2(\mathbb{R})$, we can choose the vectors on the right-hand side to form a basis, i.e.,

$$\varphi_n = \frac{1}{2k\cdots(2k+n-1)}\Big(\frac{1}{2\pi \mathrm{i}}\frac{\partial}{\partial z} - \frac{Y}{2\pi\,\mathrm{Im}\, z}\Big)^n f \tag{2.18}$$

for $n \in \mathbb{N}$. The action of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$E_+\varphi_n = (-8\pi)(2k+n)\varphi_{n+1}, \tag{2.19}$$

$$E_-\varphi_n = \frac{n}{2\pi}\varphi_{n-1}, \tag{2.20}$$

$$W\varphi_n = 2n\mathrm{i}\varphi_n. \tag{2.21}$$

We introduce an operator $\widetilde{\partial}$, such that $\widetilde{\partial}\varphi_n = \varphi_{n+1}$. Then

$$\varphi_n = \widetilde{\partial}^n\varphi_0 = \widetilde{\partial}^n f. \tag{2.22}$$

Moreover, by induction we have the following result.

Lemma 2.3 *Let f be a smooth function which satisfies the modularity condition of weight $2k$. Then,*

$$f^{(m)} := \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\Big)^m f = m!\sum_{r=0}^{m}\frac{1}{(4\pi y)^r}\frac{\widetilde{X}^{m-r}}{(m-r)!}\binom{2k+m-1}{r}f. \tag{2.23}$$

Remark 2.1 This implies exactly

$$[f,g]_n = \sum_{r=0}^{n}(-1)^r\widetilde{X}^r\binom{2k+n-1}{n-r}f\widetilde{X}^{n-r}\binom{2l+n-1}{r}g \tag{2.24}$$

for $f \in \mathcal{M}_{2k},\ g \in \mathcal{M}_{2l}$.

In fact, we have

$$\begin{aligned}
[f,g]_n &= \sum_{r=0}^{n}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}f^{(r)}g^{(n-r)}\\
&= \sum_{r=0}^{n}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}\\
&\quad\cdot\Big(r!\sum_{s=0}^{r}\frac{1}{(4\pi y)^s}\binom{2k+r-1}{s}\frac{\widetilde{X}^{r-s}}{(r-s)!}f\Big)\\
&\quad\cdot\Big((n-r)!\sum_{t=0}^{n-r}\frac{1}{(4\pi y)^t}\binom{2l+n-r-1}{t}\frac{\widetilde{X}^{n-r-t}}{(n-r-t)!}g\Big)\\
&= \sum_{s,t}\frac{1}{(4\pi y)^{s+t}}\Big(\sum_{r=s}^{n-t}(-1)^r\binom{n+2k-1}{n-r}\binom{n+2l-1}{r}\frac{r!}{(r-s)!}\frac{(n-r)!}{(n-r-t)!}\\
&\quad\cdot\binom{2k+r-1}{s}\binom{2l+n-r-1}{t}\widetilde{X}^{r-s}f\widetilde{X}^{n-r-t}g\Big).
\end{aligned}$$

It is clear that when $u = s+t,\ v = r-s$ (and so $n-r-t = n-u-v$) are all fixed, the coefficient of $\widetilde{X}^v f\widetilde{X}^{n-u-v}g$ is

$$\begin{aligned}
&\sum_{s}(-1)^{s+v}\binom{n+2k-1}{n-v-s}\binom{n+2l-1}{v+s}\\
&\quad\cdot\frac{(v+s)!}{v!}\frac{(n-v-s)!}{(n-v-u)!}\binom{2k+v+s-1}{s}\binom{2l+n-v-s-1}{u-s}\\
&= (-1)^v\sum_{s}(-1)^s\frac{(n+2k-1)!}{(2k+v+s-1)!(n-v-s)!}\frac{(n+2l-1)!}{(2l+n-v-s-1)!(v+s)!}\\
&\quad\cdot\frac{(v+s)!}{v!}\frac{(n-v-s)!}{(n-v-u)!}\frac{(2k+v+s-1)!}{s!(2k+v-1)!}\frac{(2l+n-v-s-1)!}{(u-s)!(2l+n-u-v-1)!}\\
&= (-1)^v\sum_{s}(-1)^s\frac{(n+2k-1)!(n+2l-1)!}{(2k+v-1)!v!(n-v-u)!(2l+n-u-v-1)!}\frac{1}{s!(u-s)!}\\
&= (-1)^v\frac{(n+2k-1)!(n+2l-1)!}{(2k+v-1)!v!(n-v-u)!(2l+n-u-v-1)!u!}\sum_{s}(-1)^s\frac{u!}{s!(u-s)!},
\end{aligned}$$

which is non-zero if and only if $u = 0$, i.e., $s = t = 0$. We thus get the claim of the remark.

We will see immediately a more conceptual explanation of this identity.

3 Construction of the Brackets

Given two representations of $SL_2(\mathbb{R})$ (and the corresponding derived representation of $\mathfrak{sl}_2(\mathbb{R})$ or $\mathfrak{sl}_2(\mathbb{C})$), we are interested in their tensor product. In fact, we have the following theorem of J. Repka (cf. [23]).

Theorem 3.1 *For two discrete series representations of $SL_2(\mathbb{R})$, their tensor product has the following decomposition (for $m, n \geq 1$):*

$$\pi_m \otimes \pi_n \cong \pi_{m+n} \oplus \pi_{m+n+2} \oplus \pi_{m+n+4} \oplus \cdots \cong \bigoplus_{k=0}^{\infty} \pi_{n+m+2k}. \tag{3.1}$$

To adapt (Lie algebra version of) this theorem to our situation, we give a special consideration to the representation space. More precisely, we have the following result.

Proposition 3.1 *Given two modular forms $f \in \mathcal{M}_{2k}$, $g \in \mathcal{M}_{2l}$, then in the decomposition*

$$\pi_f \otimes \pi_g = \bigoplus_{n=0} \pi_{\deg f + \deg g + 2n}, \tag{3.2}$$

a vector of the minimal K-weight of $\pi_{\deg f + \deg g + 2n}$ has the form

$$\begin{aligned} &\frac{1}{n!} \sum_{r=0} (-1)^r \binom{n}{r} \widetilde{\partial}^r f \otimes \widetilde{\partial}^{n-r} g \\ &= \frac{1}{(2k)_n (2l)_n} \sum_{r=0}^{n} (-1)^r \widetilde{X}^r \binom{2k+n-1}{n-r} f \otimes \widetilde{X}^{n-r} \binom{2l+n-1}{r} g. \end{aligned} \tag{3.3}$$

When composed the bilinear map defined by the product

$$\mathrm{M} : f \otimes g \mapsto fg, \tag{3.4}$$

this corresponds to a modular form of weight $2k + 2l + 2n$ which can be expressed as

$$\frac{1}{2k(2k+1)\cdots(2k+n-1)2l(2l+1)\cdots(2l+n-1)}[f,g]_n = \frac{1}{(2k)_n (2l)_n}[f,g]_n. \tag{3.5}$$

Proof The first part is a consequence of the fact that the space of the minimal K-weight vectors is exactly the kernel of the operator $\Delta E_- = E_- \otimes 1 + 1 \otimes E_-$, so we have

$$\begin{aligned} &\Delta E_- \Big(\sum_{r=0} (-1)^r \binom{n}{r} \widetilde{\partial}^r f \otimes \widetilde{\partial}^{n-r} g \Big) \\ &= \sum_{r=0} (-1)^r \binom{n}{r} (E_-(\widetilde{\partial}^r f) \otimes \widetilde{\partial}^{n-r} g + \widetilde{\partial}^r f \otimes E_-(\widetilde{\partial}^{n-r} g)) \\ &= \frac{1}{2\pi} \sum_{r=0} (-1)^r \binom{n}{r} (r \widetilde{\partial}^{r-1} f \otimes \widetilde{\partial}^{n-r} g + (n-r) \widetilde{\partial}^r f \otimes \widetilde{\partial}^{n-r-1} g) \\ &= \frac{1}{2\pi} \sum_{r=0} \Big((-1)^r \binom{n}{r} (n-r) + (-1)^{r+1} \binom{n}{r+1} (r+1) \Big) \widetilde{\partial}^r f \otimes \widetilde{\partial}^{n-r-1} g \\ &= 0. \end{aligned}$$

The second half is just (2.24). The operator M is an intertwining operator between the subrepresentation in the tensor product and the representation constructed from $[f,g]_n$.

N.B. In this construction, we can only determine the coefficients up to scale, as the vectors of the minimal K-weight form a subspace.

Furthermore, the formulation of Rankin-Cohen brackets using the operator $\widetilde{X}$ can be naturally generalized to any pair of functions $(f,g) \in \widetilde{\mathcal{M}}^2$, where

$$\widetilde{\mathcal{M}}(\Gamma) := \bigoplus_k \widetilde{\mathcal{M}}_{2k}(\Gamma) := \bigoplus_k \{f : \mathbb{H} \to \mathbb{C}, f \mid_{2k} \gamma = f, \ \forall \gamma \in \Gamma\} \tag{3.6}$$

is the space of smooth complex functions on the upper half plane which satisfy (only) the modularity condition.

But in this case we do not have a general discrete series interpretation as above.

Remark 3.1 In fact, the relation between the tensor products of discrete series representations and Rankin-Cohen brackets was already observed some years ago as one can find the following remark of Deligne [13] (there he talked about the discrete series of $GL(2)$):

"Remarque 2.1.4. L'espace $F(G, GL(2,\mathbb{Z}))$ ci-dessus est stable par produit. D'autre part, $D_{k-1} \otimes D_{l-1}$ contient les $D_{k+l+2m} (m \geq 0)$. Pour $m = 0$, ceci correspond au fait que le produit fg d'une forme modulaire holomorphe de poids k par une de poids l, en est une de poids $k + l$. Pour $m = 1$, en coordonnées (1.5.2) (remark: this should be 1.1.5.2), on trouve que $l\frac{\partial f}{\partial z}.g - kf.\frac{\partial g}{\partial z}$ est modulaire holomorphe de poids $k + l + 2$, et ainsi de suite. De même dans le cadre adélique."

In fact, here what we get is the modularity of $\frac{1}{k}\frac{\partial f}{\partial z}.g - f.\frac{1}{l}\frac{\partial g}{\partial z}$.

After the main part of the paper was written (as one chapter of my Ph.D. thesis in French), Weissman [30] posted on ArXiv a paper which is along the line of Deligne's remark.

Remark 3.2 We also notice that there is an interpretation of these Rankin-Cohen brackets using the theory of transvectants. Especially in a recent paper (cf. [15]), El Gradechi treated the Rankin-Cohen brackets in a very similar way as we did above.

4 Applications to Formal Deformations

In this part, we study the formal deformations constructed from the Rankin-Cohen brackets, more precisely, we are interested in the products $* : \widetilde{\mathcal{M}}(\Gamma)[[\hbar]] \times \widetilde{\mathcal{M}}(\Gamma)[[\hbar]] \to \widetilde{\mathcal{M}}(\Gamma)[[\hbar]]$ defined by $\mathbb{C}[[\hbar]]$-linearity and the following formula:

$$\begin{aligned} f * g &= \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} \Big(\sum_{r=0}^{n} (-1)^r \widetilde{X}^r \binom{2k+n-1}{n-r} f \widetilde{X}^{n-r} \binom{2l+n-1}{r} g \Big) \hbar^n \\ &= \sum \frac{A_n(\deg f, \deg g)}{(\deg f)_n (\deg g)_n} [f,g]_n \hbar^n, \end{aligned} \tag{4.1}$$

where $f, g \in \widetilde{\mathcal{M}}$, the space of smooth complex functions on the upper half plane which satisfy (only) the modularity condition. And we assume furthermore that $A_0 = 1$ and $A_1(x,y) = xy$. The main concern is to have an associative product. First, we have the following proposition.

Proposition 4.1 *If the A_n's give rise to an associative product, then for any triple (f, g, h), the coefficients of $\widetilde{X}^r f \widetilde{X}^s g \widetilde{X}^t h$ in the expansion of $(f * g) * h$ and $f * (g * h)$ are the same.*

Proof In fact, we only need to show the equality of the coefficients for $\frac{\partial^r f}{\partial z^r}\frac{\partial^s g}{\partial z^s}\frac{\partial^t h}{\partial z^t}$, and we prove this by contradiction. Assume that there are functions $f_0, g_0, h_0 \in \widetilde{\mathcal{M}}$ and an index triple (r_0, s_0, t_0), such that the coefficient of $\frac{\partial^{r_0} f}{\partial z^{r_0}}\frac{\partial^{s_0} g}{\partial z^{s_0}}\frac{\partial^{t_0} h}{\partial z^{t_0}}$ in $(f_0 * g_0) * h_0 - f_0 * (g_0 * h_0)$ is non-zero. So the associativity of the product $*$ gives rise to a differential equation which is satisfied by all $f \in \mathcal{M}_{\deg f_0},\ g \in \mathcal{M}_{\deg g_0},\ h \in \mathcal{M}_{\deg h_0}$.

Now the only constraint on these functions is their invariance under the action of Γ, which implies that we have the freedom to modify the functions in the interior of a fundamental domain. So in a small open set contained in the fundamental domain, we can have some f_1, g_1, h_1, such that $\frac{\partial^r f_1}{\partial z^r} = \frac{\partial^s g_1}{\partial z^s} = \frac{\partial^t h_1}{\partial z^t} = 0$, $0 \le r, s, t \le n$, $r \ne r_0,\ s \ne s_0,\ t \ne t_0$; and

$$\frac{\partial^{r_0} f}{\partial z^{r_0}} \ne 0, \quad \frac{\partial^{s_0} g}{\partial z^{s_0}} \ne 0, \quad \frac{\partial^{t_0} h}{\partial z^{t_0}} \ne 0.$$

But this gives us a contradiction. The proposition is then proved.

For three functions f, g and h in $\widetilde{\mathcal{M}}$, the objects $(f * g) * h$ and $f * (g * h)$ live in the vector space

$$H_{f,g,h} := \bigoplus_n H_{n;f,g,h} := \bigoplus_n \langle \widetilde{X}^r f \widetilde{X}^s g \widetilde{X}^t h\, \hbar^{r+s+t},\ r+s+t=n\rangle. \tag{4.2}$$

Generally, $H_{n;f,g,h}$ is a vector space of dimension $\frac{1}{2}(n+1)(n+2)$. So it is natural to check the identification of the coefficients with respect to the canonical base $\widetilde{X}^r f \widetilde{X}^s g \widetilde{X}^t h\ \hbar^{r+s+t}$ $(r+s+t=n)$. The problem is that in this case, for $H_{n;f,g,h}$, we will have $\sum\limits_{r=0}^{n}\sum\limits_{s=0}^{n-r}\sum\limits_{t=0}^{n-r-s} 1 = \frac{1}{2}(n+1)(n+2)$ equations, which is not very practical.

In order to reduce the number of equations to verify, we will try to determine a subspace in which live $(f * g) * h$ and $f * (g * h)$. In fact, we have already seen that when f and g are both holomorphic, $f * g$ is a series which can be written as a sum (with coefficients) of the $\hbar^n \sum (-1)^r \binom{n}{r} \widetilde{\partial}^r f \widetilde{\partial}^{n-r} g$'s, and the latter forms a basis of the kernel of the operator $\hbar^{-1}\Delta E_-$, we have the following lemma.

Lemma 4.1 *For three holomorphic functions $f, g, h \in \widetilde{\mathcal{M}}$, the kernel of the operator $\hbar^{-1}E_- : H_{f,g,h} \to H_{f,g,h}$ is generated by the vectors $(0 \le p \le n)$*

$$\xi_{n,p} = \hbar^n \sum_{s=0}^{p} (-1)^s \binom{p}{s} \frac{\widetilde{X}^s}{(2k+2l+2n)_s} \Big(\sum_{r=0}^{n-p} \binom{n-p}{r} \widetilde{\partial}^{n-p-r} f \widetilde{\partial}^r g \Big) \widetilde{\partial}^{p-s} h.$$

$(f * g) * h$ and $f * (g * h)$ belong to this kernel.

Proof We know that $H_{n;f,g,h}$ is a vector space of dimension $\frac{1}{2}(n+1)(n+2)$. We establish first the fact that the map $\mathcal{E}$ is surjective: For every vector $\hbar^{n-1}\widetilde{\partial}^r f \widetilde{\partial}^s g \widetilde{\partial}^t h$ with $r+s+t = n-1$, we have

$$\hbar^{-1}E_- \Big(\hbar^n \sum_{i=0}^{n-1-r} \frac{(-1)^i i!}{\prod\limits_{u=0}^{i}(r+1+u)} \Big[\sum_{j=0}^{i} \binom{s}{i-j} \binom{t}{j} \widetilde{\partial}^{r+1+i} f \widetilde{\partial}^{s-i+j} g \widetilde{\partial}^{t-j} h \Big] \Big)$$

$$
\begin{aligned}
&= \frac{1}{2\pi}\hbar^{n-1}\sum_{i=0}^{n-1-r}\frac{(-1)^i i!}{\prod_{u=0}^{i}(r+1+u)}\Big[\sum_{j=0}^{i}\binom{s}{i-j}\binom{t}{j}((r+1+i)\widetilde{\partial}^{r+i}f\widetilde{\partial}^{s-i+j}g\widetilde{\partial}^{t-j}h\\
&\quad + (s-i+j)\widetilde{\partial}^{r+1+i}f\widetilde{\partial}^{s-i+j-1}g\widetilde{\partial}^{t-j}h + (t-j)\widetilde{\partial}^{r+1+i}f\widetilde{\partial}^{s-i+j}g\widetilde{\partial}^{t-j-1}h)\Big]\\
&= \frac{1}{2\pi}\hbar^{n-1}\sum_{i,j}\Big[\frac{(-1)^i i!}{\prod_{u=0}^{i-1}(r+1+u)}\binom{s}{i-j}\binom{t}{j}\\
&\quad + \frac{(-1)^{i-1}(i-1)!}{\prod_{u=0}^{i-1}(r+1+u)}\binom{s}{i-j}\binom{t}{j}i\Big]\widetilde{\partial}^{r+i}f\widetilde{\partial}^{s-i+j}g\widetilde{\partial}^{t-j}h\\
&= \frac{1}{2\pi}\hbar^{n-1}\widetilde{\partial}^{r}f\widetilde{\partial}^{s}g\widetilde{\partial}^{t}h.
\end{aligned}
$$

The dimension at degree $n-1$ is $\frac{1}{2}n(n+1)$, which implies that the dimension of the kernel at degree n is $n+1$.

The vectors $\xi_{n,p}$ are in the kernel of $\hbar^{-1}E_-$: We verify first that for two functions f and g in the kernel of E_-, we have

$$E_-\widetilde{X}(fg) = 4\deg(fg)fg.$$

So by simple induction, we can get

$$
\begin{aligned}
&E_-\frac{\widetilde{X}^s}{(2k+2l+2n)_s}\Big(\sum_{r=0}^{n-p}\binom{n-p}{r}\widetilde{\partial}^{n-p-r}f\widetilde{\partial}^{r}g\Big)\\
&= \frac{\widetilde{X}^{s-1}}{(2k+2l+2n)_{(s-1)}}\Big(\sum_{r=0}^{n-p}\binom{n-p}{r}\widetilde{\partial}^{n-p-r}f\widetilde{\partial}^{r}g\Big),
\end{aligned}
\tag{4.3}
$$

which implies

$$
\begin{aligned}
\hbar^{-1}E_-\xi_{n,p} &= \frac{1}{2\pi}\hbar^{n-1}\Big[\sum_{s=0}^{p}(-1)^s\binom{p}{s}s\widetilde{\partial}^{s-1}\Big(\sum_{r=0}^{n-p}\binom{n-p}{r}\widetilde{\partial}^{n-p-r}f\widetilde{\partial}^{r}g\Big)\widetilde{\partial}^{p-s}h\\
&\quad + \sum_{s=0}^{p}(-1)^s\binom{p}{s}E_-\widetilde{\partial}^{s}\Big(\sum_{r=0}^{n-p}\binom{n-p}{r}\widetilde{\partial}^{n-p-r}f\widetilde{\partial}^{r}g\Big)(p-s)\widetilde{\partial}^{p-s-1}h\Big]\\
&= 0.
\end{aligned}
$$

Moreover, we can project $\xi_{n,p}$ onto the component whose second factor is g, and we get

$$\widetilde{\partial}^{n-p}fg\widetilde{\partial}^{p}h. \tag{4.4}$$

These functions are generally linearly independent. This proves that the $(n+1)$ $\xi_{n,p}$'s constitute a basis of the kernel of $\hbar^{-1}E_-$ at degree n.

In general, for any element $f \in \widetilde{\mathcal{M}}$, we can define an operator $\widetilde{\partial}$ by the formulae $\widetilde{\partial}\varphi_n = \varphi_{n+1}$ in the vector space generated by the basis $\big\{\varphi_n = \frac{1}{(\deg f)_n}\widetilde{X}^n f,\ n \in \mathbb{N}\big\}$, so then (3.3) is still valid. We can then define an operator $\hbar^{-1}E_- : H_{f,g,h} \to H_{f,g,h}$ by the following formula:

$$\hbar^{-1}E_-(\widetilde{\partial}^{r}f\widetilde{\partial}^{s}g\widetilde{\partial}^{t}h\ \hbar^{r+s+t})$$

$$= (r\widetilde{\partial}^{r-1} f\widetilde{\partial}^s g\widetilde{\partial}^t h + s\widetilde{\partial}^r f\widetilde{\partial}^{s-1} g\widetilde{\partial}^t h + t\widetilde{\partial}^r f\widetilde{\partial}^s g\widetilde{\partial}^{t-1} h)\hbar^{r+s+t-1}. \tag{4.5}$$

Then the above argument works without any modification.

So it is sufficient now to identify the coefficients of $\hbar^n\widetilde{\partial}^p fg\widetilde{\partial}^{n-p}h$ to obtain the associativity. In $(f*g)*h$, it is the sum of the terms (for $n-r\geq p$)

$$\frac{(-1)^r A_r(2k,2l)}{(2k)_r}\binom{n-r}{p}\frac{(-1)^{p-r}A_{n-r}(2k+2l+2r,2m)}{(2k+2l+2r)_{n-p}(2m)_p}.$$

For $f*(g*h)$, it is the sum of the terms (for $s\leq p$)

$$\frac{(-1)^p A_{n-s}(2k,2l+2m+2s)}{(2k)_{n-p}(2l+2m+2s)_p}\binom{n-s}{n-p}\frac{A_s(2l,2m)}{(2m)_s}.$$

Hence finally what we should verify are the following identities (for $p=0,1,\cdots,n$):

$$\begin{aligned}
&\sum_{r=0}\binom{n-r}{p}\frac{A_{n-r}(2k+2l+2r,2m)A_r(2k,2l)}{(2k+2l+2r)_{n-p-r}(2m)_p(2k)_r}\\
&=\sum_{s=0}\binom{n-s}{n-p}\frac{A_{n-s}(2k,2l+2m+2s)A_s(2l,2m)}{(2k)_{n-p}(2l+2m+2s)_{p-s}(2m)_s}.
\end{aligned}\tag{4.6}$$

We first look at the simplest case, the identification of the coefficient of $\hbar$. We need to verify

$$\begin{aligned}
&A_1(2k+2l,2m)\Big(\frac{1}{2k+2l}(f_{2k+2}g_{2l}h_{2m}+f_{2k}g_{2l+2}h_{2m}) - f_{2k}g_{2l}\frac{1}{2m}h_{2m+2}\Big)\\
&+A_1(2k,2l)\Big(\frac{1}{2k}f_{2k+2}g_{2l}h_{2m} - f_{2k}\frac{1}{2l}g_{2l+2}h_{2m}\Big)\\
&=A_1(2k,2l+2m)\Big(\frac{1}{2k}f_{2k+2}g_{2l}h_{2m} - \frac{1}{2l+2m}(f_{2k}g_{2l+2}h_{2m}+f_{2k}g_{2l}h_{2m+2})\Big)\\
&+A_1(2l,2m)\Big(f_{2k}\frac{1}{2l}g_{2l+2}h_{2m} - f_{2k}g_{2l}\frac{1}{2m}h_{2m+2}\Big).
\end{aligned}$$

In other words,

$$\begin{aligned}
&\frac{1}{2k+2l}A_1(2k+2l,2m)+\frac{1}{2k}A_1(2k,2l)=\frac{1}{2k}A_1(2k,2l+2m),\\
&\frac{1}{2k+2l}A_1(2k+2l,2m)-\frac{1}{2l}A_1(2k,2l)=\frac{1}{2l}A_1(2l,2m)-\frac{1}{2l+2m}A_1(2k,2l+2m),\\
&-\frac{1}{2m}A_1(2k+2l,2m)=-\frac{1}{2l+2m}A_1(2k,2l+2m)-\frac{1}{2m}A_1(2l,2m).
\end{aligned}$$

It is obvious that $A_1(2k,2l)=2k\cdot 2l$ verify these equations.

Then we pass to the next step, the identification of the coefficients of $\hbar^2$:

$$\begin{aligned}
&\frac{A_2(2k+2l,2m)}{2m(2m+1)}=\frac{A_2(2k,2l+2m)}{(2l+2m)(2l+2m+1)}+4kl+\frac{A_2(2l,2m)}{2m(2m+1)},\\
&\frac{A_2(2k+2l,2m)}{(2k+2l)2m}+(2k+2l+2)2l=\frac{A_2(2k,2l+2m)}{2k(2l+2m)}+(2l+2m+2)2l,\\
&\frac{A_2(2k+2l,2m)}{(2k+2l)(2k+2l+1)}+4lm+\frac{A_2(2k,2l)}{2k(2k+1)}=\frac{A_2(2k,2l+2m)}{2k(2k+1)}.
\end{aligned}\tag{4.7}$$

This system has a special solution

$$A_2(2k,2l)=\frac{1}{2}2k(2k+1)2l(2l+1), \tag{4.8}$$

so we need to solve the homogeneous system

$$\begin{aligned}
&\frac{A_2(2k+2l,2m)}{2m(2m+1)}=\frac{A_2(2k,2l+2m)}{(2l+2m)(2l+2m+1)}+\frac{A_2(2l,2m)}{2m(2m+1)},\\
&\frac{A_2(2k+2l,2m)}{(2k+2l)2m}=\frac{A_2(2k,2l+2m)}{2k(2l+2m)},\\
&\frac{A_2(2k+2l,2m)}{(2k+2l)(2k+2l+1)}+\frac{A_2(2k,2l)}{2k(2k+1)}=\frac{A_2(2k,2l+2m)}{2k(2k+1)}.
\end{aligned} \tag{4.9}$$

We denote by $\widetilde{A}(2k,2l)$ the function $\frac{2k+2l+1}{4kl}A(2k,2l)$. The equations that $\widetilde{A}(2k,2l)$ satisfy are

$$\begin{aligned}
&\widetilde{A}_2(2k+2l,2m)\Big(\frac{1}{2m+1}-\frac{1}{2k+2l+2m+1}\Big)\\
=&\widetilde{A}_2(2k,2l+2m)\Big(\frac{1}{2l+2m+1}-\frac{1}{2k+2l+2m+1}\Big)+\widetilde{A}_2(2l,2m)\Big(\frac{1}{2m+1}-\frac{1}{2l+2m+1}\Big),\\
&\widetilde{A}_2(2k+2l,2m)=\widetilde{A}_2(2k,2l+2m),\\
&\widetilde{A}_2(2k+2l,2m)\Big(\frac{1}{2k+2l+1}-\frac{1}{2k+2l+2m+1}\Big)+\widetilde{A}_2(2k,2l)\Big(\frac{1}{2k+1}-\frac{1}{2k+2l+1}v\Big)\\
=&\widetilde{A}_2(2k,2l+2m)\Big(\frac{1}{2k+1}-\frac{1}{2k+2l+2m+1}\Big).
\end{aligned} \tag{4.10}$$

The first two equations indicate $\widetilde{A}_2(2l,2m)=\widetilde{A}_2(2k+2l,2m)$ for any $(2k,2l,2m)$, and by using once more the second equation, we get $\widetilde{A}_2(2l,2m)=\widetilde{A}_2(2k+2l,2m)=\widetilde{A}_2(2k,2l+2m)$, i.e., $\widetilde{A}$ is a constant function. We then conclude that in our situation the degree of freedom is one, i.e., in the general formula of A_2 we can introduce a parameter c as follows:

$$A_2(2k,2l)=\frac{1}{2}2k(2k+1)2l(2l+1)+c\frac{2k2l}{2k+2l+1}. \tag{4.11}$$

Now we study some properties of a sequence A_n which defines an associative product. We assume their existence (as in the examples provided by Cohen-Manin-Zagier) and we have the following lemma.

Lemma 4.2 *Assuming their existence, the A_n's $(n\geq 3)$ are determined by $A_0,A_1,\cdots,$ A_{n-1} and the associativity.*

Proof Our aim is to determine the value of $A_n(2x,2y)$ for every pair $(x,y)\in\mathbb{N}^2\setminus\{(0,0)\}$ ($(0,0)$ is not included because in this case for all $n\geq 1$, $[f,g]_n=0$). The idea is very simple, in order to do the identification of the coefficients of $\hbar^n$, we have $n+1$ equations, indexed by p, by considering $2k,2l,2m$ as constants and assuming that A_i $(i<n)$ are already known.

If $l>0$, there is, in these equations, (at most) four unknowns: $A_n(2k,2l)$, $A_n(2l,2m)$, $A_n(2k+2l,2m)$ and $A_n(2k,2l+2m)$. The first two appear only once each: $p=0$ for $A_n(2k,2l)$, and $p=n$ for $A_n(2l,2m)$. When $n\geq 3$, we take the two equations with $p=1$ and 2. The determinant of the linear equation system with $A_n(2k+2l,2m)$ and $A_n(2k,2l+2m)$ as unknown is

$$\begin{aligned}
&\det\begin{pmatrix} \binom{n}{1}\dfrac{1}{(2k+2l)_{n-1}(2m)_1} & \binom{n}{n-1}\dfrac{1}{(2k)_{n-1}(2l+2m)_1} \\ \binom{n}{2}\dfrac{1}{(2k+2l)_{n-2}(2m)_2} & \binom{n}{n-2}\dfrac{1}{(2k)_{n-2}(2l+2m)_2} \end{pmatrix} \\
&= \binom{n}{n-1}\binom{n}{n-2}\frac{1}{(2k+2l)_{n-2}(2m)_1(2k)_{n-2}(2l+2m)_1} \\
&\cdot\Big(\frac{1}{(2k+2l+n-2)(2l+2m+1)} - \frac{1}{(2m+1)(2k+n-2)}\Big) \\
&= \binom{n}{n-1}\binom{n}{n-2}\frac{1}{(2k+2l)_{n-2}(2m)_1(2k)_{n-2}(2l+2m)_1} \\
&\cdot\frac{-(2l)^2-(2l)(2k+2m+n-1)}{(2k+2l+n-2)(2l+2m+1)(2m+1)(2k+n-2)} \neq 0,
\end{aligned} \tag{4.12}$$

following the fact that $l>0,\ n>2$, and that k,m are all positive integers.

We can therefore obtain the value of $A_n(2x,2y)$ for a pair $(2x,2y)$ which can be expressed as $(2k+2l,2m)$ or $(2k,2l+2m)$ for a certain $l>0$ without any ambiguity. This lemma is proved.

Next, we have the following lemma by induction.

Lemma 4.3 *We have $A_n(2k,2l)=A_n(2l,2k)$ and $A_n(2k,0)=0$.*

Proof We have already obtained $A_n(2k,2l)=A_n(2l,2k)$ and $A_n(2k,0)=0$ for $n=0,1,2$. Assume now that this is valid for $0,1,\cdots,n-1$. When we consider the associativity identity for three functions $f\in\widetilde{\mathcal{M}}_{2m},\ g\in\widetilde{\mathcal{M}}_{2l},\ h\in\widetilde{\mathcal{M}}_{2k}$, (4.6) becomes, for all fixed n and p,

$$\begin{aligned}
&\sum_{r=0}\binom{n-r}{p}\frac{A_{n-r}(2m+2l+2r,2k)A_r(2m,2l)}{(2m+2l+2r)_{n-p-r}(2k)_p(2m)_r} \\
&=\sum_{s=0}\binom{n-s}{n-p}\frac{A_{n-s}(2m,2l+2k+2s)A_s(2l,2k)}{(2m)_{n-p}(2l+2k+2s)_{p-s}(2k)_s}.
\end{aligned}$$

If we exchange the indices r and s, and replace p by $n-p$, we obtain

$$\begin{aligned}
&\sum_{s=0}\binom{n-s}{n-p}\frac{A_{n-s}(2m+2l+2s,2k)A_s(2m,2l)}{(2m+2l+2s)_{p-s}(2k)_{n-p}(2m)_s} \\
&=\sum_{r=0}\binom{n-r}{p}\frac{A_{n-r}(2m,2l+2k+2r)A_r(2l,2k)}{(2m)_p(2l+2k+2r)_{n-p-r}(2k)_r}.
\end{aligned}$$

For $0<p<n$, the only difference with respect to (4.6), by using the induction hypothesis, is that we have replaced $A_n(2k,2l+2m)$ (resp. $A_n(2k+2l,2m)$) by $A_n(2l+2m,2k)$ (resp. $A_n(2m,2k+2l)$). This implies that $A_n(2l+2m,2k)$ and $A_n(2m,2k+2l)$ satisfy the same linear equation system as $A_n(2k,2l+2m)$ and $A_n(2k+2l,2m)$, and the previous lemma gives $A_n(2x,2y)=A_n(2y,2x)$.

When we take $l = 0,\ k, m \neq 0$ in (4.6), the identity for $p = 0$ is simplified as

$$\sum_{r=0} \frac{A_{n-r}(2k+2r, 2m)A_r(2k, 0)}{(2k+2r)_{n-p-r}(2m)_p(2k)_r} = \frac{A_n(2k, 2m+2s)}{(2k)_n},$$

i.e.,

$$\frac{A_n(2k, 2m)}{(2k)_n} + \frac{A_n(2k, 0)}{(2k)_n} = \frac{A_n(2k, 2m)}{(2k)_n},$$

so then we have $A_n(2k, 0)$, and the lemma is established.

When we write A_n as a polynomial of $2k, 2l$ and c, then because A_0, A_1 are both of degree 0 in c, we conclude by the above argument as follows.

Lemma 4.4 *A_n is a polynomial of degree $\left[\frac{n}{2}\right]$ in c.*

In [5], the authors use only the modularity to construct what they called the invariant formal pseudodifferential operators. Therefore their results depend only on the modularity of the functions involved, which need not be holomorphic. Thus we can conclude with the following theorem.

Theorem 4.1 *Cohen-Manin-Zagier have in fact found all associative formal products of the form* (4.1)*.*

Remark 4.1 We would like to emphasize two facts as follows:

(1) Numerically, the parameter c introduced in (4.11) equals $-3+4\kappa-\kappa^2$ for the κ in (1.7).

(2) When we consider the restriction on classical modular forms, for every degree the space $\mathcal{M}_{2k}$ is of finite dimension. Our argument above does not work any more, so it is not ruled out that other formal products defined by using Rankin-Cohen brackets exist in that case.

We give next a proposition, which shows that the multiplication structure defined by the Eholzer product (or the Rankin-Cohen product for Connes-Moscovici) is somewhat "finer" than that defined by the usual product, and in fact, we have the following result.

Proposition 4.2 *Let Γ be a congruence subgroup of $SL_2(\mathbb{Z})$, such that $\mathcal{M}(\Gamma)$ admits the unique factorization property* (*for example $SL_2(\mathbb{Z})$ itself*)*, and let $F_1, F_2, G_1, G_2 \in \mathcal{M}(\Gamma)$, such that*

$$RC(F_1, G_1) = RC(F_2, G_2) \tag{4.13}$$

as formal series in $\mathcal{M}(\Gamma)[[\hbar]]$, so then there exists a constant C, such that

$$F_1 = CF_2, \quad G_2 = CG_1. \tag{4.14}$$

We prove first the following lemma.

Lemma 4.5 *Let $f \in \mathcal{M}_{2k},\ g \in \mathcal{M}_{2l},\ h \in \mathcal{M}_{2m}$ be three modular forms such that $[fg, h]_n = [f, gh]_n$ for all n. Then $l = 0$, i.e., g is a constant function.*

Proof Our data satisfy automatically $[fg, h]_0 = [f, gh]_0$. As to the case $n = 1$, we have

$$(2k+2l)fg\frac{\mathrm{d}h}{\mathrm{d}z} - 2m\frac{\mathrm{d}(fg)}{\mathrm{d}z}h = 2kf\frac{\mathrm{d}(gh)}{\mathrm{d}z} - (2l+2m)\frac{\mathrm{d}f}{\mathrm{d}z},$$

which implies that

$$(2k+2m)\frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}z} = 2l\Big(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}z} + \frac{1}{h}\frac{\mathrm{d}h}{\mathrm{d}z}\Big),$$

or in other words,

$$g^{k+m} = C^{ste}(fh)^{l} \tag{4.15}$$

for a non-zero constant. Now we write the Fourier expansions of the following three modular forms:

$$\begin{aligned} f &= \alpha_0 + \alpha_1 q + \alpha_2 q^2 + \cdots, \\ g &= \beta_0 + \beta_1 q + \beta_2 q^2 + \cdots, \\ h &= \gamma_0 + \gamma_1 q + \gamma_2 q^2 + \cdots, \end{aligned} \tag{4.16}$$

where $q = \exp(2\pi \mathrm{i} z)$. As $q\frac{\mathrm{d}}{\mathrm{d}q} = \frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}$, we have

$$\begin{aligned} \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\Big)^n f &= 0^n\alpha_0 + 1^n\alpha_1 q + 2^n\alpha_2 q^2 + \cdots, \\ \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\Big)^n g &= 0^n\beta_0 + 1^n\beta_1 q + 2^n\beta_2 q^2 + \cdots, \\ \Big(\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\Big)^n h &= 0^n\gamma_0 + 1^n\gamma_1 q + 2^n\gamma_2 q^2 + \cdots. \end{aligned} \tag{4.17}$$

This implies that in the calculation of $[fg,h]_n$ and $[f,gh]_n$ $(n \geq 1)$, there are only two terms (among the $n+1$ to sum up), which contain the term of degree 1 in q: The first and the last in the definition formula. We have then for all n,

$$\begin{aligned} &\binom{2k+2l+n-1}{n}\alpha_0\beta_0\gamma_1 + (-1)^n\binom{2m+n-1}{n}(\alpha_0\beta_1+\alpha_1\beta_0)\gamma_0 \\ &= \binom{2k+n-1}{n}\alpha_0(\beta_0\gamma_1+\beta_1\gamma_0) + (-1)^n\binom{2l+2m+n-1}{n}\alpha_1\beta_0\gamma_0. \end{aligned} \tag{4.18}$$

We have to distinguish several different cases as follows:

(1) $l = 0$, i.e., $g = \beta_0$. (4.18) is automatically valid, and it is exactly the claim of the lemma.

(2) $l > 0$, there are two possibilities.

(a) $\beta_0 \neq 0$, then following (4.15), we have $\alpha_0 \neq 0$, $\gamma_0 \neq 0$ (because the constant term of $(fh)^{k+m}$ is non-zero). By using the bilinearity of the brackets, it is possible to assume $\alpha_0 = \beta_0 = \gamma_0 = 1$. Then (4.18) becomes, for all n,

$$\begin{aligned} &\binom{2k+2l+n-1}{n}\gamma_1 + (-1)^n\binom{2m+n-1}{n}(\beta_1+\alpha_1) \\ &= \binom{2k+n-1}{n}(\gamma_1+\beta_1) + (-1)^n\binom{2l+2m+n-1}{n}\alpha_1. \end{aligned} \tag{4.19}$$

Without loss of generality, we can assume that $m \geq k$ (otherwise, we consider $[hg,f]_n = [h,gf]_n$), and now the variables $\alpha_1, \beta_1, \gamma_1$ satisfy the equations (for all n)

$$A_{n1}\alpha_1 + A_{n2}\beta_1 + A_{n3}\gamma_1 = 0, \tag{4.20}$$

where

$$\begin{aligned} A_{n1} &= (-1)^n(2m)_n - (-1)^n(2l+2m)_n, \\ A_{n2} &= (-1)^n(2m)_n - (2k)_n, \\ A_{n3} &= (2k+2l)_n - (2k)_n, \end{aligned} \tag{4.21}$$

and especially,

$$\begin{aligned} A_{11} &= 2l, \\ A_{12} &= -2k-2m, \\ A_{13} &= 2l, \\ A_{21} &= -(2m+2m+1)2l - (2l)^2 = -(4m+2l+1)(2l), \\ A_{22} &= 2m(2m+1) - 2k(2k+1) = (2m-2k)(2m+2k+1), \\ A_{23} &= (4k+2l+1)(2l), \\ A_{31} &= (2l)^3 + 3(2m+1)(2l)^2 + (3(2m)^2 + 6(2m) + 2)(2l), \\ A_{32} &= -2m(2m+1)(2m+2) - 2k(2k+1)(2k+2), \\ A_{33} &= (2l)^3 + 3(2k+1)(2l)^2 + (3(2k)^2 + 6(2k) + 2)(2l). \end{aligned} \tag{4.22}$$

The determinant of this system of linear equations is then

$$\begin{aligned} &\det A_{1\le i,j\le 3} \\ &= \det \begin{pmatrix} 2l & -2k-2m & 2l \\ -(4m+2l+1)(2l) & (2m-2k)(2m+2k+1) & (4k+2l+1)(2l) \\ A_{31} & A_{32} & A_{33} \end{pmatrix} \\ &= (2l)^2\{-6(k+l+m+1)(2m-2k)^2 \\ &\quad + (2k+2l+2m+1)[-(2k+2m)(6k+6l+2m) - 12k](2m-2k) \\ &\quad - (4k+4l+4m+2)(2k+2m)(2k+2l)(4k+2l+3)\} \\ &< 0. \end{aligned} \tag{4.23}$$

We have taken the hypothesis $m \ge k$ and that the weights of modular forms k, l, m are all positive integers and $l \ge 1$, so the last inequality is obtained because all the three terms to be summed up are nonnegative.

We can conclude that $\alpha_1 = \beta_1 = \gamma_1 = 0$. The same argument can be applied when we compare the coefficients of q^2, and we obtain a system of linear equations for $\alpha_2, \beta_2, \gamma_2$ with the same coefficient matrix, so $\alpha_2 = \beta_2 = \gamma_2 = 0$, so on and so forth. We get a contradiction.

(b) $\beta_0 = 0$, the argument in (4.15) gives $\alpha_0 = 0$ or $\gamma_0 = 0$.

So we can then assume that the first nonzero terms are $\alpha_r q^r$, $\beta_s q^s$, $\gamma_t q^t$ $(r, s, t \ge 0)$. We consider now the term of the lowest degree in q, say $r+s+t$, in the identity $[fg, h]_n = [f, gh]_n$, and we obtain, for all n,

$$\sum_{p=0}^{n} (-1)^p \binom{n}{p} (2k+2l+p)_{n-p}(2m+n-p)_p (r+s)^p t^{n-p} \alpha_r \beta_s \gamma_t$$

$$= \sum_{q=0}^{n} (-1)^q \binom{n}{q} (2k+q)_{n-1}(2l+2m+n-q)_q r^q (s+t)^{n-q} \alpha_r \beta_s \gamma_t. \tag{4.24}$$

As the $\alpha_r, \beta_s, \gamma_t$ are nonzero, dividing both sides by $\alpha_r\beta_s\gamma_t$, this becomes

$$\begin{aligned}
&\sum_{p=0}^{n} (-1)^p \binom{n}{p} (2k+2l+p)_{n-p}(2m+n-p)_p (r+s)^p t^{n-p} \\
&= \sum_{q=0}^{n} (-1)^q \binom{n}{q} (2k+q)_{n-q}(2l+2m+n-q)_q r^q (s+t)^{n-q}.
\end{aligned} \tag{4.25}$$

For $n = 1$, we have

$$(k+m)s = l(r+t).$$

By taking into account this relation, we obtain, by replacing s by $\frac{l(r+t)}{k+m}$ (if k and m are all zero, then according to (4.15), $l = 0$, too, a contradiction) for every $n \geq 2$, a homogeneous equation of degree n in r, t. For $n = 2$, this equation is

$$\begin{aligned}
0 &= [(2m)_2 - (2l+2m)_2]r^2 - 2[(2k+2l+1)(2m+1) - (2k+1)(2l+2m+1)]rt \\
&\quad + [(2k+2l)_2 - (2k)_2]t^2 + 2[(2m)_2 + (2k+1)(2l+2m+1)]rs \\
&\quad - 2[(2k+2l+1)(2m+1) + (2k)_2]st + [(2m)_2 - (2k)_2]s^2 \\
&= \frac{2l}{(k+m)^2}\{[(k+3m)(k+l+m) + (k+m)]r^2 \\
&\quad + (2m-2k)(k+l+m)rt - [(3k+m)(k+l+m) + (k+m)]t^2\}.
\end{aligned}$$

We see first that r and t are either all zero or all non-zero, because that the coefficients of r^2 and t^2 are all strictly non-zero. The case where $r = t = 0$ has already been treated above, and we assume from now on $r, s, t > 0$. The last expression has a factor $r + t$, i.e.,

$$\begin{aligned}
0 &= \frac{2l}{(k+m)^2}\{[(k+3m)(k+l+m) + (k+m)]r^2 \\
&\quad + (2m-2k)(k+l+m)rt - [(3k+m)(k+l+m) + (k+m)]t^2\} \\
&= \frac{2l}{(k+m)^2}(r+t)\{[(k+3m)(k+l+m) + (k+m)]r \\
&\quad - [(3k+m)(k+l+m) + (k+m)]t\}.
\end{aligned} \tag{4.26}$$

This implies that there exists a positive constant μ, such that

$$\begin{aligned}
t &= \mu[(k+3m)(k+l+m) + (k+m)], \\
r &= \mu[(3k+m)(k+l+m) + (k+m)], \\
s &= \mu l[4(k+l+m) + 2].
\end{aligned} \tag{4.27}$$

We calculate the equation for $n = 3$, and the difference of the two sides is, by using (4.15),

$$\begin{aligned}
&\frac{1}{(k+m)^3}\{(2k+2l)(2k+2l+1)(2k+2l+2)t^3(k+m)^3 \\
&- 3(2k+2l+1)(2k+2l+2)(2m+2)[(k+m)r + l(r+t)]t^2(k+m)^2
\end{aligned}$$

$$
\begin{aligned}
&+3(2k+2l+2)(2m+1)(2m+2)[(k+m)r+l(r+t)]^2t(k+m)\\
&-2m(2m+1)(2m+2)[(k+m)r+l(r+t)]^3\\
&-2k(2k+1)(2k+2)[l(r+t)+t(k+m)]^3\\
&+3(2k+1)(2k+2)(2l+2m+2)[l(r+t)+t(k+m)]^2r(k+m)\\
&-3(2k+2)(2l+2m+1)(2l+2m+2)[l(r+t)+t(k+m)]r^2(k+m)^2\\
&+(2l+2m)(2l+2m+1)(2l+2m+2)r^3(k+m)^3\}. \qquad (4.28)
\end{aligned}
$$

We denote by P_3 the braced quantity, as an integer coefficient polynomial of k, l, m, r, t. Taking the values of r and t as in (4.27), we obtain a polynomial in k, l, m whose coefficients are all positive (cf. Section 5 for the explicit expressions), which implies that it could not have positive integer roots in k, l, m. So this possibility is excluded.

Proof of Proposition 4.2 We do first a simplification: Let

$$
\begin{aligned}
F_1 &= f_{1,2k} + f_{1,2k+2} + f_{1,2k+4} + \cdots, \qquad & G_1 &= g_{1,2l} + g_{1,2l+2} + g_{1,2l+4} + \cdots,\\
F_2 &= f_{2,2k'} + f_{2,2k'+2} + f_{2,2k'+4} + \cdots, \qquad & G_2 &= g_{2,2l'} + g_{2,2l'+2} + g_{2,2l'+4} + \cdots
\end{aligned}
$$

be the natural grading of these modular forms. Then when we look at, for each degree in $\hbar$, the term whose coefficient is a modular form of the smallest weight, we find the terms $[f_{1,2k}, g_{1,2l}]_n\hbar^n$ and $[f_{1,2k'}, g_{1,2l'}]_n\hbar^n$. So we have

$$RC(f_{1,2k}, g_{1,2l}) = RC(f_{2,2k'}, g_{2,2l'}).$$

In other words, $[f_{1,2k}, g_{1,2l}]_n = [f_{2,2k'}, g_{2,2l'}]_n$ for all n. Using the unique factorization hypothesis, we can speak of the biggest common divisor of $f_{1,2k}$ and $f_{2,2k'}$ (resp. $g_{1,2l}$ and $g_{2,2l'}$), denoted by f_0 (resp. g_0). We see first that by adjusting constants, it is possible to have

$$
\begin{aligned}
\frac{f_{1,2k}}{f_0} &= \frac{g_{2,2l'}}{g_0} = A,\\
\frac{f_{2,2k'}}{f_0} &= \frac{g_{1,2l}}{g_0} = B.
\end{aligned}
\qquad (4.29)
$$

We have then $[f_0A, Bg_0]_i = [f_0B, Ag_0]_i$ for all i. Moreover, A, B are relatively prime as polynomials of the generators. We then use the following result.

Lemma 4.6 *Let $f \in \mathcal{M}_{2k}$, $A \in \mathcal{M}_{2l}$, $B \in \mathcal{M}_{2m}$, $g \in \mathcal{M}_{2n}$ be four modular forms such that*

$$[fA, Bg]_i = [fB, Ag]_i$$

for all i, and that A, B are relatively prime as polynomials of the generators. Then either $A = 1$, or $B = 1$.

Proof For $i = 1$, by definition,

$$(k+l)fA\frac{\mathrm{d}(Bg)}{\mathrm{d}z} - \frac{\mathrm{d}(fA)}{\mathrm{d}z}(m+n)Bg = (k+m)fB\frac{\mathrm{d}(Ag)}{\mathrm{d}z} - \frac{\mathrm{d}(fB)}{\mathrm{d}z}(l+n)Ag, \qquad (4.30)$$

i.e.,

$$(k+l)fA\Big(\frac{\mathrm{d}B}{\mathrm{d}z}g + B\frac{\mathrm{d}g}{\mathrm{d}z}\Big) - \Big(\frac{\mathrm{d}f}{\mathrm{d}z}A + f\frac{\mathrm{d}A}{\mathrm{d}z}\Big)(m+n)Bg$$

$$= (k+m)fB\Big(\frac{\mathrm{d}A}{\mathrm{d}z}g + A\frac{\mathrm{d}g}{\mathrm{d}z}\Big) - \Big(f\frac{\mathrm{d}B}{\mathrm{d}z} + \frac{\mathrm{d}f}{\mathrm{d}z}B\Big)(l+n)Ag. \tag{4.31}$$

We divide the terms by $fABg$ to obtain

$$(l-m)\Big(\frac{1}{f}\frac{\mathrm{d}f}{\mathrm{d}z} + \frac{1}{g}\frac{\mathrm{d}g}{\mathrm{d}z}\Big) = (k+2m+n)\frac{1}{A}\frac{\mathrm{d}A}{\mathrm{d}z} - (k+2l+n)\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}z}, \tag{4.32}$$

i.e.,

$$(fg)^{l-m} = \frac{A^{k+2m+n}}{B^{k+2l+n}}. \tag{4.33}$$

If $l \geq m$, then the left-hand side is a polynomial in the generators. As A, B are relatively prime, we get $B = 1$. If $l \leq m$ we get $A = 1$. The lemma is proved.

We can summarize the two lemmas above as follows:

Lemma 4.7 *For four non-zero modular forms, $f_1 \in \mathcal{M}_{2l}$, $g_1 \in \mathcal{M}_{2k}$, $f_2 \in \mathcal{M}_{2l'}$, $g_2 \in \mathcal{M}_{2k'}$, if we have*

$$[f_1, g_1]_n = [f_2, g_2]_n \tag{4.34}$$

for all n, then $k = k'$, $l = l'$, and there exists a non-zero constant C, such that

$$f_1 = Cf_2, \quad Cg_1 = g_2. \tag{4.35}$$

Proof of Proposition 4.2 (Continued) Following the Lemma 4.7, we have $k = k'$, $l = l'$ and the existence of a constant C, such that

$$f_{1,2k} = Cf_{2,2k}, \quad g_{2,2l} = Cg_{1,2l}.$$

Then we pass to the next step. We now compare the coefficient of every $\hbar^n$ in the expansion of $RC(F_1, G_1) = RC(F_2, G_2)$. And we look at the term with the second lowest weight coefficient (which is an element in $\mathcal{M}(\Gamma)$). Besides $f_{1,2k} = Cf_{2,2k}$ and $Cg_{1,2l} = g_{2,2l}$, the relevant terms in the expansion of F_1, G_1, F_2, G_2 are $f_{1,2k+2}$, $f_{2,2k+2}$, $g_{1,2l+2}$ and $g_{2,2l+2}$. We have, for all n,

$$[f_{1,2k}, g_{1,2l+2}]_n + [f_{1,2k+2}, g_{1,2l}]_n = [f_{2,2k}, g_{2,2l+2}]_n + [f_{2,2k+2}, g_{2,2l}]_n,$$

i.e.,

$$[f_{1,2k}, Cg_{1,2l+2} - g_{2,2l+2}]_n = [f_{1,2k+2} - Cf_{2,2k+2}, g_{1,2l}]_n$$

for all n and the same constant C. If $f_{1,2k+2} - Cf_{2,2k+2} \in \mathcal{M}_{2k+2}$ and $Cg_{1,2l+2} - g_{2,2l+2} \in \mathcal{M}_{2l+2}$ are non-zero, we can apply once more the Lemma 4.7 to get a contradiction. So the only possibility left is

$$f_{1,2k+2} = Cf_{2,2k+2}, \quad g_{2,2l+2} = Cg_{1,2l+2}.$$

The rest is an induction procedure. If we already have $f_{1,2k+2i} = Cf_{2,2k+2i}$ and $g_{2,2l+2i} = Cg_{1,2l+2i}$ for $0 \leq i \leq p-1$, then when we consider in $RC(F_1, G_1) = RC(F_2, G_2)$ the term that belongs to $\mathcal{M}_{2k+2l+2n+2p}\hbar^n$, we get an equality

$$\sum_i [f_{1,2k+2i}, g_{1,2l+2p-2i}]_n = \sum_i [f_{2,2k+2i}, g_{2,2l+2p-2i}]_n. \tag{4.36}$$

Using the induction hypothesis, it can be simplified to

$$[f_{1,2k}, g_{1,2l+2p}]_n + [f_{1,2k+2p}, g_{1,2l}]_n = [f_{2,2k}, g_{2,2l+2p}]_n + [f_{2,2k+2p}, g_{2,2l}]_n, \tag{4.37}$$

or, in an equivalent way,

$$[f_{1,2k}, Cg_{1,2l+2p} - g_{2,2l+2p}]_n = [f_{1,2k+2p} - Cf_{2,2k+2p}, g_{1,2l}]_n$$

for all n and the same constant C. If $f_{1,2k+2p} - Cf_{2,2k+2p} \in \mathcal{M}_{2k+2p}$ and $Cg_{1,2l+2p} - g_{2,2l+2p} \in \mathcal{M}_{2l+2p}$ are both non-zero, the Lemma 4.7 gives rise to a contradiction. So we conclude that

$$f_{1,2k+2p} = Cf_{2,2k+2p}, \quad g_{2,2l+2p} = Cg_{1,2l+2p}.$$

The proposition is established.

5 Appendix: The Value of P_3

The following are results of calculus of Mathematica.

$$\begin{aligned}
&P_3(k,l,m,r,t)\\
&= 4l(r+t)(-3k^2r^2 - 2k^3r^2 + 3klr^2 + 2kl^2r^2 - 6kmr^2 - 15k^2mr^2 - 3k^3mr^2 + 3lmr^2\\
&\quad - 9klmr^2 - 6k^2lmr^2 3kl^2mr^2 - 3m^2r^2 - 24km^2r^2 - 15k^2m^2r^2 - 9lm^2r^2\\
&\quad - 24klm^2r^2 - 9l^2m^2r^2 - 11m^3r^2 - 21km^3r^2 - 18lm^3r^2 - 9m^4r^2 + 12k^2rt + 17k^3rt\\
&\quad + 3k^4rt + 6klrt + 21k^2lrt + 6k^3lrt + 4kl^2rt + 3k^2l^2rt + 24kmrt + 51k^2mrt + 24k^3mrt\\
&\quad + 6lmrt + 42klmrt + 42k^2lmrt + 4l^2mrt + 18kl^2mrt + 12m^2rt + 51km^2rt + 42k^2m^2rt\\
&\quad + 21lm^2rt + 42klm^2rt + 3l^2m^2rt + 17m^3rt + 24km^3rt + 6lm^3rt + 3m^4rt - 3k^2t^2\\
&\quad - 11k^3t^2 - 9k^4t^2 + 3klt^2 - 9k^2lt^2 - 18k^3lt^2 + 2kl^2t^2 - 9k^2l^2t^2 - 6kmt^2 - 24k^2mt^2\\
&\quad - 21k^3mt^2 + 3lmt^2 - 9klmt^2 - 24k^2lmt^2 + 2l^2mt^2 - 3kl^2mt^2 - 3m^2t^2 - 15km^2t^2\\
&\quad - 15k^2m^2t^2 - 6klm^2t^2 - 2m^3t^2 - 3km^3t^2 + 2l^2mr^2).
\end{aligned}$$

By taking the values $t = \mu[(k+3m)(k+l+m)+(k+m)]$ and $r = \mu[(3k+m)(k+l+m)+(k+m)]$, one gets

$$\begin{aligned}
&P_3(k,l,m,\mu[(3k+m)(k+l+m)+(k+m)], \mu[(3k+m)(k+l+m)+(k+m)])\\
&= \mu^3(48k^5l + 320k^6l + 720k^7l + 672k^8l + 256k^9l + 96k^4l^2 + 960k^5l^2 + 2976k^6l^2 + 3552k^7l^2\\
&\quad + 1536k^8l^2 + 640k^4l^3 + 3792k^5l^3 + 6624k^6l^3 + 3584k^7l^3 + 1536k^4l^4 + 5280k^5l^4\\
&\quad + 4096k^6l^4 + 1536k^4l^5 + 2304k^5l^5 + 512k^4l^6 + 240k^4lm + 1920k^5lm + 5232k^6lm\\
&\quad + 5760k^7lm + 2304k^8lm + 384k^3l^2m + 4800k^4l^2m + 18240k^5l^2m + 26016k^6l^2m\\
&\quad + 12288k^7l^2m + 2560k^3l^3m + 19152k^4l^3m + 40896k^5l^3m + 25088k^6l^3m + 6144k^3l^4m\\
&\quad + 26784k^4l^4m + 24576k^5l^4m + 6144k^3l^5m + 11520k^4l^5m + 2048k^3l^6m + 480k^3lm^2\\
&\quad + 4800k^4lm^2 + 16080k^5lm^2 + 21120k^6lm^2 + 9216k^7lm^2 + 576k^2l^2m^2 + 9600k^3l^2m^2\\
&\quad + 46176k^4l^2m^2 + 80352k^5l^2m^2 + 43008k^6l^2m^2 + 3840k^2l^3m^2 + 38496k^3l^3m^2
\end{aligned}$$

$$
\begin{aligned}
&+ 103968k^4l^3m^2 + 75264k^5l^3m^2 + 9216k^2l^4m^2 + 53952k^3l^4m^2 + 61440k^4l^4m^2\\
&+ 9216k^2l^5m^2 + 23040k^3l^5m^2 + 3072k^2l^6m^2 + 480k^2lm^3 + 6400k^3lm^3 + 27120k^4lm^3\\
&+ 43392k^5lm^3 + 21504k^6lm^3 + 384kl^2m^3 + 9600k^2l^2m^3 + 61824k^3l^2m^3 + 135840k^4l^2m^3\\
&+ 86016k^5l^2m^3 + 2560kl^3m^3 + 38496k^2l^3m^3 + 139392k^3l^3m^3 + 125440k^4l^3m^3\\
&+ 6144kl^4m^3 + 53952k^2l^4m^3 + 81920k^3l^4m^3 + 6144kl^5m^3 + 23040k^2l^5m^3\\
&+ 2048kl^6m^3 + 240klm^4 + 4800k^2lm^4 + 27120k^3lm^4 + 54720k^4lm^4 + 256lm^9\\
&+ 32256k^5lm^4 + 96l^2m^4 + 4800kl^2m^4 + 46176k^2l^2m^4 + 135840k^3l^2m^4\\
&+ 107520k^4l^2m^4 + 640l^3m^4 + 19152kl^3m^4 + 103968k^2l^3m^4 + 125440k^3l^3m^4\\
&+ 1536l^4m^4 + 26784kl^4m^4 + 61440k^2l^4m^4 + 1536l^5m^4 + 11520kl^5m^4 + 512l^6m^4\\
&+ 48lm^5 + 1920klm^5 + 16080k^2lm^5 + 43392k^3lm^5 + 32256k^4lm^5 + 960l^2m^5\\
&+ 18240kl^2m^5 + 80352k^2l^2m^5 + 86016k^3l^2m^5 + 3792l^3m^5 + 40896kl^3m^5\\
&+ 75264k^2l^3m^5 + 5280l^4m^5 + 24576kl^4m^5 + 2304l^5m^5 + 320lm^6 + 5232klm^6\\
&+ 21120k^2lm^6 + 21504k^3lm^6 + 2976l^2m^6 + 26016kl^2m^6 + 43008k^2l^2m^6\\
&+ 6624l^3m^6 + 25088kl^3m^6 + 4096l^4m^6 + 720lm^7 + 5760klm^7 + 9216k^2lm^7\\
&+ 3552l^2m^7 + 12288kl^2m^7 + 3584l^3m^7 + 672lm^8 + 2304klm^8 + 1536l^2m^8).
\end{aligned}
$$

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References

[1] Bayen, F., Flato, M., Fronsdal, C., et al., Deformation theory and quantization I, Deformations of symplectic structures, *Ann. Physics*, **111**(1), 1978, 61–110.

[2] Bieliavsky, P., Tang, X. and Yao, Y. J., Rankin-Cohen brackets and quantization of foliation, Part I: formal quantization, *Advances in Mathematics* , **212**(1), 2007, 293–314.

[3] Bröcker, T. and tom Dieck, T., Representations of compact Lie groups, Translated from the German manuscript, Corrected reprint of the 1985 translation, Graduate Texts in Mathematics, Vol. 98, Springer-Verlag, New York, 1995.

[4] Cohen, H., Sums involving the values at negative integers of L-functions of quadratic characters, *Math. Ann.*, **217**, 1975, 271–285.

[5] Cohen, P. B., Manin, Y. and Zagier, D., Automorphic pseudodifferential operators, Algebraic aspects of integrable systems, Progr. Nonlinear Differential Equations Appl., Vol. 26, Birkhäuser Boston, Boston, MA, 1997, 17–47.

[6] Connes, A., Cyclic cohomology and the transverse fundamental class of a foliation, Geometric methods in operator algebras, *Pitman Res. Notes in Math.*, **123**, Longman, Harlow, 1986, 52–144.

[7] Connes, A., Noncommutative Geometry, Academic Press, Inc., San Diego, CA, 1994. ftp://ftp.alainconnes.org/book94bigpdf.pdf

[8] Connes, A. and Moscovici, H., Hopf algebras, cyclic cohomology and the transverse index theorem, *Commun. Math. Phys.*, **198**, 1998, 199–246.

[9] Connes, A. and Moscovici, H., Cyclic cohomology and Hopf algebra symmetry, *Letters Math. Phys.*, **52**, 2000, 1–28.

[10] Connes, A. and Moscovici, H., Differentiable cyclic cohomology and Hopf algebraic structures in transverse geometry, Essays on Geometry and Related Topics, Vol. 1–2, 2001, 217–255; Monogr. Enseign. Math., **38**, Enseignement Math., Geneva, 2001.

[11] Connes, A. and Moscovici, H., Modular Hecke algebras and their Hopf symmetry, *Mosc. Math. J.*, **4**(1), 2004, 67–109, 310.

[12] Connes, A. and Moscovici, H., Rankin-Cohen brackets and the Hopf algebra of transverse geometry, *Mosc. Math. J.*, **4**(1), 2004, 111–130.

[13] Deligne, P., Formes modulaires et représentations de GL(2) (French), Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972, 55–105), Lecture Notes in Math., Vol. 349, Springer-Verlag, Berlin, 1973.

[14] Fedosov, B., Deformation quantization and index theory, Mathematical Topics, **9**, Akademie-Verlag, Berlin, 1996.

[15] El Gradechi, A. M., The Lie theory of the Rankin-Cohen brackets and allied bi-differential operators, *Adv. Math.*, **207**(2), 2006, 484–531.

[16] Waldschmidt, M., Moussa, P., Luck, J. M. and Itzykson, C., From number theory to physics, Papers from the Meeting on Number Theory and Physics Held in Les Houches, March 7–16, 1989, Springer-Verlag, Berlin, 1992.

[17] Kirillov, A., Eléments de la théorie des représentations (French), Traduit du russe par A. Sossinsky, A. B. Sosinskiĭ (ed.), Editions Mir, Moscow, 1974.

[18] Knapp, A. W., Representation theory of semisimple groups, an overview based on examples, Reprint of the 1986 original, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 2001.

[19] Lang, S., $\mathrm{SL}_2(R)$, Addison-Wesley Publishing Co., Reading, Mass., London, Amsterdam, 1975.

[20] Labesse, J. P., Personal communication, 2005.

[21] Moyal, J. E., Quantum mechanics as a statistical theory, *Proc. Cambridge Philos. Soc.*, **45**, 1949, 99–124.

[22] Rankin, R. A., The construction of automorphic forms from the derivatives of a given form, *J. Indian Math. Soc.* (*N.S.*), **20**, 1956, 103–116.

[23] Repka, J., Tensor products of unitary representations of $\mathrm{SL}_2(R)$, *Amer. J. Math.*, **100**(4), 1978, 747–774.

[24] Rochberg, R., Tang, X. and Yao, Y. J., A survey on Rankin-Cohen deformations, Perspectives on Noncommutative Geometry, Fields Inst. Commun., **61**, A. M. S., Providence, RI, 2011, 133–151.

[25] Serre, J. P., A course in arithmetic, Translated from the French Graduate Texts in Mathematics, No. 7, Springer-Verlag, New York, Heidelberg, 1973.

[26] Schmid, W., Representations of semi-simple Lie groups, Representation theory of Lie groups, *London Math. Soc. Lect. Notes*, **34**, Cambridge Univ. Press., Cambridge, 1979.

[27] Sugiura, M., Unitary representations and harmonic analysis, Kodansha Ltd., Tokyo; Halstead Press (John Wiley Sons), New York, London, Sydney, 1975.

[28] Valette, A., K-Théorie pour Certaines C^*-algèbres Associées aux Groupes de Lie, Université Libre de Bruxelles, Bruxelles, 1983.

[29] Vogan, D. A., Jr., Representations of real reductive Lie groups, Progress in Mathematics, **15**, Birkhäuser, Boston, Mass., 1981.

[30] Weissman, M. H., Multiplying Modular Forms, *Modular forms on Schiermonnikoog*, Cambridge Univ. Press, Cambridge, 2008, 311–341.

[31] Zagier, D., Modular forms and differential operators, K. G. Ramanathan memorial issue, *Proc. Indian Acad. Sci. Math. Sci.*, **104**(1), 1994, 57–75.

[32] Zagier, D., Formes modulaires et opérateurs différentiels, Cours 2001-2002 au Collège de France.

[33] Zagier, D., Some combinatorial identities occuring in the theory of modular forms, in preparation.