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Existence of Generalized Heteroclinic Solutions of the Coupled Schrödinger System under a Small Perturbation*

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Abstract The following coupled Schrödinger system with a small perturbation

$$\begin{aligned} &u_{xx} + u - u^3 + \beta uv^2 + \epsilon f(\epsilon, u, u_x, v, v_x) = 0 \quad \text{in } \mathbb{R}, \\ &v_{xx} - v + v^3 + \beta u^2 v + \epsilon g(\epsilon, u, u_x, v, v_x) = 0 \quad \text{in } \mathbb{R} \end{aligned}$$

is considered, where β and ϵ are small parameters. The whole system has a periodic solution with the aid of a Fourier series expansion technique, and its dominant system has a heteroclinic solution. Then adjusting some appropriate constants and applying the fixed point theorem and the perturbation method yield that this heteroclinic solution deforms to a heteroclinic solution exponentially approaching the obtained periodic solution (called the generalized heteroclinic solution thereafter).

Keywords Coupled Schrödinger system, Heteroclinic solutions, Reversibility
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1 Introduction

The coupled nonlinear Schrödinger system was first derived by Benney and Newell [4] for two interacting nonlinear wave packets in a dispersive and conservative system, which can be written as

$$\begin{aligned} &\mathrm{i}\partial_t \phi + \Delta\phi + \mu_1 |\phi|^2 \phi + \beta |\psi|^2 \phi = 0, \\ &\mathrm{i}\partial_t \psi + \Delta\psi + \mu_2 |\psi|^2 \psi + \beta |\phi|^2 \psi = 0, \end{aligned} \tag{1.1}$$

where μ_j $(j = 1, 2)$ are constants and β is a coupling constant. In general, the sign of the parameter μ_j discriminates between the focusing and defocusing behavior of a single component, and the sign of β determines the type of interplay between the two states. The system (1.1) has applications in many physical problems such as semiconductor electronics (see [6]), optics in

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nonlinear media (see [22]), photonics (see [20]), plasmas (see [16]), fundamentation of quantum mechanics (see [33]), dynamics of accelerators (see [17]) or mean-field theory of Bose-Einstein condensates (see [14]). In some of these fields and many others, the system (1.1) appears as an asymptotic limit for a slowly varying dispersive wave envelope propagating in a nonlinear medium (see [34]). In recent years the system (1.1) has been broadly investigated in many aspects like concentration and the multi-bump phenomena for semiclassical states (see [1–2, 19, 28]), bounded solutions (see [26–27]), blow-up (see [10, 18]) and positive periodic solutions with variable coefficients and more general nonlinear terms (see [3]).

It is very important to stress that, in the particular case of standing wave solutions of (1.1), namely special solutions of (1.1) of the form

$$\phi(x,t) = \mathrm{e}^{\mathrm{i}\lambda_1 t}u(x), \quad \psi(x,t) = \mathrm{e}^{\mathrm{i}\lambda_2 t}v(x), \tag{1.2}$$

where u and v are real functions on $\mathbb{R}$, there is also an enormous literature regarding the corresponding system

$$\begin{aligned} &u_{xx} - \lambda_1 u + \mu_1 u^3 + \beta u v^2 = 0, \\ &v_{xx} - \lambda_2 v + \mu_2 v^3 + \beta u^2 v = 0. \end{aligned} \tag{1.3}$$

For instance, Yang [39] discussed the classification of the solitary waves. Pelinovsky and Yang [30] analytically and numerically studied internal modes of vector solitons. The stability of solitary waves can be found in [24, 31]. The existence of generalized homoclinic solutions (homoclinic solutions exponentially approaching the periodic solutions) under a small perturbation was proved by Deng and Guo [15] when $\lambda_1 = -\lambda_2 = \mu_1 = -\mu_2 = 1$.

In this paper, we take $\lambda_2 = \mu_2 = -\mu_1 = -\lambda_1 = 1$ and investigate the following system:

$$u_{xx} + u - u^3 + \beta u v^2 + \epsilon f(\epsilon, u, u_x, v, v_x) = 0, \tag{1.4}$$

$$v_{xx} - v + v^3 + \beta u^2 v + \epsilon g(\epsilon, u, u_x, v, v_x) = 0, \tag{1.5}$$

where β and ϵ are small parameters and the general nonlinear terms f and g satisfy the conditions given in (2.2) so that this system is reversible. For $\beta = \epsilon = 0$, this system has three saddle-center equilibriums $(u, u_x, v, v_x) = (0,0,0,0), (1,0,1,0)$ and $(-1,0,-1,0)$ (a positive eigenvalue, a negative eigenvalue and a pair of purely imaginary eigenvalues). It is easy to check that (1.4) has two heteroclinic solutions exponentially approaching $(1,0)$ and $(-1,0)$ while (1.5) has a family of periodic solutions around $(1,0)$ and $(-1,0)$, respectively. This implies that (1.4)–(1.5) may have a heteroclinic solution exponentially approaching a periodic solution at infinity (i.e., a generalized heteroclinic solution). In this paper, we will rigorously prove this. Our result is new.

There are a lot of results about the saddle-center problems if the system is conservative and Hamiltonian in particular. We mention the work: Homoclinic solutions (see [7–9, 23, 25, 29, 32, 36, 39]), generalized homoclinic solutions (see [5, 12, 32, 36–39]) and heteroclinic orbits to invariant tori (see [38]).

Our system might not be conservative. We will use a dynamic approach given in [15], which is more general and can be applied to a number of systems like the Schrödinger-KdV system since it does not require that the system have a Hamiltonian structure.

This paper is organized as follows. In Section 2, we derive the properties of heteroclinic solutions of (1.4)–(1.5) for $\beta = \epsilon = 0$. In Section 3, we use the Fourier series expansion technique to prove that the system of (1.4)–(1.5) has a periodic solution. In Section 4, we apply the fixed point theorem and the perturbation method to demonstrate that this heteroclinic solution deforms to a heteroclinic solution exponentially approaching the periodic solution obtained in Section 3 when small perturbation terms are added. This gives the existence of a generalized heteroclinic solution of (1.4)–(1.5). Section 5 is an appendix which solves an equation left in Section 4.

Throughout this paper, M denotes a positive constant and $B = O(C)$ means that $|B| \le M|C|$.

2 Preliminary

Let $u_1 = u_x$ and $v_1 = v_x$ which change (1.4)–(1.5) into

$$\begin{aligned} &u_x = u_1, \\ &u_{1x} = -u + u^3 - \beta uv^2 - \epsilon f(\epsilon, u, u_1, v, v_1), \\ &v_x = v_1, \\ &v_{1x} = v - v^3 - \beta u^2 v - \epsilon g(\epsilon, u, u_1, v, v_1). \end{aligned} \tag{2.1}$$

In this paper, we assume that f and g satisfy

$$\begin{aligned} &f(\epsilon, u, -u_1, v, -v_1) = f(\epsilon, u, u_1, v, v_1), \quad g(\epsilon, u, -u_1, v, -v_1) = g(\epsilon, u, u_1, v, v_1), \\ &f(\epsilon, -u, u_1, v, -v_1) = -f(\epsilon, u, u_1, v, v_1), \quad g(\epsilon, -u, u_1, v, -v_1) = g(\epsilon, u, u_1, v, v_1), \end{aligned} \tag{2.2}$$

and define two operators S_1 and S_2 by

$$S_1(u, u_1, v, v_1) = (u, -u_1, v, -v_1), \quad S_2(u, u_1, v, v_1) = (-u, u_1, v, -v_1), \tag{2.3}$$

respectively. From (2.2), the system (2.1) is reversible with the reverser S_i, that is, $S_iU(-x)$ is also a solution whenever $U(x) = (u(x), u_1(x), v(x), v_1(x))^{\mathrm{T}}$ is a solution for $i = 1, 2$. A solution $U(x)$ is reversible if $S_iU(-x) = U(x)$ for $i = 1, 2$. We will use the first reversibility to look for periodic solutions and the second one to construct the generalized heteroclinic solutions of the system (2.1), respectively.

When $\beta = \epsilon = 0$, the first two equations of the system (2.1) have three equilibriums $(-1, 0)$, $(0, 0)$ and $(1, 0)$. It is easy to check that $(-1, 0)$ and $(1, 0)$ are saddle points and $(0, 0)$ is a center. There exist two heteroclinic solutions

$$H_1(x) = \Big(\tanh\Big(\frac{x}{\sqrt{2}}\Big), \frac{1}{\sqrt{2}}\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\Big)^{\mathrm{T}} \tag{2.4}$$

and

$$H_2(x) = \Big(-\tanh\Big(\frac{x}{\sqrt{2}}\Big), -\frac{1}{\sqrt{2}}\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\Big)^{\mathrm{T}} \tag{2.5}$$

connecting two saddle points $(-1, 0)$ and $(1, 0)$. The last two equations of the system (2.1) also have three equilibriums $(-1, 0)$, $(0, 0)$ and $(1, 0)$. Clearly, $(-1, 0)$ and $(1, 0)$ are centers and $(0,0)$ is a saddle point. In the following we will prove that the heteroclinic solution (2.4) will deform to a generalized heteroclinic solution. By the same method, the deformation of the other heteroclinic solution (2.5) can be obtained.

Let

$$\begin{aligned} &u = \widetilde{u} + 1, \quad u_1 = \widetilde{u}_1, \quad v = \widetilde{v} + 1, \quad v_1 = \widetilde{v}_1, \\ &\widetilde{f}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1) = f(\epsilon, \widetilde{u} + 1, \widetilde{u}_1, \widetilde{v} + 1, \widetilde{v}_1), \\ &\widetilde{g}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1) = g(\epsilon, \widetilde{u} + 1, \widetilde{u}_1, \widetilde{v} + 1, \widetilde{v}_1), \end{aligned} \tag{2.6}$$

and we have from (2.2)

$$\widetilde{f}(\epsilon, \widetilde{u}, -\widetilde{u}_1, \widetilde{v}, -\widetilde{v}_1) = \widetilde{f}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1), \quad \widetilde{g}(\widetilde{\epsilon}, \widetilde{u}, -\widetilde{u}_1, \widetilde{v}, -\widetilde{v}_1) = \widetilde{g}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1). \tag{2.7}$$

Note that (2.6) changes the system (2.1) into

$$\begin{aligned} &\widetilde{u}_x = \widetilde{u}_1, \\ &\widetilde{u}_{1x} = 2\widetilde{u} + 3\widetilde{u}^2 + \widetilde{u}^3 - \beta(\widetilde{u} + 1)(\widetilde{v} + 1)^2 - \epsilon \widetilde{f}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1), \\ &\widetilde{v}_x = \widetilde{v}_1, \\ &\widetilde{v}_{1x} = -2\widetilde{v} - 3\widetilde{v}^2 - \widetilde{v}^3 - \beta(\widetilde{u} + 1)^2(\widetilde{v} + 1) - \epsilon \widetilde{g}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1). \end{aligned} \tag{2.8}$$

Symbolically, it can be written as

$$\frac{\mathrm{d}\widetilde{U}}{\mathrm{d}x} = L\widetilde{U} + N(\widetilde{U}) + \widetilde{N}(\beta, \widetilde{U}) + \epsilon R(\epsilon, \widetilde{U}), \tag{2.9}$$

where $\widetilde{U} = (\widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1)^{\mathrm{T}}$,

$$\begin{aligned} &L = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \quad N(\widetilde{U}) = \begin{pmatrix} 0 \\ 3\widetilde{u}^2 + \widetilde{u}^3 \\ 0 \\ -3\widetilde{v}^2 - \widetilde{v}^3 \end{pmatrix}, \\ &\widetilde{N}(\beta, \widetilde{U}) = \begin{pmatrix} 0 \\ -\beta(\widetilde{u} + 1)(\widetilde{v} + 1)^2 \\ 0 \\ -\beta(\widetilde{u} + 1)^2(\widetilde{v} + 1) \end{pmatrix}, \quad R(\epsilon, \widetilde{U}) = \begin{pmatrix} 0 \\ -\widetilde{f}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1) \\ 0 \\ -\widetilde{g}(\epsilon, \widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1) \end{pmatrix}. \end{aligned} \tag{2.10}$$

Note that from (2.7) the system (2.9) is still reversible with the reverser S_1 if

$$S_1(\widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1) = (\widetilde{u}, -\widetilde{u}_1, \widetilde{v}, -\widetilde{v}_1), \tag{2.11}$$

where we avoid the introduction of a new notation. We write the dominant system of (2.9) as

$$\frac{\mathrm{d}\widetilde{U}}{\mathrm{d}x} = L\widetilde{U} + N(\widetilde{U}), \tag{2.12}$$

which has a heteroclinic solution $H(x)$ given by

$$H(x) = \Big(\tanh\Big(\frac{x}{\sqrt{2}}\Big) - 1, \frac{1}{\sqrt{2}} \mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big), 0, 0 \Big)^{\mathrm{T}} \tag{2.13}$$

approaching $(0,0,0,0)^{\mathrm{T}}$ as $x \to \infty$ and $(-2,0,0,0)^{\mathrm{T}}$ as $x \to -\infty$. Moreover,

$$H(0) = \Big(-1, \frac{1}{\sqrt{2}}, 0, 0\Big)^{\mathrm{T}} \tag{2.14}$$

and $H(x)$ satisfies the following inequality:

$$|H(x)| \le M\mathrm{e}^{-\sqrt{2}x} \quad \text{for } x \in [0, +\infty). \tag{2.15}$$

In Section 4, we will prove the deformation of this heteroclinic solution $H(x)$ for the whole system (2.9). This demonstrates that the original system (2.1) has a generalized heteroclinic solution.

3 Periodic Solutions

Using the Fourier series expansion technique, we will show that (2.9) has periodic solutions which determine the forms of the generalized heteroclinic solutions at infinity. The general theory for reversible systems can be found in [21].

Let

$$C = \sqrt{2}\widetilde{v} - \mathrm{i}\widetilde{v}_1, \quad \tau = \sqrt{2}(1+r_1)x, \tag{3.1}$$

where r_1 is a small real constant to be determined later. Using the fact

$$\widetilde{v} = \frac{C+\overline{C}}{2\sqrt{2}}, \quad \widetilde{v}_1 = \mathrm{i}\frac{C-\overline{C}}{2}, \tag{3.2}$$

we can write (2.9) as

$$\begin{aligned}
\widetilde{u}_\tau &= \frac{1}{\sqrt{2}(1+r_1)}\widetilde{u}_1, \\
\widetilde{u}_{1\tau} &= \frac{\sqrt{2}}{1+r_1}\widetilde{u} + h_1(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C}), \\
C_\tau &= \frac{\mathrm{i}}{1+r_1}C + h_2(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C}), \\
\overline{C}_\tau &= \frac{-\mathrm{i}}{1+r_1}\overline{C} - h_2(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C}),
\end{aligned} \tag{3.3}$$

where h_1 is a real function, h_2 is a purely imaginary function and

$$\begin{aligned}
h_1(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C}) &= \frac{1}{\sqrt{2}(1+r_1)}\Big(3\widetilde{u}^2 + \widetilde{u}^3 - \beta(\widetilde{u}+1)\Big(\frac{C+\overline{C}}{2\sqrt{2}} + 1\Big)^2 \\
&\quad - \epsilon\widetilde{f}\Big(\epsilon, \widetilde{u}, \widetilde{u}_1, \frac{C+\overline{C}}{2\sqrt{2}}, \mathrm{i}\frac{C-\overline{C}}{2}\Big)\Big), \\
h_2(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C}) &= \frac{-\mathrm{i}}{\sqrt{2}(1+r_1)}\Big(-3\Big(\frac{C+\overline{C}}{2\sqrt{2}}\Big)^2 - \Big(\frac{C+\overline{C}}{2\sqrt{2}}\Big)^3 \\
&\quad - \beta(\widetilde{u}+1)^2\Big(\frac{C+\overline{C}}{2\sqrt{2}} + 1\Big) - \epsilon\widetilde{g}\Big(\epsilon, \widetilde{u}, \widetilde{u}_1, \frac{C+\overline{C}}{2\sqrt{2}}, \mathrm{i}\frac{C-\overline{C}}{2}\Big)\Big).
\end{aligned} \tag{3.4}$$

From (2.7) and (2.11), we may define

$$S_1(\widetilde{u}, \widetilde{u}_1, C, \overline{C}) = (\widetilde{u}, -\widetilde{u}_1, \overline{C}, C) \tag{3.5}$$

such that the system (3.3) is reversible, where we avoid again the introduction of a new notation. Assume

$$(\widetilde{u}(\tau), \widetilde{u}_1(\tau), C(\tau), \overline{C}(\tau)) = \Big(\sum_n \widetilde{u}_n \mathrm{e}^{\mathrm{i}n\tau}, \sum_n \widetilde{u}_{1,n}\mathrm{e}^{\mathrm{i}n\tau}, \sum_n C_n \mathrm{e}^{\mathrm{i}n\tau}, \sum_n \overline{C}_n \mathrm{e}^{-\mathrm{i}n\tau}\Big). \tag{3.6}$$

Plugging (3.6) into (3.3) and making the coefficient of each term in the Fourier series equal yield

$$\begin{aligned}
\widetilde{u}_n &= \frac{-(1+r_1)}{\sqrt{2}(1+r_1)^2n^2+\sqrt{2}}[h_1(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n,\\
\widetilde{u}_{1,n} &= \frac{-\mathrm{i}(1+r_1)^2 n}{(1+r_1)^2n^2+1}[h_1(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n,\\
C_n &= \frac{-\mathrm{i}(1+r_1)}{n(1+r_1)-1}[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n \quad \text{for } n \neq 1,\\
\overline{C}_n &= \frac{-\mathrm{i}(1+r_1)}{n(1+r_1)-1}[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_{-n} \quad \text{for } n \neq 1,
\end{aligned} \tag{3.7}$$

and for $n = 1$,

$$r_1 C_1 = -\mathrm{i}(1+r_1)[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_1, \tag{3.8}$$

$$r_1 \overline{C}_1 = -\mathrm{i}(1+r_1)[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_{-1}, \tag{3.9}$$

where $[f]_k$ denotes the kth Fourier coefficient of f.

Now we activate C_1, that is, we consider C_1 as a free constant to be chosen later. We first solve (3.7) for $\widetilde{u}_n, \widetilde{u}_{1,n}$, C_n and $\overline{C}_n$ $(n \neq 1)$, and then solve (3.8) for r_1.

Fix C_1 and define two spaces

$$\begin{aligned}
H_1^1(0,2\pi) &= \Big\{ f(\tau) = \sum_n f_n \mathrm{e}^{\mathrm{i}n\tau} \in H^1(0,2\pi) \mid f_1 = 0 \Big\},\\
H_{-1}^1(0,2\pi) &= \Big\{ f(\tau) = \sum_n f_n \mathrm{e}^{\mathrm{i}n\tau} \in H^1(0,2\pi) \mid f_{-1} = 0 \Big\}.
\end{aligned}$$

For $A, B \in H^1(0,2\pi) \times H^1(0,2\pi)$ and $D \in H_1^1(0,2\pi)$, use (3.7) and we define a mapping $\Theta(A, B, D, \overline{D}; \varpi)$ from $H^1(0,2\pi) \times H^1(0,2\pi) \times H_1^1(0,2\pi) \times H_{-1}^1(0,2\pi)$ to itself by

$$\Theta(A,B,D,\overline{D};\varpi) = \begin{pmatrix}
\displaystyle\sum_n \frac{-(1+r_1)}{\sqrt{2}(1+r_1)^2n^2+\sqrt{2}}[h_1(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n \mathrm{e}^{\mathrm{i}n\tau}\\
\displaystyle\sum_n \frac{-\mathrm{i}(1+r_1)^2 n}{(1+r_1)^2n^2+1}[h_1(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n \mathrm{e}^{\mathrm{i}n\tau}\\
\displaystyle\sum_{n\neq 1} \frac{-\mathrm{i}(1+r_1)}{n(1+r_1)-1}[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_n \mathrm{e}^{\mathrm{i}n\tau}\\
\displaystyle\sum_{n\neq 1} \frac{-\mathrm{i}(1+r_1)}{n(1+r_1)-1}[h_2(\beta,\epsilon,\widetilde{u},\widetilde{u}_1,C,\overline{C})]_{-n} \mathrm{e}^{-\mathrm{i}n\tau}
\end{pmatrix}, \tag{3.10}$$

where $\varpi = (\beta, \epsilon, r_1, C_1, \overline{C}_1)$. Assume that $B_r(0)$ is a ball with a radius r in the space $H^1(0, 2\pi) \times H^1(0, 2\pi) \times H^1_1(0, 2\pi) \times H^1_{-1}(0, 2\pi)$. It is easy to check the following lemma.

Lemma 3.1 *For $(A, B, D, \overline{D}), (A_1, B_1, D_1, \overline{D}_1), (A_2, B_2, D_2, \overline{D}_2) \in \overline{B}_r(0)$ and any small bounded ϖ and r, Θ is smooth in its arguments and satisfies*

$$\begin{aligned}
&\|\Theta(A, B, D, \overline{D}; \varpi)\|_1 \le M(|\beta| + |\epsilon| + \|A\|_1^2 + \|B\|_1^2 + \|D\|_1^2 + |C_1|^2),\\
&\|\Theta(A_1, B_1, D_1, \overline{D}_1; \varpi) - \Theta(A_2, B_2, D_2, \overline{D}_2; \varpi)\|_1\\
\le\ & M(|\beta| + |\epsilon| + |C_1| + \|A_1\|_1 + \|A_2\|_1 + \|B_1\|_1 + \|B_2\|_1 + \|D_1\|_1 + \|D_2\|_1)\\
&\cdot (\|A_1 - A_2\|_1 + \|B_1 - B_2\|_1 + \|D_1 - D_2\|_1).
\end{aligned}$$

Take $r = |C_1|$ and

$$\beta = \beta_1 |C_1|^{\alpha_1}, \quad \epsilon = \epsilon_1 |C_1|^{\alpha_2}, \quad \alpha_1 > 1, \quad \alpha_2 > 1, \tag{3.11}$$

where β_1, ϵ_1, α_1 and α_2 are fixed constants. Lemma 3.1 yields that Θ is a contraction mapping on $\overline{B}_r(0)$ for small C_1. Thus, Θ has a unique fixed point which is a smooth function of ϖ. Write this fixed point as

$$(u_p^0, u_{1p}^0, C_p^0, \overline{C}_p^0)(\beta, \epsilon, r_1, C_1, \overline{C}_1)(\tau), \tag{3.12}$$

which satisfies

$$\|u_p^0\|_1 + \|u_{1p}^0\|_1 + \|C_p^0\|_1 + \|\overline{C}_p^0\|_1 \le M(|\beta| + |\epsilon| + |C_1|^2). \tag{3.13}$$

Using the same argument we can show that (3.12) is in $H^m(0, 2\pi)$ and satisfies (3.13) with $H^m(0, 2\pi)$-norm for any integer $m > 0$. We use $(\widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p)(\tau)$ to denote

$$(u_p^0(\tau), u_{1p}^0(\tau), C_p^0(\tau) + C_1 \mathrm{e}^{\mathrm{i}\tau}, \overline{C}_p^0(\tau) + \overline{C}_1 \mathrm{e}^{-\mathrm{i}\tau}).$$

Now we solve (3.8) for r_1. Substitute (3.12) into (3.8) and obtain

$$-r_1 C_1 + g_1(\beta, \epsilon, r_1, C_1, \overline{C}_1) = 0, \tag{3.14}$$

where

$$g_1(\beta, \epsilon, r_1, C_1, \overline{C}_1) = -\mathrm{i}(1 + r_1)[h_1(\beta, \epsilon, \widetilde{u}, \widetilde{u}_1, C, \overline{C})]_1$$

is smooth when $\beta, \epsilon, r_1, C_1, \overline{C}_1$ are near 0.

If $(\widetilde{u}, \widetilde{u}_1, C, \overline{C})(\tau)$ is a solution of (3.3), then

$$S_1(\widetilde{u}, \widetilde{u}_1, C, \overline{C})(-\tau), \quad (\widetilde{u}, \widetilde{u}_1, C, \overline{C})(\tau + \theta)$$

are also solutions of (3.3) for any real number θ since (3.3) has the reversibility property by (3.5) and the translation invariance. Using these, we may take

$$C_1 = I > 0, \tag{3.15}$$

so that (3.11) becomes

$$\beta = \beta_1 I^{\alpha_1}, \quad \epsilon = \epsilon_1 I^{\alpha_2}, \quad \alpha_1 > 1,\ \alpha_2 > 1, \tag{3.16}$$

and (3.14) is equivalent to the following equation:

$$r_1 = \widetilde{g}_1(\beta, \epsilon, r_1, I),$$

where $\widetilde{g}_1$ is real and smooth in its arguments and is a contraction mapping satisfying $|\widetilde{g}_1| \leq M(|\beta| + |\epsilon| + I)$ under the condition (3.16) (more details can be found in [15]). By the fixed point theorem, $\widetilde{g}_1$ has a unique fixed point

$$r_1 = r_1(\beta, \epsilon, I) \tag{3.17}$$

as a smooth real function for small (β, ϵ, I) satisfying

$$|r_1| \leq M(|\beta| + |\epsilon| + I). \tag{3.18}$$

Therefore, (3.3) has a periodic solution

$$(\widetilde{u}_p(\beta, \epsilon, I)(\tau), \widetilde{u}_{1p}(\beta, \epsilon, I)(\tau), C_p(\beta, \epsilon, I)(\tau), \overline{C}_p(\beta, \epsilon, I)(\tau))$$

in $H^m(0, 2\pi)$ if $I \in (0, I_1]$ and (3.16) holds, where I_1 is a fixed small positive constant.

By the relation

$$\tau = \sqrt{2}(1 + r_1)x,$$

we write the periodic solution $(\widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p)(\tau)$ as

$$(\widetilde{u}_p(\beta, \epsilon, I)(x), \widetilde{u}_{1p}(\beta, \epsilon, I)(x), C_p(\beta, \epsilon, I)(x), \overline{C}_p(\beta, \epsilon, I))(x)$$

with the frequency

$$\omega_1(\beta, \epsilon, I) = \sqrt{2}(1 + r_1(\beta, \epsilon, I)) \tag{3.19}$$

for $I \in (0, I_1]$. Moreover, this solution is reversible since $C_1 = I$ is real, i.e.,

$$\begin{aligned} & S_1(\widetilde{u}_p(\beta, \epsilon, I), \widetilde{u}_{1p}(\beta, \epsilon, I), C_p(\beta, \epsilon, I), \overline{C}_p(\beta, \epsilon, I))(-x) \\ &= (\widetilde{u}_p(\beta, \epsilon, I), -\widetilde{u}_{1p}(\beta, \epsilon, I), \overline{C}_p(\beta, \epsilon, I), C_p(\beta, \epsilon, I))(-x) \\ &= (\widetilde{u}_p(\beta, \epsilon, I), \widetilde{u}_{1p}(\beta, \epsilon, I), C_p(\beta, \epsilon, I), \overline{C}_p(\beta, \epsilon, I))(x). \end{aligned}$$

Letting

$$C_p = \sqrt{2}\widetilde{v}_p - \mathrm{i}\widetilde{v}_{1p},$$

we have

$$\widetilde{v}_p(-x) = \widetilde{v}_p(x), \quad \widetilde{v}_{1p}(-x) = -\widetilde{v}_{1p}(x). \tag{3.20}$$

Define

$$X_{\beta,\epsilon,I}(x) = (\widetilde{u}_p, \widetilde{u}_{1p}, \widetilde{v}_p, \widetilde{v}_{1p})^{\mathrm{T}}(x) = \left(\widetilde{u}_p, \widetilde{u}_{1p}, \frac{C_p + \overline{C}_p}{2\sqrt{2}}, i\frac{C_p - \overline{C}_p}{2}\right)^{\mathrm{T}}(x), \tag{3.21}$$

which is smooth for x and small (β, ϵ, I) with the condition (3.16). Then, $X_{\beta,\epsilon,I}(x)$ is a reversible periodic solution of (2.9) under the reversor S_1 with frequency $\omega_1(\beta, \epsilon, I)$, which from (3.13) satisfies that for any integer $m > 0$,

$$\|X_{\beta,\epsilon,I}(x)\|_{H^m(0,2\pi)} \le M(|\beta| + |\epsilon| + I). \tag{3.22}$$

The Sobolev embedding theorem gives that (3.22) holds also in $C_B^m(\mathbb{R})$-norm, which is a space of continuously differentiable functions up to order m with a supreme norm.

4 Generalized Heteroclinic Solutions

In this section we demonstrate that (2.9) has a generalized heteroclinic solution exponentially approaching the periodic solution $X_{\beta,\epsilon,I}$ obtained in Section 3.

Theorem 4.1 *Suppose that the assumption* (2.2) *holds. There exist constants* $I_0 > 0$*,* β_1 *and* ϵ_1 *such that for* $I \in (0, I_0]$*, if the small parameters* $\beta = \beta_1 I^{\frac{3}{2}}$ *and* $\epsilon = \epsilon_1 I^{\frac{3}{2}}$*, then* (2.1) *has a generalized heteroclinic solution, i.e.,* (2.1) *has a solution which is reversible and exponentially approaches the periodic solution* $(1, 0, 1, 0)^{\mathrm{T}} + X_{\beta,\epsilon,I}(x + \theta)$ *as* $x \to \infty$ *and the periodic solution* $(-1, 0, 1, 0)^{\mathrm{T}} + S_2(X_{\beta,\epsilon,I}(-x + \theta))$ *as* $x \to -\infty$*, where the phase shift* θ *is a continuous function in* I*, and the operator* S_2 *is defined in* (2.3).

We divide the proof into two steps. Using the relationship between (2.1) and (2.9), we will first prove that (2.9) has a solution for $x \in [0, \infty)$, which exponentially approaches the periodic solution $X_{\beta,\epsilon,I}(x + \theta)$ for some phase shift θ as $x \to \infty$. Then we solve (2.9) for θ, and its solution is a function of β, ϵ and I. This yields that this solution can be extended to $x \in (-\infty, 0]$ by using the reversibility.

Step 1 Solution of (2.9) for $x \in [0, \infty)$.

Assume that the solution $\mathcal{U}(x)$ of (2.9) has the following form:

$$\mathcal{U}(x) = H(x) + Z(x) + \zeta(x)X_{\beta,\epsilon,I}(x + \theta), \tag{4.1}$$

where $H(x)$ and $X_{\beta,\epsilon,I}(x)$ are defined in (2.13) and (3.21) respectively, the phase shift $\theta \in S^1 = [0, 2\pi]$ is a constant, the cut-off function $\zeta(x)$ is in $C^\infty(\mathbb{R}, \mathbb{R})$ satisfying $0 \le \zeta(x) \le 1$ and

$$\zeta(x) = \begin{cases} 1, & |x| \ge 2, \\ 0, & |x| \le 1, \end{cases} \tag{4.2}$$

and $Z(x)$ is a perturbation term to be determined, which exponentially tends to 0 as $x \to \infty$ so that $\mathcal{U}(x)$ is a solution of (2.9) that approaches the periodic solution $X_{\beta,\epsilon,I}(x + \theta)$ as $x \to \infty$.

Since $H(x)$ is a solution of (2.12) and $X_{\beta,\epsilon,I}(x)$ is a solution of (2.9), plugging (4.1) into (2.9) yields

$$\frac{\mathrm{d}Z}{\mathrm{d}x} = \mathcal{L}(x)Z + \mathcal{N}(x, Z) + \epsilon\mathcal{R}(x, \epsilon, Z), \tag{4.3}$$

where

$$
\begin{aligned}
\mathcal{L}(x) &= L + \mathrm{d}N[H(x)] = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 3\tanh^2\left(\frac{x}{\sqrt{2}}\right) - 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -2 & 0 \end{pmatrix}, \\
\mathcal{N}(x, Z) &= N(H(x) + Z(x) + \zeta(x)X_{\beta,\epsilon,I}(x+\theta)) - N(H(x)) \\
&\quad - \zeta(x)N(X_{\beta,\epsilon,I}(x+\theta)) - dN[H(x)]Z(x) + \widetilde{N}(\beta, H(x) \\
&\quad + Z(x) + \zeta(x)X_{\beta,\epsilon,I}(x+\theta)) - \zeta(x)\widetilde{N}(\beta, X_{\beta,\epsilon,I}(x+\theta)), \\
\mathcal{R}(x, \epsilon, Z) &= R(\epsilon, H(x) + Z(x) + \zeta(x)X_{\beta,\epsilon,I}(x+\theta)) \\
&\quad - \zeta(x)R(\epsilon, X_{\beta,\epsilon,I}(x+\theta)) - \frac{1}{\epsilon}\zeta'(x)X_{\beta,\epsilon,I}(x+\theta),
\end{aligned}
\tag{4.4}
$$

and d means taking the Fréchet derivative.

Now we first consider $x \in [0, \infty)$ and have the following lemma by using (2.15) and (3.22).

Lemma 4.1 *If β, ϵ and I are small and $|Z| + |Z_1| + |Z_2| \le M_0$ for some positive constant M_0, then $\mathcal{N}$ and $\mathcal{R}$ satisfy that for $x \ge 0$*

$$
\begin{aligned}
&|\mathcal{N}(x, Z)| \le M[(\mathrm{e}^{-\sqrt{2}x} + |Z|)(|\beta| + |\epsilon| + I) + |Z|^2], \\
&|\mathcal{N}(x, Z_1) - \mathcal{N}(x, Z_2)| \le M(|\beta| + |\epsilon| + I + |Z_1| + |Z_2|)|Z_1 - Z_2|, \\
&|\mathcal{R}(x, \epsilon, Z)| \le M\Big(\mathrm{e}^{-\sqrt{2}x} + |Z| + \frac{|\beta| + |\epsilon| + I}{|\epsilon|}\mathrm{e}^{-\sqrt{2}x}\Big), \\
&|\mathcal{R}(x, \epsilon, Z_1) - \mathcal{R}(x, \epsilon, Z_2)| \le M|Z_1 - Z_2|.
\end{aligned}
\tag{4.5}
$$

Note that

$$
\frac{\mathrm{d}Z(x)}{\mathrm{d}x} = \mathcal{L}(x)Z(x) \tag{4.6}
$$

has four linearly independent solutions:

$$
\begin{aligned}
s_1(x) &= \frac{1}{\sqrt{2}}\Big(\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big), -\sqrt{2}\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\tanh\Big(\frac{x}{\sqrt{2}}\Big), 0, 0\Big)^{\mathrm{T}}, \\
s_2(x) &= \frac{1}{16}\Big(\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)(6\sqrt{2}x + 8\sinh(\sqrt{2}x) + \sinh(2\sqrt{2}x)) \\
&\qquad \cdot 4\Big(\sqrt{2}\cosh(\sqrt{2}x) + 3\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\Big(\sqrt{2} - x\tanh\Big(\frac{x}{\sqrt{2}}\Big)\Big)\Big), 0, 0\Big)^{\mathrm{T}}, \\
s_3(x) &= (0, 0, \cos(\sqrt{2}x), -\sqrt{2}\sin(\sqrt{2}x))^{\mathrm{T}}, \\
s_4(x) &= (0, 0, \sin(\sqrt{2}x), \sqrt{2}\cos(\sqrt{2}x))^{\mathrm{T}},
\end{aligned}
\tag{4.7}
$$

which satisfy

$$
|s_1(x)| \le M\mathrm{e}^{-\sqrt{2}x}, \quad |s_2(x)| \le M\mathrm{e}^{\sqrt{2}x}, \quad |s_3(x)| + |s_4(x)| \le M \tag{4.8}
$$

for $x \in [0, \infty)$. Moreover,

$$\begin{aligned} &s_1(0) = \Big(\frac{1}{\sqrt{2}}, 0, 0, 0\Big)^{\mathrm{T}}, \quad s_2(0) = (0, \sqrt{2}, 0, 0)^{\mathrm{T}}, \\ &s_3(0) = (0, 0, 1, 0)^{\mathrm{T}}, \quad s_4(0) = (0, 0, 0, \sqrt{2})^{\mathrm{T}}. \end{aligned} \tag{4.9}$$

The adjoint equation of (4.6) has four linearly independent solutions given by

$$\begin{aligned} s_1^*(x) &= \frac{1}{16}\Big(4\Big(\sqrt{2}\cosh(\sqrt{2}x) + 3\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\Big(\sqrt{2} - x\tanh\Big(\frac{x}{\sqrt{2}}\Big)\Big)\Big) \\ &\qquad - \mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)(6\sqrt{2}x + 8\sinh(\sqrt{2}x) + \sinh(2\sqrt{2}x)), 0, 0\Big)^{\mathrm{T}}, \\ s_2^*(x) &= \frac{1}{\sqrt{2}}\Big(\sqrt{2}\mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big)\tanh\Big(\frac{x}{\sqrt{2}}\Big), \mathrm{sech}^2\Big(\frac{x}{\sqrt{2}}\Big), 0, 0\Big)^{\mathrm{T}}, \\ s_3^*(x) &= \frac{1}{\sqrt{2}}(0, 0, \sqrt{2}\cos(\sqrt{2}x), -\sin(\sqrt{2}x))^{\mathrm{T}}, \\ s_4^*(x) &= \frac{1}{\sqrt{2}}(0, 0, \sqrt{2}\sin(\sqrt{2}x), \cos(\sqrt{2}x))^{\mathrm{T}}, \end{aligned} \tag{4.10}$$

which satisfy

$$|s_1^*(x)| \le M\mathrm{e}^{\sqrt{2}x}, \quad |s_2^*(x)| \le M\mathrm{e}^{-\sqrt{2}x}, \quad |s_3^*(x)| + |s_4^*(x)| \le M \tag{4.11}$$

for $x \in [0, \infty)$ and

$$\langle s_i(x), s_j^*(x)\rangle = 0 \quad \text{for } i \ne j, \quad \langle s_i(x), s_i^*(x)\rangle = 1, \quad i, j = 1, 2, 3, 4 \tag{4.12}$$

for each $x \in [0, \infty)$, where $\langle\,\cdot\,,\,\cdot\,\rangle$ denotes the Euclidean inner product on $\mathbb{R}^4$.

The solution of (4.3) that decays to zero at infinity can be found as

$$\begin{aligned} Z = \mathcal{F}(Z) &\triangleq \int_0^x \langle \mathcal{N}(t, Z) + \epsilon\mathcal{R}(t, \epsilon, Z), s_1^*(t)\rangle \mathrm{d}t\ s_1(x) \\ &\qquad - \sum_{j=2}^4 \int_x^\infty \langle \mathcal{N}(t, Z) + \epsilon\mathcal{R}(t, \epsilon, Z), s_j^*(t)\rangle \mathrm{d}t\ s_j(x). \end{aligned} \tag{4.13}$$

Fix $\nu \in (0, \sqrt{2})$ and consider (4.13) as a fixed point problem in a Banach space

$$E_\nu = \Big\{Z \in C([0, \infty) \times S^1) \ \Big|\ \sup_{x\in[0,\infty)}\{|Z(x, \theta)|\mathrm{e}^{\nu x}\} < \infty\Big\}$$

with the norm

$$\|Z\|_\nu = \sup\{|Z(x, \theta)|\ \mathrm{e}^{\nu x} \mid x \in [0, \infty), \theta \in S^1\}.$$

It is easy to check the following lemma by using (2.15), (3.22), (4.8), (4.11) and Lemma 4.1.

Lemma 4.2 *The function $\mathcal{F}$ satisfies*

$$\begin{aligned} &\|\mathcal{F}(Z)\|_\nu \le M[(1 + \|Z\|_\nu)(|\beta| + |\epsilon| + I) + \|Z\|_\nu^2], \\ &\|\mathcal{F}(Z_1) - \mathcal{F}(Z_2)\|_\nu \le M(|\beta| + |\epsilon| + I + \|Z_1\|_\nu + \|Z_2\|_\nu)\|Z_1 - Z_2\|_\nu \end{aligned}$$

for $Z, Z_1, Z_2 \in E_v$.

For any fixed constant

$$\gamma \in (0,1), \tag{4.14}$$

we let $r = MI^{\gamma}$ and

$$\beta = \beta_1 I^{\alpha_1}, \quad \epsilon = \epsilon_1 I^{\alpha_2}, \quad \alpha_i = \gamma + \widetilde{\alpha}_i > 1, \tag{4.15}$$

where $\widetilde{\alpha}_i$ are positive constants for $i = 1, 2$. Thus, (3.16) is satisfied. We can show from Lemma 4.2 that $\mathcal{F}$ is a contraction on $\overline{B}_r(0) \subset E_\nu$ for small I. Therefore, (4.13) has a unique solution $Z(x;\theta,\beta,\epsilon,I)$ satisfying

$$|Z(x;\theta,\beta,\epsilon,I)| \le MI^{\gamma}, \quad x \in [0,\infty). \tag{4.16}$$

Using the same argument as for (4.16) and an extension of a contraction mapping principle (see [37]), we can show that Z is smooth in its arguments. Obviously, the solution Z of (4.3) exists if x is in a finite interval and an initial condition is given. Thus, we have showed that $\mathcal{U}(x;\theta,\beta,\epsilon,I)$ defined in (4.1) exists for $x \ge \widetilde{x}_0$ with any fixed $\widetilde{x}_0 \in (-\infty,\infty)$.

Step 2 Solution of (2.9) for $x \in (-\infty,\infty)$.

Using (2.3), (4.1) and the relationship between u and $\widetilde{u}$ in (2.6), we may define

$$\widetilde{\mathcal{U}}(x) = \begin{cases} \mathcal{U}(x;\theta,\beta,\epsilon,I) + U_1 & \text{for } x \ge 0, \\ S_2 U_1 + S_2(\mathcal{U}(-x;\theta,\beta,\epsilon,I)) & \text{for } x \le 0, \end{cases} \tag{4.17}$$

where $U_1 = (1,0,1,0)^{\mathrm{T}}$. If the following equation

$$(I - S_2)(\mathcal{U}(0;\theta,\beta,\epsilon,I) + U_1) = 0 \tag{4.18}$$

is true (The basic idea is to solve this equation for θ, which is given in Section 5), the uniqueness of the solution for an initial value problem implies that $\widetilde{\mathcal{U}}$ is a generalized heteroclinic solution of (2.1) and $S_2\widetilde{\mathcal{U}}(-x) = \widetilde{\mathcal{U}}(x)$, which exponentially approaches the periodic solution

$$U_1 + X_{\beta,\epsilon,I}(x+\theta) = (1,0,1,0)^{\mathrm{T}} + X_{\beta,\epsilon,I}(x+\theta)$$

as $x \to \infty$ and the periodic solution

$$S_2(U_1 + X_{\beta,\epsilon,I}(-x+\theta)) = (-1,0,1,0)^{\mathrm{T}} + S_2(X_{\beta,\epsilon,I}(-x+\theta))$$

as $x \to -\infty$. This completes the proof of Theorem 4.1.

5 Appendix

In this section, we will solve (4.18) for θ. It is easy to check that by (2.14), the definition of $\zeta(x)$ in (4.2) and $Z = (\widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1)^{\mathrm{T}}$, (4.18) is equivalent to

$$\widetilde{u}(0) = 0, \tag{5.1}$$

$$\widetilde{v}_1(0) = 0. \tag{5.2}$$

Using (4.9) and (4.13), we know that (5.1) holds automatically. Thus, we only have to study (5.2) which can be transformed into

$$\int_0^\infty \langle \mathcal{N}(t, Z) + \epsilon\mathcal{R}(t, \epsilon, Z), s_4^*(t)\rangle \mathrm{d}t = 0. \tag{5.3}$$

Lemma 5.1 *Under the assumption in Theorem* 4.1, (5.3) *can be transformed into*

$$\theta = I^{\frac{1}{2}}\Theta(\theta, \beta, \epsilon, I), \tag{5.4}$$

where Θ *is differentiable with respect to its arguments, and* Θ *and its derivative with respect to* θ *are uniformly bounded for small bounded* β, ϵ *and* I.

Using the fixed point theorem, we can solve (5.4) for θ, and its solution is a smooth function of β, ϵ and I, so the equation (5.3) is true.

Proof Let

$$C_p = \sqrt{2}\widetilde{v}_p - \mathrm{i}\widetilde{v}_{1p}, \quad \tau = \sqrt{2}(1 + r_1)x, \tag{5.5}$$

where r_1, $\widetilde{v}_p$ and $\widetilde{v}_{1p}$ are given in (3.17) and (3.21) respectively, which yields

$$\widetilde{v}_p = \frac{C_p + \overline{C}_p}{2\sqrt{2}}, \quad \widetilde{v}_{1p} = \mathrm{i}\frac{C_p - \overline{C}_p}{2}. \tag{5.6}$$

Thus, $(\widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p)^{\mathrm{T}}(\tau)$ is a 2π-periodic solution of the following system:

$$\begin{aligned}
\widetilde{u}_{p\tau} &= \frac{1}{\sqrt{2}(1 + r_1)}\widetilde{u}_{1p}, \\
\widetilde{u}_{1p\tau} &= \frac{\sqrt{2}}{1 + r_1}\widetilde{u}_p + h_1(\beta, \epsilon, \widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p), \\
C_{p\tau} &= \frac{\mathrm{i}}{1 + r_1}C_p + h_2(\beta, \epsilon, \widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p), \\
\overline{C}_{p\tau} &= \frac{-\mathrm{i}}{1 + r_1}\overline{C}_p - h_2(\beta, \epsilon, \widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p),
\end{aligned} \tag{5.7}$$

where h_1 and h_2 are given in (3.4). We can express $C_p(\tau)$ as

$$C_p(\tau) = \mathrm{e}^{\mathrm{i}\frac{\tau}{1+r_1}}C_p(0) + w(\tau), \tag{5.8}$$

where

$$w(\tau) = \int_0^\tau \mathrm{e}^{\mathrm{i}\frac{\tau - s}{1+r_1}}h_2(\beta, \epsilon, \widetilde{u}_p, \widetilde{u}_{1p}, C_p, \overline{C}_p)\mathrm{d}s. \tag{5.9}$$

Note that the coefficient of $\mathrm{e}^{\mathrm{i}\tau}$ in $C_p(\tau)$ is $C_1 = I$ (see (3.15)). Thus,

$$\begin{aligned}
I &= \frac{1}{2\pi}\int_0^{2\pi} C_p(s)\mathrm{e}^{-\mathrm{i}s}\mathrm{d}s \\
&= \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s}(\mathrm{e}^{\mathrm{i}\frac{s}{1+r_1}}C_p(0) + w(s))\mathrm{d}s \\
&= (1 + \kappa(r_1))C_p(0) + \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s}w(s)\mathrm{d}s,
\end{aligned} \tag{5.10}$$

where $\kappa(r_1) = \frac{1+r_1}{\mathrm{i}2\pi r_1}(1 - \mathrm{e}^{-\mathrm{i}\frac{2\pi r_1}{1+r_1}}) - 1 = O(r_1)$ and $\kappa(0) = 0$, which yields

$$C_p(0) = \frac{1}{1+\kappa(r_1)}\Big(I - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s} w(s)\mathrm{d}s\Big). \tag{5.11}$$

Thus,

$$C_p(\tau) = \frac{\mathrm{e}^{\mathrm{i}\frac{\tau}{1+r_1}}}{1+\kappa(r_1)}\Big(I - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s} w(s)\mathrm{d}s\Big) + w(\tau) \tag{5.12}$$

or

$$C_p(x) = \frac{\mathrm{e}^{\mathrm{i}\sqrt{2}x}}{1+\kappa(r_1)}\Big(I - \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{i}s} w(s)\mathrm{d}s\Big) + w(\sqrt{2}(1+r_1)x). \tag{5.13}$$

(3.18), (3.22), (4.15) and the expression of h_2 in (3.4) show $w(x) = O(|\beta| + |\epsilon| + I^2)$ so that $C_p(x) = O(I)$. Therefore, we obtain by (5.6) that

$$\widetilde{v}_p(x) = \frac{1}{\sqrt{2}}\mathrm{Re}\, C_p(x) = \frac{1}{\sqrt{2}}\cos(\sqrt{2}x)I + V_1(x,\beta,\epsilon,I), \tag{5.14}$$

$$\widetilde{v}_{1p}(x) = -\mathrm{Im}\, C_p(x) = -\sin(\sqrt{2}x)I + V_2(x,\beta,\epsilon,I), \tag{5.15}$$

where $V_1(x) = O(|\beta| + |\epsilon| + I^2)$ and $V_2(x) = O(|\beta| + |\epsilon| + I^2)$.

From (2.13), (3.21), (4.4) and $Z = (\widetilde{u}, \widetilde{u}_1, \widetilde{v}, \widetilde{v}_1)^{\mathrm{T}}$, we know that the equation (5.3) becomes

$$\begin{aligned}
0 = \int_0^\infty & \frac{1}{\sqrt{2}}\Big(-3(\widetilde{v}(s) + \zeta(s)\widetilde{v}_p(s+\theta))^2 - (\widetilde{v}(s) + \zeta(s)\widetilde{v}_p(s+\theta))^3 \\
& + \zeta(s)(3\widetilde{v}_p^2(s+\theta) + \widetilde{v}_p^3(s+\theta)) \\
& - \beta\Big(\tanh\Big(\frac{s}{\sqrt{2}}\Big) + \widetilde{u}(s) + \zeta(s)\widetilde{u}_p(s+\theta)\Big)^2(\widetilde{v}(s) + \zeta(s)\widetilde{v}_p(s+\theta) + 1) \\
& + \beta\zeta(s)(\widetilde{u}_p(s+\theta)+1)^2(\widetilde{v}_p(s+\theta)+1)\Big)\cos(\sqrt{2}x) \\
& - \zeta'(s)\widetilde{v}_p(s+\theta)\sin(\sqrt{2}s) - \frac{1}{\sqrt{2}}\zeta'(s)\widetilde{v}_{1p}(s+\theta)\cos(\sqrt{2}s) \\
& - \frac{1}{\sqrt{2}}\epsilon\Big(\widetilde{g}\Big(\epsilon, \tanh\Big(\frac{s}{\sqrt{2}}\Big) - 1 + \widetilde{u}(s) + \zeta(s)\widetilde{u}_p(s+\theta), \\
& \frac{1}{2}\mathrm{sech}^2\Big(\frac{s}{\sqrt{2}}\Big) + \widetilde{u}_1(s) + \zeta(s)\widetilde{u}_{1p}(s+\theta), \\
& \widetilde{v}(s) + \zeta(s)\widetilde{v}_p(s+\theta), \widetilde{v}_1(s) + \zeta(s)\widetilde{v}_{1p}(s+\theta)\Big) \\
& - \zeta(s)\widetilde{g}(\epsilon, \widetilde{u}_p(s+\theta), \widetilde{u}_{1p}(s+\theta), \widetilde{v}_p(s+\theta), \widetilde{v}_{1p}(s+\theta))\Big)\cos(\sqrt{2}s)\mathrm{d}s.
\end{aligned} \tag{5.16}$$

For computational simplicity, we take $\gamma = \widetilde{\alpha}_k = \frac{3}{4}$ for $k = 1, 2$ such that (4.14)–(4.15) are satisfied. Thus, (5.16) is changed into

$$\begin{aligned}
0 &= I\int_0^\infty -\frac{1}{\sqrt{2}}\zeta'(s)\Big(\cos(\sqrt{2}(s+\theta))\sin(\sqrt{2}s) - \sin(\sqrt{2}(s+\theta))\cos(\sqrt{2}s)\Big)\mathrm{d}s \\
&\quad + \mathcal{P}(\theta,\beta,\epsilon,I) \\
&= \frac{1}{\sqrt{2}}\sin(\sqrt{2}\theta)I + \mathcal{P}(\theta,\beta,\epsilon,I),
\end{aligned} \tag{5.17}$$

where $\mathcal{P}(\theta, \beta, \epsilon, I) = O(I^{\frac{3}{2}})$, which is equivalent to

$$\theta = I^{\frac{1}{2}} \Psi(\theta, \beta, \epsilon, I), \tag{5.18}$$

where $\Psi(\theta, \beta, \epsilon, I) = -\frac{1}{\sqrt{2I}} \arcsin\left(\frac{\sqrt{2}\mathcal{P}(\theta,\beta,\epsilon,I)}{I}\right)$ is uniformly bounded for small β, ϵ and I. We can also check that $\Psi(\theta, \beta, \epsilon, I)$ is differentiable with respect to its arguments, and Ψ and its derivative with respect to θ are uniformly bounded for small bounded β, ϵ and I. This completes the proof of Lemma 5.1.

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