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Symplectic Group Actions on Homotopy Elliptic Surfaces*

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Abstract In this paper, the authors study the homologically trivial symplectic group actions on homotopy elliptic surfaces $E(n)$ and get some rigidity results.

Keywords Symplectic action, Homologically trivial, g-Signature theorem, Fixed-point data
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1 Introduction

Let X be a compact closed oriented simply connected topological 4-manifold, and G be a finite group. When studying actions on manifold of a finite group, one can consider an induced action on some algebraic invariants associated with the manifold, and it is often important and beneficial. Furthermore, a central problem is to describe the structure of the fixed point set (fixed point data, early noted in [7]) and the action around it. Various investigations have been conducted under different conditions and cases such as the certain manifold carries a geometric structure, or the group may be fundamental like $G \equiv \mathbb{Z}_p$ for some prime p or other simple cases. One may also see the recent surveys of [3, 6].

Moreover, a specific question may be concerned with the homological rigidity that whether an action is trivial if the induced action is trivial on homology. In the case of holomorphic actions, $K3$ surfaces provide a classical example that every homologically trivial automorphism of a $K3$ surface is trivial (see [2]). There are no locally linear, homologically trivial involutions on a homotopy $K3$ surface while a locally linear $\mathbb{Z}_p$-action is automatically homologically trivial on a homotopy $K3$ surface for $p > 23$ by the decomposition of the integral $\mathbb{Z}_p$-representation on $H^2(K3)$ (see [15, 19]). Peters extended homological rigidity of holomorphic actions to the elliptic surfaces in [18]. McCooey [16] established homological rigidity for locally linear topological actions for non-Abelian groups. Edmonds [5] showed that for any prime $p > 3$, there exists a locally linear, pseudofree and homologically trivial topological $\mathbb{Z}_p$-action.

For symplectic actions, some recent results have been obtained as well. Chen and Kwasik [4] proved that there are no homologically trivial, symplectic actions of a finite group on the standard $K3$ surface (with respect to any symplectic structure), partially extending Peters' results in [18] to the symplectic category.

Let $X = E(n)$ be the relatively minimal elliptic surface with rational base, where the elliptic surface $E(n)$ is defined as the n-fold fiber sum of copies of $E(1)$, and $E(1)$ is $\mathbb{C}P^2 \sharp 9\overline{\mathbb{C}P^2}$ being

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equipped with an elliptic fibration. Note that $E(n)$ is spin if and only if n is even. The elliptic surface $E(n)$ is known to have many exotic smooth structures, where the $E(n)$ with an exotic smooth structure is also called a homotopy elliptic surface, which is homeomorphic, but not diffeomorphic, to the standard elliptic surface. It is well-known that there are two methods to produce exotic smooth structures on $E(n)$: Logarithmic transformations (see [9]) and Fintushel-Stern's knot surgery construction (see [8]).

In [13–14], Nakamura and the second author studied the locally linear actions on elliptic surfaces $E(n)$ and proved that there exists a locally linear cyclic group action on $E(n)$ which is nonsmoothable with respect to infinitely many smooth structures on $E(n)$. In this paper, some symplectic $\mathbb{Z}_p$-actions of prime orders on homotopy elliptic surfaces $E(n)$ are studied. In the meantime, we investigate the symplectic actions on the homotopy $E(4)$ surfaces, and get some rigidity result. Our main results are stated as following.

Theorem 1.1 *Let $G = \mathbb{Z}_p$, and $X = E(n)$ be a homotpy elliptic surface with $c_1^2 = 0$ and n is even.*

(1) *For $p = 3$, there is no homologically trivial, pseudofree, symplectic G-action on X.*

(2) *For $p = 5$, if X admits a symplectic, nontrivial, homologically trivial, pseudofree action of G. Then the fixed point set of the G-action consists of $4n$ fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_5^k z_1, \mu_5^{2k} z_2)$ and $8n$ fixed points, each with local representation $(z_1, z_2) \mapsto (\mu_5^{-k} z_1, \mu_5^{4k} z_2)$ for some $k \neq 0 \bmod 5$.*

Theorem 1.2 *Let X be the symplectic 4-manifold $(E(4), \omega)$ with $c_1^2 = 0$ and $c_1(K)\cdot[\omega] < 16$, where ω defines an integral class $[\omega] \in H^2(X; \mathbb{R})$. Then there are no nontrivial homologically trivial actions of a finite group on X which preserve the symplectic structure ω.*

The current paper is organized as follows: In Section 2, we provide some preliminaries and tools. In Section 3, we initiate basic knowledge of Seiberg-Witten theory from Taubes, and restate the results of Chen and Kwasik about the structures of the union of finitely many J-holomorphic curves $\underset{i}{\cup} C_i$ and the fixed point set F under the symplectic actions. Then we give the proof of the main results of this paper in Section 4. At last, some calculation results and the mean contributions to the g-signature theorem are introduced in Section 5 as an appendix.

2 Preliminaries and Tools

In this section, we collect some results about Lefschetz fixed point theorem and g-signature theorem which we will use as the main tools (see [1, 10–11]).

Let X be a closed, oriented smooth 4-manifold, and let cyclic group $G \equiv \mathbb{Z}_p$ of prime order act on X effectively via orientation-preserving diffeomorphisms. Then the fixed point set $F = X^G$, if nonempty, will consists of isolated points and surfaces. If a generator g of G is fixed, each fixed point $m \in F$ is associated with a nonzero integers pair (a_m, b_m), where $-p < a_m, b_m < p$, and they are uniquely determined up to a change of order or a change of sign simultaneously, such that the induced g-action on the tangent space at m is given by the complex linear transformation $(z_1, z_2) \mapsto (\mu_p^{a_m} z_1, \mu_p^{b_m} z_2)$, where $\mu_p = \exp\left(\frac{2\pi \mathrm{i}}{p}\right)$. For each connected surface $Y \subset F$, the action of g on the normal bundle of Y in X is given by $z \mapsto \mu_p^{c_Y} z$ for an integer c_Y with $0 < c_Y < p$, which is uniquely determined up to a sign modulo p. The following is the Lefschetz fixed point theorem.

Theorem 2.1 (Lefschetz Fixed Point Theorem) *Let $T : X \to X$ generate an action of $\mathbb{Z}_p$*

on X, a closed, simply-connected 4-manifold. Then $L(T,X)=\chi(F)$, where $\chi(F)$ is the Euler characteristic of the fixed point set F and $L(T,X)$ is the Lefschetz number of the map T, which is defined by

$$L(T,X)=\sum_{k=0}^{4}(-1)^k\mathrm{tr}(g)|_{H^k(X;\mathbb{R})}.$$

Theorem 2.2 (G-signature Theorem) *Setting*

$$\sigma(g,X)=\mathrm{tr}(g)|_{H^{2,+}(X;\mathbb{R})}-\mathrm{tr}(g)|_{H^{2,-}(X;\mathbb{R})},$$

then

$$\sigma(g,X)=\sum_{m\in F}-\cot\Big(\frac{a_m\pi}{p}\Big)\cdot\cot\Big(\frac{b_m\pi}{p}\Big)+\sum_{Y\subset F}\csc^2\Big(\frac{c_Y\pi}{p}\Big)\cdot(Y\cdot Y),$$

where $Y\cdot Y$ denotes the self-intersection number of Y.

The weaker version of the G-signature theorem is used more often since the convenient for calculation.

Theorem 2.3 (G-signature Theorem-the Weaker Version)

$$|G|\cdot\sigma(X/G)=\sigma(X)+\sum_{m\in F}\mathrm{def}_m+\sum_{Y\subset F}\mathrm{def}_Y,$$

where the terms def_m and def_Y are called signature defects. They are given by the following formulae:

$$\mathrm{def}_m=\sum_{k=1}^{p-1}\frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})},$$

if the local representation of G at m is given by $(z_1,z_2)\mapsto(\mu_p^k z_1,\mu_p^{kq}z_2)$, and

$$\mathrm{def}_Y=\frac{p^2-1}{3}\cdot(Y\cdot Y).$$

3 Symplectic $\mathbb{Z}_p$-Actions on 4-Manifolds with $c_1^2=0$

In this section, we restate the results by Chen and Kwasik [4] in a clear way to lay a foundation of the proof to main results of this paper. We shall briefly initiate with the G-equivariant Seigberg-Witten-Taubes theory, for more details one may see [20–21].

Let X be a 4-manifold equipped a symplectic structure ω with a $G\equiv\mathbb{Z}_p$ action via symplectomorphisms on it. If X is minimal with $c_1^2=0$ and $b_2^+\geq 2$, and the induced action on $H^2(M;\mathbb{Q})$ is trivial, then, according to Taubes' results in [20–21], for any G-equivariant ω-compatible almost structure J, there exists a solution (A,ψ) to the G-equivariant Seiberg-Witten equations

$$D_A\psi=0,\quad P_+F_A=\frac{1}{4}\tau(\psi\otimes\psi^*)+\mu, \tag{3.1}$$

with perturbation term $\mu=-\frac{\mathrm{i}}{4}r\omega+P_+F_{A_0}$ for any $r>0$. Here A is a G-equivariant $U(1)$-connection and $\psi\in\Gamma(S_+)$ is a G-equivariant smooth section of S_+. S_+ and S_- are associated

$U(2)$ vector G-bundles. $D_A : \Gamma(S_+) \to \Gamma(S_-)$ is the Dirac operator. P_+ is the orthogonal projection.

Denoting $\psi = \sqrt{r}(\alpha, \beta) \in \Gamma(K \oplus \mathbb{C})$, the zero locus $\alpha^{-1}(0)$, if nonempty, will pointwise converge to a union of finitely many J-holomorphic curves with multiplicity representing the Poincare dual of $c_1(K)$ as $r \to +\infty$. We denote the union of the J-holomorphic curves as $\underset{i}{\cup} C_i$ for finitely many i, and categorize the components $\{\Lambda_\alpha\}$ of $\underset{i}{\cup} C_i$ as the following three types.

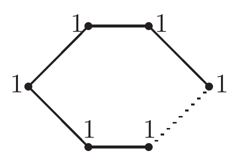
(A) $\{\Lambda_\alpha\}$ is either an embedded torus, or a nodal sphere, or a cusp sphere, all with self-intersection 0.

(B) $\{\Lambda_\alpha\}$ is a union of two embedded (-2)-spheres intersecting at a single point with tangency of order 2.

(C) $\{\Lambda_\alpha\}$ is a union of embedded (-2)-spheres intersecting transversely.

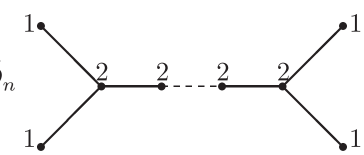
If each sphere $C_i \subset \Lambda_\alpha$ corresponds to a vertex v_i and each intersection point of $C_i \cap C_j$ corresponds to an edge connecting v_i and v_j, then the union of intersecting spheres can be represented by a graph. Moreover, we can associate a positive semi-definite matrix $Q_\alpha = (q_{ij})$, where $q_{ij} = 1$ for $i = j$, $q_{ij} = -\frac{1}{2}$ for $i \neq j$ and $C_i \cdot C_j \neq 0$ and $q_{ij} = 0$ for $i \neq j$ and $C_i \cdot C_j = 0$, noting that we assume $c_1^2 = 0$ in the main theorem implies that the annihilator exists for Lemma 2.10 in [2]. By Lemma 2.12 (ii) of [2], the graph must be one of the figures (with weights) listed as follows.

$\widetilde{A}_n$



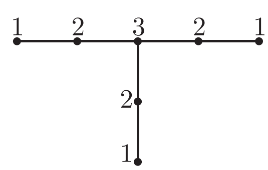
$(n+1)$ vertices, $n \geq 1$

$\widetilde{D}_n$

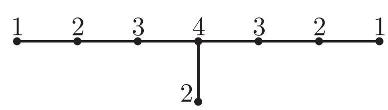


$(n+1)$ vertices, $n \geq 4$

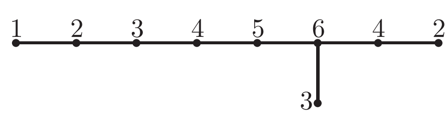
$\widetilde{E}_6$



$\widetilde{E}_7$



$\widetilde{E}_8$



The cases about the structures of $\underset{i}{\cup}C_i$ and the fixed point set F are as follows:

(1) Isolated fixed points not contained in $\underset{i}{\cup}C_i$, with local representations

$$(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$$

for some $k \neq 0 \bmod p$, i.e., with local representations contained in $SL_2(\mathbb{C})$.

(2) The connected components Λ_α of $\underset{i}{\cup}C_i$ may include:

(I) A torus with self-intersection 0, if not entirely fixed by G, contributing four isolated fixed points all with same local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \bmod p$ when $p = 2$, or three isolated fixed points all with same local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ or $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ for some $k \neq 0 \bmod p$ when $p = 3$, to X^G.

(II) A cusp sphere with self-intersection 0, contributing two isolated fixed points to X^G only when $p \geq 5$, one of which is the cusp singularity with local representation $(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2)$ for some $k \neq 0 \bmod p$, while the other with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$ for some $k \neq 0 \bmod p$.

(III) A nodal sphere with self-intersection 0, contributing one fixed point as the double point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \bmod p$.

(IV) Two (-2)-spheres intersecting at a point with order 2, contributing the intersection point to X^G with local representation $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{-2k} z_2)$ for some $k \neq 0 \bmod p$ if $p = 3$, or $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ for some $k \neq 0 \bmod p$ if $p > 3$.

(V) A union of embedded (-2)-spheres intersecting transversely with each other.

(i) Λ_α contains a 2-dimensional component of F, with $n = 4 \bmod p$ if Λ_α is represented by a type $\widetilde{D}_n$ graph and $n = -1 \bmod p$ if Λ_α is represented by a type $\widetilde{A}_n$ graph.

(ii) Λ_α is of type $\widetilde{A}_n$ and the intersection of each pair of spheres is an isolated fixed point, with rotation numbers $(1, p-1)$ associated to either sphere.

(iii) Λ_α is of type $\widetilde{A}_2$, where the three spheres intersect at a single point; there are four isolated fixed points, one occurs at the intersection point and each of the other three is contained in each one of the three spheres, with the rotation numbers associated to each sphere being $(1, 1)$ at the intersection point and $(1, |p-3|)$ at each of the other three fixed points. This case occurs only if $p \neq 3$.

(iv) Λ_α is of type $\widetilde{A}_1$ which contains four isolated fixed points. The rotation number at each fixed point is $(1, 1)$, and this case occurs only if $p = 2$.

The structure of the fixed point set for a symplectic cyclic action of prime order on a minimal symplectic 4-manifold X with $c_1^2 = 0$ and $b_2^+ \geq 2$, which induces a trival action on $H^2(X; \mathbb{Q})$, was described by Chen and Kwasik in [4]. We invoke the result for the case of pseudofree actions below.

Theorem 3.1 (see [4]) *Let X be a minimal symplectic 4-manifold with $c_1^2 = 0$ and $b_2^+ \geq 2$, which admits a nontrival, pseudofree action of $G = \mathbb{Z}_p$, where p is a prime, such that the symplectic structure is preserved under the action and the induced action on $H^2(X; \mathbb{Q})$ is trivial. Then the set of fixed points of G can be divided into groups, each of which belongs to the following five possible types*:

(1) *One fixed point with local representation $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{-k} z_2)$ for some $k \neq 0 \bmod p$, i.e., with representation contained in $SL_2(\mathbb{C})$.*

(2) *Two fixed points with local representation*

$$(z_1, z_2) \mapsto (\mu_p^{2k} z_1, \mu_p^{3k} z_2), \quad (z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{6k} z_2)$$

for some $k \neq 0 \bmod p$*, respectively. This type of fixed points occurs only when* $p > 5$.

(3) *Three fixed points, one with local representation* $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{2k} z_2)$ *and the other two with local representation* $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{4k} z_2)$ *for some* $k \neq 0 \bmod p$*. This type of fixed points occurs only when* $p > 3$.

(4) *Four fixed points, one with local representation* $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ *and the other three with local representation* $(z_1, z_2) \mapsto (\mu_p^{-k} z_1, \mu_p^{3k} z_2)$ *for some* $k \neq 0 \bmod p$*. This type of fixed points occurs only when* $p > 3$.

(5) *Three fixed points, each with local representation* $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^k z_2)$ *for some* $k \neq 0 \bmod p$*. This type of fixed points occurs only when* $p = 3$.

The rigidity for the corresponding homologically trivial actions is as follows, and it shows that symplectic symmetries are more restrictive than topological ones.

Theorem 3.2 (see [4]) *Let* X *be a minimal symplectic* 4*-manifold with* $c_1^2 = 0$ *and* $b_2^+ \geq 2$*, which admits a homologically trivial* (*over* $\mathbb{Q}$ *coefficients*)*, pseudofree, symplectic* $\mathbb{Z}_p$*-action for a prime* p*. Then the following conclusions hold:*

(a) *The action is trivial if* $p \neq 1 \bmod 4$*,* $p \neq 1 \bmod 6$*, and the signature of* X *is nonzero, then for infinitely many primes* p *the manifold* X *does not admit any such nontrivial* $\mathbb{Z}_p$*-actions.*

(b) *The action is trivial as long as there exists a fixed point of type* (1) *in Theorem* 3.1.

4 Proofs of the Main Results

Proof of Theorem 1.1 (1) First consider the $\mathbb{Z}_3$-case. The fixed points of a pseudofree $\mathbb{Z}_3$-action on X can be divided into two types by considering their local representation:

The type (+): (1,2) or (2,1).

The type (−): (1,1) or (2,2).

Let k_+, k_- be the numbers of the fixed points of the type(+), type(−) in the fixed point set separately. Let m and n be one fixed point of type (1,1) and (1,2), respectively. Then $\mathrm{def}_m = -\frac{1}{3}$, $\mathrm{def}_n = \frac{1}{3}$. The formula in Theorem 2.3 is rewritten as

$$2 \cdot \sigma(X) = \frac{2}{3}(k_+ - k_-).$$

Together with the Lefschetz fixed point theorem, we have the following inequality:

$$3|\sigma(X)| \leq \chi(F) = \chi(X) = 2 + b_2(X),$$

or

$$|\sigma(X)| \leq \frac{(b_2(X) + 2)}{3}.$$

Obviously, the elliptic surfaces do not satisfy that.

(2) By the assumption that the G-action on $X = E(n)$ is pseudofree and with the G-signature theorem, we have

$$|G| \cdot \sigma(X/G) = \sigma(X) + \sum_{m \in F} \mathrm{def}_m,$$

where $G = \mathbb{Z}_5$, and $F = X^G$ denotes the fixed point set.

The induced action on $H^2(X; \mathbb{Q})$ is trivial. Then $\sigma(X/G) = \sigma(X)$ and the Lefschetz fixed point theorem leads to $\chi(F) = \chi(X)$. The fixed point set may consist of type (1), type (3) and type (4) fixed points in Theorem 3.1, actually there are no type (1) fixed points by the assumption that the action is nontrivial and Theorem 3.2.

Let a_3, a_4 be the numbers of groups of fixed points of type (3) and type (4), respectively. Then we have

$$\begin{cases} -32n = -8a_3 - 4a_4, \\ 12n = 3a_3 + 4a_4. \end{cases} \tag{4.1}$$

Here we use the fact that $\mathrm{def}_{(3)} = -8$ and $\mathrm{def}_{(4)} = -4$. The solutions for a_3, a_4 are $a_3 = 4n$ and $a_4 = 0$. Then the fixed point set consists of $4n$ groups of type (3) fixed points in Theorem 3.1, and then the theorem follows.

Proof of Theorem 1.2 Suppose that there exists such an action, and without loss of generality, we may assume the action is periodic with prime order p. We complete the proof in two parts respectively: (i) When the action is pseudofree, and (ii) when the fixed point set F contains 2-dimensional components. The tools we use are g-signature theorem and Lefschetz fixed point formula.

For part (i), by Theorem 3.2, we may assume that there are no fixed points with local representations contained in $SL_2(\mathbb{C})$. Then we exclude the cases when $p = 2$, since each type of fixed point datum is $(k, -k)$.

When $p = 3$, we assume all the fixed points with local representations of type (k, k). With the contribution

$$I_{3,1} = -\frac{1}{3}(3-1)(3-2) = -\frac{2}{3},$$

the signature of X is $\mathrm{sign}(X) = -32$ and $|F| = \chi(X) = 48$, a calculation of the weak version of the g-signature theorem

$$(p-1)\mathrm{sign}(X) = \sum_m \mathrm{def}_m$$

follows that $-64 = -32$ which is a contradiction.

When $p > 3$, the fixed points of $(k, -k)$ type are ruled out as well. From Section 3, we denote δ_2, δ_3 and δ_4 to be the numbers of Λ_α as a cusp-sphere component, a union of two (-2)-spheres, and a union of three spheres, respectively. The action is homologically trivial. Then $|F| = \chi(X)$, and a contradiction is reached by

$$48 = \chi = |F| = 2\delta_2 + 3\delta_3 + 4\delta_4 \le 2(\delta_2 + 2\delta_3 + 3\delta_4) \le 2c_1(K)\cdot[\omega] < 32.$$

For part (ii), the fixed point set F contains a 2-dimensional component, and the g-signature theorem is equivalent to the following equation:

$$-32(p-1) = \sum_m \mathrm{def}_m + \sum_Y \mathrm{def}_Y, \tag{4.2}$$

where $\mathrm{def}_m = I_{p,q}$ determined by corresponding type, and $\mathrm{def}_Y = \frac{p^2-1}{3}(Y \cdot Y)$. Note that a toroidal fixed component makes no contribution to the signature defect. We may only consider the cases that Λ_α contains only embedded (-2)-spheres components when calculating the defects def_Y of the g-signature formula.

If $p = 2$, then each point with local representation contained in $SL_2(\mathbb{C})$ makes no contributes, and every Y contributes

$$\mathrm{def}_Y = \frac{p^2-1}{3}\cdot(Y \cdot Y) = -2$$

to the g-signature theorem. Then the equation (4.2) follows that $-32 = -2\delta_Y$, where we denote δ_Y as the number of the fixed (-2)-spheres Y. Thus we reach a contradiction since there exist at most 5 fixed (-2)-spheres under the restriction $c_1(K)\cdot[\omega] < 16$.

If $p = 3$, then

$$\mathrm{def}_Y = \frac{p^2-1}{3}\cdot(Y\cdot Y) = -\frac{16}{3}$$

for each Y, and there are two types of fixed points with local representations $(z_1, z_2) \mapsto (\mu_P^k z_1, \mu_p^k z_2)$ and $(z_1, z_2) \mapsto (\mu_P^k z_1, \mu_p^{-k} z_2)$, contributing $-\frac{2}{3}$ and $\frac{2}{3}$, respectively. The restriction $c_1(K)\cdot[\omega] < 16$ follows that the number of fixed (-2)-spheres is at most 5. Note that $\delta_m + 2\delta_Y = \chi(X) = 48$. We may reach a contradiction with a calculation to the g-signature theorem

$$-64 = -32(p-1) \geq \delta_m\cdot\Big(-\frac{2}{3}\Big) + \delta_Y\cdot\Big(-\frac{16}{3}\Big) \geq 48\cdot\Big(-\frac{2}{3}\Big) + 5\cdot\Big(-\frac{16}{3}\Big) = -\frac{176}{3}.$$

If $p = 5$, then

$$\mathrm{def}_Y = \frac{p^2-1}{3}\cdot(Y\cdot Y) = -16$$

and

$$I_{5,-1} = 4, \quad I_{5,1} = -4, \quad I_{5,2} = I_{5,3} = 0.$$

We denote δ_{Γ_α} to be the number of Γ_α components, where Γ_α stands for the corresponding type of components as $\widetilde{A}_4$, $\widetilde{A}_9$, $\widetilde{A}_{14}$, $\widetilde{D}_4$ and $\widetilde{E}_6$ components. We also denote δ_4 as the number of $\widetilde{A}_2$ components that three spheres intersecting at a single point, δ_3 as the number of components with type(B), t as the number of the fixed tori, and δ_1 as the number of isolated fixed points with representations contained in $SL_2(\mathbb{C})$. Then the restriction and the Lefschetz fixed point theorem follow that

$$\begin{aligned} 5\delta_{\widetilde{A}_4} + 10\delta_{\widetilde{A}_9} + 15\delta_{\widetilde{A}_{14}} + 6\delta_{\widetilde{D}_4} + 12\delta_{\widetilde{E}_6} + 3\delta_4 + 2\delta_3 + t &\leq c_1(K)\cdot[\omega] < 16,\\ 5\delta_{\widetilde{A}_4} + 10\delta_{\widetilde{A}_9} + 15\delta_{\widetilde{A}_{14}} + 6\delta_{\widetilde{D}_4} + 8\delta_{\widetilde{E}_6} + 4\delta_4 + 3\delta_3 + 2t + \delta_1 &= \chi(X) = 48. \end{aligned}$$

Some results may follow from the two formulae above that $0 \leq \delta_Y \leq 3$,

$$\begin{aligned} \delta_1 &= 48 - \sum(n+1)\delta_{\widetilde{A}_n} - 6\delta_{\widetilde{D}_4} - 8\delta_{\widetilde{E}_6} - 2t - 4\delta_4 - 3\delta_3 - 2\delta_2\\ &\geq 48 - 2\Big(\sum(n+1)\delta_{\widetilde{A}_n} + 6\delta_{\widetilde{D}_4} + 12\delta_{\widetilde{E}_6} + t + 3\delta_4 + 2\delta_3 + \delta_2\Big)\\ &\geq 48 - 2\cdot 15 = 18, \end{aligned}$$

and the number of other isolated points in $\underset{i}{\cup} C_i$ satisfying

$$\delta_{m'} = \chi(M) - \delta_1 - 2\delta_Y \leq 48 - 18 - 2\cdot 0 = 30.$$

Note that $\mathrm{def}_{m'} \geq \min\{I_{5,q} \mid q = 1,2,3\} = -4$. We may easily obtain the inequality

$$\begin{aligned} -128 = (p-1)\cdot\mathrm{sign}(X) &= \sum_m \mathrm{def}_m + \sum_Y \mathrm{def}_Y\\ &= \delta_1\cdot I_{5,-1} + \sum_{m'}\mathrm{def}_{m'} + \delta_Y\cdot\mathrm{def}_Y\\ &\geq 18\cdot 4 + 30\cdot(-4) + 3\cdot(-16) = -96, \end{aligned}$$

which is a contradiction.

If $p \geq 7$, there are no types more than $\widetilde{A}_n$ ($n = -1 \bmod p$), $\widetilde{D}_4$, $\widetilde{E}_6$. We may introduce a mean contribution of each type to the defect of g-signature theorem (see Section 5) to carry through the proof. If we set $\frac{\mathrm{def}_{\Gamma_\alpha}}{\mathrm{weight}_{\Gamma_\alpha}}$ as the mean contribution for each type of Γ_α, then we claim that

$$\frac{\mathrm{def}_{\Gamma_\alpha}}{\mathrm{weight}_{\Gamma_\alpha}} \geq -\frac{1}{3}(p-1)p \tag{4.3}$$

for each type of $\widetilde{A}_n$, $\widetilde{D}_4$, $\widetilde{E}_6$ with fixed 2-dimensional components (for explicit fixed point datum of certain types, one may see [17]). Note that

$$\mathrm{def}_{(k)} \geq -\frac{2k}{3}(p-1) \tag{4.4}$$

for $k = 2, 3, 4$, where $\mathrm{def}_{(k)}$ denotes the corresponding defect of the cusp sphere type, type (B) with two spheres intersecting at a point of order 2, and the $\widetilde{A}_2$ type with three spheres intersecting at a single point, respectively. The Lefschetz fixed point theorem and the restriction $c_1(K) \cdot \omega < 16$ follow that

$$6\delta_{\widetilde{D}_4} + \sum_{n=-1 \bmod p} (n+1)\delta_{\widetilde{A}_n} + 12\delta_{\widetilde{E}_6} + 3\delta_4 + 2\delta_3 + (t+\delta_2) \leq c_1(K)\cdot[\omega] < 16,$$

$$6\delta_{\widetilde{D}_4} + \sum_{n=-1 \bmod p} (n+1)\delta_{\widetilde{A}_n} + 8\delta_{\widetilde{E}_6} + 4\delta_4 + 3\delta_3 + 2(t+\delta_2) + \delta_1 = \chi(X) = 48.$$

Similar results may follow the two formulae above that $\delta_1 \geq 18$,

$$\sum_{k=2}^{4} k\delta_k \leq \chi(X) - \delta_1 \leq 30,$$

and $\sum_{\Gamma_\alpha} \mathrm{weight}t_{\Gamma_\alpha} \cdot \delta_{\Gamma_\alpha} < 16$ with Γ_α taking over all the types of $\widetilde{A}_n$, $\widetilde{D}_4$ and $\widetilde{E}_6$. Substituting equations (4.3)–(4.4) into the g-signature theorem, we may reach a contradiction as follows obviously:

$$\begin{aligned}
-32(p-1) &= \mathrm{def}_{(1)}\delta_1 + \sum_{k=2}^{4}\mathrm{def}_{(k)}\delta_k + \sum_{\Gamma_\alpha}\mathrm{def}_{\Gamma_\alpha}\delta_{\Gamma_\alpha} + \sum_{\mathrm{torus}}\mathrm{def}_{\mathrm{torus}}\cdot t\\
&= \mathrm{def}_{(1)}\delta_1 + \sum_{k=2}^{4}\frac{\mathrm{def}_{(k)}}{k}\cdot k\delta_k + \sum_{\Gamma_\alpha}\frac{\mathrm{def}_{\Gamma_\alpha}}{\mathrm{weight}_{\Gamma_\alpha}}\cdot\delta_{\Gamma_\alpha}\mathrm{weight}_{\Gamma_\alpha}\\
&\geq \frac{1}{3}(p-1)(p-2)\cdot\delta_1 + \Big(-\frac{2}{3}(p-1)\Big)\sum_{k=2}^{4}k\delta_k + \Big(-\frac{1}{3}(p-1)p\Big)\sum_{\Gamma_\alpha}\mathrm{weight}_{\Gamma_\alpha}\delta_{\Gamma_\alpha}\\
&\geq 18\cdot\frac{1}{3}(p-1)(p-2) + 30\cdot\Big(-\frac{2}{3}(p-1)\Big) + 15\cdot\Big(-\frac{1}{3}(p-1)p\Big)\\
&= (p-32)(p-1)\\
&\Leftrightarrow p \leq 0,
\end{aligned}$$

and complete the proof of Theorem 1.2.

For more details, we give the specifications of zooming the mean contribution to the g-signature theorem for each type in Section 5.

Remark 4.1 In the pseudofree case in part (i), we may replace the restriction to $c_1(K)\cdot[\omega] < 24$, and the rigidity property still remains. In fact, if the minimal symplectic manifold is $(E(n),\omega)$ with $c_1^2 = 0$, $b_2^+ \geq 2$ and $c_1(K)\cdot[\omega] < 6n$, we can maintain that there are no nontrivial homologically trivial pseudofree actions of a finite group on $E(n)$ surface, which preserve the symplectic structure ω.

Remark 4.2 In fact, for the case that $p = 5$, we can also reach a contradiction to the g-signature theorem by resizing the mean contributions in Section 5.

5 Appendix

A calculation of signature defect we use in the g-signature theorem along this paper is expressed in this section. We follow the notations above that for an isolated fixed point $m \in F$, the local representation at m is $(z_1, z_2) \mapsto (\mu_p^k z_1, \mu_p^{kq} z_2)$ for some $k \neq 0 \bmod p$ and $q \neq 0 \bmod p$, then the signature defect def_m of the g-signature theorem is given by

$$I_{p,q} \equiv \sum_{k=1}^{p-1} \frac{(1+\mu_p^k)(1+\mu_p^{kq})}{(1-\mu_p^k)(1-\mu_p^{kq})}.$$

It is obvious that $I_{p,-q} = -I_{p,q}$.

Recall the relationship between the defect and the Dedekind sum (see [11]) that $I_{p,q} = -4p\cdot s(q,p)$. Some results are collected from the direct computation of the equation

$$6p\cdot s(q,p) = (p-1)\Big(2pq - q - \frac{3p}{2}\Big) - 6f_p(q),$$

where $f_p(q) = \sum\limits_{k=1}^{p-1} k\big[\frac{kq}{p}\big]$ (see [4] for more details):

$$\begin{aligned}
I_{p,-1} &= \frac{1}{3}(p-1)(p-2),\\
I_{p,-2} &= \frac{1}{6}(p-1)(p-5),\\
I_{p,-3} &= \begin{cases} r(r-3), & \text{if } p = 3r+1,\\ r(r-1), & \text{if } p = 3r+2,\end{cases}\\
I_{p,-6} &= \begin{cases} 2r(r-6), & \text{if } p = 6r+1,\\ 2r^2+4r+4, & \text{if } p = 6r+5,\end{cases}\\
I_{p,-4} &= \begin{cases} \frac{4}{3}r(r-4), & \text{if } p = 4r+1,\\ \frac{2}{3}(2r^2+1), & \text{if } p = 4r+3,\end{cases}\\
I_{p,\frac{p+3}{2}} &= \begin{cases} 2r(2-r), & \text{if } p = 6r+1,\\ -2r^2+4r+4, & \text{if } p = 6r+5.\end{cases}
\end{aligned}$$

We take the results in a different statement and with the datum above directly evaluate the defects of type $\widetilde{A}_2$, type (B) and the cusp sphere type to the g-signature formula when $p \geq 5$, that obviously summarize (4.4) as follows.

The type $\widetilde{A}_2$ component contributes

$$\mathrm{def}_{(4)} = I_{p,1} + 3I_{p,-3} = \begin{cases} -\dfrac{8}{3}(p-1), & \text{if } p = 1 \mod 12, \\ -\dfrac{4}{3}(p-2), & \text{if } p = 5 \mod 12, \\ -\dfrac{8}{3}(p-1), & \text{if } p = 7 \mod 12, \\ -\dfrac{4}{3}(p-2), & \text{if } p = 11 \mod 12. \end{cases}$$

The type (B) component contributes

$$\mathrm{def}_{(3)} = I_{p,2} + 2I_{p,-4} = \begin{cases} -2(p-1), & \text{if } p = 1 \mod 12, \\ -2(p-1), & \text{if } p = 5 \mod 12, \\ 2, & \text{if } p = 7 \mod 12, \\ 2, & \text{if } p = 11 \mod 12. \end{cases}$$

The cusp sphere component contributes

$$\mathrm{def}_{(2)} = I_{p,\frac{p+3}{2}} + I_{p,-6} = \begin{cases} -\dfrac{4}{3}(p-1), & \text{if } p = 1 \mod 12, \\ \dfrac{4}{3}(p+1), & \text{if } p = 5 \mod 12, \\ -\dfrac{4}{3}(p-1), & \text{if } p = 7 \mod 12, \\ \dfrac{4}{3}(p+1), & \text{if } p = 11 \mod 12. \end{cases}$$

Based on the datum above, we formulate the defects of type $\widetilde{A}_n$, $\widetilde{D}_4$ and $\widetilde{E}_6$ for $p \geq 5$, which obviously support (4.3).

A $\widetilde{D}_4$ type contributes a (-2)-sphere and 4 fixed points with the same type $(1,-2)$ to X^G. With weight 6, the mean contribution

$$\frac{\mathrm{def}_{\widetilde{D}_4}}{\mathrm{weight}_{\widetilde{D}_4}} = \frac{1}{6}(\mathrm{def}_Y + 4I_{p,-2}) = \frac{1}{6}\Big(\frac{p^2-1}{3}\cdot(-2) + 4\cdot\frac{1}{6}(p-1)(p-5)\Big) = -\frac{2}{3}(p-1).$$

An $\widetilde{E}_6$ type contributes a (-2)-sphere and 6 fixed points to X^G, so that the total defects

$$\mathrm{def}_{\widetilde{E}_6} \geq \frac{p^2-1}{3}\cdot(-2) + 6\cdot\Big(-\frac{1}{3}(p-1)(p-2)\Big) = -\frac{2}{3}(p-1)(4p-5).$$

With weight 12, we have

$$\frac{\mathrm{def}_{\widetilde{E}_6}}{\mathrm{weight}_{\widetilde{E}_6}} \geq -\frac{1}{3}(p-1)\Big(\frac{2p}{3} - \frac{5}{6}\Big).$$

An $\widetilde{A}_n$ type, where $n = -1 \bmod p$, contributes a fixed (-2)-sphere and $n-1$ fixed points. Then

$$\mathrm{def}_{\widetilde{A}_n} \geq -\frac{2}{3}(p^2-1) - \frac{n-1}{3}(p-1)(p-2) = -\frac{1}{3}(p-1)((n+1)p - 2n + 4).$$

With weight $n+1$, we resize the mean contribution of $\widetilde{A_n}$ type as

$$\frac{\mathrm{def}_{\widetilde{A}_n}}{\mathrm{weight}_{\widetilde{A}_n}} \geq -\frac{1}{3}(p-1)\cdot\frac{(n+1)p-2n+4}{n+1} = -\frac{1}{3}(p-1)\Big(p - \frac{2n-4}{n+1}\Big) \geq -\frac{1}{3}(p-1)p,$$

independent of n.

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