Chin. Ann. Math.
35B(6), 2014, 885–894
DOI: 10.1007/s11401-014-0865-5

Chinese Annals of Mathematics, Series B

On Nearly SS-Embedded Subgroups of Finite Groups*

Lijun HUO[1] Wenbin GUO[2] Alexander A. MAKHNEV[3]

Abstract Let H be a subgroup of a finite group G. H is nearly SS-embedded in G if there exists an S-quasinormal subgroup K of G, such that HK is S-quasinormal in G and $H \cap K \leq H_{seG}$, where H_{seG} is the subgroup of H, generated by all those subgroups of H which are S-quasinormally embedded in G. In this paper, the authors investigate the influence of nearly SS-embedded subgroups on the structure of finite groups.

Keywords S-quasinormal subgroup, Nearly SS-embedded subgroup, Sylow subgroup, p-nilpotent group, Supersolvable group
2000 MR Subject Classification 20D10, 20D20

1 Introduction

Throughout this paper, all groups considered are finite and G denotes a finite group. Recall that a subgroup H of G is said to be S-quasinormal in G if H permutes with every Sylow subgroup of G. A subgroup H of G is said to be S-quasinormally embedded in G if every Sylow subgroup of H is a Sylow subgroup of some S-quasinormal subgroup of G. A subgroup H of G is called c-normal in G (see [21]) if there exists a normal subgroup K, such that $G = HK$ and $H \cap K \leq H_G$, where H_G is the maximal normal subgroup of G contained in H. In [9], Guo et al. gave the concept of an s-embedded subgroup as follows: A subgroup H is said to be s-embedded in G if G has an S-quasinormal subgroup T, such that $T \cap H \leq H_{sG}$ and $HT = H^{sG}$, where H_{sG} is the subgroup generated by all those subgroups of H which are S-quasinormal in G and H^{sG} is the intersection of all such S-quasinormal subgroups of G which contain H. By using the above ideas, a series of interesting results was obtained (see, e.g., [7–11, 14–16, 21–22]). In this paper, we give some new applications of S-quasinormal subgroups and S-quasinormally embedded subgroups in the theory of groups. Our main tool here is the following concept.

Definition 1.1 *Let H be a subgroup of G. We say that H is nearly SS-embedded in G if G has an S-quasinormal subgroup K such that HK is S-quasinormal in G and $H \cap K \leq H_{seG}$, where H_{seG} is the subgroup generated by all those subgroups of H which are S-quasinormally embedded in G.*

Manuscript received January 11, 2013. Revised March 4, 2014.
[1]Department of Mathematics, University of Science and Technology of China, Hefei 230026, China. E-mail: ljhuo@mail.ustc.edu.cn
[2]Corresponding author. Department of Mathematics, University of Science and Technology of China, Hefei 230026, China. E-mail: wbguo@ustc.edu.cn
[3]Institute of Mathematics and Mechanics of UB RAS, Ekaterinburg 620219, Russia. E-mail: makhnev@imm.uran.ru
*This work was supported by the National Natural Science Foundation of China (No. 11371335), the international joint research fund between NSFC and RFBR (No. 11211120148) and the Research Fund for the Doctoral Program of Higher Education of China (No. 20113402110036).

It is easy to see that all S-quasinormal, S-quasinormally embedded, c-normal and s-embedded subgroups are all nearly SS-embedded in G. However, the following examples show that the converse is not true.

Example 1.1 Suppose that G is the symmetric group S_4.

(1) Let $H = \langle(12)\rangle$. It is easy to see that $G = A_4H$ and $H \cap A_4 = 1$. Hence H is nearly SS-embedded in G. However, H is clearly not S-quasinormally embedded in G.

(2) Let $H = \langle(123)\rangle$ and $K_4 = \{(1), (12)(34), (13)(24), (14)(23)\}$. Then $HK_4 = A_4 \trianglelefteq G$ and $1 = H \cap K_4 \leq H_{\mathrm{se}G}$. Hence H is a nearly SS-embedded subgroup of G. But it is easy to check that H is not c-normal in G.

Example 1.2 Let $G = S_5 = A_5B$, where $B = \langle(12)\rangle$, and let K_4 be the group as in Example 1.1(2). Clearly, K_4 is a Sylow 2-subgroup of A_5, $K_4A_5 = A_5 \trianglelefteq G$ and $K_4 = K_4 \cap A_5 = (K_4)_{\mathrm{se}G}$. Hence K_4 is nearly SS-embedded in G, but it is not s-embedded in G.

In this paper, we investigate the influence of the nearly SS-embedded subgroups on the structure of finite groups. Some new results are obtained and some recent results are generalized.

2 Preliminaries

The following known results will be used in this paper.

Lemma 2.1 *Let G be a group and $H \leq K \leq G$.*

(1) *If H is S-quasinormal in G, then H is S-quasinormal in K (see* [13]*).*

(2) *If H is S-quasinormal in G, then H is subnormal in G (see* [13]*).*

(3) *If H and L are S-quasinormal in G, then $\langle H, L\rangle$ and $H \cap L$ are S-quasinormal in G (see* [18]*).*

(4) *If H is S-quasinormal in G and $M \leq G$, then $H \cap M$ is S-quasinormal in M (see* [3]*).*

(5) *Suppose that H is normal in G. Then K/H is S-quasinormal in G/H if and only if K is S-quasinormal in G (see* [13]*).*

(6) *If H is S-quasinormal in G, then H/H_G is nilpotent (see* [3]*).*

Lemma 2.2 *Let $A \leq K \leq G$ and $B \leq G$.*

(1) *If A is subnormal in G and B is a minimal normal subgroup of G, then $B \leq N_G(A)$ (see* [4]*).*

(2) *If A is subnormal in G and A is a π-subgroup of G, then $A \leq O_\pi(G)$. In particular, if A is a subnormal Hall subgroup of G, then A is normal in G (see* [23]*).*

Lemma 2.3 (see [10]) *Suppose that N is a normal subgroup of G and $H \leq K \leq G$. Then $H_{\mathrm{se}G} \leq H_{\mathrm{se}K}$ and $H_{\mathrm{se}G}N/N \leq (HN/N)_{\mathrm{se}(G/N)}$.*

Lemma 2.4 *Let $H \leq K \leq G$. Then*

(1) *If H is nearly SS-embedded in G, then H is nearly SS-embedded in K.*

(2) *Suppose that $H \trianglelefteq G$. If K is nearly SS-embedded in G, then K/H is nearly SS-embedded in G/H.*

(3) *If $H \trianglelefteq G$, then for every nearly SS-embedded subgroup E of G with $(|H|, |E|) = 1$, HE/H is nearly SS-embedded in G/H.*

Proof (1) Assume that there exists an S-quasinormal subgroup T of G, such that HT is S-quasinormal in G and $H \cap T \le H_{\mathrm{se}G}$. Then by Lemma 2.1(4), $T \cap K$ and $H(T \cap K) = HT \cap K$ are S-quasinormal in K. By Lemma 2.3, $H \cap (T \cap K) \le H_{\mathrm{se}G} \le H_{\mathrm{se}K}$. Hence H is nearly SS-embedded in K.

(2) Assume that there exists an S-quasinormal subgroup T of G, such that KT is S-quasinormal in G and $T \cap K \le K_{\mathrm{se}G}$. Since HTK is S-quasinormal in G by Lemma 2.1(3), $HTK/H = (HT/H)(K/H)$ is S-quasinormal in G/H by Lemma 2.1(5). On the other hand, by Lemma 2.3, $(HT/H) \cap (K/H) = (HT \cap K)/H = H(T \cap K)/H \le HK_{\mathrm{se}G}/H \le (HK/H)_{\mathrm{se}(G/H)}$ $= (K/H)_{\mathrm{se}(G/H)}$. Hence K/H is nearly SS-embedded in G/H.

(3) Assume that E is nearly SS-embedded in G. Then there exists an S-quasinormal subgroup T of G such that ET is S-quasinormal in G and $E \cap T \le E_{\mathrm{se}G}$. By Lemma 2.1(5), $(HE/H)(TH/H) = (ET)H/H$ is S-quasinormal in G/H. Since $(|H|, |E|) = 1$, $(|HE \cap T : T \cap H|, |HE \cap T : T \cap E|) = 1$. Hence $HE \cap T = (T \cap H)(T \cap E)$ (see [6, (3.8.1)]). It follows that $(HE/H) \cap (TH/H) = (HE \cap TH)/H = (HE \cap T)H/H = (E \cap T)H/H \le E_{\mathrm{se}G}H/H \le (HE/H)_{\mathrm{se}(G/H)}$ by Lemma 2.3. This shows that HE/H is nearly SS-embedded in G/H.

Lemma 2.5 (see [18]) *Let H be a p-subgroup of G for some prime p. Then H is S-quasinormal in G if and only if $O^p(G) \le N_G(H)$.*

Lemma 2.6 (see [1, Lemma 2.4]) *Let H be a subgroup of G. Then the following two statements are equivalent*:

(1) *H is an S-quasinormal nilpotent subgroup of G.*

(2) *The Sylow subgroups of H are S-quasinormal in G.*

Lemma 2.7 (see [6, (1.8.17)]) *Let N be a nontrivial solvable normal subgroup of G. If $N \cap \Phi(G) = 1$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G contained in N.*

Lemma 2.8 *Let P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every maximal subgroup of P is nearly SS-embedded in G, then G is solvable.*

Proof Suppose that the assertion is false and let G be a counterexample of the minimal order. Then $p = 2$ by Feit-Thompson's theorem. We now proceed the proof via the following steps.

(1) $O_2(G) = 1$.

Assume that $O_2(G) \ne 1$. Clearly, $P/O_2(G)$ is a Sylow 2-subgroup of $G/O_2(G)$. Let $M/O_2(G)$ be a maximal subgroup of $P/O_2(G)$. Then M is a maximal subgroup of P. By the hypothesis and Lemma 2.4(2), $M/O_2(G)$ is nearly SS-embedded in $G/O_2(G)$. The minimal choice of G implies that $G/O_2(G)$ is solvable. It follows that G is solvable. This contradiction shows that (1) holds.

(2) $O_{2'}(G) = 1$.

Suppose that $O_{2'}(G) \ne 1$. Then $PO_{2'}(G)/O_{2'}(G)$ is a Sylow 2-subgroup of $G/O_{2'}(G)$. Assume that $M/O_{2'}(G)$ is a maximal subgroup of $PO_{2'}(G)/O_{2'}(G)$. Then there exists a maximal subgroup T of P, such that $M = TO_{2'}(G)$. By the hypothesis and Lemma 2.4(3), $M/O_{2'}(G) = TO_{2'}(G)/O_{2'}(G)$ is nearly SS-embedded in $G/O_{2'}(G)$. The minimal choice of G implies that $G/O_{2'}(G)$ is solvable. By Feit-Thompson's theorem, we know that $O_{2'}(G)$ is solvable and so is G, a contradiction.

(3) P is not cyclic.

If P is cyclic, then G is 2-nilpotent by [17, (10.1.9)]. This implies that G is solvable, a contradiction.

(4) If $1 \neq N \trianglelefteq G$, then N is not solvable and $G = PN$.

Suppose that N is solvable. Then $O_2(N) \neq 1$ or $O_{2'}(N) \neq 1$. Since $O_2(N)$ char $N \trianglelefteq G$, we get $O_2(N) \leq O_2(G)$. Similarly, $O_{2'}(N) \leq O_{2'}(G)$. Hence, $O_2(G) \neq 1$ or $O_{2'}(G) \neq 1$, which contradicts (1) or (2). Therefore, N is not solvable. Assume that $PN < G$. Then by Lemma 2.4(1), every maximal subgroup of P is nearly SS-embedded in PN. Thus, PN satisfies the hypothesis. By the choice of G, we have that PN is solvable and so is N, a contradiction. Thus (4) holds.

(5) G has a unique minimal normal subgroup, and we still denote it by N.

By (4), we see that $G = PN$ for every non-identity normal subgroup N of G. It is easy to see that G/N is solvable. Since the class of all solvable groups is closed under the subdirect product, G has a unique minimal normal subgroup.

(6) Final contradiction.

If $N \cap P \leq \Phi(P)$, then N is 2-nilpotent by [12, IV, Theorem 4.7]. Let $N_{2'}$ be the normal 2-complement of N. Since $N_{2'}$ char $N \trianglelefteq G$, we have $N_{2'} \leq O_{2'}(G)$. Thus N is a 2-subgroup by (2), so N is solvable. This contradiction shows that $N \cap P \nleq \Phi(P)$. It follows that there exists a maximal subgroup P_1 of P, such that $P = P_1(P \cap N)$. By the hypothesis, there exists an S-quasinormal subgroup K, such that P_1K is S-quasinormal in G and $P_1 \cap K \leq (P_1)_{\mathrm{se}G}$. Suppose that $(P_1)_{\mathrm{se}G} \neq 1$. Let $(P_1)_{\mathrm{se}G} = \langle H_1, H_2, \cdots, H_t\rangle$, where $H_1, \cdots, H_t$ are all the nontrivial S-quasinormal embedded subgroups of G contained in P_1. Then there exist S-quasinormal subgroups $K_1, K_2, \cdots, K_t$ of G, such that $H_i \in \mathrm{Syl}_2(K_i)$ for $i = 1, 2, \cdots, t$. If $(K_i)_G \neq 1$ for some $i \in \{1, 2, \cdots, t\}$, then $N \leq (K_i)_G \leq K_i$ by (5). It is easy to see that $H_i \cap N \in \mathrm{Syl}_2(N)$ and $H_i \cap N \leq P_1 \cap N \leq P \cap N \in \mathrm{Syl}_2(N)$. Hence $H_i \cap N = P_1 \cap N = P \cap N$. Consequently, $P = (P \cap N)P_1 = (P_1 \cap N)P_1 = P_1$, a contradiction. Therefore $(K_i)_G = 1$. It follows from Lemma 2.1(6) that $K_i = K_i/(K_i)_G$ is nilpotent and S-quasinormal in G. By Lemma 2.6, H_i is S-quasinormal in G. Hence, by Lemma 2.1(3), $(P_1)_{\mathrm{se}G}$ is S-quasinormal in G, so $(P_1)_{\mathrm{se}G} \leq O_2(G) = 1$ by Lemma 2.1(2) and 2.2(2), which implies that $P_1 \cap K = 1$. If $K = 1$, then P_1 is S-quasinormal in G, so P_1 is subnormal in G by Lemma 2.1(2). It follows that $P_1 \leq O_2(G) = 1$, so P is cyclic, which contradicts (3). We may, therefore, assume that $K \neq 1$. Clearly, $|K|_2 \leq 2$. Then K is a 2-nilpotent group by [17, (10.1.9)]. Let $K_{2'}$ be a normal Hall $2'$-subgroup of K. Since K is an S-quasinormal subgroup of G, K is subnormal in G, so $K_{2'}$ is subnormal in G. Then by Lemma 2.2(2), $K_{2'} \leq O_{2'}(G) = 1$. This means that K is a group of order 2 and P_1K is a Sylow 2-subgroup of G. Since P_1K is subnormal in G, $P_1K \trianglelefteq G$ by Lemma 2.2(2), and consequently, $N \leq P_1K = P$. This implies that $G = PN = P$. The final contradiction completes the proof.

3 Main Results

Theorem 3.1 *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is nearly SS-embedded in G, then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of the minimal order. Then

(1) $O_{p'}(G) = 1$.

Suppose that $D = O_{p'}(G) \neq 1$. Clearly, PD/D is a Sylow p-subgroup of G/D and every maximal subgroup of PD/D may be written as P_1D/D, where P_1 is a maximal subgroup of P. Since P_1 is nearly SS-embedded in G, we have that P_1D/D is also nearly SS-embedded in G/D by Lemma 2.4(3). Therefore G/D satisfies the condition of the theorem. The minimal choice of G implies that G/D is p-nilpotent and consequently G is p-nilpotent, a contradiction.

(2) G is solvable.

This can be obtained by Lemma 2.8 and the Feit-Thompson theorem.

(3) G has a unique minimal normal subgroup N, such that $G = N \rtimes M$, where M is a nilpotent maximal subgroup of G, and $N = O_p(G) = F(G) = C_G(N)$.

Let N be a minimal normal subgroup of G. By (1)–(2), N is an elementary abelian p-group. If $N \neq P$, then the hypothesis still holds for G/N by Lemma 2.4(2). The choice of G implies that G/N is p-nilpotent. If $N = P$, then G/N is a p'-group and thus G/N is also p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Therefore, there exists a maximal subgroup M of G, such that $N \nleq M$. It is easy to see that $G = N \rtimes M$ and $N \subseteq O_p(G) \subseteq F(G) \subseteq C_G(N)$. Let $C = C_G(N)$. Then $C = C \cap NM = N(C \cap M)$. Clearly, $C \cap M \trianglelefteq NM = G$, which implies that $C \cap M = 1$ and thereby $C = N$. Hence (3) holds.

(4) The final contradiction.

Obviously, $P = P \cap NM = N \rtimes (P \cap M)$, where $P \cap M = M_p$ is a Sylow p-subgroup of M. Let P_1 be a maximal subgroup of P containing M_p. Clearly, $N \nleq P_1$. If $P_1 = 1$, then $|N| = |P| = p$. By (3), $G/N \cong G/C_G(N)$ is isomorphic with some subgroup of $\mathrm{Aut}(N)$, so $|G/N| \mid |\mathrm{Aut}(N)|$. Since $|\mathrm{Aut}(N)| = p - 1$ and $(|G|, p - 1) = 1$, we have that $G/N = 1$. Therefore, $G = N$ is an elementary abelian p-group, a contradiction.

Now suppose that $P_1 \neq 1$. By the hypothesis, there exists an S-quasinormal subgroup K of G, such that P_1K is S-quasinormal in G and $P_1 \cap K \leq (P_1)_{\mathrm{se}G}$. Suppose that $(P_1)_{\mathrm{se}G} \neq 1$. Let $H_1, H_2, \cdots, H_t$ be all the nontrivial subgroups of P_1 which are S-quasinormal embedded in G. Then there exist S-quasinormal subgroups $K_1, K_2, \cdots, K_t$ in G, such that H_i is a Sylow p-subgroup of K_i for $i = 1, 2, \cdots, t$. If $(K_i)_G \neq 1$ for some $i \in \{1, 2, \cdots, t\}$, then $N \leq (K_i)_G \leq K_i$, and thus $N \leq H_i \leq P_1$, a contradiction. Thus $(K_i)_G = 1$ for all $i = 1, 2, \cdots, t$. By Lemma 2.1(6), $K_i = K_i/(K_i)_G$ is an S-quasinormal nilpotent subgroup of G. It follows from Lemma 2.6 that H_i is S-quasinormal in G. Hence $(P_1)_{\mathrm{se}G}$ is S-quasinormal in G by Lemma 2.1(3). It follows from Lemma 2.2 that $(P_1)_{\mathrm{se}G} \leq P_1 \cap O_p(G) = P_1 \cap N$. Then by Lemma 2.5, $(P_1)_{\mathrm{se}G} \leq (P_1)_{\mathrm{se}G}^G = (P_1)_{\mathrm{se}G}^{O^p(G)P} = (P_1)_{\mathrm{se}G}^P \leq (P_1 \cap N)^P = P_1 \cap N \leq N$. This implies that $(P_1)_{\mathrm{se}G}^G = N = P_1 \cap N$. Consequently, $N \leq P_1$, a contradiction. Hence, $(P_1)_{\mathrm{se}G} = P_1 \cap K = 1$. If $K = 1$, then P_1 is S-quasinormal in G. By Lemma 2.5, $N \leq (P_1)^G = (P_1)^{O^p(G)P} = {P_1}^P = P_1$, a contradiction. Hence, we may assume that $K \neq 1$. Obviously, $|K_p| \leq p$, where K_p is a Sylow p-subgroup of K. If $K_p = 1$, then, clearly, K is p-nilpotent. If $|K_p| = p$, then $|\mathrm{Aut}(K_p)| = p - 1$. Since $N_K(K_p)/C_K(K_p)$ is isomorphic with some subgroup of $\mathrm{Aut}(K_p)$ and $(|G|, p - 1) = 1$, we have that $|N_K(K_p)/C_K(K_p)| = 1$. Hence by Burnside's Theorem, K is p-nilpotent. Let $K_{p'}$ be the normal p-complement of K. Then $K_{p'}$ is subnormal in G, so $K_{p'} \leq O_{p'}(G) = 1$. It follows that K is a cyclic group of order p and P_1K is a Sylow p-subgroup of G. Since P_1K is S-quasinormal in G, $P_1K \trianglelefteq G$ by Lemma 2.2(2). Therefore, $P_1K = P = O_p(G) = N$ is an elementary abelian p-group of G. By Lemma 2.5 and 2.2(1), $N \leq K^G = K^{O^p(G)P} = K^P = K^N = K$. It follows that $P_1 \leq K$ and $P_1 = 1$. The final contradiction completes the proof.

Corollary 3.1 *Let p be a prime dividing the order of G with $(|G|, p-1) = 1$ and H be a normal subgroup of G, such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H, such that every maximal subgroup of P is nearly SS-embedded in G, then G is p-nilpotent.*

Proof By Lemma 2.4(1), every maximal subgroup of P is SS-embedded in H. By Theorem 3.1, H is p-nilpotent. Now, let $H_{p'}$ be the normal p-complement of H. Then $H_{p'} \trianglelefteq G$. Assume that $H_{p'} \neq 1$, and applying Lemma 2.4 again, we see that $G/H_{p'}$ satisfies the hypothesis by induction on $|G|$. Hence $G/H_{p'}$ is p-nilpotent. It follows that G is p-nilpotent. We may, therefore, assume $H_{p'} = 1$. Then $H = P$ is a p-group. Since G/H is p-nilpotent, we may let K/H be the normal p-complement of G/H. By Schur-Zassenhaus's theorem, there exists a Hall p'-subgroup $K_{p'}$ of K such that $K = HK_{p'}$. Now by using Lemma 2.4(1) and Theorem 3.1, we see that K is p-nilpotent, so $K = H \times K_{p'}$. In this case, $K_{p'}$ is a normal p-complement of G, and thus G is p-nilpotent.

Corollary 3.2 *Suppose that every maximal subgroup of any Sylow subgroup of G is nearly SS-embedded in G. Then G is a Sylow tower group of the supersolvable type.*

Proof It follows from Theorem 3.1 and Lemma 2.4.

Theorem 3.2 *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is nearly SS-embedded in G, then G is p-nilpotent.*

Proof Suppose that the theorem is false and let G be a counterexample of the minimal order. We proceed via the following steps.

(1) $O_{p'}(G) = 1$.

Suppose that $O_{p'}(G) \neq 1$. Obviously, $PO_{p'}(G)/O_{p'}(G)$ is a Sylow p-subgroup of $G/O_{p'}(G)$. Let $T/O_{p'}(G)$ be a maximal subgroup of $PO_{p'}(G)/O_{p'}(G)$. Then $T = P_1O_{p'}(G)$ for some maximal subgroup P_1 of P. By Lemma 2.4(3) and the hypothesis, $P_1O_{p'}(G)/O_{p'}(G)$ is nearly SS-embedded in $G/O_{p'}(G)$. On the other hand, since $N_{(G/O_{p'}(G))}(PO_{p'}(G)/O_{p'}(G)) = N_G(P)O_{p'}(G)/O_{p'}(G)$ (see [6, (3.6.10)]), we see that $N_{(G/O_{p'}(G))}(PO_{p'}(G)/O_{p'}(G))$ is p-nilpotent. This shows that $G/O_{p'}(G)$ satisfies the hypothesis of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent. It follows that G is p-nilpotent, a contradiction.

(2) If M is a proper subgroup of G containing P, then M is p-nilpotent.

Firstly, clearly, $N_M(P)$ is p-nilpotent. By Lemma 2.4(1), we see that M satisfies the hypothesis. The minimal choice of G implies that M is p-nilpotent.

(3) $G = PQ$ and $O_p(G) \neq 1$, where Q is a Sylow q-subgroup of G with $q \neq p$.

Since G is not p-nilpotent, by Thompson's theorem (see [20]), there exists a nonidentity characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent. Since $N_G(P)$ is p-nilpotent, we may choose a characteristic subgroup H of P such that $N_G(H)$ is not p-nilpotent, but $N_G(K)$ is p-nilpotent for every characteristic subgroup K of P with $H < K \leq P$. Since H char

$P \trianglelefteq N_G(P)$, we have $H \trianglelefteq N_G(P)$, so $N_G(P) < N_G(H)$. Then by (2), we have $G = N_G(H)$. This shows that $H \leq O_p(G) \neq 1$ and $N_G(K)$ is p-nilpotent for any characteristic subgroup K of P with $O_p(G) < K \leq P$ (if it exists). In this case, using Thompson's theorem again, we see that $G/O_p(G)$ is p-nilpotent and then G is p-solvable. Thus for any $q \in \pi(G)$ with $q \neq p$, there exists a Sylow q-subgroup Q of G, such that PQ is a subgroup of G (see [5, (6.3.5)]). If $PQ < G$, then PQ is p-nilpotent by (2). It follows from (1) and [17, (9.3.1)] that $Q \leq C_G(O_p(G)) \leq O_p(G)$, a contradiction. Thus (3) holds.

(4) G has a unique minimal normal subgroup N, such that $G = N \rtimes M$, where M is a maximal subgroup of G, and $N = O_p(G) = C_G(N)$.

Let N be a minimal normal subgroup of G. Then by (1) and (3), N is an elementary abelian p-group, and $N \subseteq O_p(G) < P$. It is easy to see that G/N satisfies the hypothesis. Hence G/N is p-nilpotent by the choice of G. Since the class of all p-nilpotent groups is a saturated formation, N is a unique minimal normal subgroup of G and $N \nleq \Phi(G)$. Consequently, $G = N \rtimes M$ for some maximal subgroup M of G. As G is solvable by (3), we see that $C_G(N) = O_p(G) = N$. Hence, (4) holds.

(5) Conclusion.

By (4), $P = P \cap NM = N(P \cap M) = NM_p$, where $M_p = P \cap M$ is a Sylow p-subgroup of M. If $M_p = 1$, then $P = N$, so $G = N_G(N) = N_G(P)$ is p-nilpotent, a contradiction. Hence, we may assume that $M_p \neq 1$. Let P_1 be a maximal subgroup of P containing M_p. Then $P = NP_1$ and P_1 is nearly SS-embedded in G. Therefore, there exists an S-quasinormal subgroup K of G such that P_1K is S-quasinormal in G and $P_1 \cap K \leq (P_1)_{seG}$. By using the same argument as in the proof of Theorem 3.1, we get that $P_1 \cap K = 1$ and $K \neq 1$. Hence $|K_p| \leq p$, where K_p is a Sylow p-subgroup of K. If $|K_p| = 1$, then K is a q-subgroup. By Lemma 2.1(2) and Lemma 2.2(2), $K \leq O_q(G)$, which contradicts (1). Hence $|K_p| = p$. Suppose that $N \cap K = 1$. Since G is solvable by (3), the minimal normal subgroup K_1 of K is an elementary abelian p-group or a q-group. If K_1 is a p-group, then $K_1 \leq O_p(G) = N$ by Lemma 2.2(2), which contradicts $N \cap K = 1$. If K_1 is a q-group, then $K_1 \leq O_q(G)$, which contradicts (1). Hence $|N \cap K| = p$. Suppose that $K_G \neq 1$. Then $N \leq K_G \leq K$ by (4). Therefore, $|N| = |N \cap K| = p$. If $q > p$, then NQ is p-nilpotent by [17, (10.1.9)], so $Q \leq C_G(N) = N$, a contradiction. If $q < p$, then $M \cong G/N = N_G(N)/C_G(N)$ is isomorphic with some subgroup of $\mathrm{Aut}(N)$. Since $\mathrm{Aut}(N)$ is a cyclic group of order $p - 1$, we have that Q is cyclic. Then G is q-nilpotent by [17, (10.1.9)] again and thus P is normal in G. Hence $N_G(P) = G$ is p-nilpotent, a contradiction. We may, therefore, assume that $K_G = 1$. Then by Lemma 2.1(6), K is nilpotent. Hence by (1), K is a p-subgroup and $|K| = p$. This means that P_1K is a Sylow p-subgroup of G. Since P_1K is S-quasinormal in G, $P = P_1K \trianglelefteq G$ by Lemma 2.2(2). The final contradiction completes the proof.

Corollary 3.3 *Let p be a prime dividing the order of G and H be a normal subgroup of G, such that G/H is p-nilpotent. If there exists a Sylow p-subgroup P of H, such that $N_G(P)$ is p-nilpotent and every maximal subgroup of P is nearly SS-embedded in G, then G is p-nilpotent.*

Proof By Lemma 2.4(1) and Theorem 3.2, H is p-nilpotent. Let $H_{p'}$ be a normal Hall p'-subgroup of H. Then $H_{p'}$ is normal in G. By using the same argument as that in the proof of Corollary 3.1, we may assume $H_{p'} = 1$ and thus $H = P$. In this case, $G = N_G(P)$ is p-nilpotent.

Theorem 3.3 *Let G be a p-solvable group and P be a Sylow p-subgroup of G. If every maximal subgroup of P is nearly SS-embedded in G, then G is p-supersolvable.*

Proof Assume that the assertion is false and choose G to be a counterexample of the minimal order. Then

(1) $O_{p'}(G) = 1$.

See the proof of Theorem 3.1.

(2) $O_p(G) \neq 1$.

Since G is p-solvable, (2) holds by (1).

(3) G has a unique minimal normal subgroup N, such that G/N is p-supersolvable, $G = N \rtimes M$, where M is a maximal subgroup of G, $N = O_p(G) = F(G) \nleq \Phi(G)$, and $|N| > p$.

Let N be a minimal normal subgroup of G contained in $O_p(G)$. Then by Lemma 2.4, G/N satisfies the condition of the theorem, and the minimal choice of G implies that G/N is p-supersolvable. If $|N| = p$, then G is p-supersolvable, a contradiction. Hence $|N| > p$. On the other hand, since the class of all p-supersolvable groups is a saturated formation, N is the unique minimal normal subgroup of G contained in $O_p(G)$ and $O_p(G) = N = F(G) \nleq \Phi(G)$ by Lemma 2.7. Thus (3) holds.

(4) Final contradiction.

Let $P = NM_p$, where M_p is a Sylow p-subgroup of M, and P_1 is a maximal subgroup of P containing M_p. By the hypothesis, there exists an S-quasinormal subgroup K of G, such that P_1K is S-quasinormal in G and $P_1 \cap K \leq (P_1)_{\mathrm{se}G}$. It is easy to verify that $(P_1)_{\mathrm{se}G} = 1$ as the proof of Theorem 3.1. Therefore, $P_1 \cap K = 1$ and $|K_p| \leq p$. If $K_p = 1$, then K is a p'-group and $K \leq O_{p'}(G) = 1$ by Lemma 2.2(2). It follows that P_1 is S-quasinormal in G. By Lemma 2.5, $N \leq {P_1}^G = {P_1}^{O^p(G)P} = {P_1}^P = P_1$, a contradiction. Hence we may assume that $|K_p| = p$.

Suppose that $N \cap K = 1$. Since G is p-solvable, the minimal normal subgroup K_1 of K is an elementary abelian p-group by (1). Clearly, K is subnormal in G by Lemma 2.1(2). Hence $K_1 \leq O_p(G) = N$ by Lemma 2.2(2). This contradiction shows that $|N \cap K| = p$. Suppose that $K_G \neq 1$. Then $N \leq K_G \leq K$, so $|N| = |N \cap K| = p$, which contradicts (3). We may, therefore, assume that $K_G = 1$. Then $K/K_G = K$ is nilpotent by Lemma 2.1(6). Hence, the Sylow subgroups of K are S-quasinormal in G by Lemma 2.6. If K is not a p-group and $p \neq q \in \pi(K)$, then $K_q \leq O_{p'}(G)$ by Lemma 2.2(2), which contradicts (1). Thus $K_q = 1$ and so K is a group of order p. Since P_1K is S-quasinormal in G, we have that $P_1K = P \trianglelefteq G$, and consequently $N = P$. Hence by Lemma 2.5 and Lemma 2.2(1), $N \leq K^G = K^{O^p(G)P} = K^{O^p(G)N} = K^N = K$. It follows that $|N| = p$. The final contradiction completes the proof.

Corollary 3.4 *If every maximal subgroup of every Sylow subgroup of G is nearly SS-embedded in G, then G is supersolvable.*

Proof It follows directly from Lemma 2.8 and Theorem 3.3.

4 Some Applications

In the literature, one can find the following special cases of Theorems 3.1–3.3.

Corollary 4.1 (see [21]) *Let G be a finite group. Then G is solvable if every maximal subgroup of G is c-normal in G.*

Corollary 4.2 (see [11]) *Let p be the smallest prime dividing the order of G and P be a Sylow p-subgroup of G. If every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 4.3 (see [22]) *Let G be a group and p be the prime divisor of $|G|$ with $(|G|, p-1) = 1$. If G has a Sylow p-subgroup P such that every maximal subgroup of P not having a p-nilpotent supplement in G is nearly s-normal in G, then G is p-nilpotent.*

Corollary 4.4 (see [15]) *Let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is c-normal or s-quasinormally embedded in G, then G is p-nilpotent.*

Corollary 4.5 (see [7]) *Let p be the smallest prime divisor of $|G|$ and P be a Sylow p-subgroup of G. If every maximal subgroup of P is S-embedded in G, then G is p-nilpotent.*

Corollary 4.6 (see [1]) *Let G be a group and p be the smallest prime dividing $|G|$. Then G is p-nilpotent if every maximal subgroup of Sylow p-subgroups of G is S-quasinormally embedded in G.*

Corollary 4.7 (see [11]) *Let p be an odd prime dividing the order of a group G and P be a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every maximal subgroup of P is c-normal in G, then G is p-nilpotent.*

Corollary 4.8 (see [21]) *If every maximal subgroup of every Sylow subgroup of G is c-normal in G, then G is supersolvable.*

Corollary 4.9 (see [19]) *Let G be a finite group with the property that the maximal subgroups of Sylow subgroups are S-quasinormal in G. Then G is supersolvable.*

Corollary 4.10 (see [2]) *Let G be a finite group. If each maximal subgroup of Sylow subgroups of G is S-quasinormally embedded in G, then G is supersolvable.*

References

[1] Assad, M. and Heliel, A. A., On S-quasinormally embedded subgroups of finite groups, *J. Pure Appl. Algebra*, **165**, 2001, 129–135.

[2] Ballester-Bolinches, A. and Pedraza-Aguilera, M. C., Sufficient conditions for supersolubility of finite groups, *J. Pure Appl. Algebra*, **127**, 1998, 113–118.

[3] Deskins, W. E., On quasinormal subgroups of finite groups, *Math. Z.*, **82**, 1963, 125–132.

[4] Doerk, K. and Hawkes, T., Finite Solvable Groups, Walter de Gruyter, New York, 1992.

[5] Gorenstein, D., Finite Groups, Harper and Row Publishers, New York, Evanston, London, 1968.

[6] Guo, W., The Theory of Class of Groups, Science Press-Kluwer Academic Publishers, Beijing, New York, Dordrecht, Boston, London, 2000.

[7] Guo, W., Lu, Y. and Niu, W., s-embedded subgroups of finite groups, *Algebra and Logic*, **49**(4), 2010, 293–304.

[8] Guo, W., Shum, K. P. and Skiba, A. N., On solubility and supersolubility of some classes of finite groups, *Sci. China Ser. A*, **52**, 2009, 272–286.

[9] Guo, W. and Skiba, A. N., Finite groups with given s-embedded and n-embedded subgroups, *J. Algebra*, **321**, 2009, 2843–2860.

[10] Guo, W., Skiba, A. N. and Yang, N., SE-supplemented subgroups of finite groups, *Rend. Sem. Mat. Univ. Padova*, **129**, 2013, 245-263.

[11] Guo, X. and Shum, K. P., On c-normal maximal and minimal subgroups of Sylow p-subgroups of finite groups, *Arch. Math.*, **80**, 2003, 561–569.

[12] Huppert, B., Endliche Gruppen I, Springer-Verlag, Berlin, Heidelberg, New York, 1967.

[13] Kegel, O., Sylow-Gruppen and subnormalteiler endlicher gruppen, *Math. Z.*, **78**, 1962, 205–221.

[14] Li, J., Chen, G. and Chen, R., On weakly s-embedded subgroups of finite groups, *Sci. China Math.*, **54**, 2011, 1899–1908.

[15] Li, S. and Li, Y., On S-quasinormal and c-normal subgroups of a finite group, *Czechoslovak Math. J.*, **58**(133), 2008, 1083–1095.

[16] Miao, L., On weakly s-permutable subgroups of finite groups, *Bull. Braz. Math. Soc. New Series*, **41**(2), 2010, 223–235.

[17] Robinson, D. J. S., A Course in Theory of Group, Spinger-Verlag, New York, 1982.

[18] Schmid, P., Subgroups permutable with all Sylow subgroups, *J. Algebra*, **207**, 1998, 285–293.

[19] Srinivasan, S., Two sufficient conditions for supersolvability of finite groups, *Israel Journal of Mathematics*, **35**, 1980, 210–214.

[20] Thompson, J. G., Normal p-complements for finite groups, *J. Algebra*, **1**, 1964, 43–46.

[21] Wang, Y., c-normality of groups and its properties, *J. Algebra*, **180**, 1996, 954–965.

[22] Wang, Y. and Guo, W., Nearly s-normality of groups and its properties, *Comm. Algebra*, **38**, 2010, 3821–3836.

[23] Wielandt, H., Subnormal Subgroups and Permutation Groups, Lectures Given at the Ohio State University, Columbus, Ohio, 1971.