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The Brio System with Initial Conditions Involving Dirac Masses: A Result Afforded by a Distributional Product*

C. O. R. SARRICO[1]

Abstract The Brio system is a 2×2 fully nonlinear system of conservation laws which arises as a simplified model in the study of plasmas. The present paper offers explicit solutions to this system subjected to initial conditions containing Dirac masses. The concept of a solution emerges within the framework of a distributional product and represents a consistent extension of the concept of a classical solution. Among other features, the result shows that the space of measures is not sufficient to contain all solutions of this problem.

Keywords Products of distributions, Brio's system, δ-Shock waves, δ'-Shock waves, Riemann problem
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1 Introduction and Contents

We consider the 2×2 system of conservation laws:

$$u_t + \Big(\frac{u^2+v^2}{2}\Big)_x = 0, \tag{1.1}$$

$$v_t + (uv - v)_x = 0, \tag{1.2}$$

where x is the one-dimensional space variable, t is the time variable and $u(x,t)$, $v(x,t)$ are the unknown state variables. This system, proposed by Brio [4], appears in the study of plasmas and corresponds to the coupling of the fluid dynamic equations with Maxwell's equations of electrodynamics.

The study of initial value problems of Riemann type for nonlinear conservation laws in spaces of distributions which include Dirac measures was initiated in 1977 by Korchinski [14] and has been widely developed in the last years (see for instance [5, 8–10, 12, 15, 21, 31–33]). Recently, we have also given a contribution to this problem by showing that, in our framework, the solution of certain Riemann problems for the Brio system can develop a delta shock wave (see [30]).

Let δ be the Dirac measure concentrated at the origin. In the present paper we subject the Brio system to the initial conditions $u(x,0) = c_0\delta(x)$, and $v(x,0) = h_0\delta(x)$ (with $c_0, h_0 \in \mathbb{R}$, $c_0 \neq 0$, $h_0 \neq 0$). We will prove that, in a convenient space of distributions and under certain compatibility conditions, three types of solutions for this problem arise:

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[1]CMAF, University of Lisbon, Av. Prof. Gama Pinto 2, 1649-003, Lisboa, Portugal.
E-mail: csarrico@ptmat.fc.ul.pt
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(1) $u(x,t) = c_0\delta(x - st),\ v(x,t) = h_0\delta(x - st)$;
(2) $u(x,t) = c_0\delta(x - t) - \frac{h_0^2}{c_0}t\delta'(x - t),\ v(x,t) = h_0\delta(x - t)$;
(3) $u(x,t) = c_0\delta(x - t) - \big[\sqrt{\frac{2h_0^2}{c_0\lambda}}\tanh\big(\frac{h_0^2}{c_0}\sqrt{\frac{c_0\lambda}{2h_0^2}}t\big)\big]\delta'(x - t)$, $v(x,t) = h_0\delta(x - t)$.

The first one is a travelling delta wave with speed $s = \frac{c_0^2+h_0^2}{c_0^2-h_0^2}$. The second and third ones are not measures; they are singular perturbations which propagate with speed 1 (indeed, the third one is not a solution but a family of solutions parametrized by an arbitrary number λ such that $c_0\lambda > 0$). It is worth to be noted that in each of those solutions the quantities $\int_{-\infty}^{+\infty} u(x,t)\mathrm{d}x = c_0$ and $\int_{-\infty}^{+\infty} v(x,t)\mathrm{d}x = h_0$ are conserved.

Thus we can conclude that, for the Brio system with initial conditions involving Dirac masses, the space of measures is not sufficient to contain all solutions of this problem. It is interesting to note that the space of discontinuous functions is not sufficient to contain all solutions of the Brio system with discontinuous functions as initial conditions as we have proved in [30].

A very interesting theory of delta shock waves with Dirac measures developing in both state variables for a class of nonstrictly hyperbolic systems of conservation laws was recently established by Yang and Zhang in [33]. We refer the reader to this important paper which has extensive references strongly related to this subject.

In the literature different concepts of a solution can be found: The measure theoretic method (see [1–2, 5, 11]), use of smooth function nets and weighted measure spaces (see [13]), split delta functions (see [19, 22]), Colombeau generalized functions (see [20]), the weak asymptotic method (see [9–10]), and others.

Our concept of a solution is defined within the setting of our theory of distributional products; here the product of two distributions is always a distribution and it is not defined by approximation processes. Our products depend upon the choice of a certain function α that encodes the indeterminacy inherent to products. Naturally, such an indeterminacy may appear, or not, in the outcoming of differential equations or systems. We call such solutions α-solutions. The occurrence of these solutions may be interpreted as a selection made by the physical system; in certain cases, this selection can not be previously known owing to some features of the physical system that were not considered in the formulation of the model. We stress that this indeterminacy is not, in general, avoidable and in many situations it also has a physical meaning. Concerning this point let us mention [3, 6–7, 24]. It goes without saying that, also in a classical context, sometimes one is led to impose on the nonlinear model extra conditions (such as entropy conditions) to make the model more adherent to physical situations and allow a decrease of the indeterminacy.

Let us present some results we have obtained within the framework of our distributional products.

For the conservation law $u_t + [\phi(u)]_x = \psi(u)$, where ϕ, ψ are entire functions taking real values on the real axis, we have established (see [28]) necessary and sufficient conditions for the propagation of a travelling wave with a given distributional profile and we have also computed its speed. For example, for LeVeque and Yee equation ($\phi(u) = u$ and $\psi(u) = \mu u(1-u)\big(u - \frac{1}{2}\big)$, with $\mu \neq 0$) we have proved that there exist six travelling waves with the profile $c_1 + (c_2 - c_1)H$ (c_1, c_2 are constants and H stands for the Heaviside function), all of them with speed 1. When $\psi = 0$ and $\phi'' \neq 0$, we conclude that the only continuous travelling waves are the constant states. Thus, if we ask for nonconstant travelling waves for the conservative equation $u_t + [\phi(u)]_x = 0$, with $\phi'' \neq 0$ we have to seek them among distributions that are not continuous functions; for wave profiles that are C^1-functions with one jump discontinuity, our methods lead us easily to the well-known Rankine-Hugoniot conditions.

Conditions for the propagation of wave profiles $\beta + m\delta$ and $\beta + m\delta'$ (where β is a continuous function, $m \in \mathbb{R}$, and $m \neq 0$) as well as their speeds were also obtained (see [29]). For example, for the diffusionless Burgers-Fischer equation

$$u_t + a\Big(\frac{1}{2}u^2\Big)_x = ru\Big(1 - \frac{u}{k}\Big),$$

where $a > 0$, $r > 0$ and $k > 0$, the profile $b + m\delta$ (where $b \in \mathbb{R}$) can arise as a travelling wave if and only if $b = 0$ or $b = k$ with wave speed $\frac{ak}{2}$ in both cases. For the Burgers conservative equation $u_t + \left(\frac{1}{2}u^2\right)_x = 0$, the profile $\beta + m\delta'$ can arise as a travelling wave if and only if $\beta = b$ is a constant function and the wave speed is b.

In the setting of soliton wave collision, we are able to prove that delta waves under collision behave just as classical soliton collisions (as in the Korteweg-de Vries equation) in models ruled by a singular perturbation of the Burgers conservative equation (see [26]). Also, the phenomenon of gas dynamics, known as "infinitely narrow soliton solution", discovered by Maslov and his collaborators (see [16–18]), can be described directly in distributional form (see [25]).

Let us summarize the contents of the present paper. In Section 2, we present a brief survey about our products of distributions in order to introduce the reader into this framework and keep computations self-contained. These formulas are presented without proofs. The reader can obtain a general view of our distributional products in Sections 2–3 of [24]. The details are given in [23]. In Section 3, we define the concept of an α-solution for the Brio system. We stress that this concept can be viewed as a consistent extension of the concept of a classical solution. Finally, in Section 4, we define a convenient space of distributions $\mathcal{W}$ and we compute all α-solutions of the Brio system that belong to $\mathcal{W}$ and satisfy the initial conditions $u(x,0) = c_0\delta(x)$, and $v(x,0) = h_0\delta(x)$.

2 Products of Distributions

Let $\mathcal{D}$ be the space of indefinitely differentiable complex-valued functions defined on $\mathbb{R}$, with compact support, and $\mathcal{D}'$ be the space of Schwartz distributions. In our theory a certain function $\alpha \in \mathcal{D}$ with $\int_{-\infty}^{+\infty} \alpha(x)\mathrm{d}x = 1$ affords a product $T \dot{\alpha} S$ of $T, S \in \mathcal{D}'$. When agreement with the usual Schwartz product of distributions with functions is desirable, as in the sequel, the α-product must be restricted to certain spaces of distributions to be singled out, as described below.

Our α-products are bilinear and satisfy the property $\tau_a(T \dot{\alpha} S) = (\tau_a T)\dot{\alpha}(\tau_a S)$, where τ_a

means the usual translation operator in the distributional sense. If α is even the α-products are also transformed as usual by the symmetry $t \mapsto -t$ from $\mathbb{R}$ onto $\mathbb{R}$. In general associativity and commutativity do not hold (recall that the products of distributions defined by Schwartz are not in general associative). The α-products can not be completely localized: Indeed, $\operatorname{supp}(T_{\dot\alpha}S) \subset \operatorname{supp} S$ as for usual functions, but it may happen that $\operatorname{supp}(T_{\dot\alpha}S) \not\subset \operatorname{supp} T$. Thus, our α-products are to be viewed as global entities and they are consistent with Schwartz products of $\mathcal{D}'^p$-distribution with C^p-functions (if these ones are placed on the right-hand side). The α-products satisfy the usual differential rules including the Leibniz formula that must be written in the form

$$D(T_{\dot\alpha}S) = (DT)_{\dot\alpha}S + T_{\dot\alpha}(DS).$$

Here we suppose that $T_{\dot\alpha}S$, $(DT)_{\dot\alpha}S$ and $T_{\dot\alpha}(DS)$ are defined by (2.1), (2.6) or (2.7) below.

In this section we only give formulas for three kinds of α-products which will be noted by the unique symbol $T_{\dot\alpha}S$ because they are mutually compatible (details for the α-products (2.1) and (2.6) can be seen in [24]; for the α-product (2.7), see [27]).

The first one can be evaluated by the formula

$$T_{\dot\alpha}S = T\beta + (T * \alpha)f \tag{2.1}$$

for $T \in \mathcal{D}'^p$, and $S = \beta + f \in C^p \oplus \mathcal{D}'_\mu$, where $p \in \{0, 1, 2, \cdots, \infty\}$, $\mathcal{D}'^p$ is the space of distributions of order $\leq p$ in the sense of Schwartz, ($\mathcal{D}'^\infty$ means $\mathcal{D}'$), $\mathcal{D}'_\mu$ is the space of distributions whose support has Lebesgue measure zero, $T\beta$ is the usual Schwartz product of a $\mathcal{D}'^p$-distribution with a C^p-function and $(T * \alpha)f$ is the usual product of a C^∞-function with a distribution. For instance, if β is a continuous function, we have, for any α,

$$\delta_{\dot\alpha}\beta = \delta_{\dot\alpha}(\beta + 0) = \delta\beta + (\delta * \alpha)0 = \beta(0)\delta,$$

$$\beta_{\dot\alpha}\delta = \beta_{\dot\alpha}(0 + \delta) = \beta 0 + (\beta * \alpha)\delta = [(\beta * \alpha)(0)]\delta,$$

$$\delta_{\dot\alpha}\delta = \delta_{\dot\alpha}(0 + \delta) = \delta 0 + (\delta * \alpha)\delta = \alpha\delta = \alpha(0)\delta, \tag{2.2}$$

$$\delta_{\dot\alpha}\delta' = (\delta * \alpha)\delta' = \alpha\delta' = \alpha(0)\delta' - \alpha'(0)\delta, \tag{2.3}$$

$$\delta'_{\dot\alpha}\delta = (\delta' * \alpha)\delta = \alpha'\delta = \alpha'(0)\delta, \tag{2.4}$$

$$\delta'_{\dot\alpha}\delta' = (\delta' * \alpha)\delta' = \alpha'\delta' = \alpha'(0)\delta' - \alpha''(0)\delta, \tag{2.5}$$

$$H_{\dot\alpha}\delta = (H * \alpha)\delta = \Big[\int_{-\infty}^{+\infty} \alpha(-\tau)H(\tau)\mathrm{d}\tau\Big]\delta = \Big(\int_{-\infty}^{0} \alpha(\tau)\mathrm{d}\tau\Big)\delta.$$

The second one is to be computed by the formula

$$T_{\dot\alpha}S = D(TF) - (DT)F + (T * \alpha)f \tag{2.6}$$

for $T \in \mathcal{D}'^{-1}$ and $S = w + f \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_\mu$, where $\mathcal{D}'^{-1}$ stands for the space of distributions $T \in \mathcal{D}'$ such that $DT \in \mathcal{D}'^0$, and $F \in C^0$ is such that $DF = w$ (so, locally, T can be read as a function of bounded variation and F as an absolutely continuous function). In [24], we have proved that $T_{\dot\alpha}S$ given by (2.6) is independent of the choice of the function F such that $DF = w$. For instance, since $H \in \mathcal{D}'^{-1}$ and $H = H + 0 \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_\mu$, we have

$$H_{\dot\alpha}H = D(HF) - (DH)F + (H * \alpha)0 = DF - \delta F = H,$$

taking $F : \mathbb{R} \to \mathbb{R}$ defined by $F(x) = 0$ for $x \le 0$ and $F(x) = x$ for $x > 0$. More generally, if $T \in \mathcal{D}'^{-1}$ and $S \in L^1_{\mathrm{loc}}$, then we have $T \dot{_\alpha} S = TS$ (see [27, p. 1002] for a proof). We want to stress that in (2.1) or (2.6) the convolution $T * \alpha$ is not to be understood as an approximation of T. Those formulas are to be considered as exact ones.

The third α-product can be computed by the formula

$$T \dot{_\alpha} S = D(Y \dot{_\alpha} S) - Y \dot{_\alpha} (DS) \tag{2.7}$$

for $T \in \mathcal{D}'^0 \cap \mathcal{D}'_\mu$ and $S \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_c$ with $DS \in L^1_{\mathrm{loc}} \oplus \mathcal{D}'_c$, where $\mathcal{D}'_c \subset \mathcal{D}'_\mu$ is the space of distributions, the support of which is at most countable, and $Y \in \mathcal{D}'^{-1}$ is such that $DY = T$ (the products $Y \dot{_\alpha} S$ and $Y \dot{_\alpha} (DS)$ are supposed to be computed by (2.1) or (2.6)). The value of $T \dot{_\alpha} S$ given by (2.7) is independent of the choice of $Y \in \mathcal{D}'^{-1}$ such that $DY = T$ (see [27, p. 1004] for the proof). For instance, we have for any α,

$$\delta \dot{_\alpha} H = D(H \dot{_\alpha} H) - H \dot{_\alpha} (DH) = DH - H \dot{_\alpha} \delta = \delta - \Big(\int_{-\infty}^{0} \alpha(x) \mathrm{d}x \Big) \delta = \Big(\int_{0}^{+\infty} \alpha(x) \mathrm{d}x \Big) \delta.$$

3 The Concept of α-Solution for the Brio System

Let I be an interval of $\mathbb{R}$ with more than one point and let $\mathcal{F}(I)$ be the space of continuously differentiable maps $\widetilde{u} : I \to \mathcal{D}'$ in the sense of the usual topology of $\mathcal{D}'$. For $t \in I$, the notation $[\widetilde{u}(t)](x)$ is sometimes used to emphasize that the distribution $\widetilde{u}(t)$ acts on functions $\xi \in \mathcal{D}$ which depend on x.

Let $\Sigma(I)$ be the space of functions $u : \mathbb{R} \times I \to \mathbb{C}$ such that

(a) for each $t \in I$, $u(x, t) \in L^1_{\mathrm{loc}}(\mathbb{R})$;

(b) $\widetilde{u} : I \to \mathcal{D}'$ defined by $[\widetilde{u}(t)](x) = u(x, t)$ is in $\mathcal{F}(I)$.

The natural injection $u \mapsto \widetilde{u}$ of $\Sigma(I)$ into $\mathcal{F}(I)$ allows us to identify any function in $\Sigma(I)$ with a certain map in $\mathcal{F}(I)$. Since $C^1(\mathbb{R} \times I) \subset \Sigma(I)$, we can write the inclusions

$$C^1(\mathbb{R} \times I) \subset \Sigma(I) \subset \mathcal{F}(I).$$

The identification $u \mapsto \widetilde{u}$ allows us to write the Brio system (1.1)–(1.2) in the form:

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}(t) + \frac{1}{2} D[\widetilde{u}(t) \dot{_\alpha} \widetilde{u}(t) + \widetilde{v}(t) \dot{_\alpha} \widetilde{v}(t)] = 0, \tag{3.1}$$

$$\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}t}(t) + D[\widetilde{u}(t) \dot{_\alpha} \widetilde{v}(t) - \widetilde{v}(t)] = 0. \tag{3.2}$$

Definition 3.1 *Given α, the pair $(\widetilde{u}, \widetilde{v}) \in \mathcal{F}(I) \times \mathcal{F}(I)$ will be called an α-solution of the system* (3.1)–(3.2) *on I, if the α-products that appear in these equations are well defined and both equations are satisfied for all $t \in I$.*

We have the following results.

Theorem 3.1 *If (u, v) is a classical solution of* (1.1)–(1.2) *on $\mathbb{R} \times I$, then, for any α, the pair $(\widetilde{u}, \widetilde{v}) \in \mathcal{F}(I) \times \mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x, t)$ and $[\widetilde{v}(t)](x) = v(x, t)$, is an α-solution of* (3.1)–(3.2) *on I.*

Remark 3.1 By a classical solution of (1.1)–(1.2) on $\mathbb{R}\times I$, we mean a pair $(u(x,t), v(x,t))$ of C^1-functions that satisfies (1.1)–(1.2) on $\mathbb{R}\times I$.

Theorem 3.2 *If $u, v : \mathbb{R}\times I \to \mathbb{C}$ are C^1-functions and, for a certain α, the pair $(\widetilde{u}, \widetilde{v}) \in \mathcal{F}(I)\times\mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$ and $[\widetilde{v}(t)](x) = v(x,t)$, is an α-solution of* (3.1)–(3.2) *on I, then the pair (u, v) is a classical solution of* (1.1)–(1.2) *on $\mathbb{R}\times I$.*

For the proof, it is enough to observe that any C^1-function $u(x,t)$ can be read as a continuously differentiable function $\widetilde{u} \in \mathcal{F}(I)$ defined by $[\widetilde{u}(t)](x) = u(x,t)$ and to use the consistency of the α-products with the classical Schwartz products.

If we change $\widetilde{u}(t)\dot{\alpha}\widetilde{v}(t)$ to $\widetilde{v}(t)\dot{\alpha}\widetilde{u}(t)$ in the equation (3.2), we obtain the equation

$$\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}t}(t) + D[\widetilde{v}(t)\dot{\alpha}\widetilde{u}(t) - \widetilde{v}(t)] = 0, \tag{3.3}$$

which is not equivalent to (3.2), since our α-products are not, in general, commutative. However, all that we have said above for the systems (1.1)–(1.2) and (3.1)–(3.2) are also valid for the systems (1.1)–(1.2) and (3.1)–(3.3). Thus, in order to simplify the language, we will call α-solution of the system (1.1)–(1.2) an α-solution of the system (3.1)–(3.2), or of the system (3.1)–(3.3). Hence, an α-solution in this sense affords a consistent extension of the concept of a classical solution of the system (1.1)–(1.2).

4 Dirac Masses as Initial Conditions

As we have said in the introduction, we have already applied our framework to prove that the solution of a Riemann problem for the Brio system can develop a δ-shock wave. This suggests as to consider the possible formation of a δ'-shock wave from initial conditions that contain Dirac masses.

Thus, let us consider the Brio system (1.1)–(1.2) with $(x,t) \in \mathbb{R}\times\mathbb{R}$ and the unknowns $u(x,t)$ and $v(x,t)$ submitted to the initial conditions:

$$u(x,0) = c_0\delta(x), \tag{4.1}$$

$$v(x,0) = h_0\delta(x), \tag{4.2}$$

where $c_0, h_0 \in \mathbb{R}$ with $c_0 \neq 0$ and $h_0 \neq 0$. We will compute all α-solutions of this problem which belong to the following space $\mathcal{W}$: The pair $(\widetilde{u}, \widetilde{v}) \in \mathcal{W}$, if and only if $\widetilde{u}, \widetilde{v} \in \mathcal{F}(\mathbb{R})$ and there exist C^1-functions $c, f, h, \gamma : \mathbb{R} \to \mathbb{R}$ such that

$$\widetilde{u}(t) = c(t)\tau_{\gamma(t)}\delta + f(t)\tau_{\gamma(t)}(D\delta), \tag{4.3}$$

$$\widetilde{v}(t) = h(t)\tau_{\gamma(t)}\delta. \tag{4.4}$$

Clearly we can also consider more general initial conditions and more general spaces of solutions. Meanwhile, the calculations are much longer and our choice is a simple way to see that the space of measures is not sufficient to contain all α-solutions of this problem!

In the sequel, we will apply the following result.

Lemma 4.1 *Let us consider the initial value problem*

$$y'(t) = ay^2(t) + b, \tag{4.5}$$

$$y(0) = 0, \tag{4.6}$$

where $a, b \in \mathbb{R}$ and $b \neq 0$. Then this problem has a global solution $y \in C^1(\mathbb{R})$ if and only if $\frac{a}{b} \leq 0$. We also have

(a) *if $\frac{a}{b} = 0$, the unique global solution is $y(t) = bt$;*

(b) *if $\frac{a}{b} < 0$, the unique global solution is $y(t) = \sqrt{-\frac{b}{a}} \tanh\left(b\sqrt{-\frac{a}{b}}t\right)$.*

Proof If $\frac{a}{b} = 0$, the proof is trivial. For $\frac{a}{b} < 0$, let us consider the change of the unknown $y(t) \mapsto z(t)$ defined by

$$y(t) = \sqrt{-\frac{b}{a}} \tanh[z(t)]. \tag{4.7}$$

We have $y' = \sqrt{-\frac{b}{a}}(1 - \tanh^2 z)z'$ and (4.5)–(4.6) turn out to be

$$\sqrt{-\frac{b}{a}}(1 - \tanh^2 z)z' = -b \tanh^2 z + b,$$

$$\sqrt{-\frac{b}{a}} \tanh z(0) = 0.$$

This problem is equivalent to the initial value problem $z' = b\sqrt{-\frac{a}{b}}$, $z(0) = 0$ which has the unique solution $z(t) = b\sqrt{-\frac{a}{b}}t$; hence (b) follows from (4.7). For $\frac{a}{b} > 0$, (4.5) is equivalent to the equation $\frac{y'}{ay^2+b} = 1$ which can be easily solved by separation of variables. The unique solution

$$y(t) = \sqrt{\frac{b}{a}} \tan\left(b\sqrt{\frac{a}{b}}t\right)$$

defined in any interval $]-\varepsilon, \varepsilon[$ with $0 < \varepsilon < \frac{\pi}{2|b|}\sqrt{\frac{b}{a}}$, can not be extended to all $\mathbb{R}$ as a C^1-function and we have not a global solution in this case. The statement is proved.

Now, let us consider the problem (3.1)–(3.2) submitted to the initial conditions $\widetilde{u}(0) = c_0\delta$ and $\widetilde{v}(0) = h_0\delta$ which are, in $\mathcal{F}(\mathbb{R})$, the conditions corresponding to (4.1)–(4.2):

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}(t) + \frac{1}{2}D[\widetilde{u}(t) \dot{\alpha} \widetilde{u}(t) + \widetilde{v}(t) \dot{\alpha} \widetilde{v}(t)] = 0, \tag{4.8}$$

$$\frac{\mathrm{d}\widetilde{v}}{\mathrm{d}t}(t) + D[\widetilde{u}(t) \dot{\alpha} \widetilde{v}(t) - \widetilde{v}(t)] = 0, \tag{4.9}$$

$$\widetilde{u}(0) = c_0\delta, \tag{4.10}$$

$$\widetilde{v}(0) = h_0\delta. \tag{4.11}$$

We have the following result.

Theorem 4.1 *Given α, the initial value problem* (4.8)–(4.11) *has an α-solution in $\mathcal{W}$ precisely if and only if any of the following three conditions* (*each one of them excludes the others*) *is satisfied*:

(I) $c_0^2 \neq h_0^2$, $\frac{2c_0}{c_0^2-h_0^2} = \alpha(0)$;
(II) $\frac{2}{c_0} = \alpha(0)$, $\alpha'(0) = 0, \alpha''(0) = 0$;
(III) $\frac{2}{c_0} = \alpha(0)$, $\alpha'(0) = 0$, $c_0\alpha''(0) > 0$.

Under (I) *the unique* α*-solution in* $\mathcal{W}$ *is given by* $(\widetilde{u}, \widetilde{v})$ *with*

$$\widetilde{u}(t) = c_0\, \tau_{st}\, \delta, \quad \widetilde{v}(t) = h_0\, \tau_{st}\, \delta,$$

where $s = \frac{c_0^2+h_0^2}{c_0^2-h_0^2}$.

Under (II) *the unique* α*-solution in* $\mathcal{W}$ *is given by* $(\widetilde{u}, \widetilde{v})$ *with*

$$\widetilde{u}(t) = c_0\, \tau_t\delta - \frac{h_0^2}{c_0} t\, \tau_t(D\delta), \quad \widetilde{v}(t) = h_0\, \tau_t\delta.$$

Under (III) *the unique* α*-solution in* $\mathcal{W}$ *is given by* $(\widetilde{u}, \widetilde{v})$ *with*

$$\begin{aligned}\widetilde{u}(t) &= c_0\, \tau_t\delta - \Big[\sqrt{\frac{2h_0^2}{c_0\alpha''(0)}} \tanh\Big(\frac{h_0^2}{c_0}\sqrt{\frac{c_0\alpha''(0)}{2h_0^2}} t\Big)\Big]\tau_t(D\delta),\\ \widetilde{v}(t) &= h_0\, \tau_t\delta.\end{aligned}$$

Proof Clearly, (I) excludes (II) and (III) since the equality $\frac{2c_0}{c_0^2-h_0^2} = \frac{2}{c_0}$ implies $h_0 = 0$ which is impossible. The other exclusions are obvious.

Let us suppose $(\widetilde{u}, \widetilde{v}) \in \mathcal{W}$. Then by (4.3)–(4.4), we have

$$\begin{aligned}\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}(t) &= c'(t)\tau_{\gamma(t)}\delta + c(t)\tau_{\gamma(t)}(D\delta)[-\gamma'(t)]\\ &\quad + f'(t)\tau_{\gamma(t)}(D\delta) + f(t)\tau_{\gamma(t)}(D^2\delta)[-\gamma'(t)],\\ \frac{\mathrm{d}\widetilde{v}}{\mathrm{d}t}(t) &= h'(t)\tau_{\gamma(t)}\delta + h(t)\tau_{\gamma(t)}(D\delta)[-\gamma'(t)].\end{aligned}$$

Applying (2.2)–(2.5) when necessary, we also have

$$\begin{aligned}\widetilde{u}(t)\,\dot{\alpha}\,\widetilde{u}(t) &= c^2(t)\alpha(0)\tau_{\gamma(t)}\delta + c(t)f(t)\tau_{\gamma(t)}[\alpha(0)(D\delta) - \alpha'(0)\delta]\\ &\quad + f(t)c(t)\tau_{\gamma(t)}[\alpha'(0)\delta] + f^2(t)\tau_{\gamma(t)}[\alpha'(0)(D\delta) - \alpha''(0)\delta]\\ &= [c^2(t)\alpha(0) - f^2(t)\alpha''(0)]\tau_{\gamma(t)}\delta + [c(t)f(t)\alpha(0) + f^2(t)\alpha'(0)]\tau_{\gamma(t)}(D\delta),\\ \widetilde{v}(t)\,\dot{\alpha}\,\widetilde{v}(t) &= h^2(t)\alpha(0)\tau_{\gamma(t)}\delta,\\ \widetilde{u}(t)\,\dot{\alpha}\,\widetilde{u}(t) + \widetilde{v}(t)\,\dot{\alpha}\,\widetilde{v}(t) &= [c^2(t)\alpha(0) - f^2(t)\alpha''(0) + h^2(t)\alpha(0)]\tau_{\gamma(t)}\delta\\ &\quad + [c(t)f(t)\alpha(0) + f^2(t)\alpha'(0)]\tau_{\gamma(t)}(D\delta),\\ \frac{1}{2}D[\widetilde{u}(t)\,\dot{\alpha}\,\widetilde{u}(t) + \widetilde{v}(t)\,\dot{\alpha}\,\widetilde{v}(t)] &= \Big[c^2(t)\frac{\alpha(0)}{2} - f^2(t)\frac{\alpha''(0)}{2}\Big]\tau_{\gamma(t)}(D\delta)\\ &\quad + \Big[c(t)f(t)\frac{\alpha(0)}{2} + f^2(t)\frac{\alpha'(0)}{2}\Big]\tau_{\gamma(t)}(D^2\delta),\\ \widetilde{u}(t)\,\dot{\alpha}\,\widetilde{v}(t) &= c(t)h(t)\alpha(0)\tau_{\gamma(t)}\delta + f(t)h(t)\alpha'(0)\tau_{\gamma(t)}\delta\\ &= [c(t)h(t)\alpha(0) + f(t)h(t)\alpha'(0)]\tau_{\gamma(t)}\delta,\\ D[\widetilde{u}(t)\,\dot{\alpha}\,\widetilde{v}(t) - \widetilde{v}(t)] &= [c(t)h(t)\alpha(0) + f(t)h(t)\alpha'(0) - h(t)]\tau_{\gamma(t)}(D\delta).\end{aligned}$$

Thus, the problem (4.8)–(4.11) has an α-solution in $\mathcal{W}$ if and only if, for all $t \in \mathbb{R}$, the following four equations are satisfied:

$$c'(t)\tau_{\gamma(t)}\delta + \Big[- c(t)\gamma'(t) + f'(t) + c^2(t)\frac{\alpha(0)}{2} + f^2(t)\frac{\alpha''(0)}{2} + h^2(t)\frac{\alpha(0)}{2}\Big]\tau_{\gamma(t)}(D\delta)$$

$$+ \Big[- f(t)\gamma'(t) + c(t)f(t)\frac{\alpha(0)}{2} + f^2(t)\frac{\alpha'(0)}{2}\Big]\tau_{\gamma(t)}(D^2\delta) = 0, \tag{4.12}$$

$$h'(t)\tau_{\gamma(t)}\delta + [-h(t)\gamma'(t) + c(t)h(t)\alpha(0) + f(t)h(t)\alpha'(0) - h(t)]\tau_{\gamma(t)}(D\delta) = 0, \tag{4.13}$$

$$c(0)\tau_{\gamma(0)}\delta + f(0)\tau_{\gamma(0)}(D\delta) = c_0\delta, \tag{4.14}$$

$$h(0)\tau_{\gamma(0)}\delta = h_0\delta. \tag{4.15}$$

As a consequence, the problem (4.8)–(4.11) has an α-solution in $\mathcal{W}$ if and only if, for all $t \in \mathbb{R}$, the following conditions are satisfied:

$$c'(t) = 0, \tag{4.16}$$

$$- c(t)\gamma'(t) + f'(t) + c^2(t)\frac{\alpha(0)}{2} - f^2(t)\frac{\alpha''(0)}{2} + h^2(t)\frac{\alpha(0)}{2} = 0, \tag{4.17}$$

$$- f(t)\gamma'(t) + c(t)f(t)\frac{\alpha(0)}{2} + f^2(t)\frac{\alpha'(0)}{2} = 0, \tag{4.18}$$

$$h'(t) = 0, \tag{4.19}$$

$$- h(t)\gamma'(t) + c(t)h(t)\alpha(0) + f(t)h(t)\alpha'(0) - h(t) = 0. \tag{4.20}$$

From (4.14) we conclude that $f(0) = 0$. Thus, $c(0)\tau_{\gamma(t)}\delta = c_0\delta$ and it is easy to see that $c(0) = c_0$ and $\gamma(0) = 0$. From (4.15) we conclude that $h(0) = h_0$ and $\gamma(0) = 0$ as well. Conditions (4.16) and (4.19) jointly with $c(0) = c_0$ and $h(0) = h_0$ enable us to conclude that $c(t) = c_0$ and $h(t) = h_0$ for all $t \in \mathbb{R}$. Thus, (4.17)–(4.18) and (4.20) turn out to be

$$- c_0\gamma'(t) + f'(t) + c_0^2\frac{\alpha(0)}{2} - \frac{\alpha''(0)}{2}f^2(t) + h_0^2\frac{\alpha(0)}{2} = 0, \tag{4.21}$$

$$- f(t)\gamma'(t) + c_0\frac{\alpha(0)}{2}f(t) + \frac{\alpha'(0)}{2}f^2(t) = 0, \tag{4.22}$$

$$- h_0\gamma'(t) + c_0h_0\alpha(0) + h_0\alpha'(0)f(t) - h_0 = 0. \tag{4.23}$$

From (4.23), it follows

$$\gamma'(t) = \alpha'(0)f(t) + c_0\alpha(0) - 1, \tag{4.24}$$

and (4.21)–(4.22) turn out to be

$$f'(t) = \frac{\alpha''(0)}{2}f^2(t) + c_0\alpha'(0)f(t) + c_0^2\frac{\alpha(0)}{2} - c_0 - h_0^2\frac{\alpha(0)}{2}, \tag{4.25}$$

$$- \frac{\alpha'(0)}{2}f^2(t) - c_0\frac{\alpha(0)}{2}f(t) + f(t) = 0. \tag{4.26}$$

Taking the derivative of (4.26), we have, for all $t \in \mathbb{R}$,

$$-\alpha'(0)f(t)f'(t) - c_0\frac{\alpha(0)}{2}f'(t) + f'(t) = 0,$$

and from the particular value $t = 0$ we obtain

$$\Big[-c_0 \frac{\alpha(0)}{2} + 1 \Big] f'(0) = 0.$$

Thus, we have two possibilities: $f'(0) = 0$ or $\alpha(0) = \frac{2}{c_0}$.

Let us suppose $f'(0) = 0$. Then putting $t = 0$ in (4.25) we obtain $\frac{\alpha(0)}{2}(c_0^2 - h_0^2) = c_0$ and this means that $c_0^2 \neq h_0^2$ and $\alpha(0) = \frac{2c_0}{c_0^2 - h_0^2}$. Thus, (4.24)–(4.26) give

$$\gamma'(t) = \alpha'(0) f(t) + \frac{c_0^2 + h_0^2}{c_0^2 - h_0^2}, \tag{4.27}$$

$$f'(t) = \frac{\alpha''(0)}{2} f^2(t) + c_0 \alpha'(0) f(t), \tag{4.28}$$

$$-\frac{\alpha'(0)}{2} f^2(t) - \frac{h_0^2}{c_0^2 - h_0^2} f(t) = 0. \tag{4.29}$$

Now, if $\alpha'(0) = 0$, from (4.29) we have $f(t) = 0$ for all $t \in \mathbb{R}$ and from (4.27) we obtain $\gamma'(t) = \frac{c_0^2 + h_0^2}{c_0^2 - h_0^2}$. If $\alpha'(0) \neq 0$, from (4.29) we have

$$f^2(t) = -\frac{2h_0^2}{\alpha'(0)(c_0^2 - h_0^2)} f(t)$$

and (4.28) turns out to be

$$f'(t) = \Big[-\frac{\alpha''(0) h_0^2}{\alpha'(0)(c_0^2 - h_0^2)} + c_0 \alpha'(0) \Big] f(t);$$

this differential equation with the initial condition $f(0) = 0$ has the unique solution $f(t) = 0$ for all $t \in \mathbb{R}$, which leads once more to $\gamma'(t) = \frac{c_0^2 + h_0^2}{c_0^2 - h_0^2}$; since $\gamma(0) = 0$, case (I) follows.

Let us suppose $\alpha(0) = \frac{2}{c_0}$. Then (4.24)–(4.26) turn out to be

$$\gamma'(t) = \alpha'(0) f(t) + 1, \tag{4.30}$$

$$f'(t) = \frac{\alpha''(0)}{2} f^2(t) + c_0 \alpha'(0) f(t) - \frac{h_0^2}{c_0}, \tag{4.31}$$

$$-\frac{\alpha'(0)}{2} f^2(t) = 0. \tag{4.32}$$

From (4.31) we know that $f(t)$ does not vanish identically and $\alpha'(0) = 0$ follows from (4.32); then, we can write (4.30)–(4.31) in the form:

$$\gamma'(t) = 1, \tag{4.33}$$

$$f'(t) = \frac{\alpha''(0)}{2} f^2(t) - \frac{h_0^2}{c_0}. \tag{4.34}$$

From (4.33) and $\gamma(0) = 0$ we conclude that $\gamma(t) = t$; from (4.34) and $f(0) = 0$ we conclude, applying Lemma 4.1, that:

(a) If $\alpha''(0) = 0$, then $f(t) = -\frac{h_0^2}{c_0} t$;

(b) If $c_0 \alpha''(0) > 0$, then $f(t) = -\sqrt{\frac{2h_0^2}{c_0 \alpha''(0)}} \tanh\big(\frac{h_0^2}{c_0} \sqrt{\frac{c_0 \alpha''(0)}{2h_0^2}} t\big)$.

Cases (II) and (III) follow immediately.

This theorem furnishes three types of α-solutions of the problem (1.1)–(1.2), (4.1)–(4.2). These solutions, subjected to the admissibility conditions already specified, read thus:

(1) $u(x,t) = c_0\delta\big(x - \frac{c_0^2+h_0^2}{c_0^2-h_0^2}t\big),\ v(x,t) = h_0\delta\big(x - \frac{c_0^2+h_0^2}{c_0^2-h_0^2}t\big)$;

(2) $u(x,t) = c_0\delta(x-t) - \frac{h_0^2}{c_0}t\delta'(x-t),\ v(x,t) = h_0\delta(x-t)$;

(3) $u(x,t) = c_0\delta(x-t) - \Big[\sqrt{\frac{2h_0^2}{c_0\alpha''(0)}}\tanh\Big(\frac{h_0^2}{c_0}\sqrt{\frac{c_0\alpha''(0)}{2h_0^2}}t\Big)\Big]\delta'(x-t),\ v(x,t) = h_0\delta(x-t)$.

The first one is a travelling delta wave with speed $\frac{c_0^2+h_0^2}{c_0^2-h_0^2}$, and the second and third ones are not measures. Also remark that the first and second ones do not depend on the choice of α; the third one is a family of solutions parametrized by an arbitrary number $\alpha''(0)$ such that $c_0\alpha''(0) > 0$.

About the interpretation of the physical occurrence of an α-solution we have already mentioned it in the introduction.

The next theorem deals with the same problem for the Brio system by reversing in (4.9) the order of factors $\widetilde{u}, \widetilde{v}$; it is an interesting feature that only solutions of the first type are obtained. Thus, let us solve in $\mathcal{W}$ the problem (4.8), (3.3), (4.10), (4.11), which, due to the noncommutativity of our product, does not coincides with the problem (4.8)–(4.11) we have considered above. We have the following result.

Theorem 4.2 *Given α, the initial value problem* (4.8), (3.3), (4.10), (4.11) *has an α-solution in $\mathcal{W}$ if and only if $c_0^2 \neq h_0^2$ and $\frac{2c_0}{c_0^2-h_0^2} = \alpha(0)$. This α-solution is unique in $\mathcal{W}$ and is given by $(\widetilde{u}, \widetilde{v})$ with*

$$\widetilde{u}(t) = c_0\,\tau_{st}\,\delta \quad \text{and} \quad \widetilde{v}(t) = h_0\,\tau_{st}\,\delta,$$

where $s = \frac{c_0^2+h_0^2}{c_0^2-h_0^2}$.

Proof Let us suppose $(\widetilde{u}, \widetilde{v}) \in \mathcal{W}$. Then, by applying (2.2)–(2.3), we have

$$\begin{aligned}
\widetilde{v}(t)\dot{\alpha}\widetilde{u}(t) &= h(t)c(t)\alpha(0)\tau_{\gamma(t)}\delta + h(t)f(t)\tau_{\gamma(t)}[\alpha(0)(D\delta) - \alpha'(0)\delta] \\
&= [h(t)c(t)\alpha(0) - h(t)f(t)\alpha'(0)]\tau_{\gamma(t)}\delta + h(t)f(t)\alpha(0)\tau_{\gamma(t)}(D\delta), \\
D[\widetilde{v}(t)\dot{\alpha}\widetilde{u}(t) - \widetilde{v}(t)] &= [h(t)c(t)\alpha(0) - h(t)f(t)\alpha'(0) - h(t)]\tau_{\gamma(t)}D\delta \\
&\quad + h(t)f(t)\alpha(0)\tau_{\gamma(t)}(D^2\delta),
\end{aligned}$$

so that (3.3) turns out to be

$$\begin{aligned}
&h'(t)\tau_{\gamma(t)}\delta + h(t)(-\gamma'(t))\tau_{\gamma(t)}D\delta + [h(t)c(t)\alpha(0) - h(t)f(t)\alpha'(0) - h(t)]\tau_{\gamma(t)}D\delta \\
&+ h(t)f(t)\alpha(0)\tau_{\gamma(t)}D^2\delta = 0;
\end{aligned}$$

this means that the problem (4.8), (3.3), (4.10), (4.11) has an α-solution in $\mathcal{W}$ if and only if the conditions (4.12) and (4.14)–(4.15) are satisfied together with the condition

$$\begin{aligned}
&h'(t)\tau_{\gamma(t)}\delta + [-h(t)\gamma'(t) + h(t)c(t)\alpha(0) - h(t)f(t)\alpha'(0) - h(t)]\tau_{\gamma(t)}D\delta \\
&+ h(t)f(t)\alpha(0)\tau_{\gamma(t)}D^2\delta = 0.
\end{aligned}$$

As a consequence, we conclude that $f(0)=0$, $c(0)=c_0$, $\gamma(0)=0$, $h(0)=0$ and also that the following conditions are to be satisfied (they correspond to conditions (4.16)–(4.20) of the previous theorem):

$$c'(t)=0, \tag{4.35}$$

$$-c(t)\gamma'(t)+f'(t)+c^2(t)\frac{\alpha(0)}{2}-f^2(t)\frac{\alpha''(0)}{2}+h^2(t)\frac{\alpha(0)}{2}=0, \tag{4.36}$$

$$-f(t)\gamma'(t)+c(t)f(t)\frac{\alpha(0)}{2}+f^2(t)\frac{\alpha'(0)}{2}=0, \tag{4.37}$$

$$h'(t)=0, \tag{4.38}$$

$$h(t)[-\gamma'(t)+c(t)\alpha(0)-f(t)\alpha'(0)-1]=0, \tag{4.39}$$

$$h(t)f(t)\alpha(0)=0. \tag{4.40}$$

From (4.35) and (4.38) we conclude that $c(t)=c_0$ and $h(t)=h_0$, and (4.36)–(4.37) and (4.39)–(4.40) turn out to be

$$-c_0\gamma'(t)+f'(t)+\frac{1}{2}c_0^2\alpha(0)-\frac{\alpha''(0)}{2}f^2(t)+\frac{1}{2}h_0^2\alpha(0)=0, \tag{4.41}$$

$$-f(t)\gamma'(t)+\frac{1}{2}c_0\alpha(0)f(t)+\frac{1}{2}\alpha'(0)f^2(t)=0, \tag{4.42}$$

$$h_0[-\gamma'(t)+c_0\alpha(0)-f(t)\alpha'(0)-1]=0, \tag{4.43}$$

$$h_0\alpha(0)f(t)=0. \tag{4.44}$$

From (4.44) we have $\alpha(0)=0$ or $f(t)=0$ for all $t\in\mathbb{R}$. Now, we will see that we can not have $\alpha(0)=0$. Indeed, if $\alpha(0)=0$, (4.41)–(4.43) would read

$$f'(t)=\frac{\alpha''(0)}{2}f^2(t)+c_0\gamma'(t), \tag{4.45}$$

$$-f(t)\gamma'(t)+\frac{1}{2}\alpha'(0)f^2(t)=0, \tag{4.46}$$

$$\gamma'(t)=-f(t)\alpha'(0)-1. \tag{4.47}$$

Applying (4.47) we can write (4.46) in the form:

$$f(t)+\frac{3}{2}\alpha'(0)f^2(t)=0,$$

and taking the derivative, we have, for all $t\in\mathbb{R}$, $f'(t)+3\alpha'(0)f(t)f'(t)=0$. Putting $t=0$ we conclude that $f'(0)=0$. Now, putting $t=0$ in (4.45) and (4.47), we obtain $0=c_0\gamma'(0)$ and $\gamma'(0)=-1$, which is a contradiction.

As a consequence, we have $f(t)=0$ for all $t\in\mathbb{R}$, and from (4.41) and (4.43), we obtain

$$-c_0\gamma'(t)+\frac{1}{2}c_0^2\alpha(0)+\frac{1}{2}h_0^2\alpha(0)=0,$$
$$\gamma'(t)=c_0\alpha(0)-1.$$

This linear system in the unknowns $\alpha(0)$ and $\gamma'(t)$ has a solution, just,

$$\alpha(0)=\frac{2c_0}{c_0^2-h_0^2},\quad \gamma'(t)=\frac{c_0^2+h_0^2}{c_0^2-h_0^2},$$

if and only if $c_0^2 \neq h_0^2$. The statement follows immediately.

Thus, this theorem does not introduce more α-solutions for the problem (1.1), (1.2), (4.1), (4.2) and we conclude that all α-solutions for this problem were given in Theorem 4.1.

A final remark. We know that there exist nonlinear systems for which the singular solutions, as obtained by approximation, are exactly the same as those obtained within our distributional framework. However, for the Brio system, we don't know if it is possible to obtain the results of the present paper by approximation. In this setting, and in several cases, the solutions are strongly affected by the process of approximation chosen.

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