Chin. Ann. Math.
35B(6), 2014, 955–968
DOI: 10.1007/s11401-014-0861-9

Chinese Annals of Mathematics, Series B

Evolution Equations of Curvature Tensors Along the Hyperbolic Geometric Flow

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Abstract The author considers the hyperbolic geometric flow $\frac{\partial^2}{\partial t^2}g(t) = -2\mathrm{Ric}_{g(t)}$ introduced by Kong and Liu. Using the techniques and ideas to deal with the evolution equations along the Ricci flow by Brendle, the author derives the global forms of evolution equations for Levi-Civita connection and curvature tensors under the hyperbolic geometric flow. In addition, similarly to the Ricci flow case, it is shown that any solution to the hyperbolic geometric flow that develops a singularity in finite time has unbounded Ricci curvature.

Keywords Hyperbolic geometric flow, Evolution equations, Singularity
2010 MR Subject Classification 53C21, 53C44, 58J45

1 Introduction

Geometric flows are important in many fields of mathematics and physics. A geometric flow is an evolution of a geometric structure under a differential equation related to a functional on a manifold, usually associated with some curvatures. The most popular geometric flows in mathematics are the heat flow (see [12, 18]), the Ricci flow (see [3, 6, 21]), the mean curvature flow (see [26]) and the Yamabe flow (see [1–2, 5, 20, 25]).

The subject of Hamilton's Ricci flow (see [9]): $\frac{\partial}{\partial t}g(t) = -2\mathrm{Ric}_{g(t)}$ lies in the more general field of geometric flows, which in turn lies in the even more general field of geometric analysis. In Ricci flow we see the unity of geometry and analysis. As a fully nonlinear system of parabolic partial differential equations of second order (see [22]), the Ricci flow in many respects appears to be very natural equation.

Note that most of the contributions from geometric analysis to date have involved either elliptic or parabolic equation. The elliptic and parabolic partial differential equations have been successfully applied to differential geometry and physics. A natural and important question is whether the well-developed theory of hyperbolic differential equations may be applied to solve problems in differential geometry and theoretical physics. In geometry, a good example is the singularity developed in general relativity, but the major problem here is that one has very little understanding of the global behavior of nonlinear hyperbolic systems when the spatial dimension is greater than one (see [19]). In fact, there are plenty of existing problems that involve hyperbolic equations. To this end, Kong and Liu [14] introduced a new flow called hyperbolic geometric flow (HGF, for short) in 2007. HGF is helpful to understand the wave

Manuscript received September 12, 2012. Revised September 10, 2013.
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character of the metrics, wave phenomenon of the curvatures, the evolution of manifolds and their structures (see [8, 13–14, 17, 23]).

It is generally known that HGF equation is quite difficult to solve in all generality unlike the nonlinear elliptic problems with a well-developed regularity theory. Although the short time existence of solutions is guaranteed by hyperbolic nature of the equations, their (long time) convergence to canonical geometric structures is analyzed under various conditions. For some recent work and the related results in this direction, see [7, 10–11, 15–16, 23].

In [8], the authors studied the local forms of the evolutions under the HGF. They proved the following result.

Theorem 1.1 (see [8]) *Assume that $(M, g(t))$, $t \in [0, T)$ is a family of complete Riemannian manifolds evolving under hyperbolic geometric flow $\frac{\partial^2}{\partial t^2} g(t)_{ij} = -2\mathrm{Ric}_{ij}$, the curvature tensors satisfy the evolution equations*

$$\begin{aligned}\frac{\partial^2}{\partial t^2} R_{ijkl} &= \Delta R_{ijkl} + 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) \\ &\quad - g^{pq}(R_{pjkl}\mathrm{Ric}_{qi} + R_{ipkl}\mathrm{Ric}_{qj} + R_{ijpl}\mathrm{Ric}_{qk} + R_{ijkp}\mathrm{Ric}_{ql}) \\ &\quad + 2g_{pq}\Big(\frac{\partial \Gamma^p_{il}}{\partial t}\frac{\partial \Gamma^q_{jk}}{\partial t} - \frac{\partial \Gamma^p_{jl}}{\partial t}\frac{\partial \Gamma^q_{ik}}{\partial t}\Big), \end{aligned} \tag{1.1}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathrm{Ric}_{ik} &= \Delta \mathrm{Ric}_{ik} + 2g^{pr}g^{qs}R_{piqk}\mathrm{Ric}_{rs} - 2g^{pq}\mathrm{Ric}_{pi}\mathrm{Ric}_{qk} \\ &\quad + 2g^{jl}g_{pq}\Big(\frac{\partial \Gamma^p_{il}}{\partial t}\frac{\partial \Gamma^q_{jk}}{\partial t} - \frac{\partial \Gamma^p_{jl}}{\partial t}\frac{\partial \Gamma^q_{ik}}{\partial t}\Big) \\ &\quad - 2g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} + 2g^{jp}g^{rq}g^{sl}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}R_{ijkl}, \end{aligned} \tag{1.2}$$

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathrm{Scal} &= \Delta \mathrm{Scal} + 2|\mathrm{Ric}|^2 \\ &\quad + 2g^{ik}g^{jl}g_{pq}\Big(\frac{\partial \Gamma^p_{il}}{\partial t}\frac{\partial \Gamma^q_{jk}}{\partial t} - \frac{\partial \Gamma^p_{jl}}{\partial t}\frac{\partial \Gamma^q_{ik}}{\partial t}\Big) \\ &\quad - 2g^{ik}g^{jp}g^{lq}\frac{\partial g_{pq}}{\partial t}\frac{\partial}{\partial t}R_{ijkl} \\ &\quad - 2g^{ip}g^{kq}\frac{\partial g_{pq}}{\partial t}\frac{\partial \mathrm{Ric}_{ik}}{\partial t} + 4\mathrm{Ric}_{ik}g^{ip}g^{rq}g^{sk}\frac{\partial g_{pq}}{\partial t}\frac{\partial g_{rs}}{\partial t}, \end{aligned} \tag{1.3}$$

where all those components of metric, curvatures, etc. are components with respect to a local normal coordinate system around a fixed point $p \in M$, $B_{ijkl} = g^{pr}g^{qs}R_{piqj}R_{rksl}$, and Δ is the Laplacian with respect to the evolving metric $g(t)$.

Here "Ric" and "Scal" denote the Ricci and scalar curvatures of $(M, g(t))$, respectively. Analogous results for the Einstein's hyperbolic geometric flow (2.3) and the dissipative hyperbolic geometric flow (2.4) occur in [7, 10], respectively.

The purpose of this paper is to derive the global forms of evolution equations for Riemannian, Ricci and scalar curvatures under HGF. Motivated by the techniques and ideas concerning the Ricci flow in Brendle's paper [3–4], we give and prove the following main results.

Theorem 1.2 *Let X, Y, Z, W be fixed vector fields on M. Then under the HGF* (2.1)*, we have*

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R(X,Y,Z,W) &= -\mathrm{Ric}_{g(t)}(R_{X,Y}Z, W) + \mathrm{Ric}_{g(t)}(R_{X,Y}W, Z) \\ &\quad + (D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) - (D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z)\end{aligned}$$

$$
\begin{aligned}
&- (D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W) + (D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z)\\
&+ 2\frac{\partial g(t)}{\partial t}((D_XB)(Y,Z),W) - 2\frac{\partial g(t)}{\partial t}((D_YB)(X,Z),W)\\
&- 2g(t)(B(X,B(Y,Z)),W) + 2g(t)(B(Y,B(X,Z)),W), \qquad (1.4)
\end{aligned}
$$

where $B(X,Y) := \frac{\partial}{\partial t}(D_XY)$, *and the alternative notation* $R(X,Y)Z$ *for* $R_{X,Y}Z$ *is convenient.*

Theorem 1.3 *Let* X,Y,Z,W *be arbitrary vector fields on* M. *Then under the HGF* (2.1), *the curvature tensors satisfy the evolution equations*

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}R(X,Y,Z,W) = {}&(\Delta R)(X,Y,Z,W) + Q(R)(X,Y,Z,W)\\
&- \mathrm{Ric}_{g(t)}(X,R_{Z,W}Y) + \mathrm{Ric}_{g(t)}(Y,R_{Z,W}X)\\
&+ \mathrm{Ric}_{g(t)}(Z,R_{X,Y}W) - \mathrm{Ric}_{g(t)}(W,R_{X,Y}Z)\\
&+ 2\frac{\partial g(t)}{\partial t}((D_XB)(Y,Z),W) - 2\frac{\partial g(t)}{\partial t}((D_YB)(X,Z),W)z\\
&- 2g(t)(B(X,B(Y,Z)),W) + 2g(t)(B(Y,B(X,Z)),W), \qquad (1.5)
\end{aligned}
$$

where $Q(R) := R^2 + R^\#$ *is a curvature tensor satisfying the first Bianchi identity, and with respect to a given local orthonormal basis* $\{e_1,\cdots,e_n\}$, R^2 *and* $R^\#$ *are defined by*

$$
\begin{aligned}
R^2(X,Y,Z,W) &:= \sum_{p,q=1}^n R(X,Y,e_p,e_q)R(e_p,e_q,Z,W),\\
R^\#(X,Y,Z,W) &:= 2\sum_{p,q=1}^n \begin{vmatrix} R(X,e_p,Z,e_q) & R(X,e_p,W,e_q)\\ R(Y,e_p,Z,e_q) & R(Y,e_p,W,e_q)\end{vmatrix},
\end{aligned} \qquad (1.6)
$$

respectively.

Theorem 1.4 *Let* X,Y *be arbitrary vector fields on* M. *Then under the HGF* (2.1), *the Ricci curvature tensor* $\mathrm{Ric}_{g(t)}$ *and scalar curvature* $\mathrm{Scal}_{g(t)}$ *satisfy the evolution equations, respectively,*

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathrm{Ric}_{g(t)}(X,Y) = {}&(\Delta \mathrm{Ric}_{g(t)})(X,Y) + 2\sum_{i,j=1}^n R(X,e_i,Y,e_j)\mathrm{Ric}_{g(t)}(e_i,e_j)\\
&+ 2\sum_{i=1}^n\Big(\frac{\partial g(t)}{\partial t}((D_XB)(e_i,Y),e_i) - \frac{\partial g(t)}{\partial t}((D_{e_i}B)(X,Y),e_i)\Big)\\
&- 2\sum_{i=1}^n(g(t)(B(X,B(e_i,Y)),e_i) - g(t)(B(e_i,B(X,Y)),e_i)), \qquad (1.7)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2}{\partial t^2}\mathrm{Scal}_{g(t)} = {}&\Delta\mathrm{Scal}_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2\\
&+ 2\sum_{i,j=1}^n\Big(\frac{\partial g(t)}{\partial t}((D_{e_j}B)(e_i,e_j),e_i)) - \frac{\partial g(t)}{\partial t}((D_{e_i}B)(e_j,e_j),e_i)\Big)\\
&- 2\sum_{i,j=1}^n(g(t)(B(e_j,B(e_i,e_j)),e_i) - g(t)(B(e_i,B(e_j,e_j)),e_i)), \qquad (1.8)
\end{aligned}
$$

where $\{e_1,\cdots,e_n\}$ *is a local orthonormal basis of* M, *and* $|\mathrm{Ric}|^2 = \sum_{i,j=1}^n(\mathrm{Ric}(e_i,e_j))^2$.

On the other hand, hyperbolic differential equations model things like wave or vibrations, which may never reach equilibrium. These equations are more prone to singularities than parabolic ones (see [24]). Based on Theorem 14.1 in [9] which states that a maximal solution to the Ricci flow that develops a singularity in finite time must have unbounded curvature, we conjecture that there is a corresponding conclusion in the HGF case, see Theorem 4.1.

The structure of the paper is as follows. In Section 2, we state the related concepts and present some examples of specific solutions to the HGF. In Section 3, we give evolution equations for the Levi-Civita connection and prove the main results in the introduction. In Section 4, we consider Ricci curvature blow-up at finite-time singularities. Section 5 presents some problems for further study.

2 The Hyperbolic Geometric Flow

The hyperbolic geometric flow (HGF, for short) differs from the Hamilton's Ricci flow, although these two flows share a common Ricci term $-2\mathrm{Ric}_{g(t)}$. Its definition is as follows.

Definition 2.1 (see [14]) *Let M be a manifold. The hyperbolic geometric flow is the evolution equation*

$$\frac{\partial^2}{\partial t^2} g(t) = -2\mathrm{Ric}_{g(t)} \tag{2.1}$$

for a one-parameter family of Riemannian metrics $g(t)$, $t \in [0, T)$ on M. We say that $g(t)$ is a solution to the hyperbolic geometric flow if it satisfies (2.1).

Similarly to the Ricci flow $\frac{\partial}{\partial t} g(t) = -2\mathrm{Ric}_{g(t)}$, the HGF equation (2.1) is an unnormalized evolution equation. In [14], Kong and Liu also considered the normalized version of hyperbolic geometric flow, which preserves the volume of the flow. Considering the HGF and the normalized HGF differ only by a change of scale in space M and a change of time t. The normalized HGF equation takes the form

$$\frac{\partial^2}{\partial t^2} g(t) = -2\mathrm{Ric}_{g(t)} + a(t)\frac{\partial}{\partial t} g(t) + b(t) g(t), \tag{2.2}$$

where $a(t)$ and $b(t)$ are certain functions of t.

In order to further understand the relationship between the Einstein equation and the HGF, Kong and Liu also introduced the so-called Einstein's hyperbolic geometric flow.

Definition 2.2 (see [7]) *Let $\mathbb{R} \times M$ be a space-time with the Lorentzian metric $\mathrm{d}s^2 = \mathrm{d}t^2 + g_{ij}(x,t)\mathrm{d}x^i \mathrm{d}x^j$. Suppose that the Einstein equations in the vacuum, which correspond to the metric $\mathrm{d}s^2$, have the form*

$$\frac{\partial^2}{\partial t^2} g(t)_{ij} = -2\mathrm{Ric}_{ij} - \frac{1}{2} g^{pq} \frac{\partial g_{ij}}{\partial t} \frac{\partial g_{pq}}{\partial t} + g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t}. \tag{2.3}$$

The equation (2.3) *is called the Einstein's hyperbolic geometric flow.*

Motivated by the well-developed theory of the dissipative hyperbolic equations, the paper [7] introduced a new geometric analytical tool called dissipative hyperbolic geometric flow defined by

$$\begin{aligned} \frac{\partial^2}{\partial t^2} g(t)_{ij} = {} & -2\mathrm{Ric}_{ij} + 2g^{pq} \frac{\partial g_{ip}}{\partial t} \frac{\partial g_{jq}}{\partial t} - \Big(d + 2g^{pq} \frac{\partial g_{pq}}{\partial t}\Big) \frac{\partial g_{ij}}{\partial t} \\ & + \frac{1}{n-1} \Big(\Big(g^{pq} \frac{\partial g_{pq}}{\partial t} \Big)^2 + \frac{\partial g^{pq}}{\partial t} \frac{\partial g_{pq}}{\partial t} \Big) g_{ij}, \end{aligned} \tag{2.4}$$

where d is a positive constant. The reason that (2.4) is chosen as the equation form of dissipative hyperbolic geometric flow is that, in the case it possesses a simpler equation satisfied by the scalar curvature.

Next we describe some examples of specific solutions (see [7–8, 10, 14–15]) so that we may get a better sense of the HGF.

Example 2.1 (i) (Trivial example) If the initial metric is Ricci flat, so that $\mathrm{Ric}_{ij} = 0$, then clearly the metric does not change under (2.1). Hence any Ricci flat metric $g(t)$ is a stationary solution to the HGF. This happens, for example, on the flat torus or on any K3-surface with a Calabi-Yau metric.

(ii) (Non-trivial example) A typical example of the Einstein metric is

$$\mathrm{d}s_0^2 = \frac{1}{1-\kappa r^2}\mathrm{d}r^2 + r^2\mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\varphi^2,$$

where κ is a constant taking its value $-1, 0$ or 1. We can prove that the metric

$$\mathrm{d}s^2 = (-2\kappa t^2 + c_1 t + c_2)\mathrm{d}s_0^2$$

is a solution to the HGF (2.1), where c_1 and c_2 are two constants.

Example 2.2 Consider a solution in conformal class with the following form:

$$g_{ij}(t,x) = \rho(t)g_{ij}(0,x). \tag{2.5}$$

Suppose that the initial metric $g_{ij}(0,x)$ is Einstein, that is, there exists some constant λ such that

$$\mathrm{Ric}_{ij}(0,x) = \lambda g_{ij}(0,x), \quad \forall x \in M.$$

Then (2.5) with $\rho(t) = -\lambda t^2 + vt + 1$ and a real number v standing for the initial velocity, is a solution to the equation (2.1).

3 Evolution Equations and Proofs of the Main Results

In this section, we use the techniques of Brendle [3–4] in studying evolution equations along the Ricci flow to derive evolution equations with global forms along the HGF.

From now on, we assume that $(M, g(t))$, $t \in [0,T)$, is a family of complete Riemannian manifolds evolving under HGF.

3.1 Evolution of the Levi-Civita connection

Let X, Y be fixed vector fields on M (that is, X, Y are independent of t). We define

$$A(X,Y) := \frac{\partial^2}{\partial t^2}(D_X Y), \quad B(X,Y) := \frac{\partial}{\partial t}(D_X Y).$$

Observe that the difference of two connections is always a tensor, consequently, A, B are tensors.

Proposition 3.1 *Let X, Y, Z be fixed vector fields on M. Then*

$$\begin{aligned} g(t)(A(X,Y),Z) = &-(D_X\mathrm{Ric}_{g(t)})(Y,Z) - (D_Y\mathrm{Ric}_{g(t)})(X,Z) \\ &+ (D_Z\mathrm{Ric}_{g(t)})(X,Y) - 2\frac{\partial g(t)}{\partial t}(B(X,Y),Z). \end{aligned} \tag{3.1}$$

Proof Differentiating $g(t)(D_XY,Z)$ twice with respect to t, we have

$$\frac{\partial^2}{\partial t^2}(g(t)(D_XY,Z))=\frac{\partial^2 g(t)}{\partial t^2}(D_XY,Z)+2\frac{\partial g(t)}{\partial t}(B(X,Y),Z)+g(t)(A(X,Y),Z). \tag{3.2}$$

Since the Levi-Civita connection satisfies

$$\begin{aligned}2g(t)(D_XY,Z)&=X(g(t)(Y,Z))+Y(g(t)(Z,X))-Z(g(t)(X,Y))\\&\quad-g(t)(X,[Y,Z])+g(t)(Y,[Z,X])+g(t)(Z,[X,Y]),\end{aligned}$$

(3.2) can be rewritten in the following form:

$$\begin{aligned}g(t)(A(X,Y),Z)&=X\Big(\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(Y,Z)\Big)+Y\Big(\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(Z,X)\Big)\\&\quad-Z\Big(\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(X,Y)\Big)-\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(X,[Y,Z])\\&\quad+\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(Y,[Z,X])+\frac{1}{2}\frac{\partial^2 g(t)}{\partial t^2}(Z,[X,Y])\\&\quad-\frac{\partial^2 g(t)}{\partial t^2}(D_XY,Z)-2\frac{\partial g(t)}{\partial t}(B(X,Y),Z).\end{aligned}$$

By the definition of HGF, we have

$$\begin{aligned}g(t)(A(X,Y),Z)&=-X(\mathrm{Ric}_{g(t)}(Y,Z))-Y(\mathrm{Ric}_{g(t)}(Z,X))+Z(\mathrm{Ric}_{g(t)}(X,Y))\\&\quad+\mathrm{Ric}_{g(t)}(X,[Y,Z])-\mathrm{Ric}_{g(t)}(Y,[Z,X])-\mathrm{Ric}_{g(t)}(Z,[X,Y])\\&\quad+2\mathrm{Ric}_{g(t)}(D_XY,Z)-2\frac{\partial g(t)}{\partial t}(B(X,Y),Z).\end{aligned} \tag{3.3}$$

Noting that A is a tensor, we conclude that

$$\begin{aligned}g(t)(A(X,Y),Z)&=-(D_X\mathrm{Ric}_{g(t)}(Y,Z)-(D_Y\mathrm{Ric}_{g(t)})(Z,X)\\&\quad+(D_Z\mathrm{Ric}_{g(t)})(X,Y)-2\frac{\partial g(t)}{\partial t}(B(X,Y),Z),\end{aligned} \tag{3.4}$$

as claimed.

3.2 Proof of Theorem 1.2

Now we return to compute the evolution equation for the curvature tensor. For convenience we need the second order covariant derivative $D^2_{X,Y}Z$ defined by

$$D^2_{X,Y}Z:=D_XD_YZ-D_{D_XY}Z,$$

from which we have

$$R(X,Y)Z=R_{X,Y}Z:=D_XD_YZ-D_YD_XZ-D_{[X,Y]}Z=D^2_{X,Y}Z-D^2_{Y,X}Z.$$

Proof of Theorem 1.2 The second derivative of $R_{X,Y}Z$ yields

$$\begin{aligned}\frac{\partial^2}{\partial t^2}R_{X,Y}Z&=\frac{\partial^2}{\partial t^2}D_X(D_YZ)+2\frac{\partial}{\partial t}D_X\Big(\frac{\partial}{\partial t}D_YZ\Big)+D_X\Big(\frac{\partial^2}{\partial t^2}D_YZ\Big)\\&\quad-\frac{\partial^2}{\partial t^2}D_Y(D_XZ)-2\frac{\partial}{\partial t}D_Y\Big(\frac{\partial}{\partial t}D_XZ\Big)-D_Y\Big(\frac{\partial^2}{\partial t^2}D_YZ\Big)-\frac{\partial^2}{\partial t^2}D_{[X,Y]}Z\\&=A(X,D_YZ)+2B(X,B(Y,Z))+D_XA(Y,Z)-A(Y,D_XZ)\\&\quad-2B(Y,B(X,Z))-D_YA(X,Z)-A(D_XY-D_YX,Z)\\&=(D_XA)(Y,Z)-(D_YA)(X,Z)+2B(X,B(Y,Z))-2B(Y,B(X,Z)).\end{aligned} \tag{3.5}$$

This implies

$$
\begin{aligned}
&\frac{\partial^2}{\partial t^2}R(X,Y,Z,W)\\
&=\frac{\partial^2}{\partial t^2}(g(t)(-R_{X,Y}Z,W)\\
&=-\frac{\partial^2 g(t)}{\partial t^2}(R_{X,Y}Z,W)-g(t)\Big(\frac{\partial^2}{\partial t^2}R_{X,Y}Z,W\Big)\\
&=2\mathrm{Ric}_{g(t)}(R_{X,Y}Z,W)-g(t)((D_XA)(Y,Z),W)+g(t)((D_YA)(X,Z),W)\\
&\quad-2g(t)(B(X,B(Y,Z)),W)+2g(t)(B(Y,B(X,Z)),W). \qquad (3.6)
\end{aligned}
$$

Applying Proposition 3.1, we obtain

$$
\begin{aligned}
&g(t)((D_XA)(Y,Z),W)\\
&=X(g(t)(A(Y,Z),W))-g(t)(A(Y,Z),D_XW)\\
&\quad-g(t)(A(D_XY,Z),W)-g(t)(A(Y,D_XZ),W)\\
&=X\Big(-(D_Y\mathrm{Ric}_{g(t)})(Z,W)-(D_Z\mathrm{Ric}_{g(t)})(Y,W)+(D_W\mathrm{Ric}_{g(t)})(Y,Z)\\
&\quad-2\frac{\partial g(t)}{\partial t}(B(Y,Z),W)\Big)+(D_Y\mathrm{Ric}_{g(t)})(Z,D_XW)+(D_Z\mathrm{Ric}_{g(t)})(Y,D_XW)\\
&\quad-(D_{D_XW}\mathrm{Ric}_{g(t)})(Y,Z)+2\frac{\partial g(t)}{\partial t}(B(Y,Z),D_XW)\\
&\quad+(D_{D_XY}\mathrm{Ric}_{g(t)})(Z,W)+(D_Z\mathrm{Ric}_{g(t)})(D_XY,W)\\
&\quad-(D_W\mathrm{Ric}_{g(t)})(D_XY,Z)+2\frac{\partial g(t)}{\partial t}(B(D_XY,Z),W)\\
&\quad+(D_Y\mathrm{Ric}_{g(t)})(D_XZ,W)+(D_{D_XZ}\mathrm{Ric}_{g(t)})(Y,W)\\
&\quad-(D_W\mathrm{Ric}_{g(t)})(Y,D_XZ)+2\frac{\partial g(t)}{\partial t}(B(Y,D_XZ),W)\\
&=-(D_XD_Y\mathrm{Ric}_{g(t)}-D_{D_XY}\mathrm{Ric}_{g(t)})(Z,W)-(D_XD_Z\mathrm{Ric}_{g(t)}-D_{D_XZ}\mathrm{Ric}_{g(t)})(Y,W)\\
&\quad+(D_XD_W\mathrm{Ric}_{g(t)}-D_{D_XW}\mathrm{Ric}_{g(t)})(Y,Z)-2\frac{\partial g(t)}{\partial t}((D_XB)(Y,Z),W)\\
&=-(D^2_{X,Y}\mathrm{Ric}_{g(t)})(Z,W)-(D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W)\\
&\quad+(D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z)-2\frac{\partial g(t)}{\partial t}((D_XB)(Y,Z),W). \qquad (3.7)
\end{aligned}
$$

Interchanging the roles of X and Y yields

$$
\begin{aligned}
&g(t)((D_YA)(X,Z),W)\\
&=-(D^2_{Y,X}\mathrm{Ric}_{g(t)})(Z,W)-(D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W)\\
&\quad+(D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z)-2\frac{\partial g(t)}{\partial t}((D_YB)(X,Z),W). \qquad (3.8)
\end{aligned}
$$

Moreover, we have

$$
\begin{aligned}
&(D^2_{X,Y}\mathrm{Ric}_{g(t)})(Z,W)-(D^2_{Y,X}\mathrm{Ric}_{g(t)})(Z,W)\\
&=((D^2_{X,Y}-D^2_{Y,X})\mathrm{Ric}_{g(t)})(Z,W)\\
&=(R(X,Y)\mathrm{Ric}_{g(t)})(Z,W)\\
&=\mathrm{Ric}_{g(t)}(R_{X,Y}Z,W)+\mathrm{Ric}_{g(t)}(R_{X,Y}W,Z). \qquad (3.9)
\end{aligned}
$$

Substituting (3.7)–(3.9) into (3.6) yields

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}R(X,Y,Z,W)\\
&= -\mathrm{Ric}_{g(t)}(R_{X,Y}Z,W) + \mathrm{Ric}_{g(t)}(R_{X,Y}W,Z)\\
&\quad + (D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) - (D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z)\\
&\quad - (D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W) + (D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z)\\
&\quad + 2\frac{\partial g(t)}{\partial t}((D_XB)(Y,Z),W) - 2\frac{\partial g(t)}{\partial t}((D_YB)(X,Z),W)\\
&\quad - 2g(t)(B(X,B(Y,Z)),W) + 2g(t)(B(Y,B(X,Z)),W),
\end{aligned} \tag{3.10}$$

as claimed.

3.3 Proof of Theorem 1.3

We will show that we can rewrite the right-hand side of (3.10) so that the Laplacian of the curvature tensor term will appear up to lower order terms. To this end, we first give the following lemma which is independent of any evolution.

Lemma 3.1 (see [3]) *Let X,Y,Z,W be arbitrary vector fields on M. Then*

$$\begin{aligned}
&(D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) - (D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z)\\
&- (D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W) + (D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z)\\
&= (\Delta R)(X,Y,Z,W) + Q(R)(X,Y,Z,W) - \mathrm{Ric}_{g(t)}(X,R_{Z,W}Y) + \mathrm{Ric}_{g(t)}(Y,R_{Z,W}X),
\end{aligned} \tag{3.11}$$

where $Q(R) := R^2 + R^\#$ is a curvature tensor satisfying the first Bianchi identity, and with respect to a local orthonormal basis $\{e_1,\cdots,e_n\}$, R^2 and $R^\#$ are difined by (1.6) *in Theorem* 1.3.

Proof Under an orthonormal basis $\{e_1,\cdots,e_n\}$, it is easy to show that

$$\begin{aligned}
(D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) &= \sum_{k=1}^n (D^2_{X,Z}R)(e_k,Y,e_k,W),\\
(D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z) &= \sum_{k=1}^n (D^2_{X,W}R)(e_k,Y,e_k,Z).
\end{aligned} \tag{3.12}$$

Using the second Bianchi identity, after a direct computation, we obtain

$$(D^2_{X,Z}R)(e_k,Y,e_k,W) - (D^2_{X,W}R)(e_k,Y,e_k,Z) = (D^2_{X,e_k}R)(e_k,Y,Z,W). \tag{3.13}$$

Thus we have

$$(D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) - (D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z) = \sum_{k=1}^n (D^2_{X,e_k}R)(e_k,Y,Z,W) \tag{3.14}$$

and

$$(D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W) - (D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z) = \sum_{k=1}^n (D^2_{Y,e_k}R)(e_k,X,Z,W). \tag{3.15}$$

Therefore, (3.14)–(3.15) imply

$$\begin{aligned} I :=&(D^2_{X,Z}\mathrm{Ric}_{g(t)})(Y,W) - (D^2_{X,W}\mathrm{Ric}_{g(t)})(Y,Z) \\ &- (D^2_{Y,Z}\mathrm{Ric}_{g(t)})(X,W) + (D^2_{Y,W}\mathrm{Ric}_{g(t)})(X,Z) \\ =&\sum_{k=1}^{n}(D^2_{X,e_k}R)(e_k,Y,Z,W) - (D^2_{Y,e_k}R)(e_k,X,Z,W)). \end{aligned} \tag{3.16}$$

Now we consider to put $Q(R)$ into (3.16). Note that

$$\begin{aligned} &\sum_{k=1}^{n}(D^2_{X,e_k}R - D^2_{e_k,X}R)(e_k,Y,Z,W) \\ &= \sum_{k,l=1}^{n}(R(X,e_k,e_k,e_l)R(e_l,Y,Z,W) + R(X,e_k,Y,e_l)R(e_k,e_l,Z,W) \\ &\quad + R(X,e_k,Z,e_l)R(e_k,Y,e_l,W) + R(X,e_k,W,e_l)R(e_k,Y,Z,e_l)) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} &\sum_{k=1}^{n}(D^2_{Y,e_k}R - D^2_{e_k,Y}R)(e_k,X,Z,W) \\ &= \sum_{k,l=1}^{n}(R(Y,e_k,e_k,e_l)R(e_l,X,Z,W) + R(Y,e_k,X,e_l)R(e_k,e_l,Z,W) \\ &\quad + R(Y,e_k,Z,e_l)R(e_k,X,e_l,W) + R(Y,e_k,W,e_l)R(e_k,X,Z,e_l)), \end{aligned} \tag{3.18}$$

we have

$$\begin{aligned} &\sum_{k=1}^{n}((D^2_{X,e_k}R - D^2_{e_k,X}R)(e_k,Y,Z,W) - (D^2_{Y,e_k}R - D^2_{e_k,Y}R)(e_k,X,Z,W)) \\ &= \sum_{l=1}(\mathrm{Ric}(X,e_l)R(e_l,Y,Z,W) - \mathrm{Ric}(Y,e_l)R(e_l,X,Z,W)) \\ &\quad + \sum_{k,l=1}^{n}(R(X,e_k,Y,e_l) - R(Y,e_k,X,e_l))R(e_k,e_l,Z,W) \\ &\quad + 2\sum_{k,l=1}^{n}(R(X,e_k,Z,e_l)R(e_k,Y,e_l,W) - R(X,e_k,W,e_l)R(Y,e_k,Z,e_l)). \end{aligned} \tag{3.19}$$

By definitions of R^2 and $R^{\#}$, together with the first Bianchi identity

$$R(X,e_k,Y,e_l) - R(Y,e_k,X,e_l) = R(X,Y,e_k,e_l),$$

(3.19) can be reduced as

$$\begin{aligned} &\sum_{k=1}^{n}((D^2_{X,e_k}R - D^2_{e_k,X}R)(e_k,Y,Z,W) - (D^2_{Y,e_k}R - D^2_{e_k,Y}R)(e_k,X,Z,W)) \\ &= -\mathrm{Ric}(X,R_{Z,W}Y) + \mathrm{Ric}(Y,R_{Z,W}X) + (R^2+R^{\#})(X,Y,Z,W). \end{aligned} \tag{3.20}$$

Hence, from (3.16) and (3.20) we have

$$I = Q(R)(X,Y,Z,W) + \sum_{k=1}^{n} (D^2_{e_k,X} R)(e_k,Y,Z,W) - (D^2_{e_k,Y} R)(e_k,X,Z,W)) - \mathrm{Ric}(X, R_{Z,W}Y) + \mathrm{Ric}(Y, R_{Z,W}X). \tag{3.21}$$

Next we consider to put ΔR into (3.20). Similarly to (3.13), we have

$$\sum_{k=1}^{n} (D^2_{e_k,e_k} R)(X,Y,Z,W) = \sum_{k=1}^{n} ((D^2_{e_k,X} R)(e_k,Y,Z,W) - (D^2_{e_k,Y} R)(e_k,X,Z,W)).$$

Moreover, recall that

$$\Delta R = \sum_{k=1}^{n} D^2_{e_k,e_k} R.$$

After putting these facts together, the desired equation (3.11) follows immediately.

Proof of Theorem 1.3 By Theorem 1.2 and Lemma 3.1, we obtain the following wave equation for curvature tensor:

$$\begin{aligned}\frac{\partial^2}{\partial t^2} R(X,Y,Z,W) = {} & (\Delta R)(X,Y,Z,W) + Q(R)(X,Y,Z,W) \\ & - \mathrm{Ric}_{g(t)}(X, R_{Z,W}Y) + \mathrm{Ric}_{g(t)}(Y, R_{Z,W}X) \\ & + \mathrm{Ric}_{g(t)}(Z, R_{X,Y}W) - \mathrm{Ric}_{g(t)}(W, R_{X,Y}Z) \\ & + 2\frac{\partial g(t)}{\partial t}((D_X B)(Y,Z), W) - 2\frac{\partial g(t)}{\partial t}((D_Y B)(X,Z), W) \\ & - 2g(t)(B(X, B(Y,Z)), W) + 2g(t)(B(Y, B(X,Z)), W).\end{aligned} \tag{3.22}$$

3.4 Proof of Theorem 1.4

Recall that the HGF is an evolution equation on the metric $g_{ij}(t)$. The evolution for the metric implies a nonlinear wave equation not only for the Riemannian curvature tensor, but also for the Ricci curvature tensor and the scalar curvature. This result (Theorems 1.4) is stated in the introduction. Now we give its proof.

Proof of Theorem 1.4 Similarly to (3.12), we have

$$\begin{aligned}(\Delta \mathrm{Ric}_{g(t)})(X,Y) &= \Big(\sum_{j=1}^{n} (D^2_{e_j,e_j} \mathrm{Ric})\Big)(X,Y) \\ &= \sum_{i,j=1}^{n} (D^2_{e_j,e_j} R)(X,e_i,Y,e_i) \\ &= \sum_{i=1}^{n} (\Delta R)(X,e_i,Y,e_i).\end{aligned} \tag{3.23}$$

By the definition of $Q(R)$, we have

$$\begin{aligned}\sum_{i=1}^{n} Q(R)(X,e_i,Y,e_i) = \sum_{i,j,k=1} & (R(X,e_i,e_j,e_k) R(Y,e_i,e_j,e_k) \\ & + 2R(X,e_j,Y,e_k) R(e_i,e_j,e_i,e_k) \\ & - 2R(e_i,e_j,Y,e_k) R(X,e_j,e_i,e_k)).\end{aligned} \tag{3.24}$$

Using the first Bianchi identity, we obtain

$$\begin{aligned}
&-2\sum_{i,j,k=1} R(e_i,e_j,Y,e_k)R(X,e_j,e_i,e_k)\\
&= -\sum_{i,j,k=1} R(X,e_j,e_i,e_k)(R(Y,e_k,e_i,e_j)-R(Y,e_i,e_k,e_j))\\
&= -\sum_{i,j,k=1} R(X,e_j,e_i,e_k)R(Y,e_j,e_i,e_k)\\
&= -\sum_{i,j,k=1} R(X,e_i,e_j,e_k)R(Y,e_i,e_j,e_k).
\end{aligned}$$

Hence (3.24) can be reduced as

$$\begin{aligned}
\sum_{i=1}^{n} Q(R)(X,e_i,Y,e_i) &= 2\sum_{i,j,k=1} R(X,e_j,Y,e_k)R(e_i,e_j,e_i,e_k)\\
&= 2\sum_{j,k=1} R(X,e_j,Y,e_k)\mathrm{Ric}_{g(t)}(e_j,e_k). \qquad (3.25)
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
&\sum_{i=1}^{n}[-\mathrm{Ric}_{g(t)}(X,R_{Y,e_i}e_i)+\mathrm{Ric}_{g(t)}(e_i,R_{Y,e_i}X)\\
&\quad+\mathrm{Ric}_{g(t)}(Y,R_{X,e_i}e_i)-\mathrm{Ric}_{g(t)}(e_i,R_{X,e_i}Y)]\\
&= \sum_{i,j=1}^{n}[-R(Y,e_i,e_i,e_j)\mathrm{Ric}_{g(t)}(X,e_j)+R(Y,e_i,X,e_j)\mathrm{Ric}_{g(t)}(e_i,e_j)\\
&\quad+R(X,e_i,e_i,e_j)\mathrm{Ric}_{g(t)}(Y,e_j)-R(X,e_i,Y,e_j)\mathrm{Ric}_{g(t)}(e_i,e_j)]\\
&= \sum_{j=1}^{n}[\mathrm{Ric}_{g(t)}(Y,e_j)\mathrm{Ric}_{g(t)}(X,e_j)-\mathrm{Ric}_{g(t)}(X,e_j)\mathrm{Ric}_{g(t)}(Y,e_j)]\\
&= 0. \qquad (3.26)
\end{aligned}$$

Using Theorem 1.3, or by (3.22), together with (3.23), (3.25)–(3.26), we get

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\mathrm{Ric}_{g(t)}(X,Y) = \sum_{i=1}^{n}\frac{\partial^2}{\partial t^2}R(X,e_i,Y,e_i)\\
&= (\Delta\mathrm{Ric}_{g(t)})(X,Y)+2\sum_{j,k=1} R(X,e_j,Y,e_k)\mathrm{Ric}_{g(t)}(e_j,e_k)\\
&\quad+2\sum_{i=1}^{n}\Big[\frac{\partial g(t)}{\partial t}((D_XB)(e_i,Y),e_i))-\frac{\partial g(t)}{\partial t}((D_{e_i}B)(X,Y),e_i)\Big]\\
&\quad-2\sum_{i=1}^{n}[g(t)(B(X,B(e_i,Y)),e_i)-g(t)(B(e_i,B(X,Y)),e_i)], \qquad (3.27)
\end{aligned}$$

which is the first claimed equation (1.7).

As for the second assertion (1.8), noting that

$$\frac{\partial^2}{\partial t^2}\mathrm{Scal}_{g(t)} = \sum_{i=1}^{n}\frac{\partial^2}{\partial t^2}\mathrm{Ric}_{g(t)}(e_i,e_i),$$

$$\sum_{i=1}^{n}(\Delta\mathrm{Ric}_{g(t)})(e_i,e_i) = \Delta\mathrm{scal}_{g(t)},$$

$$\sum_{i,j,k=1} R(e_i,e_j,e_i,e_k)\mathrm{Ric}_{g(t)}(e_j,e_k) = \sum_{j,k=1}\mathrm{Ric}_{g(t)}(e_j,e_k)\mathrm{Ric}_{g(t)}(e_j,e_k) = |\mathrm{Ric}_{g(t)}|^2,$$

it follows from (3.27) that

$$\begin{aligned}\frac{\partial^2}{\partial t^2}\mathrm{Scal}_{g(t)} = {}& \Delta\mathrm{Scal}_{g(t)} + 2|\mathrm{Ric}_{g(t)}|^2 \\ & + 2\sum_{i,j=1}^{n}\Big[\frac{\partial g(t)}{\partial t}((D_{e_j}B)(e_i,e_j),e_i)) - \frac{\partial g(t)}{\partial t}((D_{e_i}B)(e_j,e_j),e_i)\Big] \\ & - 2\sum_{i,j=1}^{n}[g(t)(B(e_j,B(e_i,e_j)),e_i) - g(t)(B(e_i,B(e_j,e_j)),e_i)],\end{aligned}$$

as claimed.

4 Curvature Blow-up at Finite-Time Singularities

In this section, we consider a maximal solution to the HGF which is defined on a finite interval $[0,T)$. Similarly to the result in Hamilton's paper (see [9, Theorem 14.1]), we show that such a solution must have unbounded curvature. Our proof is inspired in part by the proof of Theorem 6.1 in [23] and is greatly improved.

Theorem 4.1 *Let M be a compact manifold, and let $g(t)$, $t\in[0,T)$, be a maximal solution to the HGF* (2.1) *on M. Moreover, suppose that $T<\infty$. Then*

$$\limsup_{t\to T}(\sup_M|\mathrm{Ric}_{g(t)}|) = \infty.$$

Proof We argue by contradiction. Assuming that the Ricci tensor of $g(t)$ is uniformly bounded for all $t\in[0,T)$, that is, there exists a positive constant m_0 such that $|\mathrm{Ric}_{g(t)}|\le m_0$, $t\in[0,T)$. So the solution $g(t)$ can be extended to a larger time interval $[0,T+\varepsilon)$, where ε is a positive real number. Indeed, by (2.1), we have the relations

$$\begin{aligned}\frac{\partial g_{ij}}{\partial t}(t,x) &= \frac{\partial g_{ij}}{\partial t}(0,x) - 2\int_0^t \mathrm{Ric}_{ij}(s,x)\mathrm{d}s, \\ g_{ij}(t,x) &= g_{ij}(0,x) + t\frac{\partial g_{ij}}{\partial t}(0,x) - 2\int_0^t\int_0^u \mathrm{Ric}_{ij}(s,x)\mathrm{d}s\mathrm{d}u, \quad t\in[0,T),\end{aligned} \tag{4.1}$$

which imply

$$g_{ij}(t_1,x) - g_{ij}(t_2,x) = (t_1-t_2)\frac{\partial g_{ij}}{\partial t}(0,x) - 2\int_{t_1}^{t_2}\int_0^u \mathrm{Ric}_{ij}(s,x)\mathrm{d}s\mathrm{d}u, \quad t_1,t_2\in[0,T).$$

Hence we obtain

$$|g_{ij}(t_1,x) - g_{ij}(t_2,x)| \le \Big(\Big|\frac{\partial g_{ij}}{\partial t}(0,x)\Big| + 2m_0T\Big)|t_1-t_2|.$$

Therefore for any sufficiently small number $\varepsilon > 0$, there exists $\delta = \frac{\varepsilon}{|\frac{\partial g_{ij}}{\partial t}(0,x)|+2m_0 T}$ such that as $|t_1 - t_2| < \delta$, always holds

$$|g_{ij}(t_1, x) - g_{ij}(t_2, x)| < \varepsilon.$$

By the Cauchy Criterion, $\lim\limits_{t\to T} g_{ij}(t,x)$ exists for all i, j, which implies that $\lim\limits_{t\to T} g(t,x)$ exists. Since $\lim\limits_{t\to T} \mathrm{Ric}(t,x)$ exists (see Definition 2.1), together with (4.1), we have that $\lim\limits_{t\to T} \frac{\partial g}{\partial t}(t,x)$ and $\lim\limits_{t\to T} \frac{\partial^2 g}{\partial t^2}(t,x)$ exist. Consequently, $g(t) := g(t,x)$, $t \in [0,T]$ is the solution to a HGF.

In this case, $g(t)$ may be extended from being a C^∞ solution on $[0,T)$ to a C^∞ solution on $[0,T]$. Then we take $g(T)$, $\frac{\partial g}{\partial t}(T)$ to be an initial metric by a short-time existence theorem (see [14, Theorem 1]) in order to extend the solution to a HGF for $t \in [0, T+\varepsilon)$. This contradicts the assumption that $[0,T)$ is a maximal time interval. Therefore, we complete the proof.

5 Some Problems for Further Study

To conclude this paper, we raise two questions for a future study. Since we have the global forms of evolution equations (1.4)–(1.5), (1.7)–(1.8) along the HGF (2.1), in the future we are interested in the following problems:

1. For fixed $(t,p) \in [0,T] \times M$, we denote by $K_{\max}(t,p)/K_{\min}(t,p)$ the maximum/minimum sectional curvature of $g(t)$ at the point p. Moreover, for abbreviation, we define

$$K_{\max}(t) = \sup_{p\in M} K_{\max}(t,p), \quad K_{\min}(t) = \inf_{p\in M} K_{\min}(t,p).$$

Let $\{t_k\}$ be a sequence of times such that $\lim\limits_{k\to\infty} t_k = T$ and $K_{\max}(t_k) \geq \frac{1}{2} \sup\limits_{t\in[0,t_k]} K_{\max}(t)$ for all k. Then by Theorem 4.1, does the following relation hold:

$$\limsup_{k\to\infty} \frac{K_{\min}(t_k)}{K_{\max}(t_k)} < 1,$$

or

$$\lim_{t\to T} \frac{K_{\min}(t)}{K_{\max}(t)} = 1?$$

2. (Preserved curvature conditions by the HGF) We know that if we want to study the global properties of HGF, then it is important to find curvature conditions that are preserved under the evolution. How to develop such techniques? For instance, suppose that M is a compact manifold, and let $g(t)$, $t \in [0,T)$ be a solution to HGF on M, and consider an appropriate ODE$(*)$ $\frac{\mathrm{d}^2}{\mathrm{d}t^2} R(t) = Q(R(t)) + (\text{certain term})$. Can we claim that the nonnegative isotropic curvature (see [3]) is preserved by the ODE$(*)$?

Acknowledgement The author thanks Professor Kefeng Liu for offering him tremendous information and constructive suggestions. He would like to thank Professor Hongwei Xu for a helpful discussion on a course based Brendle's book Ricci Flow and the Sphere Theorem.

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