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Schur Convexity for Two Classes of Symmetric Functions and Their Applications*

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Abstract For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n_+ \cup \mathbb{R}^n_-$, the symmetric functions $F_n(x, r)$ and $G_n(x, r)$ are defined by

$$F_n(x, r) = F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}}$$

and

$$G_n(x, r) = G_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 - x_{i_j}}{x_{i_j}},$$

respectively, where $r = 1, 2, \cdots, n$, and $i_1, i_2, \cdots, i_n$ are positive integers. In this paper, the Schur convexity of $F_n(x, r)$ and $G_n(x, r)$ are discussed. As applications, by a bijective transformation of independent variable for a Schur convex function, the authors obtain Schur convexity for some other symmetric functions, which subsumes the main results in recent literature; and by use of the theory of majorization establish some inequalities. In particular, the authors derive from the results of this paper the Weierstrass inequalities and the Ky Fan's inequality, and give a generalization of Safta's conjecture in the n-dimensional space and others.

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1 Introduction

We use the following notations throughout this paper: Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space over the field of real numbers ($n \ge 2$), $\mathbb{R}^n_+ = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, 2, \cdots, n\}$, $\mathbb{R}^n_- = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : x_i < 0, i = 1, 2, \cdots, n\}$, $R = (-\infty, +\infty)$ and $N = \{1, 2, \cdots, n, \cdots\}$. For $x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \cdots, y_n) \in \mathbb{R}^n$ and $\alpha \in R$, we denote by

$$x + y = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n),$$
$$xy = (x_1 y_1, x_2 y_2, \cdots, x_n y_n),$$

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$$\alpha x = (\alpha x_1, \alpha x_2, \cdots, \alpha x_n),$$
$$\frac{1}{x} = \Big(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\Big).$$

Schur in [1] gave the definition of the Schur convex function as in Definition 1.1, which can be found in [2].

Definition 1.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a set, a real-valued function f on Ω is said to be Schur convex if*

$$f(x_1, x_2, \cdots, x_n) \le f(y_1, y_2, \cdots, y_n)$$

for each pair of n-tuples $x = (x_1, x_2, \cdots, x_n)$ and $y = (y_1, y_2, \cdots, y_n)$ in Ω, such that $x \prec y$, that is

$$\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}, \quad k = 1, 2, \cdots, n-1$$

and

$$\sum_{i=1}^{n} x_{[i]} = \sum_{i=1}^{n} y_{[i]},$$

where $x_{[i]}$ denotes the i-th largest component of x. f is called Schur concave if $-f$ is Schur convex.

The above Schur convexity has many important applications in analytic inequalities (see [3–13]), linear regression (see [14]), combinatorial optimization (see [15]), graphs and matrices (see [16]), gamma and digamma functions (see [17]), reliability and availability (see [18]) and other related fields. Hardy, Littlewood, and Polya were also interested in some inequalities that are related to the Schur convex functions (see [19]).

Recently, Schur convexity of the some symmetric functions and their applications have been investigated by many authors, see for instance (see [3, 20–24]) and the references therein.

In [3], Guan defined the following symmetric functions:

$$K_n(x, r) = K_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{x_{i_j}}{1 - x_{i_j}} \tag{1.1}$$

for $x = (x_1, \cdots, x_n) \in [0, 1)^n = \{x = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n : 0 \le x_i < 1, i = 1, 2, \cdots, n\}$, $r \in N$ and $r \le n$, where $i_1, i_2, \cdots, i_n$ are positive integers. The Schur convexity and Schur geometric convexity for $K_n(x, r)$ were discussed and some inequalities were established by use of the theory of majorization in [3].

In [20], Chu et al. had a further discussion on the Schur comvexity of $K_n(x, r)$, in particular, solved an open problem proposed by Guan in [3].

Xia and Chu in [21–22] defined the symmetric functions $M_n(x, r)$ and $N_n(x, r)$ as follows:

$$M_n(x, r) = M_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{1 - x_{i_j}}, \tag{1.2}$$

$$N_n(x, r) = N_n(x_1, x_2, \cdots, x_n; r) = \sum_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod_{j=1}^{r} \frac{1 - x_{i_j}}{1 + x_{i_j}} \tag{1.3}$$

for $x = (x_1, \cdots, x_n) \in (0,1)^n$, $r \in N$ and $r \leq n$, where $i_1, i_2, \cdots, i_n$ are positive integers. Xia et al. [23] defined the symmetric function

$$T_n(x,r) = T_n(x_1, x_2, \cdots, x_n; r) = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{x_{i_j}}{1+x_{i_j}} \tag{1.4}$$

for $x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n$, $r \in N$ and $r \leq n$, where $i_1, i_2, \cdots, i_n$ are positive integers. In [21–23], they discussed the Schur convexity, Schur multiplicative convexity and Schur harmonic convexity for $M_n(x,r)$, $N_n(x,r)$ and $T_n(x,r)$, respectively, and established some inequalities.

In this paper, motivated by ideas in [3, 20–23], we define the following symmetric functions:

$$F_n(x,r) = F_n(x_1, x_2, \cdots, x_n; r) = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1+x_{i_j}}{x_{i_j}}, \tag{1.5}$$

$$G_n(x,r) = G_n(x_1, x_2, \cdots, x_n; r) = \sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1-x_{i_j}}{x_{i_j}} \tag{1.6}$$

for $x = (x_1, \cdots, x_n) \in \mathbb{R}_+^n \cup \mathbb{R}_-^n$, $r \in N$ and $r \leq n$, where $i_1, i_2, \cdots, i_n$ are positive integers. The main purpose of this paper is to discuss the Schur convexity for the symmetric function $F_n(x,r)$, As applications, by a bijective transformation of independent variable for a Schur convex function, we obtain the Schur convexity of $G_n(x,r)$ and the Schur convexity for the symmetric functions $M_n(x,r)$, $N_n(x,r)$ and $T_n(x,r)$, which subsumes the main results in [3, 20–23]; establish some inequalities by use of the theory of majorization. In particular, we derive from our results the Weierstrass inequalities (see [25, P. 260]) and the Ky Fan's inequality (see [26]), and give a generalization of Safta's conjecture (see [27–28]) in the n-dimensional space and others.

This paper, except for the introduction, is divided into three sections. In Section 2, we introduce and establish some lemmas. By using the results of Section 2, we will give the main results in Section 3. Finally, some applications are given by use of the theory of majorization.

2 Some Lemmas

In this section, we introduce and establish some lemmas, which will be used in the proof of our main results.

Lemma 2.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior* $\operatorname{int}\Omega$*, and $f : \Omega \to R$ be a Schur convex (or concave, respectively) function on Ω. If the transformation $T : \Omega' \to \Omega$ defined by $x = ay + b$ $(a \neq 0)$ for $y \in \Omega', a, b \in R$ is bijective, put $\varphi(y) = f(ay+b) = f(x)$, then $\varphi : \Omega' \to R$ is a Schur convex (or concave, respectively) function on Ω'. Here Ω is a symmetric set which means that $x \in \Omega$ implies $Px \in \Omega$ for any $n \times n$ permutation matrix P.*

Proof We give only the proof in the case of φ being Schur convex on Ω', since the proof in the others case is similar.

It is easy to derive that Ω' is a symmetric convex set with nonempty interior. For any $y', y'' \in \Omega', y' \prec y''$, since T is bijective, there exist only point $x', x'' \in \Omega$ such that $x' = ay' + b$, $x'' = ay'' + b$ and $x' = ay' + b \prec ay'' + b = x''$. Noting that f is Schur convex on Ω.

We have $f(x') \le f(x'')$. Further we obtain

$$\varphi(y') = f(x') \le f(x'') = \varphi(y'').$$

Therefore, φ is Schur convex on Ω'.

Lemma 2.2 (see [29]) *Let $\Omega \subseteq \mathbb{R}^n$ be a symmetric convex set with nonempty interior* $\operatorname{int}\Omega$, *and $f : \Omega \to R$ be a continuous symmetry function on Ω. If f is differentiable on* $\operatorname{int}\Omega$, *then f is Schur convex on Ω if and only if*

$$(x_i - x_j)\Big(\frac{\partial f(x)}{\partial x_i} - \frac{\partial f(x)}{\partial x_j}\Big) \ge 0 \tag{2.1}$$

for $i, j = 1, 2, \cdots, n$ and all $x = (x_1, \cdots, x_n) \in \operatorname{int}\Omega$. And f is Schur concave on Ω if and only if the inequality (2.1) *is reversed. Here f is a symmetric function in Ω which means that $f(Px) = f(x)$ for any $x \in \Omega$ and any $n \times n$ permutation matrix P.*

Remark 2.1 Since f is symmetric, the Schur's condition in Lemma 2.2, i.e. (2.1) can be reduced as

$$(x_1 - x_2)\Big(\frac{\partial f(x)}{\partial x_1} - \frac{\partial f(x)}{\partial x_2}\Big) \ge 0.$$

For $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n_+ \cup \mathbb{R}^n_- (n \ge 2)$ and $r \in \{1, 2, \cdots, n\}$, the r-th order elementary symmetric function $E_n(t, r)$ (see [30]) is defined as

$$E_n(t, r) = E_n(t_1, t_2, \cdots, t_n; r) = \begin{cases} \sum\limits_{1 \le i_1 < i_2 < \cdots < i_r \le n} \prod\limits_{j=1}^{r} t_{i_j}, & r = 1, 2, \cdots, n, \\ 1, & r = 0. \end{cases}$$

where $i_1, i_2, \cdots, i_n$ are positive integers.

Lemma 2.3 *Let $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n_+ \cup \mathbb{R}^n_-$. If $1 \le r \le n-1$, then*

$$E_n^2(t, r) \ge E_n(t, r-1)E_n(t, r+1).$$

Proof For $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n_+$, the proof of Lemma 2.3 can be found in [20, 31]. Applying the known results for $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n_+$, we derive that Lemma 2.3 is valid for $t = (t_1, t_2, \cdots, t_n) \in \mathbb{R}^n_-$.

Lemma 2.4 *Let $n \ge 3$, $1 \le r \le n-1$. Then the function*

$$\varphi_n(x_1, x_2, \cdots, x_n; r) = \frac{F_n(x_1, x_2, \cdots, x_n; r+1)}{F_n(x_1, x_2, \cdots, x_n; r)}$$

is decreasing with respect to x_i $(i = 1, 2, \cdots, n)$ on $(-1, 0)$.

Proof Since $\varphi_n(x_1, x_2, \cdots, x_n; r)$ is symmetric with respect to $(x_1, x_2, \cdots, x_n) \in (-1, 0)^n$, we give only the proof in the case of $\varphi_n(x_1, x_2, \cdots, x_n; r)$ being decreasing with respect to x_1 on $(-1, 0)$. The proof is divided into three cases.

Case 1 If $r = 1$, then

$$\varphi_n(x_1, x_2, \cdots, x_n; 1) = \frac{\frac{1+x_1}{x_1} \sum\limits_{i=2}^{n} \frac{1+x_i}{x_i} + \sum\limits_{2 \le i < j \le n} \frac{(1+x_i)(1+x_j)}{x_i x_j}}{\sum\limits_{i=1}^{n} \frac{1+x_i}{x_i}}$$

and

$$\frac{\partial\varphi_n(x_1,x_2,\cdots,x_n;1)}{\partial x_1}=\frac{-\Big(\sum\limits_{i=2}^{n}\frac{1+x_i}{x_i}\Big)^2+\sum\limits_{2\le i<j\le n}\frac{(1+x_i)(1+x_j)}{x_ix_j}}{\Big[x_1\sum\limits_{i=1}^{n}\frac{1+x_i}{x_i}\Big]^2}\le 0.$$

Case 2 If $r=n-1$, then

$$\varphi_n(x_1,x_2,\cdots,x_n;n-1)=\frac{\prod\limits_{i=1}^{n}\frac{1+x_i}{x_i}}{\sum\limits_{1\le i_1<i_2<\cdots<i_{n-1}\le n}\prod\limits_{j=1}^{n-1}\frac{1+x_{i_j}}{x_{i_j}}}=\frac{1}{\sum\limits_{i=1}^{n}\frac{x_i}{1+x_i}}$$

and

$$\frac{\partial\varphi_n(x_1,x_2,\cdots,x_n;n-1)}{\partial x_1}=-\frac{1}{\Big[(1+x_1)\sum\limits_{i=1}^{n}\frac{x_i}{1+x_i}\Big]^2}\le 0.$$

Case 3 If $n\ge 4$ and $2\le r\le n-2$, then

$$\varphi_n(x_1,x_2,\cdots,x_n;r)=\frac{\frac{1+x_1}{x_1}F_{n-1}(x_2,x_3,\cdots,x_n;r)+F_{n-1}(x_2,x_3,\cdots,x_n;r+1)}{\frac{1+x_1}{x_1}F_{n-1}(x_2,x_3,\cdots,x_n;r-1)+F_{n-1}(x_2,x_3,\cdots,x_n;r)}$$

and

$$\begin{aligned}&\frac{\partial\varphi_n(x_1,x_2,\cdots,x_n;r)}{\partial x_1}\\ =&\frac{-F_{n-1}^2(x_2,x_3,\cdots,x_n;r)+F_{n-1}(x_2,x_3,\cdots,x_n;r-1)F_{n-1}(x_2,x_3,\cdots,x_n;r+1)}{\Delta^2}\\ =&\frac{-E_{n-1}^2(t_1,t_2,\cdots,t_{n-1};r)+E_{n-1}(t_1,t_2,\cdots,t_{n-1};r-1)E_{n-1}(t_1,t_2,\cdots,t_{n-1};r+1)}{\Delta^2},\end{aligned}\tag{2.2}$$

where $\Delta=(1+x_1)F_{n-1}(x_2,x_3,\cdots,x_n;r-1)+x_1F_{n-1}(x_2,x_3,\cdots,x_n;r)$, $t_i=\frac{1+x_{i+1}}{x_{i+1}}$, $i=1,2,\cdots,n-1$. From Lemma 2.3 and (2.2), we have

$$\frac{\partial\varphi_n(x_1,x_2,\cdots,x_n;r)}{\partial x_1}\le 0.$$

This completes the proof of Lemma 2.4.

Lemma 2.5 (see [10]) *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_+^n$, and $\sum\limits_{i=1}^{n}x_i=s_n$. If $c\ge s_n$, then*

$$\frac{c-x}{\frac{nc}{s_n}-1}=\Big(\frac{c-x_1}{\frac{nc}{s_n}-1},\frac{c-x_2}{\frac{nc}{s_n}-1},\cdots,\frac{c-x_n}{\frac{nc}{s_n}-1}\Big)\prec(x_1,x_2,\cdots,x_n)=x.$$

From Lemma 2.5, we have the following lemma.

Lemma 2.6 *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}_-^n$, and $\sum\limits_{i=1}^{n}x_i=s_n$. If $c\le s_n$, then*

$$\frac{c-x}{\frac{nc}{s_n}-1}=\Big(\frac{c-x_1}{\frac{nc}{s_n}-1},\frac{c-x_2}{\frac{nc}{s_n}-1},\cdots,\frac{c-x_n}{\frac{nc}{s_n}-1}\Big)\prec(x_1,x_2,\cdots,x_n)=x.$$

Lemma 2.7 (see [10]) *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_+$, and $\sum\limits_{i=1}^n x_i=s_n$. If $c\geq 0$, then*

$$\frac{c+x}{\frac{nc}{s_n}+1}=\Big(\frac{c+x_1}{\frac{nc}{s_n}+1},\frac{c+x_2}{\frac{nc}{s_n}+1},\cdots,\frac{c+x_n}{\frac{nc}{s_n}+1}\Big)\prec(x_1,x_2,\cdots,x_n)=x.$$

From Lemma 2.7, we derive the following lemma.

Lemma 2.8 *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_-$, and $\sum\limits_{i=1}^n x_i=s_n$. If $c\leq 0$, then*

$$\frac{c+x}{\frac{nc}{s_n}+1}=\Big(\frac{c+x_1}{\frac{nc}{s_n}+1},\frac{c+x_2}{\frac{nc}{s_n}+1},\cdots,\frac{c+x_n}{\frac{nc}{s_n}+1}\Big)\prec(x_1,x_2,\cdots,x_n)=x.$$

Lemma 2.9 (see [32]) *Let $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n$, and $\sum\limits_{i=1}^n x_i=s_n$. If $0\leq\lambda\leq 1$, then*

$$\frac{s_n-\lambda x}{n-\lambda}=\Big(\frac{s_n-\lambda x_1}{n-\lambda},\frac{s_n-\lambda x_2}{n-\lambda},\cdots,\frac{s_n-\lambda x_n}{n-\lambda}\Big)\prec(x_1,x_2,\cdots,x_n)=x.$$

3 Main Results

Theorem 3.1 *For $n\geq 2$, the function $F_n(x,1)$ is Schur concave in $\mathbb{R}^n_-$, and Schur convex in $\mathbb{R}^n_+$.*

Proof From (1.5), we have

$$F_n(x,1)=\sum_{i=1}^n\frac{1+x_i}{x_i},$$
$$\frac{\partial F_n(x,1)}{\partial x_i}=-\frac{1}{x_i^2},\quad i=1,2.$$

When $x=(x_1,x_2,\cdots,x_n)\in\mathbb{R}^n_-$, we derive

$$(x_1-x_2)\Big(\frac{\partial F_n(x,1)}{\partial x_1}-\frac{\partial F_n(x,1)}{\partial x_2}\Big)=\frac{(x_1-x_2)^2(x_1+x_2)}{x_1^2x_2^2}\leq 0.$$

Applying Lemma 2.2 and Remark 2.1, we obtain that $F_n(x,1)$ is Schur concave in $\mathbb{R}^n_-$. Similarly, it is easy to see that $F_n(x,1)$ is Schur convex in $\mathbb{R}^n_+$.

Theorem 3.2 *For $n\geq 2$ and $2\leq r\leq n$,*

(1) *if r is an even integer (or odd integer, respectively), then $F_n(x,r)$ is Schur convex (or concave, respectively) in $\big[-\frac{2n-r-1}{2(n-1)},0\big)^n$;*

(2) *if r is an even integer (or odd integer, respectively), then $F_n(x,r)$ is Schur concave (or convex, respectively) in $\big[-1,-\frac{2n-r-1}{2(n-1)}\big]^n$;*

(3) *$F_n(x,r)$ is Schur concave in $(-\infty,-1]^n$, and Schur convex in $\mathbb{R}^n_+$.*

Proof Here we give only the proof in the case of r being an even integer, since the proof in the case of r being an odd integer is similar.

(1) According to Lemma 2.2 and Remark 2.1, we only need to prove that

$$(x_1-x_2)\Big(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\Big)\geq 0 \tag{3.1}$$

for $x = (x_1, x_2, \cdots, x_n) \in \big(-\frac{2n-r-1}{2(n-1)}, 0\big)^n$.

To prove (3.1), we divide the proof into five cases.

Case 1 If $n \geq 2$, $r = n$, then from (1.5), we have

$$F_n(x, n) = \prod_{i=1}^{n} \frac{1 + x_i}{x_i}. \tag{3.2}$$

By (3.2), we obtain

$$(x_1 - x_2)\Big(\frac{\partial F_n(x, n)}{\partial x_1} - \frac{\partial F_n(x, n)}{\partial x_2}\Big) = \Delta_1, \tag{3.3}$$

where $\Delta_1 = \frac{(x_1-x_2)^2(x_1+x_2+1)}{x_1x_2(1+x_1)(1+x_2)} F_n(x, n)$. When r is even integer and $x \in \big(-\frac{1}{2}, 0\big)^n$, we derive $\Delta_1 \geq 0$. Therefore, from (3.3) we get that (3.1) holds.

Case 2 If $n = 3$, $r = 2$, then (1.5) yields

$$F_3(x, 2) = \frac{(1 + x_1)(1 + x_2)}{x_1x_2} + \frac{(1 + x_1)(1 + x_3)}{x_1x_3} + \frac{(1 + x_2)(1 + x_3)}{x_2x_3}. \tag{3.4}$$

From (3.4), we have

$$(x_1 - x_2)\Big(\frac{\partial F_3(x, 2)}{\partial x_1} - \frac{\partial F_3(x, 2)}{\partial x_2}\Big) = \frac{(x_1 - x_2)^2}{x_1^2 x_2^2}\Delta_2, \tag{3.5}$$

where $\Delta_2 = (x_1 + x_2 + 1) + \frac{1+x_3}{x_3}(x_1 + x_2)$. When $x \in \big(-\frac{3}{4}, 0\big)^n$, we obtain $\frac{1+x_3}{x_3} < -\frac{1}{3}$. Hence,

$$\begin{aligned}\Delta_2 &= (x_1 + x_2 + 1) + \frac{1 + x_3}{x_3}(x_1 + x_2)\\ &> (x_1 + x_2 + 1) - \frac{1}{3}(x_1 + x_2)\\ &= 1 + \frac{2}{3}(x_1 + x_2) > 0.\end{aligned}$$

Therefore, from (3.5), we get that (3.1) holds.

Case 3 If $n \geq 4$, $r = 2$, then from (1.5) we have

$$F_n(x, 2) = \frac{(1 + x_1)(1 + x_2)}{x_1x_2} + \Big(\frac{1 + x_1}{x_1} + \frac{1 + x_2}{x_2}\Big)\sum_{i=3}^{n} \frac{1 + x_i}{x_i} + \sum_{3 \leq i < j \leq n} \frac{(1 + x_i)(1 + x_j)}{x_ix_j}. \tag{3.6}$$

Hence, from (3.6), we get

$$(x_1 - x_2)\Big(\frac{\partial F_n(x, 2)}{\partial x_1} - \frac{\partial F_n(x, 2)}{\partial x_2}\Big) = \frac{(x_1 - x_2)^2}{x_1^2 x_2^2}\Delta_3, \tag{3.7}$$

where $\Delta_3 = (x_1 + x_2 + 1) + (x_1 + x_2)\sum\limits_{i=3}^{n} \frac{1+x_i}{x_i}$. When $x \in \big(-\frac{2n-3}{2(n-1)}, 0\big)^n$, we have $\sum\limits_{i=3}^{n} \frac{1+x_i}{x_i} < \frac{n-2}{-2n+3}$. Thus

$$\Delta_3 > (x_1 + x_2 + 1) - (x_1 + x_2)\frac{n - 2}{2n - 3} = \frac{2n - 3 + (n - 1)(x_1 + x_2)}{2n - 3} \geq 0.$$

Therefore, from (3.7) we derive that (3.1) is valid.

Case 4 If $n \geq 4$, $r = n-1$, from (1.5) we have

$$F_n(x, n-1) = \Big[\frac{(1+x_1)(1+x_2)}{x_1 x_2}\sum_{i=3}^{n}\frac{x_i}{1+x_i} + \frac{1+x_1}{x_1} + \frac{1+x_2}{x_2}\Big]\prod_{i=3}^{n}\frac{1+x_i}{x_i}, \tag{3.8}$$

$$\begin{cases} \dfrac{\partial F_n(x,n-1)}{\partial x_1} = \Big(-\dfrac{1+x_2}{x_1^2 x_2}\displaystyle\sum_{i=3}^{n}\frac{x_i}{1+x_i} - \frac{1}{x_1^2}\Big)\prod_{i=3}^{n}\frac{1+x_i}{x_i}, \\ \dfrac{\partial F_n(x,n-1)}{\partial x_2} = \Big(-\dfrac{1+x_1}{x_1 x_2^2}\displaystyle\sum_{i=3}^{n}\frac{x_i}{1+x_i} - \frac{1}{x_2^2}\Big)\prod_{i=3}^{n}\frac{1+x_i}{x_i}. \end{cases}$$

Hence, we obtain

$$(x_1 - x_2)\Big(\frac{\partial F_n(x,n-1)}{\partial x_1} - \frac{\partial F_n(x,n-1)}{\partial x_2}\Big) = \frac{(x_1-x_2)^2}{x_1^2 x_2^2}\Delta_4, \tag{3.9}$$

where $\Delta_4 = \big(\sum\limits_{i=3}^{n}\frac{x_i}{1+x_i}\big)\big(\prod\limits_{i=3}^{n}\frac{1+x_i}{x_i}\big)\big[(x_1+x_2+1) + \frac{x_1+x_2}{\sum\limits_{i=3}^{n}\frac{x_i}{1+x_i}}\big]$. When r is an even integer and $x \in \big(-\frac{n}{2(n-1)}, 0\big)^n$, we have $-n < \sum\limits_{i=3}^{n}\frac{x_i}{1+x_i} < 0$, $\prod\limits_{i=3}^{n}\frac{1+x_i}{x_i} < 0$,

$$\begin{aligned}\Delta_4 &> \Big(\sum_{i=3}^{n}\frac{x_i}{1+x_i}\Big)\Big(\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Big)\Big[(x_1+x_2+1) - \frac{1}{n}(x_1+x_2)\Big] \\ &= \frac{1}{n}\Big(\sum_{i=3}^{n}\frac{x_i}{1+x_i}\Big)\Big(\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Big)[(n-1)(x_1+x_2)+n] > 0.\end{aligned}$$

Thus, from (3.9), we have that (3.1) holds.

Case 5 If $n \geq 5$, $3 \leq r \leq n-2$, from (1.5), we have

$$\begin{aligned}F_n(x,r) = {}& \frac{(1+x_1)(1+x_2)}{x_1x_2}F_{n-2}(x_3, x_4, \cdots, x_n; r-2) \\ &+ \Big(\frac{1+x_1}{x_1} + \frac{1+x_2}{x_2}\Big)F_{n-2}(x_3, x_4, \cdots, x_n; r-1) + F_{n-2}(x_3, x_4, \cdots, x_n; r),\end{aligned}$$

$$\begin{cases} \dfrac{\partial F_n(x,r)}{\partial x_1} = \dfrac{(1+x_2)F_{n-2}(x_3, x_4, \cdots, x_n; r-2)}{x_1^2 x_2} - \dfrac{F_{n-2}(x_3, x_4, \cdots, x_n; r-1)}{x_1^2}, \\ \dfrac{\partial F_n(x,r)}{\partial x_2} = \dfrac{(1+x_1)F_{n-2}(x_3, x_4, \cdots, x_n; r-2)}{x_1 x_2^2} - \dfrac{F_{n-2}(x_3, x_4, \cdots, x_n; r-1)}{x_2^2}. \end{cases}$$

Hence, we obtain

$$(x_1 - x_2)\Big(\frac{\partial F_n(x,r)}{\partial x_1} - \frac{\partial F_n(x,r)}{\partial x_2}\Big) = \frac{(x_1-x_2)^2}{x_1^2 x_2^2}\Delta_5, \tag{3.10}$$

where $\Delta_5 = F_{n-2}(x_3, x_4, \cdots, x_n; r-2)\big[(x_1+x_2+1) + \frac{F_{n-2}(x_3,x_4,\cdots,x_n;r-1)}{F_{n-2}(x_3,x_4,\cdots,x_n;r-2)}(x_1+x_2)\big]$. When r is an even integer and $x \in \big(-\frac{2n-r-1}{2(n-1)}, 0\big)^n$, we have $F_{n-2}(x_3, x_4, \cdots, x_n; r-2) > 0$. From (1.5) and Lemma 2.4, we derive

$$\Delta_5 \geq F_{n-2}(x_3, x_4, \cdots, x_n; r-2)\Big[(x_1+x_2+1) + \frac{\frac{(n-2)!}{(r-1)!(n-r-1)!}}{\frac{(n-2)!}{(r-2)!(n-r)!}}\frac{1-\frac{2n-r-1}{2(n-1)}}{-\frac{2n-r-1}{2(n-1)}}(x_1+x_2)\Big]$$

$$= F_{n-2}(x_3, x_4, \cdots, x_n; r-2)\Big[\frac{n-1}{2n-r-1}(x_1+x_2)+1\Big] > 0.$$

Therefore, from (3.10), we derive that (3.1) is valid.

(2) By the notations in the proof of (1), from Lemma 2.2 and Remark 2.1, we only need to prove that

$$(x_1-x_2)\Big(\frac{\partial F_n(x,r)}{\partial x_1}-\frac{\partial F_n(x,r)}{\partial x_2}\Big) \le 0 \tag{3.11}$$

for $x=(x_1,x_2,\cdots,x_n)\in\big(-1,-\frac{2n-r-1}{2(n-1)}\big)^n$.

To prove (3.11), using the discussion similar to that of (1), we also divide the proof into five cases.

Case 1 If $n\ge 2$, $r=n$, then from (3.3) we derive $\Delta_1\le 0$ for $x\in\big(-1,-\frac{1}{2}\big)^n$. Therefore, we obtain that (3.11) holds.

Case 2 If $n=3$, $r=2$, and $x\in\big(-1,-\frac{3}{4}\big)^n$, then $-\frac{1}{3}<\frac{1+x_3}{x_3}<0$. Hence,

$$\begin{aligned}\Delta_2 &= (x_1+x_2+1)+\frac{1+x_3}{x_3}(x_1+x_2)\\ &< (x_1+x_2+1)-\frac{1}{3}(x_1+x_2) = 1+\frac{2}{3}(x_1+x_2) < 0.\end{aligned}$$

Therefore, from (3.5) we get that (3.11) holds.

Case 3 If $n\ge 4$, $r=2$, then when $x\in\big(-1,-\frac{2n-3}{2(n-1)}\big)^n$, we get $\frac{n-2}{-2n+3}<\sum\limits_{i=3}^{n}\frac{1+x_i}{x_i}<0$. Thus

$$\begin{aligned}\Delta_3 &= (x_1+x_2+1)+(x_1+x_2)\sum_{i=3}^{n}\frac{1+x_i}{x_i}\\ &< (x_1+x_2+1)-(x_1+x_2)\frac{n-2}{2n-3} = \frac{2n-3+(n-1)(x_1+x_2)}{2n-3} < 0.\end{aligned}$$

Hence, from (3.7) we obtain that (3.11) is valid.

Case 4 If $n\ge 4$, $r=n-1$, and $x\in\big(-1,-\frac{n}{2(n-1)}\big)^n$, then $\sum\limits_{i=3}^{n}\frac{x_i}{1+x_i}<-n$, $\prod\limits_{i=3}^{n}\frac{1+x_i}{x_i}<0$,

$$\begin{aligned}\Delta_4 &= \Big(\sum_{i=3}^{n}\frac{x_i}{1+x_i}\Big)\Big(\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Big)\Big[(x_1+x_2+1)+\frac{x_1+x_2}{\sum\limits_{i=3}^{n}\frac{x_i}{1+x_i}}\Big]\\ &< \Big(\sum_{i=3}^{n}\frac{x_i}{1+x_i}\Big)\Big(\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Big)\Big[(x_1+x_2+1)-\frac{1}{n}(x_1+x_2)\Big]\\ &= \frac{1}{n}\Big(\sum_{i=3}^{n}\frac{x_i}{1+x_i}\Big)\Big(\prod_{i=3}^{n}\frac{1+x_i}{x_i}\Big)[(n-1)(x_1+x_2)+n] < 0.\end{aligned}$$

Therefore, from (3.9), we have that (3.11) holds.

Case 5 If $n\ge 5$, $3\le r\le n-2$, and $x\in\big(-1,-\frac{2n-r-1}{2(n-1)}\big]^n$, then $F_{n-2}(x_3,x_4,\cdots,x_n;r-2)>0$. From (1.5) and Lemma 2.4, we derive

$$\Delta_5 = F_{n-2}(x_3,x_4,\cdots,x_n;r-2)\Big[(x_1+x_2+1)+\frac{F_{n-2}(x_3,x_4,\cdots,x_n;r-1)}{F_{n-2}(x_3,x_4,\cdots,x_n;r-2)}(x_1+x_2)\Big]$$

$$\le F_{n-2}(x_3,x_4,\cdots,x_n;r-2)\Big[(x_1+x_2+1)+\frac{\frac{(n-2)!}{(r-1)!(n-r-1)!}}{\frac{(n-2)!}{(r-2)!(n-r)!}}\frac{1-\frac{2n-r-1}{2(n-1)}}{-\frac{2n-r-1}{2(n-1)}}(x_1+x_2)\Big]$$
$$=F_{n-2}(x_3,x_4,\cdots,x_n;r-2)\Big[\frac{n-1}{2n-r-1}(x_1+x_2)+1\Big]<0.$$

Therefore, from (3.10), we get that (3.11) is valid.

(3) According to Lemma 2.2 and Remark 2.1, from (3.3), (3.5), (3.7), (3.9)–(3.10), it is easy to see that $F_n(x,r)$ is Schur concave in $(-\infty,-1]^n$, and Schur convex in $\mathbb{R}^n_+$.

This completes the proof of Theorem 3.2.

Applying Lemma 2.1 to Theorems 3.1–3.2, we can derive the Schur convexity of the symmetric functions $G_n(x,r)$, $M_n(x,r)$, $N_n(x,r)$ and $T_n(x,r)$ as follows.

Theorem 3.3 *For $n\ge 2$, the function $G_n(x,1)$ is Schur concave in $\mathbb{R}^n_-$, and Schur convex in $\mathbb{R}^n_+$.*

Proof By replacing x_{i_j} by $-x_{i_j}$ in (1.5), we get

$$F_n(-x,r)=(-1)^r\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}\frac{1-x_{i_j}}{x_{i_j}}=(-1)^rG_n(x,r),$$

where $G_n(x,r)$ is the symmetric function in (1.6). From Lemma 2.1 and Theorem 3.1 we obtain that Theorem 3.3 is valid.

Similarly, by replacing x_{i_j} by $-x_{i_j}$ in (1.5), from Lemma 2.1 and Theorem 3.2, we can derive the following Theorem 3.4.

Theorem 3.4 *For $n\ge 2$, and $2\le r\le n$,*

(1) $G_n(x,r)$ is Schur convex in $\big(0,\frac{2n-r-1}{2(n-1)}\big]^n$, and Schur concave in $\big[\frac{2n-r-1}{2(n-1)},1\big]^n$;

(2) if r is an even integer (or odd integer, respectively), then $G_n(x,r)$ is Schur convex (or concave, respectively) in $\mathbb{R}^n_-$;

(3) if r is an even integer (or odd integer, respectively), then $G_n(x,r)$ is schur concave (or convex, respectively) in $[1,\infty)^n$.

By substituting x_{i_j} by $-1+x_{i_j}$ in (1.5), we get

$$F_n(-1+x,r)=(-1)^r\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}\frac{x_{i_j}}{1-x_{i_j}}=(-1)^rK_n(x,r),$$

where $K_n(x,r)$ is the symmetric function in (1.1). From Lemma 2.1 and Theorems 3.1–3.2, we can derive the following Corollaries 3.1–3.2.

Corollary 3.1 *For $n\ge 2$, the function $K_n(x,1)$ is Schur concave in $(1,\infty)^n$, and Schur convex in $(-\infty,1)^n$.*

Corollary 3.2 *For $n\ge 2$, and $2\le r\le n$,*

(1) the function $K_n(x,r)$ is Schur convex in $\big[\frac{r-1}{2(n-1)},1\big)^n$, and Schur concave in $\big[0,\frac{r-1}{2(n-1)}\big]^n$;

(2) if r is an even integer (or odd integer, respectively), then $K_n(x,r)$ is Schur convex (or concave, respectively) in $(1,\infty)^n$;

(3) if r is an even integer (or odd integer, respectively), then $K_n(x,r)$ is Schur concave (or convex, respectively) in $(-\infty,0]^n$.

Remark 3.1 It is easy to see that Corollaries 3.1–3.2 are the generalization of the Schur convexity of $K_n(x,r)$ in $(0,1)^n$, which is obtained by Guan [3] and Chu et al. [20].

Replacing x_{i_j} by $-1-x_{i_j}$ in (1.5), we get

$$F_n(-1-x,r)=\sum_{1\le i_1<i_2<\cdots<i_r\le n}\prod_{j=1}^{r}\frac{x_{i_j}}{1+x_{i_j}}=T_n(x,r),$$

where $T_n(x,r)$ is the symmetric function in (1.4). From Lemma 2.1 and Theorems 3.1–3.2, we can derive the following Corollaries 3.3–3.4.

Corollary 3.3 *For $n\ge 2$, the function $T_n(x,1)$ is Schur concave in $(-1,\infty)^n$, and Schur convex in $(-\infty,-1)^n$.*

Corollary 3.4 *For $n\ge 2$, and $2\le r\le n$,*

(1) *if r is an even integer (or odd integer, respectively), then $T_n(x,r)$ is Schur convex (or concave, respectively) in $\big(-1,-\frac{r-1}{2(n-1)}\big]^n$;*

(2) *if r is an even integer (or odd integer, respectively), then $T_n(x,r)$ is Schur concave (or convex, respectively) in $\big[-\frac{r-1}{2(n-1)},0\big]^n$;*

(3) *$T_n(x,r)$ is Schur convex in $(-\infty,-1)^n$, and Schur concave in $[0,\infty)^n$.*

Remark 3.2 It is easy to see that Corollaries 3.3–3.4 are the generalization of the Schur convexity of $T_n(x,r)$ in $\mathbb{R}_+^n$, which is obtained by Xia et al. [23].

By replacing x_{i_j} by $\frac{-1+x_{i_j}}{2}$ in (1.5), from Lemma 2.1 and Theorems 3.1–3.2, we have the following Corollaries 3.5–3.6.

Corollary 3.5 *For $n\ge 2$, the function $M_n(x,1)$ is Schur concave in $(1,\infty)^n$, and Schur convex in $(-\infty,1)^n$.*

Corollary 3.6 *For $n\ge 2$, and $2\le r\le n$,*

(1) *$M_n(x,r)$ is Schur convex in $\big[\frac{r-n}{n-1},1\big)^n$, and Schur concave in $\big[-1,\frac{r-n}{n-1}\big]^n$;*

(2) *if r is an even integer (or odd integer, respectively), then $M_n(x,r)$ is Schur convex (or concave, respectively) in $(1,\infty)^n$;*

(3) *if r is an even integer (or odd integer, respectively), then $M_n(x,r)$ is Schur concave (or convex, respectively) in $(-\infty,-1]^n$.*

Remark 3.3 It is easy to see that Corollaries 3.5–3.6 are the generalization of the Schur convexity of $M_n(x,r)$ in $(0,1)^n$, which is obtained by Xia and Chu [21].

By replacing x_{i_j} by $\frac{-1-x_{i_j}}{2}$ in (1.5), from Lemma 2.1 and Theorems 3.1–3.2, we get the following Corollaries 3.7–3.8.

Corollary 3.7 *For $n\ge 2$, the function $N_n(x,1)$ is Schur concave in $(-\infty,-1)^n$, and Schur convex in $(-1,\infty)^n$.*

Corollary 3.8 *For $n\ge 2$, and $2\le r\le n$,*

(1)*$N_n(x,r)$ is Schur convex in $\big(-1,\frac{n-r}{n-1}\big]^n$, and Schur concave in $\big[\frac{n-r}{n-1},1\big]^n$;*

(2) *if r is an even integer (or odd integer, respectively), then $N_n(x,r)$ is Schur convex (or concave, respectively) in $(-\infty,-1)^n$;*

(3) *if r is an even integer (or odd integer, respectively), then $N_n(x,r)$ is Schur concave (or convex, respectively) in $[1,\infty)^n$.*

Remark 3.4 It is easy to see that Corollaries 3.7–3.8 are the generalization of the Schur convexity of $N_n(x,r)$ in $(0,1)^n$, which is established by Xia and Chu [22].

Theorem 3.5 *For $n \geq 2$, and $1 \leq r \leq n$, $G_n(x,r)$ is Schur convex in $D_n = \big\{x = (x_1, x_2, \cdots, x_n) \mid x_i > 0, \sum\limits_{i=1}^{n} x_i \leq 1\big\}$.*

Proof According to Lemma 2.2 and Remark 2.1, we only need to prove that

$$(x_1 - x_2)\Big(\frac{\partial G_n(x,r)}{\partial x_1} - \frac{\partial G_n(x,r)}{\partial x_2}\Big) \geq 0 \tag{3.12}$$

for $x \in D_n$.

The proof is divided into seven cases.

Case 1 If $n \geq 2$, $r = 1$, from (1.6), we have

$$G_n(x,1) = \sum_{i=1}^{n} \frac{1-x_i}{x_i}.$$

Thus we derive

$$(x_1 - x_2)\Big(\frac{\partial G_n(x,1)}{\partial x_1} - \frac{\partial G_n(x,1)}{\partial x_2}\Big) = \frac{(x_1-x_2)^2(x_1+x_2)}{x_1^2 x_2^2} \geq 0.$$

Case 2 If $n = 2$, $r = 2$, then from (1.6), we have

$$G_2(x,2) = \frac{(1-x_1)(1-x_2)}{x_1 x_2}.$$

Thus we get

$$(x_1 - x_2)\Big(\frac{\partial G_n(x,2)}{\partial x_1} - \frac{\partial G_n(x,2)}{\partial x_2}\Big) = \frac{(x_1-x_2)^2[1-(x_1+x_2)]}{x_1^2 x_2^2} \geq 0.$$

Case 3 If $n = 3$, $r = 2$, then from (1.6) we have

$$G_3(x,2) = \frac{(1-x_1)(1-x_2)}{x_1 x_2} + \frac{(1-x_1)(1-x_3)}{x_1 x_3} + \frac{(1-x_2)(1-x_3)}{x_2 x_3}.$$

Thus we obtain

$$(x_1 - x_2)\Big(\frac{\partial G_3(x,2)}{\partial x_1} - \frac{\partial G_3(x,2)}{\partial x_2}\Big) = \frac{(x_1-x_2)^2}{x_1^2 x_2^2}\Big[1-(x_1+x_2)+(x_1+x_2)\frac{1-x_3}{x_3}\Big] \geq 0.$$

Case 4 If $n \geq 4$, $r = n-1$, then from (1.6) we have

$$G_n(x,n-1) = \Big[\frac{(1-x_1)(1-x_2)}{x_1 x_2}\sum_{i=3}^{n}\frac{x_i}{1-x_i} + \frac{1-x_1}{x_1} + \frac{1-x_2}{x_2}\Big]\prod_{i=3}^{n}\frac{1-x_i}{x_i}.$$

Thus we derive

$$(x_1 - x_2)\Big(\frac{\partial G_n(x,n-1)}{\partial x_1} - \frac{\partial G_n(x,n-1)}{\partial x_2}\Big)$$

$$= \frac{(x_1-x_2)^2}{x_1^2x_2^2}\prod_{i=3}^{n}\frac{1-x_i}{x_i}\Big[(1-x_1-x_2)\sum_{i=3}^{n}\frac{x_i}{1-x_i}+(x_1+x_2)\Big]\geq 0.$$

Case 5 If $n\geq 4$, $r=2$, then from (1.6) we have

$$G_n(x,2)=\frac{(1-x_1)(1-x_2)}{x_1x_2}+\Big(\frac{1-x_1}{x_1}+\frac{1-x_2}{x_2}\Big)\sum_{i=3}^{n}\frac{1-x_i}{x_i}+\sum_{3\leq i<j\leq n}\frac{(1-x_i)(1-x_j)}{x_ix_j}.$$

Thus we get

$$(x_1-x_2)\Big(\frac{\partial G_n(x,2)}{\partial x_1}-\frac{\partial G_n(x,2)}{\partial x_2}\Big)=\frac{(x_1-x_2)^2}{x_1^2x_2^2}\Big[1-(x_1+x_2)+(x_1+x_2)\sum_{i=3}^{n}\frac{1-x_i}{x_i}\Big]\geq 0.$$

Case 6 If $n\geq 3$, $r=n$, then from (1.6), we have

$$G_n(x,n)=\prod_{i=1}^{n}\frac{1-x_i}{x_i}.$$

Thus we obtain

$$(x_1-x_2)\Big(\frac{\partial G_n(x,n)}{\partial x_1}-\frac{\partial G_n(x,n)}{\partial x_2}\Big)=\frac{(x_1-x_2)^2[1-(x_1+x_2)]}{x_1x_2(1-x_1)(1-x_2)}G_n(x,n)\geq 0.$$

Case 7 If $n\geq 5$, $3\leq r\leq n-2$, then from (1.6), we have

$$\begin{aligned}G_n(x,r)=&\frac{(1-x_1)(1-x_2)}{x_1x_2}G_{n-2}(x_3,x_4,\cdots,x_n;r-2)\\&+\Big(\frac{1-x_1}{x_1}+\frac{1-x_2}{x_2}\Big)G_{n-2}(x_3,x_4,\cdots,x_n;r-1)+G_{n-2}(x_3,x_4,\cdots,x_n;r).\end{aligned}$$

Thus we derive

$$\begin{aligned}&(x_1-x_2)\Big(\frac{\partial G_n(x,r)}{\partial x_1}-\frac{\partial G_n(x,r)}{\partial x_2}\Big)\\=&\frac{(x_1-x_2)^2}{x_1^2x_2^2}[(1-x_1-x_2)G_{n-2}(x_3,x_4,\cdots,x_n;r-2)\\&+(x_1+x_2)G_{n-2}(x_3,x_4,\cdots,x_n;r-1)]\geq 0.\end{aligned}$$

Therefore, (3.12) follows from cases 1–7, and the proof of Theorem 3.5 is completed.

4 Applications

In this section, we establish some inequalities by use of Theorems 3.1–3.5 and the theory of majorization.

Theorem 4.1 *If $n\geq 2$, $x=(x_1,x_2,\cdots,x_n)$, and $s_n=\sum_{i=1}^{n}x_i$, $0\leq\lambda\leq 1$, then*

(1) $F_n(x,1)\leq F_n(\frac{s_n-\lambda x}{n-\lambda},1)$ *for* $x\in\mathbb{R}^n_-$;

(2) $F_n(x,1)\geq F_n(\frac{s_n-\lambda x}{n-\lambda},1)$ *for* $x\in\mathbb{R}^n_+$.

Proof Theorem 4.1 follows from Theorem 3.1 and Lemma 2.9.

Taking $\lambda=0$ in Theorem 4.1, we have the following corollary.

Corollary 4.1 *If* $n \geq 2$, $x = (x_1, x_2, \cdots, x_n)$, $s_n = \sum\limits_{i=1}^{n} x_i$, *then*

(1) $\sum\limits_{i=1}^{n} \frac{1+x_i}{x_i} \leq \frac{n(n+s_n)}{s_n}$ *for* $x \in \mathbb{R}_-^n$;

(2) $\sum\limits_{i=1}^{n} \frac{1+x_i}{x_i} \geq \frac{n(n+s_n)}{s_n}$ *for* $x \in \mathbb{R}_+^n$.

From Theorem 3.2 and Lemmas 2.5–2.6, we have the following result.

Theorem 4.2 *Let* $n \geq 2$, $2 \leq r \leq n$, *and* $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum\limits_{i=1}^{n} x_i$.

(1) *Suppose that* $x \in \big[-\frac{2n-r-1}{2(n-1)}, 0\big)^n$ *and* $c \leq s_n$. *If* r *is even, then*

$$F_n(x, r) \geq F_n\Big(\frac{c - x}{\frac{nc}{s_n} - 1}, r\Big), \tag{4.1}$$

while if r *is odd, the inequality* (4.1) *is reversed.*

(2) *Suppose that* $x \in \big[-1, -\frac{2n-r-1}{2(n-1)}\big]^n$ *and* $c \leq s_n$. *If* r *is even, then* (4.1) *is reversed, while if* r *is odd, then* (4.1) *is holds.*

(3) *If* $x \in \mathbb{R}_+^n$ *and* $c \geq s_n$, *then* (4.1) *holds, while if* $x \in (-\infty, -1]^n$ *and* $c \leq s_n$, *then* (4.1) *is reversed.*

Similarly, the following Theorems 4.3–4.5 can be derived from Theorem 3.2 and Lemmas 2.7–2.9 together with the fact that

$$\frac{s_n + \lambda x}{n + \lambda} = \Big(\frac{s_n + \lambda x_1}{n + \lambda}, \frac{s_n + \lambda x_2}{n + \lambda}, \cdots, \frac{s_n + \lambda x_n}{n + \lambda}\Big) \prec (x_1, x_2, \cdots, x_n) = x. \tag{4.2}$$

Theorem 4.3 *Let* $n \geq 2$, $2 \leq r \leq n$, *and* $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum\limits_{i=1}^{n} x_i$.

(1) *Suppose that* $x \in \big[-\frac{2n-r-1}{2(n-1)}, 0\big)^n$ *and* $c \leq 0$. *If* r *is even, then*

$$F_n(x, r) \geq F_n\Big(\frac{c + x}{\frac{nc}{s_n} + 1}, r\Big), \tag{4.3}$$

while if r *is odd, then* (4.3) *is reversed.*

(2)*Suppose that* $x \in \big[-1, -\frac{2n-r-1}{2(n-1)}\big]^n$ *and* $c \leq 0$. *If* r *is even, then* (4.3) *is reversed, while if* r *is odd, then* (4.3) *holds.*

(3) *If* $x \in \mathbb{R}_+^n$ *and* $c \geq 0$, *then* (4.3) *holds, while if* $x \in (-\infty, -1]^n$ *and* $c \leq 0$, *then* (4.3) *is reversed.*

Theorem 4.4 *Let* $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum\limits_{i=1}^{n} x_i$, *and* $0 \leq \lambda \leq 1$.

(1) *Suppose that* $x \in \big[-\frac{2n-r-1}{2(n-1)}, 0\big)^n$. *If* r *is even, then*

$$F_n(x, r) \geq F_n\Big(\frac{s_n - \lambda x}{n - \lambda}, r\Big), \tag{4.4}$$

while if r *is odd, then* (4.4) *is reversed.*

(2) *Suppose that* $x \in \big[-1, -\frac{2n-r-1}{2(n-1)}\big]^n$. *If* r *is even, then* (4.4) *is reversed, while if* r *is odd, then* (4.4) *holds.*

(3) *If* $x \in \mathbb{R}_+^n$, *then* (4.4) *holds, while if* $x \in (-\infty, -1]^n$, *then* (4.4) *is reversed.*

Theorem 4.5 *Let $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ with $s_n = \sum_{i=1}^{n} x_i$, and $0 \leq \lambda \leq 1$.*

(1) *Suppose that $x \in \big[-\frac{2n-r-1}{2(n-1)}, 0\big)^n$. If r is even, then*

$$F_n(x, r) \geq F_n\Big(\frac{s_n + \lambda x}{n + \lambda}, r\Big), \tag{4.5}$$

while if r is odd, then (4.5) *is reversed.*

(2) *Suppose that $x \in \big[-1, -\frac{2n-r-1}{2(n-1)}\big]^n$. If r is even, then* (4.5) *is reversed, while if r is odd, then* (4.5) *holds.*

(3) *If $x \in \mathbb{R}_+^n$, then* (4.5) *holds, while if $x \in (-\infty, -1]^n$, then* (4.5) *is reversed.*

Taking $\lambda = 0$ in Theorem 4.4 or Theorem 4.5, we derive the following theorem.

Theorem 4.6 *Let $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$, with $A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$.*

(1) *Suppose that $x \in \big[-\frac{2n-r-1}{2(n-1)}, 0\big)^n$. If r is even, then*

$$\sum_{1 \leq i_1 < i_2 < \cdots < i_r \leq n} \prod_{j=1}^{r} \frac{1 + x_{i_j}}{x_{i_j}} \geq \frac{n!}{r!(n-r)!}\Big[\frac{A_n(1+x)}{A_n(x)}\Big]^r, \tag{4.6}$$

while if r is odd, then (4.6) *is reversed.*

(2) *Suppose that $x \in \big[-1, -\frac{2n-r-1}{2(n-1)}\big]^n$. If r is even, then* (4.6) *is reversed, while if r is odd, then* (4.6) *holds.*

(3) *If $x \in \mathbb{R}_+^n$, then* (4.6) *holds, while if $x \in (-\infty, -1]^n$, then* (4.6) *is reversed.*

Remark 4.1 Taking $r = n$ and $\sum_{i=1}^{n} x_i = 1$ in Theorem 4.6(3), we obtain the Weierstrass inequality (see [25, p. 260])

$$\prod_{i=1}^{n}\Big(\frac{1}{x_i} + 1\Big) \geq (n+1)^n. \tag{4.7}$$

From Theorems 3.3–3.4, Lemmas 2.5–2.9, and (4.2), by an argument similar to that used in the proof of Theorems 4.1–4.5, we have the following Theorems 4.7–4.11.

Theorem 4.7 *Let $n \geq 2$, $x = (x_1, x_2, \cdots, x_n)$ with $s_n = \sum_{i=1}^{n} x_i$, and $0 \leq \lambda \leq 1$. If $x \in \mathbb{R}_-^n$, then*

$$G_n(x, 1) \leq G_n\Big(\frac{s_n - \lambda x}{n - \lambda}, 1\Big), \tag{4.8}$$

while if $x \in \mathbb{R}_+^n$, then (4.8) *is reversed.*

Taking $\lambda = 0$ in Theorem 4.7, we have the following corollary.

Corollary 4.2 *Let $n \geq 2$, $x = (x_1, x_2, \cdots, x_n)$ with $s_n = \sum_{i=1}^{n} x_i$. If $x \in \mathbb{R}_-^n$, then*

$$\sum_{i=1}^{n} \frac{1 - x_i}{x_i} \leq \frac{n(n - s_n)}{s_n}, \tag{4.9}$$

while if $x \in \mathbb{R}_+^n$, then (4.9) *is reversed.*

Theorem 4.8 *Let* $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum_{i=1}^{n} x_i$.

(1) *Suppose that* $c \geq s_n$. *If* $x \in \left(0, \frac{2n-r-1}{2(n-1)}\right]^n$, *then*

$$G_n(x, r) \geq G_n\Big(\frac{c - x}{\frac{nc}{s_n} - 1}, r\Big), \tag{4.10}$$

while if $x \in \left[\frac{2n-r-1}{2(n-1)}, 1\right]^n$, *then* (4.10) *is reversed.*

(2) *Suppose that* $x \in \mathbb{R}_-^n$ *and* $c \leq s_n$. *If* r *is even, then* (4.10) *holds, while if* r *is odd, then* (4.10) *is reversed.*

(3) *Suppose that* $x \in [1, +\infty)^n$ *and* $c \leq s_n$. *If* r *is even, then* (4.10) *is reversed, while if* r *is odd, then* (4.10) *holds.*

Theorem 4.9 *Let* $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum_{i=1}^{n} x_i$.

(1) *Suppose that* $c \geq 0$. *If* $x \in \left(0, \frac{2n-r-1}{2(n-1)}\right]^n$, *then*

$$G_n(x, r) \geq G_n\Big(\frac{c + x}{\frac{nc}{s_n} + 1}, r\Big), \tag{4.11}$$

while if $x \in \left[\frac{2n-r-1}{2(n-1)}, 1\right]^n$, *then* (4.11) *is reversed.*

(2) *Suppose that* $x \in \mathbb{R}_-^n$ *and* $c \leq 0$. *If* r *is even, then* (4.11) *is holds, while if* r *is odd, then* (4.11) *is reversed.*

(3) *Suppose that* $x \in [1, +\infty)^n$ *and* $c \geq 0$. *If* r *is even, then* (4.11) *is reversed, while if* r *is odd, then* (4.11) *holds.*

Theorem 4.10 *Let* $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum_{i=1}^{n} x_i$, *and* $0 \leq \lambda \leq 1$.

(1) *If* $x \in \left(0, \frac{2n-r-1}{2(n-1)}\right]^n$, *then*

$$G_n(x, r) \geq G_n\Big(\frac{s_n - \lambda x}{n - \lambda}, r\Big), \tag{4.12}$$

while if $x \in \left[\frac{2n-r-1}{2(n-1)}, 1\right]^n$, *then* (4.12) *is reversed.*

(2) *Suppose that* $x \in \mathbb{R}_-^n$. *If* r *is even, then* (4.12) *holds, while if* r *is odd, then* (4.12) *is reversed.*

(3) *Suppose that* $x \in [1, +\infty)^n$. *If* r *is even, then* (4.12) *is reversed, while if* r *is odd, then* (4.12) *holds.*

Theorem 4.11 *Let* $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$ *with* $s_n = \sum_{i=1}^{n} x_i$, *and* $0 \leq \lambda \leq 1$.

(1) *If* $x \in \left(0, \frac{2n-r-1}{2(n-1)}\right]^n$, *then*

$$G_n(x, r) \geq G_n\Big(\frac{s_n + \lambda x}{n + \lambda}, r\Big), \tag{4.13}$$

while if $x \in \left[\frac{2n-r-1}{2(n-1)}, 1\right]^n$, *then* (4.13) *is reversed.*

(2) *Suppose that* $x \in \mathbb{R}_-^n$. *If* r *is even, then* (4.13) *holds, while if* r *is odd, then* (4.13) *is reversed;*

(3) *Suppose that* $x \in [1, +\infty)^n$. *If* r *is even, then* (4.13) *is reversed, while if* r *is odd, then* (4.13) *holds.*

Taking $\lambda = 0$ in Theorem 4.10 or Theorem 4.11, we derive the following theorem.

Theorem 4.12 *Let $n \geq 2$, $2 \leq r \leq n$, $x = (x_1, x_2, \cdots, x_n)$, with $A_n(x) = \frac{1}{n}\sum\limits_{i=1}^{n} x_i$.*

(1) *If $x \in \big(0, \frac{2n-r-1}{2(n-1)}\big]^n$, then*

$$\sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1-x_{i_j}}{x_{i_j}} \geq \frac{n!}{r!(n-r)!}\Big[\frac{A_n(1-x)}{A_n(x)}\Big]^r, \tag{4.14}$$

while if $x \in \big[\frac{2n-r-1}{2(n-1)}, 1\big]^n$, then (4.14) is reversed.

(2) *Suppose that $x \in \mathbb{R}^n_-$. If r is even, then (4.14) holds, while if r is odd, then (4.14) is reversed.*

(3) *Suppose that $x \in [1, +\infty)^n$. If r is even, then (4.14) is reversed, while if r is odd, then (4.14) holds.*

Remark 4.2 Taking $r = n$ in Theorem 4.12(1), we have the Ky Fan's inequality (see [26])

$$\Big(\prod_{i=1}^{n} \frac{x_i}{1-x_i}\Big)^{\frac{1}{n}} \leq \frac{A_n(x)}{A_n(1-x)} \tag{4.15}$$

for $x \in (0, \frac{1}{2}]^n$. Inequality (4.15) has evoked the interest of several mathematicians, and different proofs as well as many extension, sharpenings, and variants have been published, see the survey paper [33] and the references therein. It is easy to see that Theorem 4.12 is a generalizations of the Ky Fan's inequality (4.15).

Theorem 4.13 *Let $n \geq 2$, $x_i \geq 0$ $(i = 1, 2, \cdots, n)$, with $\sum\limits_{i=1}^{n} x_i = 1$. Then*

$$\sum_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod_{j=1}^{r} \frac{1-x_{i_j}}{x_{i_j}} \geq \frac{n!}{r!(n-r)!}(n-1)^r.$$

Proof Theorem 4.13 follows from Theorem 3.5 and the following fact

$$\Big(\frac{1}{n}, \frac{1}{n}, \cdots, \frac{1}{n}\Big) \prec (x_1, x_2, \cdots, x_n).$$

Remark 4.3 Taking $r = n$ and $\sum\limits_{i=1}^{n} x_i = 1$ in Theorem 4.13, we obtain the Weierstrass inequality (see [25, p. 260])

$$\prod_{i=1}^{n} \Big(\frac{1}{x_i} - 1\Big) \geq (n-1)^n.$$

Theorem 4.14 *Suppose that $A \in M_n(C)$ $(n \geq 2)$ is a complex matrix, and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of A. If A is a positive Hermitian matrix, then*

(1) $\sum\limits_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod\limits_{j=1}^{r} \big[\frac{2(n-1)\lambda_1}{(2n-r-1)\lambda_{i_j}} - 1\big] \geq \frac{n!}{r!(n-r)!}\big[\frac{2n(n-1)\lambda_1}{(2n-r-1)\operatorname{tr} A} - 1\big]^r$;

(2) $\sum\limits_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod\limits_{j=1}^{r} \big[\frac{\lambda_{i_j}}{2(n-1)\lambda_1-(r-1)\lambda_{i_j}}\big] \leq \frac{n!}{r!(n-r)!}\big[\frac{\operatorname{tr} A}{2n(n-1)\lambda_1-(r-1)\operatorname{tr} A}\big]^r$;

(3) $\sum\limits_{1\leq i_1<i_2<\cdots<i_r\leq n} \prod\limits_{j=1}^{r} \big(\lambda_{i_j}^{-1} + 1\big) \geq \frac{n!}{r!(n-r)!}\big(1 + \frac{n}{\operatorname{tr} A}\big)^r$;

(4) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n}\prod\limits_{j=1}^{r}\frac{\lambda_{i_j}}{1+\lambda_{i_j}}\le\frac{n!}{r!(n-r)!}\big(\frac{\operatorname{tr}A}{n+\operatorname{tr}A}\big)^r$;

(5) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n}\prod\limits_{j=1}^{r}\frac{\operatorname{tr}A+\lambda_{i_j}}{\lambda_{i_j}}\ge\frac{n!}{r!(n-r)!}(n+1)^r$;

(6) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n}\prod\limits_{j=1}^{r}\frac{\operatorname{tr}A-\lambda_{i_j}}{\lambda_{i_j}}\ge\frac{n!}{r!(n-1)!}(n-1)^r$.

Proof It is easy to know that

$$\begin{aligned}&\Big(-\frac{(2n-r-1)\operatorname{tr}A}{2n(n-1)\lambda_1},-\frac{(2n-r-1)\operatorname{tr}A}{2n(n-1)\lambda_1},\cdots,-\frac{(2n-r-1)\operatorname{tr}A}{2n(n-1)\lambda_1}\Big)\\ \prec&\Big(-\frac{(2n-r-1)\lambda_1}{2(n-1)\lambda_1},-\frac{(2n-r-1)\lambda_2}{2(n-1)\lambda_1},\cdots,-\frac{(2n-r-1)\lambda_n}{2(n-1)\lambda_1}\Big),\end{aligned}\tag{4.16}$$

$$-\frac{(2n-r-1)\lambda_i}{2(n-1)\lambda_1}\in\Big[-\frac{2n-r-1}{2(n-1)},0\Big),\quad i=1,2,\cdots,n,\tag{4.17}$$

$$\begin{aligned}&\Big(-1+\frac{(r-1)\operatorname{tr}A}{2n(n-1)\lambda_1},-1+\frac{(r-1)\operatorname{tr}A}{2n(n-1)\lambda_1},\cdots,-1+\frac{(r-1)\operatorname{tr}A}{2n(n-1)\lambda_1}\Big)\\ \prec&\Big(-1+\frac{(r-1)\lambda_1}{2(n-1)\lambda_1},-1+\frac{(r-1)\lambda_2}{2(n-1)\lambda_1},\cdots,-1+\frac{(r-1)\lambda_n}{2(n-1)\lambda_1}\Big),\end{aligned}\tag{4.18}$$

$$-1+\frac{(r-1)\lambda_i}{2(n-1)\lambda_1}\in\Big(-1,-\frac{2n-r-1}{2(n-1)}\Big),\quad i=1,2,\cdots,n,\tag{4.19}$$

$$\Big(\frac{\operatorname{tr}A}{n},\frac{\operatorname{tr}A}{n},\cdots,\frac{\operatorname{tr}A}{n}\Big)\prec(\lambda_1,\lambda_2,\cdots,\lambda_n),\tag{4.20}$$

$$\Big(-1-\frac{\operatorname{tr}A}{n},-1-\frac{\operatorname{tr}A}{n},\cdots,-1-\frac{\operatorname{tr}A}{n}\Big)\prec(-1-\lambda_1,-1-\lambda_2,\cdots,-1-\lambda_n),\tag{4.21}$$

$$\Big(\frac{1}{n},\frac{1}{n},\cdots,\frac{1}{n}\Big)\prec\Big(\frac{\lambda_1}{\operatorname{tr}A},\frac{\lambda_2}{\operatorname{tr}A},\cdots,\frac{\lambda_n}{\operatorname{tr}A}\Big).\tag{4.22}$$

Therefore, Theorem 4.14(1) follows from (4.16)–(4.17) and Theorem 3.2(1). Theorem 4.14(2) follows from (4.18)–(4.19) and Theorem 3.2(2). Theorem 4.14(3) and Theorem 4.14(4) follow from (4.20) and (4.21) together with Theorem 3.2(3), respectively, while Theorem 4.14(5) and Theorem 4.14(6) follow from (4.22) together with Theorem 3.2(3) and Theorem 3.5, respectively.

For the proofs of Theorem 4.14(1)and Theorem 4.14(2), the reader is also referred to [20].

In 1981, Safta [27–28] proposed the following conjecture:

Let AA_1, BB_1, CC_1 be any Cevian lines in $\triangle ABC$, where the points A_1, B_1, C_1 lie on sides BC, CA, AB, respectively. If $AA_1\cap B_1C_1=P, BB_1\cap A_1C_1=Q, CC_1\cap A_1B_1=R$, then

$$\frac{AP}{PA_1}+\frac{BQ}{QB_1}+\frac{CR}{RC_1}\ge 3.\tag{4.23}$$

In [34], Zhang proved the inequality (4.23), and obtained a generalization and an improvement of this conjecture in the n-dimensional space. In this paper we give a generalization of Safta's conjecture in the n-dimensional space, and obtain more extensive resuls than the main results in [34].

Theorem 4.15 *Let $\mathcal{A}=A_1A_2\cdots A_{n+1}$ be an n-dimensional simples in $\mathbb{R}^n$ and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum\limits_i=A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$ $(i=1,2,\cdots,n+1)$, and C_i is the intersection* *point of straight line* A_iP *and the hyperplane* $\Omega_i = B_1B_2\cdots B_{i-1}B_{i+1}\cdots B_{n+1}$ $(i=1,2,\cdots,n+1)$, *then for* $r\in\{1,2,\cdots,n+1\}$,

$$\sum_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{A_{i_j}C_{i_j}}{C_{i_j}B_{i_j}}\ge\frac{(n+1)!}{r!(n-r+1)!}(n-1)^r, \tag{4.24}$$

$$\sum_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod_{j=1}^{r}\frac{A_{i_j}B_{i_j}}{C_{i_j}B_{i_j}}\ge\frac{(n+1)!}{r!(n-r+1)!}n^r. \tag{4.25}$$

Proof Let $(\lambda_1,\lambda_2,\cdots,\lambda_{n+1})$ be the barycentric coordinates of the point P. It is easy to know that the barycentric coordinates of the point B_i are $B_i(\lambda_1,\lambda_2,\cdots,\lambda_{i-1},0,\lambda_{i+1},\cdots,\lambda_{n+1})$.

Suppose that $\frac{A_iC_i}{C_iB_i}=\theta_i$ $(i=1,2,\cdots,n,n+1)$. Then we have that the barycentric coordinates of the point C_i are

$$C_i\Big(\frac{\theta_1}{1+\theta_1}\cdot\frac{\lambda_1}{1-\lambda_i},\cdots,\frac{\theta_{i-1}}{1+\theta_{i-1}}\cdot\frac{\lambda_{i-1}}{1-\lambda_i},\frac{1}{1+\theta_i},\frac{\theta_{i+1}}{1+\theta_{i+1}}\cdot\frac{\lambda_{i+1}}{1-\lambda_i},\cdots,\frac{\theta_{n+1}}{1+\theta_{n+1}}\cdot\frac{\lambda_{n+1}}{1-\lambda_i}\Big),$$

Since the point C_i lie on the hyperplane Ω_i, thus we have

$$(i)\quad\begin{vmatrix}0&\lambda_2&\lambda_3&\cdots&\lambda_i&\cdots&\lambda_{n+1}\\ \lambda_1&0&\lambda_3&\cdots&\lambda_i&\cdots&\lambda_{n+1}\\ \vdots&\vdots&\vdots&&\vdots&&\vdots\\ \lambda_1&\lambda_2&\lambda_3&\cdots&\frac{1-\lambda_i}{\theta_i}&\cdots&\lambda_{n+1}\\ \vdots&\vdots&\vdots&&\vdots&&\vdots\\ \lambda_1&\lambda_2&\lambda_3&\cdots&\lambda_i&\cdots&0\end{vmatrix}=0. \tag{4.26}$$

Applying the property of determinant, we can derive

$$\theta_i=\frac{n-1}{n}\cdot\frac{1-\lambda_i}{\lambda_i},\quad i=1,2,\cdots,n+1. \tag{4.27}$$

Noting that $\lambda_i>0$ and $\sum\limits_{i=1}^{n+1}\lambda_i=1$, from (4.27), Theorem 3.5 and the fact

$$\Big(\frac{1}{n+1},\frac{1}{n+1},\cdots,\frac{1}{n+1}\Big)\prec(\lambda_1,\lambda_2,\cdots,\lambda_{n+1}),$$

we have that (4.24) holds.

From (4.27), and noting the obvious fact that

$$A_iB_i=A_iC_i+C_iB_i,$$

we have

$$\frac{A_iB_i}{C_iB_i}=\frac{1}{n}\Big(\frac{n-1}{\lambda_i}+1\Big),\quad i=1,2,\cdots,n+1. \tag{4.28}$$

Applying (4.28), Theorems 3.1–3.2 and the fact

$$\Big(\frac{1}{n^2-1},\frac{1}{n^2-1},\cdots,\frac{1}{n^2-1}\Big)\prec\Big(\frac{\lambda_1}{n-1},\frac{\lambda_2}{n-1},\cdots,\frac{\lambda_{n+1}}{n-1}\Big),$$

we get that the inequality (4.25) holds.

This completes the proof of Theorem 4.15.

Remark 4.4 It is easy to see that Safta's conjecture is the special case of (4.24) with $n = 2$ and $r = 1$, while the case of $r = n$ in Theorem 4.15 is the results of Theorems 1–2 in [34]. Thus we give a generalization of Safta's conjecture in the n-dimensional space, and obtain more extensive results than the main results in [34].

Theorem 4.16 *Let $\mathcal{A} = A_1A_2\cdots A_{n+1}$ be an n-dimensional simples in $\mathbb{R}^n$ and P be an arbitrary point in the interior of $\mathcal{A}$. If B_i is the intersection point of straight line A_iP and hyperplane $\sum_i = A_1A_2\cdots A_{i-1}A_{i+1}\cdots A_{n+1}$ $(i = 1, 2, \cdots, n+1)$, then for $r \in \{1, 2, \cdots, n+1\}$,*

(1) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\left[\frac{2(n-1)A_{i_j}B_{i_j}}{(2n-r-1)PB_{i_j}}-1\right] \ge \frac{(n+1)!}{r!(n-r+1)!}\left[\frac{2(n^2-1)}{2n-r-1}-1\right]^r;$

(2) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\left[\frac{PB_{i_j}}{2(n-1)A_{i_j}B_{i_j}-(r-1)PB_{i_j}}\right] \le \frac{(n+1)!}{r!(n-r+1)!(2n^2-r-1)^r};$

(3) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\left[\frac{2(n-1)A_{i_j}B_{i_j}}{(2n-r-1)PA_{i_j}}-1\right] \ge \frac{(n+1)!}{r!(n-r+1)!}\left[\frac{2(n^2-1)}{n(2n-r-1)}-1\right]^r;$

(4) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\left[\frac{PA_{i_j}}{2(n-1)A_{i_j}B_{i_j}-(r-1)PA_{i_j}}\right] \le \frac{(n+1)!}{r!(n-r+1)!}\left(\frac{n}{2n^2-nr+n-2}\right)^r;$

(5) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\frac{A_{i_j}B_{i_j}+PB_{i_j}}{PB_{i_j}} \ge \frac{(n+1)!}{r!(n-r+1)!}(n+2)^r;$

(6) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\frac{PB_{i_j}}{A_{i_j}B_{i_j}+PB_{i_j}} \le \frac{(n+1)!}{r!(n-r+1)!}\frac{1}{(n+2)^r};$

(7) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\frac{A_{i_j}B_{i_j}+PA_{i_j}}{PA_{i_j}} \ge \frac{(n+1)!}{r!(n-r+1)!}\left(\frac{2n+1}{n}\right)^r;$

(8) $\sum\limits_{1\le i_1<i_2<\cdots<i_r\le n+1}\prod\limits_{j=1}^{r}\frac{PA_{i_j}}{A_{i_j}B_{i_j}+PA_{i_j}} \le \frac{(n+1)!}{r!(n-r+1)!}\left(\frac{n}{2n+1}\right)^r.$

Proof It is easy to see that $\sum\limits_{i=1}^{n+1}\frac{PB_i}{A_iB_i} = 1$ and $\frac{PA_i}{A_iB_i} = 1 - \frac{PB_i}{A_iB_i}$, $i = 1, 2, \cdots, n+1$, these imply that

$$\begin{aligned}&\Big(-\frac{2n-r-1}{2(n^2-1)}, -\frac{2n-r-1}{2(n^2-1)}, \cdots, -\frac{2n-r-1}{2(n^2-1)}\Big)\\ &\prec \Big(-\frac{(2n-r-1)PB_1}{2(n-1)A_1B_1}, -\frac{(2n-r-1)PB_2}{2(n-1)A_2B_2}, \cdots, -\frac{(2n-r-1)PB_{n+1}}{2(n-1)A_{n+1}B_{n+1}}\Big), \qquad (4.29)\end{aligned}$$

$$-\frac{(2n-r-1)PB_i}{2(n-1)A_iB_i} \in \Big(-\frac{2n-r-1}{2(n-1)}, 0\Big), \quad i = 1, 2, \cdots, n+1, \qquad (4.30)$$

$$\begin{aligned}&\Big(-\frac{2n^2-r-1}{2(n^2-1)}, -\frac{2n^2-r-1}{2(n^2-1)}, \cdots, -\frac{2n^2-r-1}{2(n^2-1)}\Big)\\ &\prec \Big(-1+\frac{(r-1)PB_1}{2(n-1)A_1B_1}, -1+\frac{(r-1)PB_2}{2(n-1)A_2B_2}, \cdots, -1+\frac{(r-1)PB_{n+1}}{2(n-1)A_{n+1}B_{n+1}}\Big), \qquad (4.31)\end{aligned}$$

$$-1+\frac{(r-1)PB_i}{2(n-1)A_iB_i} \in \Big(-1, -\frac{2n-r-1}{2(n-1)}\Big), \quad i = 1, 2, \cdots, n+1, \qquad (4.32)$$

$$\begin{aligned}&\Big(-\frac{2n^2-nr-n}{2(n^2-1)}, -\frac{2n^2-nr-n}{2(n^2-1)}, \cdots, -\frac{2n^2-nr-n}{2(n^2-1)}\Big)\\ &\prec \Big(-\frac{(2n-r-1)PA_1}{2(n-1)A_1B_1}, -\frac{(2n-r-1)PA_2}{2(n-1)A_2B_2}, \cdots, -\frac{(2n-r-1)PA_{n+1}}{2(n-1)A_{n+1}B_{n+1}}\Big), \qquad (4.33)\end{aligned}$$

$$-\frac{(2n-r-1)PA_i}{2(n-1)A_iB_i} \in \Big(-\frac{2n-r-1}{2(n-1)}, 0\Big), \quad i = 1, 2, \cdots, n+1, \qquad (4.34)$$

$$\Big(-\frac{2n^2-n(r-1)-2}{2(n^2-1)},-\frac{2n^2-n(r-1)-2}{2(n^2-1)},\cdots,-\frac{2n^2-n(r-1)-2}{2(n^2-1)}\Big)$$

$$\prec\Big(-1+\frac{(r-1)PA_1}{2(n-1)A_1B_1},-1+\frac{(r-1)PA_2}{2(n-1)A_2B_2},\cdots,-1+\frac{(r-1)PA_{n+1}}{2(n-1)A_{n+1}B_{n+1}}\Big), \tag{4.35}$$

$$-1+\frac{(r-1)PA_i}{2(n-1)A_iB_i}\in\Big(-1,-\frac{2n-r-1}{2(n-1)}\Big),\quad i=1,2,\cdots,n+1, \tag{4.36}$$

$$\Big(\frac{1}{n+1},\frac{1}{n+1},\cdots,\frac{1}{n+1}\Big)\prec\Big(\frac{PB_1}{A_1B_1},\frac{PB_2}{A_2B_2},\cdots,\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\Big), \tag{4.37}$$

$$\Big(-1-\frac{1}{n+1},-1-\frac{1}{n+1},\cdots,-1-\frac{1}{n+1}\Big)$$

$$\prec\Big(-1-\frac{PB_1}{A_iB_1},-1-\frac{PB_2}{A_2B_2},\cdots,-1-\frac{PB_{n+1}}{A_{n+1}B_{n+1}}\Big), \tag{4.38}$$

$$\Big(\frac{n}{n+1},\frac{n}{n+1},\cdots,\frac{n}{n+1}\Big)\prec\Big(\frac{PA_1}{A_1B_1},\frac{PA_2}{A_2B_2},\cdots,\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\Big), \tag{4.39}$$

$$\Big(-1-\frac{n}{n+1},-1-\frac{n}{n+1},\cdots,-1-\frac{n}{n+1}\Big)$$

$$\prec\Big(-1-\frac{PA_1}{A_1B_1},-1-\frac{PA_2}{A_2B_2},\cdots,-1-\frac{PA_{n+1}}{A_{n+1}B_{n+1}}\Big). \tag{4.40}$$

Therefore, Theorem 4.16(1) and Theorem 4.16(3) follow from (4.29)–(4.30) and (4.33)–(4.34) together with Theorem 3.2(1), respectively, Theorem 4.16(2) and Theorem 4.16(4) follow from (4.31)–(4.32) and (4.35)–(4.36) together with Theorem 3.2(2), respectively, while Theorem 4.16(5)–(8) follow from (4.37)–(4.40) and Theorem 3.2(3), respectively.

For the proofs of Theorem 4.16(1)–(4), the reader is also referred to [20], while the other proofs of Theorem 4.16(6) and Theorem 4.16(8) can be found in [23].

Remark 4.5 Mitrinović et al. (see [27, pp. 473–479]) established a series of inequalities for $\frac{PA_i}{A_iB_i}$ and $\frac{PB_i}{A_iB_i}$ $(i=1,2,\cdots,n,n+1)$. Obviously, our inequalities in Theorem 4.16(5) and Theorem 4.16(7) are different from theirs.

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