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Weak Finite Determinacy of Relative Map-Germs*

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Abstract The weak finite determinacy of relative map-germs is studied. The authors first give the concept of weak finite determination, and then give several sufficient conditions for a relative map-germ to be weak finitely determined, which is an important complement to Mather's work. Moreover, as an application, it is proven that the relative stable map-germs are weak finitely determined.

Keywords Relative map-germs, Weak finite determinacy, Relative stability
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1 Introduction

Singularity theory is a young branch of analysis which currently occupies a central place in mathematics; it is the crossroads of paths leading from very abstract subjects of mathematics, such as commutation algebra, Lie group, differential geometry and topology. Moreover, singularity theory offers an extremely useful approach to most applied areas, such as differential geometry (see [1–2]), the theory of bifurcation (see [3–4]) and physics (see [5]). Because every finitely determined germ is equivalent to its Taylor polynomial of some degree, the analysis of the conditions for a germ to be finitely determined involves the most important local aspects of the singularity theory. Therefore, the study of finite determinacy of smooth map-germs is an important subject in singularity theory, and it has been widely studied. The foundation of the study is laid in an important paper [6]. In [6], Mather gave both algebraic and geometric characterizations of finitely determined germs with respect to the groups $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$. There are also numerous useful results on the determinacy of germs due to Gaffney, du Pless and Wall, etc., for instance in [7, 8].

In the present paper, we shall work in the space of differentiable map-germs between Euclidean spaces with the constraint that a fixed submanifold is mapped into another fixed submanifold, and then naturally we encounter the relative map-germs. The concept of relative finite determination was introduced by Porto and Loibel [9]. More recently there are more and more papers studying the notions of singularity theory in the relative case, for instance, finite determinacy, stability and universality (see [9–14]). However, the study of relative finite

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determinacy of map-germs is very few, and in all the references, the relative finite determinacy is only related to the Taylor polynomial at the origin of map-germs. In this paper we generalize this concept for a proper submanifold without boundary of the source space, and we call it the weak finite determination. The purpose of this paper is to obtain algebraic characterization of weak finitely determined relative map-germs with respect to two equivalent relations. The first is relative right-left equivalence, and the second is relative contact equivalence (i.e., $\mathcal{A}_{S,T}$-equivalence and $\mathcal{K}_{S,T}$-equivalence).

The paper is organized as follows. Section 2 deals with the weak finite $\mathcal{A}_{S,T}$-determinacy of relative map-germs. We achieve a sufficient condition to characterize weak finite $\mathcal{A}_{S,T}$-determinacy of relative map-germs (see Theorem 2.1) and two results for estimating the precise degree of determinacy (see Theorems 2.2–2.3). In Section 3 we give a sufficient condition for a relative map-germ to be weak finitely $\mathcal{K}_{S,T}$-determined (see Theorem 3.1). Finally, as an application, we prove that $\mathcal{A}_{S,T}$-stable map-germs are weak $(m+1)$-$\mathcal{A}_{S,T}$-determined and weak $(m+1)$-$\mathcal{K}_{S,T}$-determined.

We shall use the notations as follows.

Let S and T be submanifolds without boundary of $\mathbb{R}^n$ and $\mathbb{R}^m$ respectively, both containing the origin. Since this paper is concerned with a local study, without loss of generality, we may assume that

$$S = \{0\} \times \mathbb{R}^{n-s} \subset \mathbb{R}^n, \quad T = \{0\} \times \mathbb{R}^{m-t} \subset \mathbb{R}^m \quad (s,\, t \geq 1).$$

A relative map-germ $f : (\mathbb{R}^n, S) \to (\mathbb{R}^m, T)$ is a differential map-germ at $0 \in \mathbb{R}^n$ with $f(S) \subset T$ and $f(0) = 0$. Denote by E the space of relative map-germs.

Let E_n denote the ring of smooth function-germs at the origin in $\mathbb{R}^n$, and let $\mathcal{M}_n$ denote its unique maximal ideal. Let $C_S(\mathbb{R}^n)$ be the local ring $\{f \in E_n : f|_S \equiv \text{constant}\}$, and let $\varepsilon(S, n) = \{f \in C_S(\mathbb{R}^n) \mid f(S) \equiv 0\}$, which is the maximal ideal of $C_S(\mathbb{R}^n)$.

For any $f \in E$, it induces a homomorphism $f^* : C_T(\mathbb{R}^m) \to C_S(\mathbb{R}^n)$ defined by $f^*(h) = h \circ f$; this allows us to consider every $C_S(\mathbb{R}^n)$-module as a $C_T(\mathbb{R}^m)$-module via f^*. Let $f^*\varepsilon(T, m) = \langle f_1, f_2, \cdots, f_t\rangle$ be the ideal generated by the components $f_1, \cdots, f_t$, and let $f^*(\varepsilon(T, m))$ denote the image of $\varepsilon(T, m)$ under f, which is not (in general) an ideal of $C_S(\mathbb{R}^n)$.

Let $e_1, e_2, \cdots, e_m$ be the canonical basis of the vector space $\mathbb{R}^m$, and they define a system of generators of the $C_S(\mathbb{R}^n)$-module

$$C_S(\mathbb{R}^n)^m = C_S(\mathbb{R}^n)\langle e_1, e_2, \cdots, e_m\rangle.$$

Similarly, we have $\varepsilon(S, n)^m = \varepsilon(S, n)\langle e_1, e_2, \cdots, e_m\rangle$.

Let $\mathcal{R}$ denote the group of germs at the origin of local diffeomorphisms of $\mathbb{R}^n$, and let $\mathcal{R}_S(n) = \{\phi \in \mathcal{R} : \phi|_S \equiv \mathrm{id}_S\}$, where id denotes the identity. We also observe that $\mathcal{R}_S(n)$ is a subgroup of $\mathcal{R}$.

Now, let $\mathcal{A}_{S,T} = \mathcal{R}_S(n) \times \mathcal{R}_T(m)$ denote the relative right-left equivalent group, and let

$$\mathcal{K}_{S,T} = \{(M, h) \mid M : (\mathbb{R}^n, 0) \to (\mathrm{GL}(m, \mathbb{R}), M(0)) \text{ is a } C^\infty \text{ map-germ}, h \in \mathcal{R}_S(n)\}$$

denote the relative contact equivalent group.

The two groups act on E in the following way:

If $f \in E$, $(\phi, \psi) \in \mathcal{A}_{S,T}$ and $(M, h) \in \mathcal{K}_{S,T}$, then $(\phi, \psi) \cdot f$ and $(M, h) \cdot f$ are defined respectively by

$$(\phi, \psi) \cdot f = \psi \circ f \circ \phi^{-1} \quad \text{and} \quad (M, h) \cdot f = M(x) \cdot f(h(x)).$$

Let $\mathcal{G}$ be a group which acts on E.

Definition 1.1 (see [8]) *Two relative map-gems f, g are called $\mathcal{G}$-equivalent if f and g belong to the same orbit of $\mathcal{G}$ on E.*

For $f, g \in E$, we say that f and g have the same k-jet at S if g has the same k-jet as f at each point of S. We write $j^k f$ for the k-jet at S of f.

Definition 1.2 *A relative map-germ f is called weak k-$\mathcal{G}$-determined if every relative map-germ having the same k-jet at S as f is $\mathcal{G}$-equivalent to f. If f is weak k-$\mathcal{G}$-determined for some $k < \infty$, then it is weak finitely $\mathcal{G}$-determined, and the least such k is the degree of determinacy.*

Remark 1.1 (i) If f and g are $\mathcal{A}_{S,T}$-equivalent, then $f|_S = g|_S$, that is, the value of a germ at S is an invariant of the action of $\mathcal{A}_{S,T}$. Set $E_f = \{g \in E : f|_S = g|_S\}$.

(ii) Taking $s = n$ and $t = m$, then $S = T = \{0\}$. In this case, the weak $\mathcal{G}$-determinacy actually is the $\mathcal{G}$-determinacy.

2 Weak Finite $\mathcal{A}_{S,T}$-Determinacy of Relative Map-Germs

Definition 2.1 (see [10]) *Let $f \in E$ be a relative map-germ. We define*

$$Tf = \varepsilon(S, n)\langle \mathrm{d}f\rangle + f^*(\varepsilon(T, m))\langle e_1, \cdots, e_m\rangle,$$

where $\langle \mathrm{d}f\rangle = \langle \frac{\partial f}{\partial x_1}, \cdots, \frac{\partial f}{\partial x_n}\rangle$.

In addition, denote $\overline{Tf} = C_S(\mathbb{R}^n)\langle \mathrm{d}f\rangle + C_T(\mathbb{R}^m)\langle e_1, \cdots, e_m\rangle$.

Definition 2.2 (see [10]) *Let $F : (\mathbb{R}^p \times \mathbb{R}^n, \mathbb{R}^p \times S) \to (\mathbb{R}^p \times \mathbb{R}^m, \mathbb{R}^p \times T)$ be a germ of the form $F(t, x) = (t, f(t, x))$. Then we say that F is a p-parameter unfolding of $f_0 : (\mathbb{R}^n, S) \to (\mathbb{R}^m, T)$, if $f(0, x) = f_0(x)$ and $f(t, x) = f(0, x)$ for all $t \in \mathbb{R}^p$ and $x \in S$.*

In order to characterize weak finite $\mathcal{A}_{S,T}$-determinacy for relative map-germs, we need the following lemmas which can be obtained in a similar way as [6]. So these results are stated without proofs here.

Lemma 2.1 *Suppose that g has the same k-jet at S as f. Then*

(1) $Tf + \varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m = Tg + \varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m$;

(2) $\overline{Tf} + \varepsilon^{k}(S, n) \cdot C_S(\mathbb{R}^n)^m = \overline{Tg} + \varepsilon^{k}(S, n) \cdot C_S(\mathbb{R}^n)^m$,

where $\varepsilon^k(S, n)$ denotes the k-th power of $\varepsilon(S, n)$.

Lemma 2.2 *Let $F, G : (\mathbb{R} \times \mathbb{R}^n, \mathbb{R} \times S) \to (\mathbb{R} \times \mathbb{R}^m, \mathbb{R} \times T)$ be level preserving C^∞ map-germs such that $G - F \in \varepsilon^\ell(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m$. Then*

$$TF + \varepsilon^\ell(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m = TG + \varepsilon^\ell(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m,$$

where $TF = \varepsilon(\mathbb{R}\times S, 1+n)\langle \mathrm{d}f_t(x)\rangle + F^*(\varepsilon(\mathbb{R}\times T, 1+m))\langle e_1, \cdots, e_m\rangle$ and $F(t, x) = (t, f_t(x))$.

Lemma 2.3 *Let $F_0(t, x) = (t, f(x))$ be the constant unfolding of f. Then $\varepsilon^\ell(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tf$ if and only if $\varepsilon^\ell(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m \subset TF_0$.*

Lemma 2.4 (see [10]) *Let $F(t, x) = (t, f(t, x))$ be a one-parameter unfolding of f_0. Then F is trivial if and only if $\frac{\partial F}{\partial t} \in TF$.*

Lemma 2.5 *Let f be a relative map-germ, A and B be a finitely generated $C_S(\mathbb{R}^n)$-module and a $C_T(\mathbb{R}^m)$-module respectively, and $M \subset C_S(\mathbb{R}^n)$ be a finitely generated ideal. If $B \subset f^*\varepsilon(T, m) \cdot A$, then $M \cdot A \subset B + M^2 \cdot A$ implies $M \cdot A \subset B$.*

Lemma 2.6 (see [15]) *Let $h : \mathcal{U} \to \mathcal{U}'$ be a G-submersion, where G is a Lie group. Let $u' \in \mathcal{U}'$ and $V = h^{-1}(u')$. Suppose that V is connected. Then the necessary and sufficient condition for V to be contained in a single orbit of G is that*

$$T_v V = T_v(G \cdot v) \quad \textit{for all } v \in V.$$

Lemma 2.7 *Let $F : (\mathbb{R} \times \mathbb{R}^n, \mathbb{R} \times S) \to (\mathbb{R} \times \mathbb{R}^m, \mathbb{R} \times T)$ be given by $F(t, x) = (t, f_t(x))$, which is a C^∞ level preserving map-germ. Then each germ f_t has the same k-jet at S if and only if*

$$\frac{\partial F}{\partial t} \in \varepsilon^{k+1}(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m.$$

Lemma 2.8 *Let $f \in E$ and $V = \{g \in E_f \mid j^k g = j^k f\}$. Then*

$$T_f V = \varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m.$$

Theorem 2.1 *Let $f \in E$. If $\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tf$, then f is weak $(2k+1)$-$\mathcal{A}_{S,T}$-determined.*

Proof Let $g \in E_f$ such that $j^\ell g = j^\ell f$, where $\ell = 2k+1$. Let

$$F : (\mathbb{R} \times \mathbb{R}^n, \mathbb{R} \times S) \to (\mathbb{R} \times \mathbb{R}^m, \mathbb{R} \times T)$$

be given by $F(t, x) = (t, f_t(x))$, where $f_t(x) = (1-t)f(x) + tg(x)$. So F is a level preserving C^∞ map-germ with $f_0(x) = f(x)$ and $f_1(x) = g(x)$. To prove that f is $\mathcal{A}_{S,T}$-equivalent to g, it is enough to show that F is a trivial unfolding of f. However, by using the assumption that f and g have the same ℓ-jet at S, we get

$$\frac{\partial F}{\partial t} = \frac{\partial f_t(x)}{\partial t} \in \varepsilon^{\ell+1}(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m.$$

According to Lemma 2.4, it suffices to prove that

$$\varepsilon^{\ell+1}(S, n) \cdot C_{\mathbb{R}\times S}(\mathbb{R} \times \mathbb{R}^n)^m \subset TF. \tag{2.1}$$

The hypothesis in the theorem gives

$$\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tf. \tag{2.2}$$

By Lemma 2.1, we have

$$Tf + \varepsilon^{\ell+1}(S, n) \cdot C_S(\mathbb{R}^n)^m = Tg + \varepsilon^{\ell+1}(S, n) \cdot C_S(\mathbb{R}^n)^m. \tag{2.3}$$

So

$$\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tg + \varepsilon^{\ell+1}(S, n) \cdot C_S(\mathbb{R}^n)^m. \tag{2.4}$$

Set

$$\begin{aligned}
A &= C_S(\mathbb{R}^n)^m/\varepsilon(S,\,n)\langle \mathrm{d}g\rangle,\\
B &= f^*(\varepsilon(T,\,m))\langle e_1,\,\cdots,\,e_m\rangle/\varepsilon(S,\,n)\langle \mathrm{d}g\rangle \cap f^*(\varepsilon(T,\,m))\langle e_1,\,\cdots,\,e_m\rangle,\\
M &= \varepsilon^{k+1}(S,\,n).
\end{aligned}$$

Then A is a finitely generated $C_S(\mathbb{R}^n)$-module, B is a finitely generated $C_T(\mathbb{R}^m)$-module, and M is an ideal of $C_S(\mathbb{R}^n)$. Note that

$$\begin{aligned}
B &= [f^*(\varepsilon(T,\,m))\langle e_1,\,\cdots,\,e_m\rangle + \varepsilon(S,\,n)\langle \mathrm{d}g\rangle]/\varepsilon(S,\,n)\langle \mathrm{d}g\rangle\\
&\subset [(f^*\varepsilon(T,\,m)\cdot C_S(\mathbb{R}^n)^m + \varepsilon(S,\,n)\langle \mathrm{d}g\rangle)]/\varepsilon(S,\,n)\langle \mathrm{d}g\rangle\\
&= f^*\varepsilon(T,\,m)\cdot A.
\end{aligned}$$

From (2.4) we get

$$M\cdot A \subset B + M^2\cdot A. \tag{2.5}$$

Thus (2.5) implies $M\cdot A\subset B$ by Lemma 2.5. This shows that

$$\varepsilon^{k+1}(S,\,n)\cdot C_S(\mathbb{R}^n)^m \subset Tg. \tag{2.6}$$

Since each germ f_t has the same ℓ-jet at S as f, it follows that

$$\varepsilon^{k+1}(S,\,n)\cdot C_S(\mathbb{R}^n)^m \subset Tf_t. \tag{2.7}$$

For fixed $t_0\in\mathbb{R}$, we let $F^{t_0}(t,\,x) = (t,\,f_{t_0}(x))$. Applying Lemma 2.3 and (2.7), we get

$$\varepsilon^{k+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m \subset TF^{t_0}.$$

$F(t,\,x) - F^{t_0}(t,\,x) = (0,\,(t-t_0)[g(x)-f(x)])$, then

$$F - F^{t_0} \in \varepsilon^{\ell+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m.$$

Thus,

$$TF + \varepsilon^{\ell+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m = TF^{t_0} + \varepsilon^{\ell+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m,$$

and we again use the algebraic argument as (2.6) to conclude that

$$\varepsilon^{k+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m \subset TF.$$

Then (2.1) holds.

Theorem 2.2 *Let f be a relative map-germ. If any relative map-germ g which has the same k-jet at S as f satisfies*

$$\varepsilon^{k+1}(S,\,n)\cdot C_S(\mathbb{R}^n)^m \subset Tg + \varepsilon^{2k+2}(S,\,n)\cdot C_S(\mathbb{R}^n)^m, \tag{2.8}$$

then f is weak k-$\mathcal{A}_{S,\,T}$-determined.

Proof Let $\ell = 2k+1$. We denote by J^ℓ the set of ℓ-jets at S of elements of E_f, and $G = \mathcal{A}^\ell_{S,T}$. Let $V = \pi^{-1}(j^k f)$, where $\pi : J^\ell \to J^k$ is the canonical projection. Then G is a Lie group, and J^ℓ is a C^∞ manifold with a vector space structure, so V is a subspace of J^ℓ. Therefore, V is a connected C^∞ manifold of J^ℓ.

First, we show that V is in a single orbit of the action of G on J^ℓ. By Lemma 2.6, it suffices to show that

$$T_v V \subset T_v(G \cdot v) \tag{2.9}$$

for all $v \in V$. Let $\pi^\ell : E_f \to J^\ell$ be the canonical projection. From (2.8), we get

$$\pi^\ell(\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m) \subset \pi^\ell(Tg).$$

Note that $\pi^\ell(\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m) = T_{j^\ell g}V$ and $\pi^\ell(Tg) = T_{j^\ell g}(G \cdot j^\ell g)$, thus

$$T_{j^\ell g}V \subset T_{j^\ell g}(G \cdot j^\ell g) \quad \text{for all } j^\ell g \in V.$$

So (2.9) holds. Hence $j^\ell g$ and $j^\ell f$ are in the same G-orbit. This means that there exists $\Phi \in \mathcal{A}_{S,T}$ such that $j^\ell\Phi \cdot j^\ell g = j^\ell f$, that is, $j^\ell(\Phi \cdot g) = j^\ell f$.

Since f also satisfies (2.8), we get

$$\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tf + \varepsilon^{2k+2}(S, n) \cdot C_S(\mathbb{R}^n)^m. \tag{2.10}$$

From the proof of Theorem 2.1, we have

$$\varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m \subset Tf. \tag{2.11}$$

Then f is weak $(2k+1)$-$\mathcal{A}_{S,T}$-determined. Noting that $j^{2k+1}(\Phi \cdot g) = j^{2k+1}f$, it follows that $\Phi \cdot g$ is $\mathcal{A}_{S,T}$-equivalent to f. Clearly g and $\Phi \cdot g$ are $\mathcal{A}_{S,T}$-equivalent. Therefore, g and f are $\mathcal{A}_{S,T}$-equivalent.

Theorem 2.3 *Let f be a relative map-germ. If D is a $C_S(\mathbb{R}^n)$-module such that*

(a) $D \subset \overline{Tf} + \varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m$,

(b) $\varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m \subset \varepsilon(S, n)\langle \mathrm{d}f\rangle + f^*\varepsilon(T, m) \cdot D + \varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m$,

then f is weak k-$\mathcal{A}_{S,T}$-determined.

Proof It suffices to show that (a) and (b) imply the condition (2.8), for any $g \in E_f$ which has the same k-jet at S as f.

By Lemma 2.1, we see that (a), (b) also hold with g replacing f:

(a′) $D \subset \overline{Tg} + \varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m$,

(b′) $\varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m \subset \varepsilon(S, n)\langle \mathrm{d}g\rangle + g^*\varepsilon(T, m) \cdot D + \varepsilon^{k+1}(S, n) \cdot C_S(\mathbb{R}^n)^m$.

Set $A = [\varepsilon(S, n)\langle \mathrm{d}g\rangle + g^*\varepsilon(T, m) \cdot D + \varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m]/[\varepsilon(S, n)\langle \mathrm{d}g\rangle + g^*\varepsilon(T, m) \cdot D + \varepsilon^{2k+2}(S, n) \cdot C_S(\mathbb{R}^n)^m]$. It follows from (b′) that

$$A \subset \varepsilon(S, n) \cdot A.$$

Besides, $\varepsilon^{k+2}(S, n) \cdot A = 0$. Thus $A = 0$. So

$$\varepsilon^k(S, n) \cdot C_S(\mathbb{R}^n)^m \subset \varepsilon(S, n)\langle \mathrm{d}g\rangle + g^*\varepsilon(T, m) \cdot D + \varepsilon^{2k+2}(S, n) \cdot C_S(\mathbb{R}^n)^m. \tag{2.12}$$

Combining (a′) and (2.12), we have

$$\begin{aligned}\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m &\subset \varepsilon(S,\,n)\langle \mathrm{d}g\rangle + g^*\varepsilon(T,\,m)[\overline{Tg}+\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m] \\ &\quad + \varepsilon^{2k+2}(S,\,n)\cdot C_S(\mathbb{R}^n)^m \\ &\subset Tg + g^*\varepsilon(T,\,m)\cdot\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m + \varepsilon^{2k+2}(S,\,n)\cdot C_S(\mathbb{R}^n)^m. \qquad (2.13)\end{aligned}$$

Set $B=[Tg+\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m)]/[Tg+\varepsilon^{2k+2}(S,\,n)\cdot C_S(\mathbb{R}^n)^m]$. Then (2.13) implies

$$B\subset g^*\varepsilon(T,\,m)\cdot B.$$

Note that $\varepsilon^{k+2}(S,\,n)\cdot B=0$, and $g^*\varepsilon(T,\,m)\subset\varepsilon(S,\,n)$. So

$$B\subset\varepsilon(S,\,n)\cdot B\subset\varepsilon^{k+2}(S,\,n)\cdot B=0.$$

Therefore, $B=0$. It follows that

$$\varepsilon^{k+1}(S,\,n)\cdot C_S(\mathbb{R}^n)^m\subset\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m\subset Tg+\varepsilon^{2k+2}(S,\,n)\cdot C_S(\mathbb{R}^n)^m.$$

Applying Theorem 2.2, we complete the proof.

3 Weak Finite $\mathcal{K}_{S,T}$-Determinacy of Relative Map-Germs

For a map-germ f, define

$$T\mathcal{K}_{S,\,T}f=\varepsilon(S,\,n)\cdot\langle\mathrm{d}f\rangle+f^*\varepsilon(T,\,m)\cdot C_S(\mathbb{R}^n)^m.$$

This is a $C_S(\mathbb{R}^n)$-submodule of $C_S(\mathbb{R}^n)^m$.

Theorem 3.1 *Let f be a relative map-germ. If*

$$\varepsilon^k(S,\,n)\cdot C_S(\mathbb{R}^n)^m\subset T\mathcal{K}_{S,\,T}f, \qquad (3.1)$$

then f is weak k-$\mathcal{K}_{S,\,T}$-determined.

Proof Let g be a relative map-germ which has the same k-jet at S as f. Define

$$F:(\mathbb{R}\times\mathbb{R}^n,\,\mathbb{R}\times S)\to(\mathbb{R}^m,\,T)$$

by $F(\lambda,\,x)=f(x)+\lambda h(x)$ and $F_\lambda(x)=F(\lambda,\,x)$, where $h=g-f$.

For fixed $\lambda_0\in\mathbb{R}$, it is clear that we only have to prove that F_λ is $\mathcal{K}_{S,\,T}$-equivalent to F_{λ_0} for any λ sufficiently close to λ_0 in $\mathbb{R}$.

Since $F_{\lambda_0}-f=\lambda_0h\in\varepsilon^{k+1}(S,\,n)\cdot C_S(\mathbb{R}^n)^m$, from (3.1), it follows that

$$T\mathcal{K}_{S,\,T}F_{\lambda_0}\subset T\mathcal{K}_{S,\,T}f$$

and

$$T\mathcal{K}_{S,\,T}f\subset T\mathcal{K}_{S,\,T}F_{\lambda_0}+\varepsilon(S,\,n)\cdot T\mathcal{K}_{S,\,T}f.$$

By Nakayama's lemma, we get

$$T\mathcal{K}_{S,\,T}F_{\lambda_0}=T\mathcal{K}_{S,\,T}f. \qquad (3.2)$$

Hence $T\mathcal{K}_{S,\,T}F_{\lambda_0}$ also satisfies (3.1). Now we identify $C_S(\mathbb{R}^n)$ with the subring of $C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)$ composed of functions independent of λ, so $C_S(\mathbb{R}^n)\subset C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)$. From (3.1)–(3.2), we get

$$\varepsilon^k(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m\subset\varepsilon(\mathbb{R}\times S,\,1+n)\cdot\langle \mathrm{d}F_{\lambda_0}\rangle+F_{\lambda_0}^*\varepsilon(T,\,m)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m. \tag{3.3}$$

Also, since $F-F_{\lambda_0}=(\lambda-\lambda_0)h\in\varepsilon^{k+1}(S,\,n)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m$, by the same argument just as (3.2), we have

$$\begin{aligned}&\varepsilon(\mathbb{R}\times S,\,1+n)\cdot\langle \mathrm{d}F_{\lambda_0}\rangle+F_{\lambda_0}^*\varepsilon(T,\,m)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m\\&=\varepsilon(\mathbb{R}\times S,\,1+n)\cdot\Big\langle\frac{\partial F}{\partial x_1},\,\cdots,\,\frac{\partial F}{\partial x_n}\Big\rangle+F^*\varepsilon(T,\,m)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m.\end{aligned} \tag{3.4}$$

Combining (3.3) and (3.4), it shows that there exist germs $X_i\in\varepsilon(\mathbb{R}\times S,\,1+n)$, $i=1,\,\cdots,\,n$, such that

$$h+\sum_{i=1}^n X_i(\lambda,x)\frac{\partial F}{\partial x_i}\in F^*\varepsilon(T,\,m)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m. \tag{3.5}$$

From (3.5), we can find a germ of the vector field X in $\mathbb{R}\times\mathbb{R}^n$ of the following form:

$$\frac{\partial}{\partial\lambda}+\sum_{i=1}^n X_i(\lambda,x)\frac{\partial}{\partial x_i},\quad \text{where } X_i\in\varepsilon(\mathbb{R}\times S,\,1+n)$$

such that $DF(X)\in F^*\varepsilon(T,\,m)\cdot C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)^m$. By integrating the vector field X, we get a one-parameter family of diffeomorphisms $\phi_\lambda\in\mathcal{R}_S(n)$. So we can find a $m\times m$ matrix $A(\lambda,\,x)$ with entries in $C_{\mathbb{R}\times S}(\mathbb{R}\times\mathbb{R}^n)$ such that

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}F(\lambda,\,\phi_\lambda(x))=DF(\lambda,\,\phi_\lambda(x))\cdot X(\lambda,\,\phi_\lambda(x))=A(\lambda,\,\phi_\lambda(x))\cdot F(\lambda,\,\phi_\lambda(x)), \tag{3.6}$$

where matrix $A(\lambda,\,x)$ has the form:

$$\begin{pmatrix} * & 0_{m\times(m-t)}\end{pmatrix}_{m\times m}.$$

Hence, for fixed $x\in\mathbb{R}^n$, $F(\lambda,\,\phi_\lambda(x))$ is a solution of the differential equation $\dot{y}=A(\lambda,\,\phi_\lambda(x))y$ with initial condition $y(\lambda_0,\,x)=F_{\lambda_0}(\phi_{\lambda_0}(x))$.

Since the solution of this differential equation is unique and of the form:

$$y(\lambda,\,x)=M(\lambda,\,x)\cdot y(\lambda_0,\,x),$$

where $M(\lambda,\,x)$ is an invertible matrix, we can conclude that

$$F(\lambda,\,\phi_\lambda(x))=M(\lambda,\,x)\cdot(F_{\lambda_0}\circ\phi_{\lambda_0}(x)). \tag{3.7}$$

Therefore, $f+\lambda h$ and $f+\lambda_0 h$ are $\mathcal{K}_{S,\,T}$-equivalent for λ close to λ_0. By the connection of $[0,\,1]$, we obtain that f and g are $\mathcal{K}_{S,\,T}$-equivalent.

4 Application

Definition 4.1 *A relative map-germ f is $\mathcal{A}_{S,T}$-stable if all its unfoldings are trivial with respect to group $\mathcal{A}_{S,T}$.*

Remark 4.1 The argument given in [10] shows that a relative map-germ f is $\mathcal{A}_{S,T}$-stable if and only if $\overline{Tf} = C_S(\mathbb{R}^n)^m$.

Theorem 4.1 *Suppose that a relative map-germ f is $\mathcal{A}_{S,T}$-stable. Then f is weak $(m+1)$-$\mathcal{A}_{S,T}$-determined and weak $(m+1)$-$\mathcal{K}_{S,T}$-determined.*

Proof Set $A = C_S(\mathbb{R}^n)\langle \mathrm{d}f\rangle + f^*\varepsilon(T, m)\cdot C_S(\mathbb{R}^n)^m$. This is a $C_S(\mathbb{R}^n)$-submodule of $C_S(\mathbb{R}^n)^m$. Consider the sequence of inclusion of $C_S(\mathbb{R}^n)$-module:

$$\begin{aligned} C_S(\mathbb{R}^n)^m &\supset A+\varepsilon(S, n)C_S(\mathbb{R}^n)^m \supset A+\varepsilon(S, n)\mathcal{M}_n C_S(\mathbb{R}^n)^m \cdots \\ &\supset A+\varepsilon(S, n)\mathcal{M}_n^m C_S(\mathbb{R}^n)^m \supset \cdots . \end{aligned}$$

Denote by c_i the codimension of $A+\varepsilon(S, n)\mathcal{M}_n^i C_S(\mathbb{R}^n)^m$ in $A+\varepsilon(S, n)\mathcal{M}_n^{i-1} C_S(\mathbb{R}^n)^m$, $i \geq 1$ and let c_0 denote the codimension of $A+\varepsilon(S, n)C_S(\mathbb{R}^n)^m$ in $C_S(\mathbb{R}^n)^m$. Thus the codimension of $A+\varepsilon(S, n)\mathcal{M}_n^i C_S(\mathbb{R}^n)^m$ in $C_S(\mathbb{R}^n)^m$ is equal to

$$c_0 + c_1 + \cdots + c_i. \tag{4.1}$$

Note that $f^*(\varepsilon(T, m))\cdot C_S(\mathbb{R}^n)^m \subset A$. By Remark 4.1, we have

$$\begin{aligned} &\dim C_S(\mathbb{R}^n)^m/(A+\varepsilon(S, n)\mathcal{M}_n^m C_S(\mathbb{R}^n)^m) \\ &\leq \dim f^*(C_T(\mathbb{R}^m))\langle e_1, \cdots, e_m\rangle / f^*(\varepsilon(T, m))\langle e_1, \cdots, e_m\rangle \\ &\leq m. \end{aligned}$$

Therefore, by (4.1), $c_0 + c_1 + \cdots + c_m \leq m$, thus $c_m = 0$. It follows that

$$\varepsilon(S, n)\mathcal{M}_n^{m-1} C_S(\mathbb{R}^n)^m \subset A.$$

So, $\varepsilon^m(S, n)C_S(\mathbb{R}^n)^m \subset A$.

Then it follows that

(a) $\varepsilon^m(S, n)\cdot C_S(\mathbb{R}^n)^m \subset C_S(\mathbb{R}^n)\langle \mathrm{d}f\rangle + f^*\varepsilon(T, m)\cdot C_S(\mathbb{R}^n)^m$.

Multiplying through (a) by $\varepsilon(S, n)$, we have

(b) $\varepsilon^{m+1}(S, n)\cdot C_S(\mathbb{R}^n)^m \subset \varepsilon(S, n)\langle \mathrm{d}f\rangle + f^*\varepsilon(T, m)\cdot \varepsilon(S, n)^m$.

On the other hand, by the definition of $\mathcal{A}_{S,T}$-stable map-germ, we get

(c) $\varepsilon(S, n)^m = Tf \subset \overline{Tf}$.

Using Theorem 2.3 with $D = \varepsilon(S, n)^m$, we can conclude that f is $(m+1)$-$\mathcal{A}_{S,T}$-determined.

From (b), we get

(d) $\varepsilon^{m+1}(S, n)\cdot C_S(\mathbb{R}^n)^m \subset T\mathcal{K}_{S,T}f$.

Thus f is weak $(m+1)$-$\mathcal{K}_{S,T}$-determined, by applying Theorem 3.1.

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