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Symmetric q-Deformed KP Hierarchy*

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Abstract Based on the analytic property of the symmetric q-exponent $e_q(x)$, a new symmetric q-deformed Kadomtsev-Petviashvili (q-KP for short) hierarchy associated with the symmetric q-derivative operator ∂_q is constructed. Furthermore, the symmetric q-CKP hierarchy and symmetric q-BKP hierarchy are defined. The authors also investigate the additional symmetries of the symmetric q-KP hierarchy.

Keywords q-Derivative, Symmetric q-KP hierarchy, Additional symmetries
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1 Introduction

The origin of q-calculus (quantum calculus) (see [1–2]) traces back to the early 20th century. Many mathematicians have important works in the area of q-calculus, q-hypergeometric series and quantum group. There exist two different forms of q-derivative operators, which are defined respectively by

$$D_q(f(x)) = \frac{f(qx) - f(x)}{(q-1)x}, \quad q \neq 1 \tag{1.1}$$

and

$$\partial_q(f(x)) = \frac{f(qx) - f(q^{-1}x)}{(q-q^{-1})x}, \quad q \neq 1. \tag{1.2}$$

The so-called q-deformation of the integrable system (or q-deformed integrable system) started in the 1990s by means of the first q-derivative D_q in (1.1) instead of the usual derivative ∂ with respect to x in the classical system. As we know, the q-deformed integrable system reduces to a classical integrable system as q goes to 1. Several q-deformed integrable systems have been presented, for example, the q-deformation of the KdV hierarchy (see [3–6]), the q-Toda equation (see [7]), the q-Calogero-Moser equation (see [8]) and so on. The q-deformed Kadomtsev-Petviashvili (q-KP for short) hierarchy is also a subject of intensive study in the literature [9–17]. Indeed, it is worth pointing out that there exist two variants of the q-deformed integrable system, one belonging to Frenkel [3] and the other to Zhang et al. [4–17].

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It has been known for some time that different sub-hierarchies of the KP hierarchy can be obtained by adding different reduction conditions on Lax operator L. Two important sub-hierarchies of the KP hierarchy are CKP hierarchy (see [18]) through a restriction $L^* = -L$ and BKP hierarchy (see [19]) through a restriction $L^* = -\partial L \partial^{-1}$. However, to the best of our knowledge, there has been no result on the q-deformed CKP hierarchy and the q-deformed BKP hierarchy so far. The difficulty to define them is the conjugate operation "$*$" of q-derivative D_q in (1.1). In fact, $D_q^* \neq -D_q$ but $D_q^* = -D_q\theta^{-1} = -\frac{1}{q}D_{\frac{1}{q}}$. This paper shows a quite interesting fact as $\partial_q^* = -\partial_q$, where the symmetric q-derivative operator ∂_q is defined by (1.2). In what follows, we shall fill the gap by constructing the new symmetric q-deformed KP hierarchy based on the symmetric q-derivative operator ∂_q.

This paper is organized as follows. Some basic results of the symmetric q-derivative operator ∂_q are given in Section 2, and one formula for the symmetric q-exponent $e_q(x)$ is established. Then a new symmetric q-KP hierarchy is stated in Sections 3 similar to the classical KP hierarchy (see [20]), and also a symmetric q-CKP hierarchy and a symmetric q-BKP hierarchy are given in this section. We further study the additional symmetries for the symmetric q-KP hierarchy in Section 4. Section 5 is devoted to conclusions and discussions.

2 Symmetric Quantum Calculus

We give some useful facts about the symmetric q-derivative operator ∂_q in the form of (1.2) based on the literature [2]. We work in an associative ring of functions which includes a q-variable x and infinite time variables $t_i \in \mathbb{R}$,

$$F = f = f(x; t_1, t_2, t_3, \cdots).$$

The q-shift operator is defined by

$$\theta(f(x)) = f(qx). \tag{2.1}$$

Note that θ does not commute with ∂_q. Indeed, the relation

$$(\partial_q \theta^k(f)) = q^k \theta^k(\partial_q f), \quad k \in \mathbb{Z}$$

holds. The limit of $\partial_q(f(x))$ as q approaches to 1 is the ordinary differentiation $\partial_x(f(x))$. We denote the formal inverse of ∂_q as ∂_q^{-1}.

Theorem 2.1 *The conjugate of ∂_q can be defined as*

$$\partial_q^* = -\partial_q.$$

Proof The first step is to prove $\theta^* = q^{-1}\theta^{-1}$. According to the definition, we have

$$\partial_q(fg) = (\theta f)(\partial_q g) + (\partial_q f)(\theta^{-1} g) = (\theta g)(\partial_q f) + (\partial_q g)(\theta^{-1} f).$$

Calculating the quantum integration $\int \cdot \mathrm{d}_q x$ for the above two formulas separately, it follows that

$$\int (\theta f)(\partial_q g)\mathrm{d}_q x = -\int (\partial_q f)(\theta^{-1} g)\mathrm{d}_q x, \tag{2.2}$$

$$\int (\theta g)(\partial_q f)\mathrm{d}_q x = -\int (\partial_q g)(\theta^{-1} f)\mathrm{d}_q x. \tag{2.3}$$

Let $g \to \theta^{-2}g$ in (2.3), and it now yields

$$\int(\theta^{-1}g)(\partial_q f)\mathrm{d}_q x = -\int(\partial_q\theta^{-2}g)(\theta^{-1}f)\mathrm{d}_q x.$$

Comparing it with (2.2), the above equation becomes

$$\int(\theta f)(\partial_q g)\mathrm{d}_q x = \int(\partial_q\theta^{-2}g)(\theta^{-1}f)\mathrm{d}_q x.$$

It can now be written in the form

$$\langle\theta f, \partial_q g\rangle = \langle\theta^{-1}f, q^{-2}\theta^{-2}\partial_q g\rangle.$$

By letting $g \to \theta^{-2}g$ and $f \to \theta f$ in the above equation, we find that

$$\langle\theta^2 f, g\rangle = \langle f, q^{-2}\theta^{-2}g\rangle,$$

so one can choose $\theta^* = q^{-1}\theta^{-1}$.

We will now proceed to prove $\partial_q^* = -\partial_q$. Let $f \to \theta^{-1}f$ and $g \to \theta g$ in (2.2), and it now reads

$$\langle\partial_q\theta^{-1}f, g\rangle = -\langle f, \partial_q\theta g\rangle.$$

This implies

$$(\partial_q\theta)^* = -\partial_q\theta^{-1}.$$

According to the equation $\theta^* = q^{-1}\theta^{-1}$, we get

$$\partial_q^* = -q\theta\partial_q\theta^{-1} = -\partial_q.$$

The following q-deformed Leibnitz rule holds:

$$\partial_q^n \circ f = \sum_{k\geq 0}\binom{n}{k}_q \theta^{n-k}(\partial_q^k f)\theta^{-k}\partial_q^{n-k}, \quad n \in Z, \tag{2.4}$$

where the q-number

$$(n)_q = \frac{q^n - q^{-n}}{q - q^{-1}}$$

and the q-binomial is introduced as

$$\begin{aligned}\binom{n}{0}_q &= 1,\\ \binom{n}{k}_q &= \frac{(n)_q(n-1)_q\cdots(n-k+1)_q}{(1)_q(2)_q\cdots(k)_q}, \quad n\in\mathbb{Z},\ k\in\mathbb{Z}_+.\end{aligned}$$

To illustrate the q-deformed Leibnitz rule, the following examples are given:

$$\begin{aligned}\partial_q \circ f &= \theta(f)\partial_q + (\partial_q f)\theta^{-1},\\ \partial_q^2 \circ f &= (q+q^{-1})\theta(\partial_q f)\theta^{-1}\partial_q + \theta^2(f)\partial_q^2 + (\partial_q^2 f)\theta^{-2},\\ \partial_q^3 \circ f &= (q^2+q^{-2}+1)\theta(\partial_q^2 f)\theta^{-2}\partial_q + (q^2+q^{-2}+1)\theta^2(\partial_q f)\theta^{-1}\partial_q^2 + (\partial_q^3 f)\theta^{-3} + \theta^3(f)\partial_q^3,\\ \partial_q^{-1} \circ f &= \theta^{-1}(f)\partial_q^{-1} - \theta^{-2}(\partial_q f)\theta^{-1}\partial_q^{-2} + \cdots + (-1)^k\theta^{-k-1}(\partial_q^k f)\theta^{-k}\partial_q^{-k-1} + \cdots.\end{aligned}$$

Using the Taylor's formula, we can get the following proposition for the symmetric q-exponent $e_q(x)$, which is crucial to developing the tau function of the symmetric q-KP hierarchy and to researching the interaction of q-solitons in the future.

Theorem 2.2 *The q-exponent $e_q(x)$ is defined as*

$$e_q(x)=\sum_{n=0}^{\infty}\frac{x^n}{(n)_q!},\tag{2.5}$$

where

$$(n)_q!=(n)_q(n-1)_q(n-2)_q\cdots(1)_q,$$

and then the formula

$$e_q(x)=\exp\Big(\sum_{k=1}^{\infty}c_kx^k\Big)\tag{2.6}$$

holds, where

$$c_k=\sum_{i=1}^{k}(-1)^{i-1}\frac{1}{i}\sum_{\substack{v_1+v_2+\cdots+v_i=k\\ v_1,v_2,\cdots,v_i\in\mathbb{Z}_+}}\frac{1}{(v_1)_q!(v_2)_q!\cdots(v_i)_q!}.\tag{2.7}$$

Proof From the definition of $e_q(x)$ and Taylor's formula, it follows that

$$\begin{aligned}
e_q(x)=&1+\sum_{n=1}^{\infty}\frac{x^n}{(n)_q!}\\
=&\exp\Big(\ln\Big(1+\sum_{n=1}^{\infty}\frac{x^n}{(n)_q!}\Big)\Big)\\
=&\exp\Big(\sum_{i=1}^{\infty}(-1)^{i-1}\frac{1}{i}\Big(\sum_{n=1}^{\infty}\frac{x^n}{(n)_q!}\Big)^i\Big)\\
=&\exp\Big(\sum_{k=1}^{\infty}\sum_{i=1}^{k}(-1)^{i-1}\frac{1}{i}\sum_{\substack{v_1+v_2+\cdots+v_i=k\\ v_1,v_2,\cdots,v_i\in\mathbb{Z}_+}}\frac{x^k}{(v_1)_q!(v_2)_q!\cdots(v_i)_q!}\Big)\\
=&\exp\Big(\sum_{k=1}^{\infty}c_kx^k\Big),
\end{aligned}$$

where c_k is given by (2.7).

Several explicit forms of q-exponent $e_q(x)$ can be written out as follows:

$$\begin{aligned}
c_1=&1,\\
c_2=&-\frac{(q-1)^2}{2(q^2+1)},\\
c_3=&\frac{(q-1)^2(q^4-q^3-q^2-q+1)}{3(q^2+1)(q^4+q^2+1)},\\
c_4=&-\frac{(q-1)^4(q^4-q^3-2q^2-q+1)}{4(q^2-q+1)(q^6+q^4+q^2+1)},\\
c_5=&\frac{(q-1)^4(q^{14}-2q^{13}-2q^{11}+q^{10}-2q^9+5q^8+q^7+5q^6-2q^5+q^4-2q^3-2q-1)}{5(q^2+1)(q^2-q+1)(q^6+q^4+q^2+1)(q^8+q^6+q^4+q^2+1)},\\
c_6=&-\frac{(q-1)^6((q^{12}+1)(q^2-3q+1)+q^2(q^8+1)(q+1)-4q^5(q^3-1)(q-1)+2q^7)}{6(q^2-q+1)(q^6+q^4+q^2+1)(q^4-q^3+q^2-q+1)(q^8-q^7+q^6+q^2-q+1)}.
\end{aligned}$$

For the case $D_q(f(x)) = \frac{f(qx)-f(x)}{(q-1)x}$ and $(\widetilde{n})_q = \frac{q^n-1}{q-1}$, the q-exponent function $\widetilde{e}_q(x)$ is defined as $\widetilde{e}_q(x) = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{(\widetilde{n})_q!}$, and then

$$\widetilde{e}_q(x) = \exp\Big(\sum_{k=1}^{\infty} \widetilde{c}_k x^k\Big), \tag{2.8}$$

where

$$\widetilde{c}_k = \frac{(1-q)^k}{k(1-q^k)}. \tag{2.9}$$

Recall that the q-exponent function $e_q(x)$ is the eigenfunction of the operator ∂_q, i.e.,

$$\partial_q e_q(x) = e_q(x).$$

Furthermore, from

$$e_q(xz) = \sum_{n=0}^{\infty} \frac{(xz)^n}{(n)_q!}$$

one obtains immediately the formula

$$\partial_q^m e_q(xz) = z^m e_q(xz), \quad m = 1, 2, 3, \cdots,$$

which is useful for defining the q-wave function of the symmetric q-KP hierarchy in the following section.

3 Symmetric q-Deformed KP Hierarchy

Similar to the classical KP hierarchy (see [19–20]), we will define a new symmetric q-deformed KP hierarchy. The Lax operator L of the symmetric q-KP hierarchy is given by

$$L = \partial_q + u_1 + u_2\partial_q^{-1} + u_3\partial_q^{-2} + \cdots, \tag{3.1}$$

where $u_i = u_i(x; t_1, t_2, t_3, \cdots)$, $i = 1, 2, 3, \cdots$. The corresponding Lax equation of the symmetric q-KP hierarchy is defined by

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots, \tag{3.2}$$

where the differential part $B_n = (L^n)_+ = \sum\limits_{i=0}^{n} b_i\partial_q^i$ and the integral part $(L^n)_- = L^n - (L^n)_+$.

The first few B_n and flow equations in (3.2) for dynamical variables $\{u_1, u_2, u_3, \cdots\}$ can be written out as follows:

$$\begin{aligned}
B_1 &= \partial_q + u_1,\\
B_2 &= \partial_q^2 + v_1\partial_q + v_0,\\
B_3 &= \partial_q^3 + w_2\partial_q^2 + w_1\partial_q + w_0,
\end{aligned}$$

where $L^2 = B_2 + v_{-1}\partial_q^{-1} + \cdots$ and

$$\begin{aligned}
v_1 &= \theta(u_1) + u_1,\\
v_0 &= (\partial_q u_1)\theta^{-1} + \theta(u_2) + u_1^2 + u_2,\\
v_{-1} &= (\partial_q u_2)\theta^{-1} + \theta(u_3) + u_1u_2 + u_2\theta^{-1}(u_1) + u_3,\\
w_2 &= \theta(v_1) + u_1,\\
w_1 &= (\partial_q v_1)\theta^{-1} + \theta(v_0) + u_1v_1 + u_2,\\
w_0 &= (\partial_q v_0)\theta^{-1} + \theta(v_{-1}) + u_1v_0 + u_2\theta^{-1}(v_1) + u_3.
\end{aligned}$$

The first flow equations are

$$\begin{aligned}
\frac{\partial u_1}{\partial t_1} &= \theta(u_2) - u_2,\\
\frac{\partial u_2}{\partial t_1} &= (\partial_q u_2)\theta^{-1} + \theta(u_3) + u_1u_2 - u_2\theta^{-1}(u_1) - u_3,\\
\frac{\partial u_3}{\partial t_1} &= (\partial_q u_3)\theta^{-1} + \theta(u_4) + u_1u_3 + u_2(\theta^{-2}(\partial_q u_1))\theta^{-1} - u_3\theta^{-2}(u_1) - u_4,\\
\frac{\partial u_4}{\partial t_1} &= (\partial_q u_4)\theta^{-1} + \theta(u_5) + u_1u_4 - u_2(\theta^{-3}(\partial_q^2 u_1))\theta^{-2} - u_4\theta^{-3}(u_1) - u_5\\
&\quad + (2)_q u_3(\theta^{-3}(\partial_q u_1))\theta^{-1}.
\end{aligned}$$

The Lax operator L in (3.1) can be generated by a pseudo-difference operator $S = 1 + \sum_{k=1}^{\infty} s_k\partial_q^{-k}$ in the following way:

$$L = S\partial_q S^{-1}. \tag{3.3}$$

Here S is called a dressing operator or a wave operator of the symmetric q-KP hierarchy.

Theorem 3.1 *The dressing operator S of the symmetric q-KP hierarchy satisfies the Sato equation*

$$\frac{\partial S}{\partial t_j} = -(L^j)_- S, \quad j = 1, 2, 3, \cdots. \tag{3.4}$$

Proof From the Lax equation, $\frac{\partial L}{\partial t_n} = [B_n, L]$, which is followed by

$$\begin{aligned}
\frac{\partial L}{\partial t_j} &= [B_j, L] = (L^j)_+L - L(L^j)_+\\
&= (L^j - (L^j)_-)L - L(L^j - (L^j)_-)\\
&= -(L^j)_-L + L(L^j)_-.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\frac{\partial L}{\partial t_j} &= \frac{\partial}{\partial t_j}(S\partial_q S^{-1}) = \frac{\partial S}{\partial t_j}\partial_q S^{-1} + S\partial_q\frac{\partial S^{-1}}{\partial t_j}\\
&= \frac{\partial S}{\partial t_j}S^{-1}S\partial_q S^{-1} + S\partial_q\Big(-S^{-1}\frac{\partial S}{\partial t_j}S^{-1}\Big) = \frac{\partial S}{\partial t_j}S^{-1}L - L\frac{\partial S}{\partial t_j}S^{-1},
\end{aligned}$$

and then

$$\frac{\partial L}{\partial t_j} = -(L^j)_-L + L(L^j)_- = \frac{\partial S}{\partial t_j}S^{-1}L - L\frac{\partial S}{\partial t_j}S^{-1}.$$

The above equation implies that

$$\frac{\partial S}{\partial t_j} S^{-1} = -(L^j)_-, \quad j = 1, 2, 3, \cdots,$$

which ends the proof.

Definition 3.1 *The q-wave function $w_q(x, t; z)$ for the symmetric q-KP hierarchy* (3.2) *with the wave operator S in* (3.3) *is given by*

$$w_q(x, t; z) = S e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big), \tag{3.5}$$

where $t = (t_1, t_2, t_3, \cdots)$.

Theorem 3.2 *The q-wave function $w_q(x, t; z)$ of the symmetric q-KP hierarchy satisfies the following linear q-differential equations:*

$$L w_q = z w_q, \quad \partial_m w_q = (L^m)_+ w_q,$$

where $\partial_m = \frac{\partial}{\partial t_m}$.

Proof Using the equation $\partial_q e_q(xz) = z e_q(xz)$, then

$$L w_q = S \partial_q S^{-1} S e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) = S \partial_q e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) = z w_q.$$

From the Sato equation $\partial_m S = -(L^m)_- S$, it follows that

$$\begin{aligned}
\partial_m w_q &= \partial_m \Big(S e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big)\Big) \\
&= (\partial_m S) e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) + S e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) z^m \\
&= -(L^m)_- S e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) + S \partial_q^m e_q(xz) \exp\Big(\sum_{i=1}^{\infty} t_i z^i\Big) \\
&= -(L^m)_- w_q + (L^m)_+ w_q \\
&= (L^m)_+ w_q.
\end{aligned}$$

Furthermore, we would like to give the definitions of the symmetric q-CKP hierarchy and the symmetric q-BKP hierarchy respectively to answer the previous question proposed in the introduction.

Definition 3.2 *Let the operator L in* (3.1) *be the Lax operator for the symmetric q-KP hierarchy associated with* (3.2)*, if L satisfies the reduction condition $L^* = -L$, and then we call it the symmetric q-CKP hierarchy.*

Definition 3.3 *Let the operator L in* (3.1) *be the Lax operator for the symmetric q-KP hierarchy associated with* (3.2)*, if L satisfies the reduction condition $L^* = -\theta^{-\frac{1}{2}} \partial_q L \partial_q^{-1} \theta^{\frac{1}{2}}$, and then it is the symmetric q-BKP hierarchy.*

4 Additional Symmetries of the Symmetric q-KP Hierarchy

Another main goal of this paper is to consider the additional symmetries of the symmetric q-KP hierarchy. First, let us define Γ_q and Orlov-Shulman's M operator as

$$\Gamma_q = \sum_{i=1}^{\infty}(it_i + ic_i x^i)\partial_q^{i-1}, \quad M = S\Gamma_q S^{-1},$$

respectively, where c_i is given by (2.7). Then the additional flows of the symmetric q-KP hierarchy for each pair $\{m, n\}$ are defined by

$$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S. \tag{4.1}$$

Theorem 4.1 *The additional flows act on L and M of the symmetric q-KP hierarchy as*

$$\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L], \tag{4.2}$$

$$\frac{\partial M}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M]. \tag{4.3}$$

Proof By performing the derivative $\frac{\partial}{\partial t^*_{m,n}}$ on $L = S\partial_q S^{-1}$ and using (4.1), we observe that

$$\begin{aligned}
\frac{\partial L}{\partial t^*_{m,n}} &= \frac{\partial S}{\partial t^*_{m,n}}\partial_q S^{-1} + S\partial_q \frac{\partial S^{-1}}{\partial t^*_{m,n}} \\
&= -(M^m L^n)_- S\partial_q S^{-1} + S\partial_q\Big(-S^{-1}\frac{\partial S}{\partial t^*_{m,n}}S^{-1}\Big) \\
&= -(M^m L^n)_- L + S\partial_q S^{-1}(M^m L^n)_- \\
&= -[(M^m L^n)_-, L].
\end{aligned}$$

For the action on $M = S\Gamma_q S^{-1}$, there exists a similar derivation as $\frac{\partial L}{\partial t^*_{m,n}}$, and then

$$\begin{aligned}
\frac{\partial M}{\partial t^*_{m,n}} &= \frac{\partial S}{\partial t^*_{m,n}}\Gamma_q S^{-1} + S\Gamma_q \frac{\partial S^{-1}}{\partial t^*_{m,n}} \\
&= -(M^m L^n)_- S\Gamma_q S^{-1} + S\Gamma_q\Big(-S^{-1}\frac{\partial S}{\partial t^*_{m,n}}S^{-1}\Big) \\
&= -(M^m L^n)_- M + S\Gamma_q S^{-1}(M^m L^n)_- \\
&= -[(M^m L^n)_-, M].
\end{aligned}$$

In the above calculation, the fact that Γ_q does not depend on the additional flow variables $t^*_{m,n}$ has been used.

Theorem 4.2

$$\frac{\partial L^k}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L^k], \tag{4.4}$$

$$\frac{\partial M^k}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M^k], \tag{4.5}$$

$$\frac{\partial M^k L^l}{\partial t^*_{m,n}} = -[(M^m L^n)_-, M^k L^l], \tag{4.6}$$

$$\frac{\partial M^k L^l}{\partial t_n} = [B_n, M^k L^l]. \tag{4.7}$$

Proof We present only the proof of the first equation here. The others can be proved in a similar way.

$$\begin{aligned}
\frac{\partial L^k}{\partial t^*_{m,n}} &= \frac{\partial L}{\partial t^*_{m,n}} L^{k-1} + L \frac{\partial L}{\partial t^*_{m,n}} L^{k-2} + \cdots + L^{k-2} \frac{\partial L}{\partial t^*_{m,n}} L + L^{k-1} \frac{\partial L}{\partial t^*_{m,n}} \\
&= \sum_{l=1}^{k} L^{l-1} \frac{\partial L}{\partial t^*_{m,n}} L^{k-l} \\
&= \sum_{l=1}^{k} L^{l-1} (-[(M^m L^n)_-, L]) L^{k-l} \\
&= -[(M^m L^n)_-, L^k],
\end{aligned}$$

where we have used the formula $\frac{\partial L}{\partial t^*_{m,n}} = -[(M^m L^n)_-, L]$ in Theorem 4.1.

Theorem 4.3 *The additional flows* $\partial^*_{mn} = \frac{\partial}{\partial t^*_{m,n}}$ *commute with the hierarchy* $\partial_k = \frac{\partial}{\partial t_k}$, *i.e.,*

$$[\partial^*_{mn}, \partial_k] = 0,$$

and thus we call them additional symmetries of the symmetric q-KP hierarchy.

Proof According to the definition and Theorem 4.2, it equals

$$\begin{aligned}
[\partial^*_{mn}, \partial_k] S &= \partial^*_{mn}(\partial_k S) - \partial_k(\partial^*_{mn} S) \\
&= \partial^*_{mn}(-(L^k)_- S) - \partial_k(-(M^m L^n)_- S) \\
&= -(\partial^*_{mn} L^k)_- S - (L^k)_-(\partial^*_{mn} S) + (\partial_k M^m L^n)_- S + (M^m L^n)_-(\partial_k S) \\
&= [(M^m L^n)_-, L^k]_- S + (L^k)_-(M^m L^n)_- S + [(L^k)_+, M^m L^n]_- S - (M^m L^n)_-(L^k)_- S \\
&= [(M^m L^n)_-, L^k]_- S - [(M^m L^n)_-, (L^k)_+] S + [(L^k)_-, (M^m L^n)_-] S \\
&= [(M^m L^n)_-, (L^k)_-]_- S + [(L^k)_-, (M^m L^n)_-] S \\
&= 0.
\end{aligned}$$

$[(L^k)_+, (M^m L^n)]_- = [(L^k)_+, (M^m L^n)_-]_-$ and $[(M^m L^n)_-, (L^k)_-]_- = [(M^m L^n)_-, (L^k)_-]$ have been used in the above derivation.

5 Conclusions and Discussions

To summarize, we have derived the antisymmetric property of ∂_q in Theorem 2.1 and a crucial expression of $e_q(x)$ by the usual exponential in Theorem 2.2. The analytic property of symmetric $e_q(x)$ in Theorem 2.2 is used to define the wave function of the symmetric q-KP hierarchy. After introducing the dressing operator and the q-wave function of the symmetric q-KP hierarchy in Section 3, we also give the definitions of the symmetric q-CKP hierarchy and the symmetric q-BKP hierarchy. The additional symmetries of the symmetric q-KP hierarchy are obtained in Section 4. The above results of this paper show obviously that the symmetric q-KP hierarchy is different from the q-KP hierarchy (see [8–17]) based on $D_q(f(x))$.

In comparison with the known interesting results of the KP hierarchy (see [18–20]) and the q-KP hierarchy based on the $D_q(f(x))$ (see [8–17]), the symmetric q-KP hierarchy defined in this paper deserves further study from several aspects including the tau function and its Hirota bilinear identity, the Hamiltonian structure, the gauge transformation, the symmetry analysis and the interaction of q-solitons. Furthermore, it is highly nontrivial to consider the above topics of the symmetric q-CKP (or q-BKP) hierarchy because of the reduction condition $L^* = -L$ (or $L^* = -\partial_q L \partial_q^{-1}$) and the complexity of the ∂_q.

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