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Eigenvalue Comparison Theorems on Finsler Manifolds*

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Abstract Cheng-type inequality, Cheeger-type inequality and Faber-Krahn-type inequality are generalized to Finsler manifolds. For a compact Finsler manifold with the weighted Ricci curvature bounded from below by a negative constant, Li-Yau's estimation of the first eigenvalue is also given.

Keywords The first eigenvalue, Finsler-Laplacian, Ricci curvature, S-Curvature
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1 Introduction

In recent years, Finsler geometry has developed rapidly in its global and analytic aspects. The present main work is to generalize and improve some famous theorems of Riemann geometry to the Finsler setting. Among these issues, Finsler-Laplacian is one of the most important and interesting projects. As is well-known, there are several definitions of Finsler-Laplacian in Finsler geometry including nonlinear Laplacian, mean-value Laplacian and so on. With regard to nonlinear Finsler-Laplacian, some Laplacian comparison theorems, volume comparison theorems and various estimations on the first eigenvalue have been established (see [9, 12–13, 15]).

For a Riemannian n-manifold with $\mathrm{Ric} \geq (n-1)k$, Cheng [5] gave an upper bound estimation of the first Dirichlet eigenvalue of a geodesic ball. By using the weighted Ricci curvature condition, we extend this result to Finsler manifolds in this article (see Theorem 3.1). Besides, we also obtain an upper bound estimation of the first Neumann eigenvalue on a compact Finsler manifold in terms of the reversibility, the diameter and the lower bound of the weighted Ricci curvature (see Theorem 3.2). It is worth mentioning that in [12] and [4], Shen and Chen discussed these problems under the condition of Ricci curvature and S-curvature. However, in [12], the upper bound dose not have an explicit expression, and in [4] the result only focuses on the manifolds with Ricci curvature $\mathrm{Ric} \geq (n-1)k,\ k \leq 0$, while in this paper we will give an explicit expression of the upper bound on Finsler manifolds with weighted Ricci curvature $\mathrm{Ric}_N \geq (n-1)k,\ \forall k$. Here Ric_N is defined in Definition 2.1 below.

It is well-known that Cheeger's constant is estimated from below by a positive constant which depends only on the diameter and the lower bound of Ricci curvature of M. By defining

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Cheeger's constant on Finsler manifolds, we generalize the Cheeger's inequality into the Finsler setting (see Theorem 4.1). As to the Faber-Krahn inequality, Shen extended it in [6] to the domains in a Minkowski space. In the present paper, we will consider the Randers manifolds with constant flag curvature and obtain another Finsler version of Faber-Krahn inequality (see Theorem 4.2).

In the Riemannian case, by using the gradient estimate, Li and Yau [7] gave the lower bound of the first eigenvalue of manifolds with Ricci curvature bounded from below by a negative constant. Recently, Wang and Xia [14] generalized it to Finsler manifolds by the method of one-dimensional model. To give an explicit expression of the lower bound, we should use some weighted-linear operators. The technique is based on a comparison theorem on the gradient of the eigenfunction. Then we follow step by step the work of Li and Yau to get the result (see Theorem 4.3).

The contents of this paper are arranged as follows. In Section 2, some fundamental formulas which are necessary for the present paper are given, where some lemmas are contained. In Section 3, the upper bounds of the first eigenvalue of Finsler-Laplacian such as Cheng's type of inequalities are obtained. In Section 4, the lower bounds of the first eigenvalue of Finsler-Laplacian such as Cheeger type inequality, Faber-Krahn type inequality and Li-Yau's estimation are shown.

2 Preliminaries

Let M be an n-dimensional smooth manifold and $\pi : TM \to M$ be the natural projection from the tangent bundle TM. Let (x, y) be a point of TM with $x \in M$, $y \in T_xM$, and let (x^i, y^i) be the local coordinates on TM with $y = y^i \frac{\partial}{\partial x^i}$. A Finsler metric on M is a function $F : TM \to [0, +\infty)$ satisfying the following properties:

(i) (Regularity) $F(x, y)$ is smooth in $TM \setminus \{0\}$.

(ii) (Positive homogeneity) $F(x, \lambda y) = \lambda F(x, y)$ for $\lambda > 0$.

(iii) (Strong convexity) The fundamental quadratic form

$$g := g_{ij}(x, y)\mathrm{d}x^i \otimes \mathrm{d}x^j, \quad g_{ij} := \frac{1}{2}[F^2]_{y^i y^j}$$

is positively definite.

Let $X = X^i \frac{\partial}{\partial x^i}$ be a vector field. Then the covariant derivative of X by $v \in T_xM$ with reference vector $w \in T_xM\setminus\{0\}$ is defined by (see [2])

$$D_v^w X(x) := \Big\{ v^j \frac{\partial X^i}{\partial x^j}(x) + \Gamma^i_{jk}(w) v^j X^k(x) \Big\} \frac{\partial}{\partial x^i},$$

where Γ^i_{jk} denote the coefficients of the Chern connection.

Given two linearly independent vectors $V, W \in T_xM\setminus\{0\}$, the flag curvature is defined by

$$K(V, W) := \frac{g_V(R^V(V, W)W, V)}{g_V(V, V)g_V(W, W) - g_V(V, W)^2},$$

where R^V is the Chern curvature:

$$R^V(X, Y)Z = D_X^V D_Y^V Z - D_Y^V D_X^V Z - D_{[X,Y]}^V Z.$$

Then the Ricci curvature for (M, F) is defined as

$$\mathrm{Ric}(V) = \sum_{i=1}^{n-1} K(V, e_i),$$

where $e_1, \cdots, e_{n-1}, \frac{V}{F(V)}$ form an orthonormal basis of T_xM with respect to g_V.

For a given volume form $\mathrm{d}\mu = \sigma(x)\mathrm{d}x$ and a vector $y \in T_xM\backslash\{0\}$, the distortion of $(M, F, \mathrm{d}\mu)$ is defined by

$$\tau(y) := \ln \frac{\sqrt{\det(g_{ij}(y))}}{\sigma}.$$

To measure the rate of changes of the distortion along geodesics, we define

$$S(y) := \frac{\mathrm{d}}{\mathrm{d}t}[\tau(\dot{c}(t))]_{t=0},$$

where $c(t)$ is the geodesic with $\dot{c}(0) = y$. S is called the S-curvature.

Now we introduce the weighted Ricci curvature on Finsler manifolds, which was defined by Ohta. In the present paper, we reform it as follows.

Definition 2.1 (see [8]) *Let $(M, F, \mathrm{d}\mu)$ be a Finsler n-manifold with volume form $\mathrm{d}\mu$. Given a vector $V \in T_xM$, let $\gamma : (-\varepsilon, \varepsilon) \to M$ be a geodesic with $\gamma(0) = x$ and $\dot{\gamma}(0) = V$. Define*

$$\dot{S}(V) := F^{-2}(V)\frac{\mathrm{d}}{\mathrm{d}t}[S(\gamma(t), \dot{\gamma}(t))]_{t=0},$$

where $S(V)$ denotes the S-curvature at (x, V).

The weighted Ricci curvature of $(M, F, \mathrm{d}\mu)$ is defined by

$$\begin{cases} \mathrm{Ric}_n(V) := \begin{cases} \mathrm{Ric}(V) + \dot{S}(V) & \text{for } S(V) = 0, \\ -\infty, & \text{otherwise}, \end{cases} \\ \mathrm{Ric}_N(V) := \mathrm{Ric}(V) + \dot{S}(V) - \dfrac{S(V)^2}{(N-n)F(V)^2}, \quad \forall N \in (n, \infty), \\ \mathrm{Ric}_\infty(V) := \mathrm{Ric}(V) + \dot{S}(V). \end{cases}$$

For a smooth function $u : M \to R$, the gradient vector of u at x is defined as

$$\nabla u(x) := \begin{cases} g^{ij}(x, \nabla u)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i}, & \mathrm{d}u(x) \neq 0, \\ 0, & \mathrm{d}u(x) = 0. \end{cases}$$

Set $M_V := \{x \in M \mid V(x) \neq 0\}$ for a vector field V on M, and $M_u := M_{\nabla u}$. For a smooth vector field V on M and $x \in M_V$, we define $\nabla V(x) \in T_x^*M \otimes T_xM$ by using the covariant derivative as

$$\nabla V(v) := D_v^V V(x) \in T_xM, \quad v \in T_xM.$$

For a smooth function $u : M \to R$ and $x \in M_u$, we set $\nabla^2 u(x) := \nabla(\nabla u)(x)$. Let $\{e_a\}_{a=1}^n$ be a local orthonormal basis with respect to $g_{\nabla u}$ on M_u and put $u_{ab} = g_{\nabla u}(D_{e_a}^{\nabla u}\nabla u, e_b)$. Then we have

$$u_{ab} = u_{ba}, \quad \forall a, b.$$

Let $V = V^i\frac{\partial}{\partial x^i}$ be a C^∞ vector field on M. The divergence of V with respect to an arbitrary volume form $\mathrm{d}\mu$ is defined by

$$\mathrm{div}V := \sum_{i=1}^n \Big(\frac{\partial V^i}{\partial x^i} + V^i\frac{\partial \Phi}{\partial x^i}\Big),$$

where $\mathrm{d}\mu = \mathrm{e}^{\Phi}\mathrm{d}x$. Then the Finsler-Laplacian of u can be defined by

$$\Delta u := \operatorname{div}(\nabla u).$$

Given a vector field V such that $V \neq 0$ on M_u, the weighted gradient vector and the weighted Laplacian on the weighted Riemannian manifold (M, g_V) are defined by

$$\nabla^V u := \begin{cases} g^{ij}(V)\dfrac{\partial u}{\partial x^j}\dfrac{\partial}{\partial x^i} & \text{on } M_u, \\ 0 & \text{on } M\backslash M_u, \end{cases} \qquad \Delta^V u := \operatorname{div}(\nabla^V u).$$

We note that$\nabla^{\nabla u} u = \nabla u$, and $\Delta^{\nabla u} u = \Delta u$.

Let (M, F) be a Finsler manifold. Define the distance function by

$$d(p,q) := \inf_{\gamma} \int_0^1 F(\gamma, \dot{\gamma}(t))\mathrm{d}t,$$

where the infimum is taken over all differentiable curves $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Define reversibility $\eta = \eta(M, F)$ as

$$\eta := \sup_{X \in TM\backslash\{0\}} \frac{F(-X)}{F(X)}.$$

Obviously, $\eta \in [1, \infty)$, and $\eta = 1$ if and only if (M, F) is reversible.

Lemma 2.1 (see [9]) *Let $(M, F, \mathrm{d}\mu)$ be a Finsler manifold with volume form $\mathrm{d}\mu$. If its weighted Ricci curvature satisfies $\mathrm{Ric}_N \geq K$, $K \in R$, $N \in (n, \infty)$, then the Laplacian of the distance function $\rho(x) = d(p,x)$ from any given point $p \in M$ can be estimated as follows:*

$$\Delta\rho \leq (N-1)\mathrm{ct}_{\frac{K}{N-1}}(\rho)$$

pointwise on $M\backslash(\{p\} \cup \mathrm{Cut}(p))$ and in the sense of distributions on $M\backslash\{z\}$, where

$$\mathrm{ct}_c(\rho) = \begin{cases} \sqrt{c} \cdot \cot(\sqrt{c}\rho), & c > 0, \\ \dfrac{1}{\rho}, & c = 0, \\ \sqrt{-c} \cdot \coth(\sqrt{-c}\rho), & c < 0. \end{cases}$$

Lemma 2.2 (see [11]) *Let $(M, F, \mathrm{d}\mu)$ be a Finsler metric measure space with volume form $\mathrm{d}\mu$. Let φ be a piecewise C^1 function on M such that every $\varphi^{-1}(t)$ is compact. Then for any continuous function f on M,*

$$\int_M f F(\nabla\varphi)\mathrm{d}\mu = \int_{-\infty}^{\infty} \Big(\int_{\varphi^{-1}(t)} f \mathrm{d}\nu \Big) \mathrm{d}t.$$

Lemma 2.3 (see [8]) *Let $(M, F, \mathrm{d}\mu)$ be a Finsler n-manifold with volume form $\mathrm{d}\mu$. Given $u \in C^\infty(M)$, we have*

$$\Delta^{\nabla u}\Big(\frac{F(\nabla u)^2}{2}\Big) - D(\Delta u)(\nabla u) = |\nabla u|^2 \mathrm{Ric}_\infty(\nabla u) + |\nabla^2 u|^2_{\mathrm{HS}(\nabla u)}$$

pointwise on M_u, where $|\nabla^2 u|^2_{\mathrm{HS}(\nabla u)}$ stands for the Hilbert-Schmidt norm with respect to $g_{\nabla u}$.

Lemma 2.4 (see [15]) *Let $(M, F, \mathrm{d}\mu)$ be a Finsler n-manifold with volume form $\mathrm{d}\mu$ and $u : M \to R$ be a smooth function. Then on M_u we have*

$$\Delta u = \mathrm{tr}_{g_{\nabla u}}(\nabla^2 u) - S(\nabla u) = \sum_a u_{aa} - S(\nabla u),$$

where $u_{aa} = g_{\nabla u}(\nabla^2 u(e_a), e_a)$ and $\{e_a\}_{a=1}^n$ is a local $g_{\nabla u}$-orthonormal basis on M_u.

3 The Upper Bound Estimation of the First Eigenvalue

For a positive number N, we denote by $[N]$ the integer part of N. Then $\overline{N}$ is defined by

$$\overline{N} = \begin{cases} N, & N \in \mathbb{N}, \\ [N] + 1, & N \in \mathbb{R}^+ \backslash \mathbb{N}. \end{cases} \tag{3.1}$$

Theorem 3.1 *Let $(M, F, \mathrm{d}\mu)$ be a complete Finsler n-manifold with volume form $\mathrm{d}\mu$. If the weighted Ricci curvature satisfies $\mathrm{Ric}_N \geq (N-1)k$, $N \in (n, \infty)$, then the first Dirichlet eigenvalue*

$$\lambda_1(B_p(r)) \leq \lambda_1(V_{\overline{N}}(k, r)),$$

where $V_{\overline{N}}(k, r)$ denotes a geodesic ball with radius r in the $\overline{N}$-dim simply connected space form with sectional curvature k, and $\overline{N}$ is defined in (3.1).

Proof Let φ be the nonpositive first eigenfunction of $\overline{V_{\overline{N}}(k, r)}$. Since all simply connected space forms are two-point homogenous, φ is a radial function. Namely, $\varphi(x) = \varphi(d(x_0, x))$, where x_0 is the center of $\overline{V_{\overline{N}}(k, r)}$. Moreover, we have (see [5])

$$\begin{cases} \varphi''(t) + (\overline{N} - 1)\mathrm{ct}_k(t)\varphi'(t) + \lambda_1(V_{\overline{N}}(k, r))\varphi(t) = 0, \\ \varphi(r) = 0, \ \varphi'(t) > 0, \quad t \in (0, r). \end{cases}$$

Let $\rho(x) = d_F(p, x)$ be the distance function of (M, F), $u(x) = \varphi(\rho(x))$. Since $\mathrm{d}u = \varphi' \mathrm{d}\rho$ and $\varphi' > 0$, we find $\nabla u = \varphi' \nabla \rho$. Using Lemma 2.1 and noting that $F(\nabla \rho) = 1$, we have

$$\begin{aligned} \Delta u &= \mathrm{div}(\nabla^{\nabla \rho} u) = \mathrm{div}(\nabla^{\nabla \rho} \varphi(\rho(x))) = \mathrm{div}(\varphi' \nabla \rho) = \varphi'(\rho)\Delta \rho + \nabla \rho(\varphi') = \varphi''(\rho) + \varphi'(\rho)\Delta \rho \\ &\leq \varphi''(\rho) + (\overline{N} - 1)\mathrm{ct}_k(t)\varphi'(\rho) = -\lambda_1(V_{\overline{N}}(k, r))u. \end{aligned}$$

Note that $u|_{B_p(r)} < 0$ and $u|_{\partial B_p(r)} = 0$. It follows that

$$\begin{aligned} \int_{B_p(r)} (F^*(\mathrm{d}u))^2 \mathrm{d}\mu &= \int_{B_p(r)} \mathrm{d}u(\nabla u)\mathrm{d}\mu = -\int_{B_p(r)} u \Delta u \mathrm{d}\mu \\ &\leq \lambda_1(V_{\overline{N}}(k, r)) \int_{B_p(r)} u^2 \mathrm{d}\mu. \end{aligned} \tag{3.2}$$

Thus, $\lambda_1(B_p(r)) \leq \frac{\int_{B_p(r)} (F^*(\mathrm{d}u))^2 \mathrm{d}\mu}{\int_{B_p(r)} u^2 \mathrm{d}\mu} \leq \lambda_1(V_{\overline{N}}(k, r))$.

Corollary 3.1 *Let $(M, F, \mathrm{d}\mu)$ be a complete Finsler n-manifold volume form $\mathrm{d}\mu$. If the Ricci curvature satisfies $\mathrm{Ric} \geq (n-1)k$ and S-curvature $S = 0$, then the first Dirichlet eigenvalue*

$$\lambda_1(B_p(r)) \leq \lambda_1(V_n(k, r)),$$

where $V_n(k, r)$ denotes a geodesic ball with radius r in the n-dim simply connected space form with sectional curvature k.

For a complete simply connected Finsler n-manifold with finite reversibility η, the authors obtained in [16]

$$\lambda_1(M) \geq \frac{(n-1)^2}{4\eta^2} a^2$$

under the condition that $S \leq 0$ and $K \leq -a^2 < 0$. On the other hand, from Corollary 3.1 and Cheng's estimation on $V_n(k,r)$, we have

$$\lambda_1(B_p(r)) \leq \frac{(n-1)^2}{4} a^2 + \frac{C(n)}{r^2}$$

for $k = -a^2$. By the discussion above, we can state the following result.

Corollary 3.2 *Let $(M, F, \mathrm{d}\mu)$ be a complete and simply connected reversible Finsler n-manifold with volume form $\mathrm{d}\mu$. If $S = 0$ and the flag curvature $K = -a^2$, then*

$$\lambda_1(M) = \frac{(n-1)^2}{4} a^2.$$

Remark 3.1 The result of Corollary 3.2 was also obtained by Chen [4].

Let $(M, F, \mathrm{d}\mu)$ be a compact Finsler n-manifold with volume form $\mathrm{d}\mu$ and boundary ∂M. If there exists a function $u \in W^{1,2}(M)$ satisfying

$$\Delta u = -\mu_1 u \quad \text{in } M$$

with a boundary condition

$$\nabla u \in T_x(\partial M),$$

we call μ_1 the first Neumann eigenvalue of the Finsler-Laplacian (see also [14]).

Theorem 3.2 *Let $(M, F, \mathrm{d}\mu)$ be a compact Finsler n-manifold with volume form $\mathrm{d}\mu$ and the finite reversibility η. Suppose that the weighted Ricci curvature satisfies $\mathrm{Ric}_N \geq (N-1)k,\ N \in (n, \infty)$. Then the first Neumann or closed eigenvalue*

$$\mu_1(M) \leq \eta^2 \lambda_1\Big(V_{\overline{N}}\Big(k, \frac{d}{1+\eta}\Big)\Big),$$

where d denotes the diameter of M.

Proof Let $p_1, p_2 \in M$ so that $d_F(p_1, p_2) = d$. Choosing $r = \frac{d}{1+\eta}$, then $B_{p_1}(r) \cap B_{p_2}(r) = \emptyset$. In fact, if there exists a point $q \in B_{p_1}(r) \cap B_{p_2}(r)$, then

$$d = d_F(p_1, p_2) \leq d_F(p_1, q) + d_F(q, p_2) \leq d_F(p_1, q) + \eta d_F(p_2, q) < r(1+\eta) = d,$$

which is a contradiction. Thus from (3.2) we conclude

$$\int_{B_{p_i}(\frac{d}{1+\eta})} (F^*(\mathrm{d}u_i))^2 \mathrm{d}\mu \leq \lambda_1\Big(V_{\overline{N}}\Big(k, \frac{d}{1+\eta}\Big)\Big) \int_{B_{p_i}(\frac{d}{1+\eta})} u_i^2 \mathrm{d}\mu, \quad i = 1, 2,$$

where u_i are defined as in the proof of Theorem 3.1. Extend u_i to be zero outside $B_{p_i}(\frac{d}{1+\eta})$ and take $a_1 = \int_M u_2 \mathrm{d}\mu,\ a_2 = -\int_M u_1 \mathrm{d}\mu$. Then $a_1 < 0,\ a_2 > 0$ and

$$u := a_1 u_1 + a_2 u_2 \neq 0, \quad \int_M u \mathrm{d}\mu = 0.$$

Define $\eta^* := \sup\limits_{\theta\in T^*M\setminus\{0\}} \frac{F^*(-\theta)}{F^*(\theta)}$. Then we know from [4] that $\eta^* = \eta$. Hence,

$$
\begin{aligned}
\mu_1(M)\int_M u^2\mathrm{d}\mu &\le \int_M (F^*(\mathrm{d}u))^2\mathrm{d}\mu = \sum_{i=1}^2 \int_{B_{p_i}(\frac{d}{1+\eta})} (F^*(a_i\mathrm{d}u_i))^2\mathrm{d}\mu \\
&\overset{a_1<0<a_2}{=} a_1^2\int_{B_{p_1}(\frac{d}{1+\eta})} (F^*(-\mathrm{d}u_1))^2\mathrm{d}\mu + a_2^2\int_{B_{p_2}(\frac{d}{1+\eta})} (F^*(\mathrm{d}u_2))^2\mathrm{d}\mu \\
&\le \eta^2\sum_{i=1}^2 a_i^2\int_{B_{p_i}(\frac{d}{1+\eta})} (F^*(\mathrm{d}u_i))^2\mathrm{d}\mu \\
&\le \eta^2\lambda_1\Big(V_{\overline{N}}\Big(k,\frac{d}{1+\eta}\Big)\Big)\sum_{i=1}^2 a_i^2\int_{B_{p_i}(\frac{d}{1+\eta})} u_i^2\mathrm{d}\mu \\
&= \eta^2\lambda_1\Big(V_{\overline{N}}\Big(k,\frac{d}{1+\eta}\Big)\Big)\int_M u^2\mathrm{d}\mu.
\end{aligned}
$$

This completes the proof.

In [5], Cheng obtained the upper bound estimation of the first eigenvalue of a geodesic ball in the space form (see also in [10]). Thus, we give the following proposition.

Proposition 3.1 *Let $(M,F,\mathrm{d}\mu)$ be a compact Finsler n-manifold with volume form $\mathrm{d}\mu$ and the finite reversibility η. If the weighted Ricci curvature satisfies $\mathrm{Ric}_N \ge (N-1)k,\ N\in(n,\infty)$, then the first Neumann eigenvalue*

$$
\mu_1(M) \le \begin{cases} \dfrac{\eta^2(1+\eta)^2\overline{N}\pi^2}{4d^2}, & k=1,\\ \dfrac{\overline{N}(\overline{N}+4)(1+\eta)^2\eta^2}{2d^2}, & k=0,\\ \dfrac{(\overline{N}-1)^2\eta^2}{4}(-k) + \dfrac{C(\overline{N})(1+\eta)^2\eta^2}{d^2}, & k<0. \end{cases}
$$

4 The Lower Bound Estimation of the First Eigenvalue

Let $(M,F,\mathrm{d}\mu)$ be a compact Finsler n-manifold with volume form $\mathrm{d}\mu$ and boundary ∂M. For any $x\in\partial M$, there exist exactly two unit norm vectors $\mathbf{n}$ characterized by

$$T_x(\partial M) = \{V\in T_xM \mid g_{\mathbf{n}}(\mathbf{n},V)=0,\ g_{\mathbf{n}}(\mathbf{n},\mathbf{n})=1\}.$$

We remark here that if $\mathbf{n}$ is a norm vector, then $-\mathbf{n}$ may not be a norm vector unless F is reversible. Denote by $\mathbf{n}_+$ (resp. $\mathbf{n}_-$) the normal vector that points outwards (resp. inwards) ∂M. Then the induced volume forms of ∂M with respect to $\mathbf{n}_+$ and $\mathbf{n}_-$ are

$$\mathrm{d}\nu_+ = i^*(\mathbf{n}_+\rfloor\mathrm{d}\mu) \quad\text{and}\quad \mathrm{d}\nu_- = i^*(\mathbf{n}_-\rfloor\mathrm{d}\mu).$$

Consequently, the volumes of ∂M with respect to $\mathbf{n}_+$ and $\mathbf{n}_-$ are

$$\mathrm{vol}_+(\partial M) = \int_{\partial M}\mathrm{d}\nu_+ = \int_{\partial M} i^*(\mathbf{n}_+\rfloor\mathrm{d}\mu) \quad\text{and}\quad \mathrm{vol}_-(\partial M) = \int_{\partial M}\mathrm{d}\nu_- = \int_{\partial M} i^*(\mathbf{n}_-\rfloor\mathrm{d}\mu).$$

Note that $\mathrm{vol}_+(\partial M)\ne\mathrm{vol}_-(\partial M)$ in general. In fact, the ratio $\frac{\mathrm{vol}_+(\partial M)}{\mathrm{vol}_-(\partial M)}$ or $\frac{\mathrm{vol}_-(\partial M)}{\mathrm{vol}_+(\partial M)}$ may be very large (see [4] for details).

In the Riemannian case, Cheeger [3] defined a constant and gave an original estimation on the lower bound of the first eigenvalue. Now we define Cheeger constant $h(\Omega)$ of Ω on Finsler manifolds by

$$h(\Omega) := \inf_{\Omega'} \frac{\min\{\mathrm{vol}_+(\partial\Omega'), \mathrm{vol}_-(\partial\Omega')\}}{\mathrm{vol}(\Omega')},$$

where Ω' ranges over all open submanifolds of Ω with compact closure in Ω and smooth boundary $\partial\Omega'$, $\mathrm{vol}(\Omega')$ denotes the volume of Ω', and $\mathrm{vol}_+(\partial\Omega')$ and $\mathrm{vol}_-(\partial\Omega')$ denote the volumes of $\partial\Omega'$ with respect to outwards normal vectors and inwards normal vectors, respectively.

Theorem 4.1 *Let $(M, F, \mathrm{d}\mu)$ be a complete Finsler n-manifold with volume form $\mathrm{d}\mu$. For any bounded domain Ω with a piecewise smooth boundary in M, the first Dirichlet eigenvalue satisfies*

$$\lambda_1 \geq \frac{1}{4} h(\Omega)^2.$$

Proof By using Co-Area formula in Lemma 2.2, we have $\forall \varphi \in C^1(\Omega)$, provided that $\varphi|_\Omega > 0$, $\varphi|_{\partial\Omega} = 0$,

$$\begin{aligned}
\int_\Omega |\nabla\varphi| \mathrm{d}\mu &= \int_0^\infty \Big(\int_{\varphi^{-1}(t)} \mathrm{d}\nu \Big) \mathrm{d}t = \int_0^\infty \mathrm{vol}_-(\varphi = t) \mathrm{d}t \\
&= \int_0^\infty \frac{\mathrm{vol}_-(\varphi = t)}{\mathrm{vol}(\varphi \geq t)} \mathrm{vol}(\varphi \geq t) \mathrm{d}t \\
&\geq \inf_t \frac{\mathrm{vol}_-(\varphi = t)}{\mathrm{vol}(\varphi \geq t)} \int_0^\infty \mathrm{vol}(\varphi \geq t) \mathrm{d}t \\
&\geq h(\Omega) \int_\Omega |\varphi| \mathrm{d}\mu.
\end{aligned}$$

Let f be the first eigenfunction with the Dirichlet boundary condition. Then from [12], we know that $f \in C^{1,\alpha}(\Omega)$. If $f > 0$ in Ω, then let $\varphi = f^2$ and we have

$$\int_\Omega |\nabla f^2| \mathrm{d}\mu \geq h(\Omega) \int_\Omega f^2 \mathrm{d}\mu. \tag{4.1}$$

On the other hand, it follows from Legendre transform that

$$\nabla f^2 = \mathcal{L}(2f\mathrm{d}f) = 2f\mathcal{L}(\mathrm{d}f) = 2f\nabla f.$$

Using Hölder's inequality, one gets

$$\int_\Omega |\nabla f^2| \mathrm{d}\mu = 2\int_\Omega f|\nabla f| \mathrm{d}\mu \leq 2\Big(\int_\Omega f^2 \mathrm{d}\mu \Big)^{\frac{1}{2}} \Big(\int_\Omega |\nabla f|^2 \mathrm{d}\mu \Big)^{\frac{1}{2}}. \tag{4.2}$$

Substituting (4.2) into (4.1), one has

$$\lambda_1 = \frac{\int_\Omega |\nabla f|^2 \mathrm{d}\mu}{\int_\Omega f^2 \mathrm{d}\mu} \geq \frac{1}{4} h(\Omega)^2.$$

If $f < 0$ in Ω, we choose $\varphi = -f^2$. Then by a similar argument as above, we have

$$\int_\Omega |\nabla\varphi| \mathrm{d}\mu \geq h(\Omega) \int_\Omega |\varphi| \mathrm{d}\mu.$$

In this case, $\mathrm{d}\varphi = -2f\mathrm{d}f$, which yields

$$\int_\Omega |\nabla^{\nabla f} f^2| \mathrm{d}\mu \geq h(\Omega) \int_\Omega f^2 \mathrm{d}\mu.$$

Since $\nabla^{\nabla f} f^2 = 2f\nabla f$, a similar argument gives

$$\lambda_1 = \frac{\int_\Omega |\nabla f|^2 \mathrm{d}\mu}{\int_\Omega f^2 \mathrm{d}\mu} \geq \frac{1}{4} h(\Omega)^2.$$

If f changes its sign in Ω, then we can choose a domain $\widetilde{\Omega} \subset \Omega$ such that $f > 0$ in $\widetilde{\Omega}$ and $f|_{\partial\widetilde{\Omega}} = 0$. Define

$$f_1 := f|_{\widetilde{\Omega}}.$$

Then

$$\begin{cases} \Delta f_1 = -\lambda_1 f_1 & \text{in } \widetilde{\Omega}, \\ f_1 = 0 & \text{on } \partial\widetilde{\Omega}. \end{cases}$$

From the above formula we find that f_1 is the eigenfunction of $\widetilde{\Omega}$ and λ_1 is an eigenvalue of $\widetilde{\Omega}$ which implies $\lambda_1 \geq \lambda_1(\widetilde{\Omega})$. On the other hand, since $\widetilde{\Omega} \subset \Omega$, $\lambda_1(\widetilde{\Omega}) \geq \lambda_1(\Omega) = \lambda_1$. Thus $\lambda_1(\widetilde{\Omega}) = \lambda_1$. According to the discussion above, we have

$$\lambda_1 \geq \frac{1}{4} h(\widetilde{\Omega})^2 \geq \frac{1}{4} h(\Omega)^2.$$

Combining three cases above, we complete the proof.

Using Zermelo's navigation idea, we can express a Randers metric $F = \alpha + \beta$ in terms of a Riemannian metric $h = \sqrt{h_{ij}(x) y^i y^j}$ and a vector field $W = W^i \frac{\partial}{\partial x^i}$ by

$$F = \frac{\sqrt{\lambda h^2 + W_0^2}}{\lambda} - \frac{W_0}{\lambda}, \quad W_0 = W_i y^i, \tag{4.3}$$

where $W_i = h_{ij} W^i$ and

$$\lambda := 1 - W_i W^i = 1 - h(x, W)^2.$$

Theorem 4.2 *Let $(M, F, \mathrm{d}\mu)$ be a Randers manifold with Busemann-Hausdorff volume form and the finite reversibility η, where F is expressed by (4.3) and has constant flag curvature. Let $\Omega \subset M$ be a domain and B be a geodesic ball of Riemannian manifold (M, h). If $\mathrm{vol}^{\mathrm{d}\mu}(\Omega) = \mathrm{vol}^{\mathrm{d}\mu}(B)$, then the first Dirichlet eigenvalue satisfies*

$$\lambda_1(\Omega) \geq \frac{1}{\eta^2} \lambda_1(B),$$

where the equality holds if and only if $F = h$ and $\Omega = B$.

Proof Firstly we have that if (M, F) has constant flag curvature, then (M, h) has constant sectional curvature (see [1]). Let f be the first Dirichlet eigenfunction corresponding to the first eigenvalue λ_1 in Ω, that is,

$$\begin{cases} \Delta f = -\lambda_1 f & \text{in } \Omega, \\ f = 0 & \text{on } \partial\Omega. \end{cases}$$

When $f>0$ in Ω, we set $\Omega_t := \{x\in\Omega \mid f(x)>t\}$ and $\Gamma_t := \{x\in\Omega \mid f(x)=t\}$. Using a symmetrization procedure, we construct the geodesic ball B_t in (M,h) such that $\mathrm{vol}^{\mathrm{d}\mu}(B_t)=\mathrm{vol}^{\mathrm{d}\mu}(\Omega_t)$ for each t, and $B=B_0$. We define a function $g: B\to R^+$ such that g is a radially decreasing function and $\partial B_t=\{x\in B \mid g(x)=t\}$.

Notice the important fact that $\mathrm{d}\mu=\mathrm{d}V_h$ where $\mathrm{d}V_h$ is the Riemaniann volume form of (M,h) (see [1, p. 12]). Using Co-Area formula in Lemma 2.2, we have

$$\begin{aligned}\int_\Omega f^2\mathrm{d}\mu &= \int_0^\infty\int_{\Gamma_t}\frac{f^2}{F(\nabla f)}\mathrm{d}\nu_t\mathrm{d}t=\int_0^\infty t^2\Big(\int_{\Gamma_t}\frac{\mathrm{d}\nu_t}{F(\nabla f)}\Big)\mathrm{d}t\\ &= -\int_0^\infty t^2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}^{\mathrm{d}\mu}(\Omega_t)\mathrm{d}t=-\int_0^\infty t^2\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}^{\mathrm{d}V_h}(B_t)\mathrm{d}t\\ &= \int_0^\infty t^2\Big(\int_{B_t}\frac{\mathrm{d}A_t}{h(\nabla^h g)}\Big)\mathrm{d}t=\int_0^\infty\int_{\partial B_t}\frac{g^2}{h(\nabla^h g)}\mathrm{d}A_t\mathrm{d}t\\ &= \int_B g^2\mathrm{d}V_h=\int_B g^2\mathrm{d}V_F, \end{aligned}\tag{4.4}$$

where $\mathrm{d}\nu_t$ and $\mathrm{d}A_t$ denote the volume elements on Γ_t and ∂B_t respectively, and $\nabla^h g$ denotes the gradient of g with respect to h. Here we have used the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}^{\mathrm{d}\mu}(\Omega_t)=-\int_{\Gamma_t}\frac{\mathrm{d}\nu_t}{F(\nabla f)}.$$

From the discussion in [10, p. 28], we find

$$\mathrm{d}A_t=i^*(\mathbf{n}_h\rfloor\mathrm{d}\mu)=g_{\mathbf{n}}(\mathbf{n},\mathbf{n}_h)\mathrm{d}\nu_t,$$

where $\mathbf{n}=\frac{\nabla f}{F(\nabla f)}$ and $\mathbf{n}_h=\frac{\nabla^h f}{h(\nabla^h f)}$ denote the unit normal vector fields of $\partial\Omega_t$ with respect to F and h respectively. Since $\|W\|_h := \sup\limits_{y\in TM}\frac{W_iy^i}{h}\le b_0<1$, for an arbitrary vector X, one gets

$$\begin{aligned} h(X) &= \frac{\lambda h(X)}{\sqrt{\lambda h(X)^2+W(X)^2}-W(X)}F(X)\\ &= \Big(\sqrt{\lambda+\frac{W(X)^2}{h(X)^2}}+\frac{W(X)}{h(X)}\Big)F(X)\\ &\ge \inf_{y\in TM}\Big(\sqrt{\lambda+\frac{W_0^2}{h^2}}+\frac{W_0}{h}\Big)F(X)\\ &\ge (1-b_0)F(X). \end{aligned}\tag{4.5}$$

Using Hölder's inequality, the isoperimetric inequality and (4.5), we obtain

$$\begin{aligned}\int_{\Gamma_t}F(\nabla f)\mathrm{d}\nu_t\int_{\Gamma_t}\frac{\mathrm{d}\nu_t}{F(\nabla f)} &\ge \Big(\int_{\Gamma_t}\mathrm{d}\nu_t\Big)^2=\Big(\int_{\Gamma_t}\frac{\mathrm{d}A_t}{g_{\mathbf{n}}(\mathbf{n},\mathbf{n}_h)}\Big)^2\\ &\ge (1-b_0)^2\Big(\int_{\Gamma_t}\mathrm{d}A_t\Big)^2\\ &\ge (1-b_0)^2\Big(\int_{\partial B_t}\mathrm{d}A_t\Big)^2, \end{aligned}$$

which yields

$$\begin{aligned}\int_{\Gamma_t} F(\nabla f)\mathrm{d}\nu_t &\geq \frac{(\int_{\Gamma_t}\mathrm{d}\nu_t)^2}{-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}^{\mathrm{d}\mu}(\Omega_t)} \geq (1-b_0)^2\frac{(\int_{\partial B_t}\mathrm{d}A_t)^2}{-\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}^{\mathrm{d}V_h}(B_t)}\\ &\geq (1-b_0)^2\frac{(\int_{\partial B_t}\mathrm{d}A_t)^2}{\int_{\partial B_t}\frac{\mathrm{d}A_t}{h(\nabla^h g)}} = (1-b_0)^2\int_{\partial B_t} h(\nabla^h g)\mathrm{d}A_t. \end{aligned}\tag{4.6}$$

Recall that the dual metric of (4.3) is

$$F^* := h^* + W^* = \sqrt{h^{ij}\xi_i\xi_j} + W^i\xi_i,$$

where $(h^{ij}) = (h_{ij})^{-1}$ and $W^i = W_j h^{ij}$. Thus for a C^1 function f, we have

$$F(\nabla f) = F^*(\mathrm{d}f) = h^*(\mathrm{d}f) + W^i f_i = h(\nabla^h f) + W^i f_i,$$

from which we conclude

$$(1-b_0)h(\nabla^h f) \leq F(\nabla f) \leq (1+b_0)h(\nabla^h f). \tag{4.7}$$

From (4.6)–(4.7), we obtain

$$\begin{aligned}\int_\Omega F(\nabla f)^2\mathrm{d}\mu &= \int_0^\infty \Big(\int_{\Gamma_t} F(\nabla f)\mathrm{d}\nu_t\Big)\mathrm{d}t \geq (1-b_0)^2\int_0^\infty \Big(\int_{\partial B_t} h(\nabla^h g)\mathrm{d}A_t\Big)\mathrm{d}t\\ &= (1-b_0)^2\int_B h(\nabla^h g)^2\mathrm{d}V_h \geq \frac{(1-b_0)^2}{(1+b_0)^2}\int_B F(\nabla g)^2\mathrm{d}\mu. \end{aligned}\tag{4.8}$$

By using (4.4) and (4.8), it follows that

$$\lambda_1(\Omega) \geq \frac{\lambda_1(B)}{\eta^2}.$$

Here we have used the fact that $\eta = \frac{1+b_0}{1-b_0}$.

If the equality holds, then the above inequalities become equalities. In particular,

$$\begin{cases} F(\nabla f) = (1+b_0)h(\nabla^h f),\\ h(\nabla^h f) = (1-b_0)F(\nabla^h f),\\ g_{\mathbf{n}}(\mathbf{n}, \mathbf{n}_h) = F(\mathbf{n}_h).\end{cases}$$

The third formula implies $h(\nabla^h f)^2 = F(\nabla f)F(\nabla^h f)$, which together with the first two formulas yields $b_0 = 0$. In this case, $F = h$ and $\Omega = B$.

When $f < 0$ in Ω, we set $\Omega_t := \{x \in \Omega \mid f(x) < t\}$ and $\Gamma_t := \{x \in \Omega \mid f(x) = t\}$. Then by a similar argument as the above, we also give the conclusion.

When f changes its sign in Ω, we can choose a domain $\widetilde{\Omega} \subset \Omega$ such that $f|_{\widetilde{\Omega}} > 0$ and $f|_{\partial\widetilde{\Omega}} = 0$. Define

$$f_1 := f|_{\widetilde{\Omega}}.$$

Then f_1 is the first eigenfunction of $\widetilde{\Omega}$ corresponding to λ_1. Using the conclusion of the first case, we have

$$\lambda_1(\widetilde{\Omega}) \geq \frac{1}{\eta^2}\lambda_1(\widetilde{B}),$$

where $\widetilde{B} \subset B$ is a geodesic ball of Riemannian manifold (M, h) with $\mathrm{vol}^{\mathrm{d}\mu}(\widetilde{\Omega}) = \mathrm{vol}^{\mathrm{d}\mu}(\widetilde{B})$. Since $\lambda_1(\widetilde{\Omega}) = \lambda_1$ and $\lambda_1(\widetilde{B}) \geq \lambda_1(B)$, we obtain

$$\lambda_1(\Omega) \geq \frac{1}{\eta^2}\lambda_1(B).$$

Theorem 4.3 *Let $(M, F, \mathrm{d}\mu)$ be a compact Finsler n-manifold with volume form $\mathrm{d}\mu$, without boundary. If the weighted Ricci curvature $\mathrm{Ric}_N \geq (n-1)(-k)$ for $k \geq 0$ and $N \in (n, \infty)$, then the first closed eigenvalue*

$$\mu_1(M) \geq \frac{2 + 2\sqrt{1 + 3.3(n-1)(N-1)d^2k}}{3.3(N-1)d^2}\exp\{-[1 + \sqrt{1 + 3.3(n-1)(N-1)d^2k}]\},$$

where d denotes the diameter of M.

Proof Let u be the first eigenfunction of Finsler-Laplacian. Then $\int_M u\mathrm{d}\mu = 0$. Without loss of generality, we assume that $-m = \inf u < \sup u = 1$, where $m \leq 1$. Let $\beta > 1$ be a real number. We define the function

$$G(x) := \frac{|\nabla u|^2}{(\beta - u)^2}. \tag{4.9}$$

Suppose that x_0 is a maximum point on M. Then $x_0 \in M_u$ and

$$\nabla^{\nabla u} G(x_0) = 0, \quad \Delta^{\nabla u} G(x_0) \leq 0. \tag{4.10}$$

From (4.9), we get

$$\Delta^{\nabla u} G \cdot (\beta - u)^2 + 2g_{\nabla u}(\nabla^{\nabla u} G, \nabla^{\nabla u}(\beta - u)^2) + G\Delta^{\nabla u}(\beta - u)^2 = \Delta^{\nabla u}|\nabla u|^2.$$

Then at x_0 we have

$$\Delta^{\nabla u}|\nabla u|^2 - G\Delta^{\nabla u}(\beta - u)^2 \leq 0.$$

Using Lemma 2.3, one gets

$$|\nabla^2 u|^2_{\mathrm{HS}(\nabla u)} + g_{\nabla u}\left(\nabla^{\nabla u}\Delta u, \nabla u\right) + |\nabla u|^2 \mathrm{Ric}_\infty(\nabla u) - \frac{1}{2}G[\mathrm{div}(\nabla^{\nabla u}((\beta - u)^2))] \leq 0,$$

where

$$\mathrm{div}(\nabla^{\nabla u}((\beta - u)^2)) = 2\mathrm{div}((u - \beta)\nabla u) = 2(u - \beta)\Delta u + 2|\nabla u|^2.$$

Since u is the eigenfunction of Laplacian, we have

$$|\nabla^2 u|^2_{\mathrm{HS}(\nabla u)} - \mu_1|\nabla u|^2 + |\nabla u|^2\mathrm{Ric}_\infty(\nabla u) - G[\mu_1 u(\beta - u) + |\nabla u|^2] \leq 0. \tag{4.11}$$

A direct calculation on the first formula of (4.10) yields

$$\nabla^2 u(\nabla u) = -\frac{|\nabla u|^2}{\beta - u}\nabla u.$$

Choosing a $g_{\nabla u}$-orthonormal basis at x_0 such that $e_1 = \frac{\nabla u}{|\nabla u|}, e_2, \cdots, e_n$, we have

$$\begin{cases} u_{11} = -\dfrac{|\nabla u|^2}{\beta - u}, \\ u_{1a} = 0, \quad a > 1. \end{cases} \tag{4.12}$$

By using the standard inequality and Lemma 2.4, one has for $0 < \varepsilon < 1$ that

$$
\begin{aligned}
\sum_{a,b=2}^{n} u_{ab}^2 \geq \sum_{a=2}^{n} u_{aa}^2 \geq \frac{1}{n-1}\Big(\sum_{a=2}^{n} u_{aa}\Big)^2 \\
&= \frac{1}{n-1}(\Delta u - u_{11} + S(\nabla u))^2 \\
&= \frac{(\Delta u - u_{11})^2}{N-1} - \frac{S(\nabla u)^2}{N-n} + \frac{(N-n)(N-1)}{n-1}\Big(\frac{\Delta u - u_{11}}{N-1} + \frac{S(\nabla u)}{N-n}\Big)^2 \\
&\geq \frac{(\Delta u - u_{11})^2}{N-1} - \frac{S(\nabla u)^2}{N-n} \\
&\geq \frac{1-\varepsilon}{N-1}u_{11}^2 - \frac{1-\varepsilon}{\varepsilon(N-1)}\mu_1^2 u^2 - \frac{S(\nabla u)^2}{N-n}. \qquad (4.13)
\end{aligned}
$$

Using (4.11)–(4.13) and noting that $\mathrm{Ric}_N \geq (n-1)(-k)$, we obtain

$$
\frac{1-\varepsilon}{N-1}\frac{|\nabla u|^4}{(\beta-u)^2} - \frac{1-\varepsilon}{\varepsilon(N-1)}\mu_1^2 u^2 - (\mu_1 + (n-1)k)|\nabla u|^2 - \mu_1 u \frac{|\nabla u|^2}{\beta - u} \leq 0. \qquad (4.14)
$$

Set $\alpha := \frac{u}{\beta-u}$. Then $-1 \leq \alpha \leq \frac{1}{\beta-1}$. Thus (4.14) can be changed into

$$
\frac{1-\varepsilon}{N-1}G^2(x_0) - \frac{1-\varepsilon}{\varepsilon(N-1)}\mu_1^2\alpha^2 - (\mu_1 + (n-1)k + \mu_1\alpha)G(x_0) \leq 0. \qquad (4.15)
$$

Viewing (4.15) as a quadratic inequality and noting that $|\alpha| \leq \max\{1, \frac{1}{\beta-1}\} \leq \frac{\beta}{\beta-1}$, we have

$$
\begin{aligned}
G(x_0) &\leq \frac{N-1}{2(1-\varepsilon)}\Big\{2(\mu_1 + (n-1)k + \mu_1\alpha) + \frac{2(1-\varepsilon)}{(N-1)\sqrt{\varepsilon}}\mu_1|\alpha|\Big\} \\
&\leq \Big(\frac{1}{1-\varepsilon} + \frac{1}{\sqrt{\varepsilon}}\Big)(N-1)\Big((n-1)k + \frac{\mu_1\beta}{\beta-1}\Big) \\
&\overset{\varepsilon=\frac{1}{3}}{\leq} 3.3(N-1)\Big((n-1)k + \frac{\mu_1\beta}{\beta-1}\Big).
\end{aligned}
$$

Therefore, for an arbitrary point $x \in M$, it is concluded that

$$
|\nabla u| \leq \sqrt{3.3(N-1)\Big((n-1)k + \frac{\mu_1\beta}{\beta-1}\Big)}\,(\beta - u).
$$

Let x_1, x_2 be two points such that $u(x_1) = 0$ and $u(x_2) = 1$. Consider a regular minimal geodesic $\gamma(s)$ joining x_1 and x_2, and then

$$
\begin{aligned}
\log\frac{\beta}{\beta-1} &= \int_0^1 \frac{\mathrm{d}u}{\beta-u} = \int_\gamma \frac{\dot\gamma(u)}{\beta-u}\mathrm{d}s \\
&= \int_\gamma \frac{g_{\nabla u}(\nabla u, \dot\gamma)}{\beta-u}\mathrm{d}s \leq \int_\gamma \frac{|\nabla u|}{\beta-u}\mathrm{d}s \\
&\leq \sqrt{3.3(N-1)\Big((n-1)k + \frac{\mu_1\beta}{\beta-1}\Big)}\, d,
\end{aligned}
$$

where d is the diameter of M. Hence

$$
\mu_1 \geq \frac{\beta-1}{\beta}\Big[\frac{1}{3.3(N-1)d^2}\Big(\log\frac{\beta}{\beta-1}\Big)^2 - (n-1)k\Big]. \qquad (4.16)
$$

Let $f(y) := y[a(\log y)^2 - b]$. Then $f(y)$ attains its maximum when $y = \exp\big(-1-\sqrt{1+\frac{b}{a}}\big)$. Putting $a = \frac{1}{3.3(N-1)d^2}$, $b = (n-1)k$ and $y = \frac{\beta-1}{\beta}$ into (4.16), we have

$$\mu_1 \geq \frac{2+2\sqrt{1+3.3(n-1)(N-1)d^2k}}{3.3(N-1)d^2}\exp\{-[1+\sqrt{1+3.3(n-1)(N-1)d^2k}]\}.$$

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