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Non-degeneracy of Extremal Points*

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Abstract For a family of smooth functions, the author shows that, under certain generic conditions, all extremal (minimal and maximal) points are non-degenerate.

Keywords Non-degeneracy, Extremal point, Generic condition
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1 Introduction

It is well-known that the set of Morse functions is residual in $C^r(M,\mathbb{R})$ space, where M is a closed manifold and $r \geq 2$. Let us extend this issue to a family of smooth functions. Let $F_\lambda : M \to \mathbb{R}$ be a family of smooth functions continuously depending on a parameter $\lambda \in [0,1]$, it is natural to ask that whether there exists a residual set $\mathfrak{R} \subset C^r(M,\mathbb{R})$ such that for each $V \in \mathfrak{R}$ and for each $\lambda \in [0,1]$, the function $F_\lambda + V$ is a Morse function. Unfortunately, it is not true even if F_λ smoothly depends on the parameter. Here is a counterexample. Let $F_\lambda : \mathbb{T} \to \mathbb{R}$ be a family of functions such that for each $\lambda \in [0,1]$, F_λ takes its maximum at $x = \frac{1}{2}\pi$, takes its minimum at $x = -\frac{1}{2}\pi$ and $F_\lambda = \frac{1}{3}x^3 - \big(\frac{1}{2} - \lambda\big)x$ when the variables (λ, x) are restricted in a suitably small neighborhood of the origin $\big(\frac{1}{2}, 0\big)$. Clearly, the point $x = \pm\big(\frac{1}{2} - \lambda\big)^{\frac{1}{2}}$ is the non-degenerate critical point of F_λ for $\lambda > 0$. There exists no critical point of F_λ in the neighborhood of $x = 0$ for $\lambda > \frac{1}{2}$. For $\lambda = \frac{1}{2}$, the point $x = 0$ is a degenerate critical point. We note that the third derivative of F_λ is bounded away from zero for all λ when x remains close to the origin and the second derivative monotonously increases with respect to λ. Therefore, for any C^3-small perturbation V, certain λ_V exists such that $\big|\lambda_V - \frac{1}{2}\big|$ is small and $F_{\lambda_V} + V$ has a degenerate critical point close to the origin.

However, we are in different situation if we only consider the minimal as well as the maximal points of functions. Let $[a]$ denote the integer part of the real number a. The following theorem is the main result of this paper.

Theorem 1.1 *Let $F_\lambda : \mathbb{T} \to \mathbb{R}$ be a family of C^r-functions depending on the parameter $\lambda \in [0,1]$.*

(1) *If $r \geq 4$ and F_λ is Lipschitz in the parameter λ, there exists an open-dense set $\mathfrak{O} \subset C^r(\mathbb{T},\mathbb{R})$ such that for each $V \in \mathfrak{O}$ and each $\lambda \in [0,1]$, each global minimum as well as each global maximum of $F_\lambda - V$ is non-degenerate.*

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(2) *More generally, if F_λ is α-Hölder continuous in λ $(0<\alpha\le 1)$ and*

$$k=\Big[\frac{1}{4}\Big(\frac{2}{\alpha}+1+\sqrt{\Big(\frac{2}{\alpha}+1\Big)^2+16}\Big)\Big]-1,$$

then there exists an open-dense set $\mathfrak{O}\subset C^r(\mathbb{T},\mathbb{R})$ $(r\ge 2k+2)$ such that for each $V\in\mathfrak{O}$ and each $\lambda\in[0,1]$, certain weak non-degeneracy condition holds at each global minimum as well as each global maximum of $F_\lambda - V$: Some integer $1\le \ell\le k$ exists such that $\frac{\partial^{2\ell}(F_\lambda-V)}{\partial x^{2\ell}}\neq 0$.

For the function of Lagrange action, the non-degeneracy of critical points corresponds to the hyperbolicity of periodic orbits of Lagrange flow (see [1]). It is closely related to the Mañé conjecture (see [2]), one can refer to [3] for some new progress in this problem.

We feel that the result can be extended to functions defined on closed smooth manifold with finite dimensions.

Conjecture 1.1 Let $F_\lambda\in C^4(M,\mathbb{R})$ be a family of smooth functions, where M is a closed smooth manifold and $\lambda\in[0,1]$. If F_λ is Lipschitz in the parameter λ, then some open-dense set $\mathfrak{O}\subset C^4(M,\mathbb{R})$ exists such that for each $V\in\mathfrak{O}$ and each $\lambda\in[0,1]$, each global minimum as well as each global maximum of $F_\lambda - V$ is non-degenerate.

2 Proof

We only need to prove the second part of the theorem, the first part is a special case of the second one. Obviously, the set $\mathfrak{O}$ is the open set, as weak non-degeneracy of the critical point survives small perturbation. Therefore, we only need to show the density. Also, we only need to prove the non-degeneracy of the minimum, it is the same for the non-degeneracy of the maximum. Towards this goal, we introduce a set of small perturbations with $2k+2$ parameters:

$$\mathfrak{V}=\Big\{V=\epsilon\sum_{i=1}^{k+1}(A_i\cos ix+B_i\sin ix):(A_1,B_1,\cdots,A_{k+1},B_{k+1})\in\mathbb{I}^{2k+2}\Big\},$$

where $\mathbb{I}=[1,2]$. Let

$$M=\frac{2}{(2k+2)!}\sup_{x,\lambda}|\partial_x^{2k+2}F_\lambda|.$$

We are going to show that, for any small numbers $\epsilon, d>0$, there exists $(A_1,B_1,\cdots,A_{k+1},B_{k+1})$ $\in\mathbb{I}^{2k+2}$ such that

$$(F_\lambda-V)(x)-\min_x(F_\lambda-V)\ge M|x-x^*|^{2k+2},\quad \forall x\in[x^*-d,x^*+d] \tag{2.1}$$

holds for each $\lambda\in[0,1]$ whenever the point x^* is a global minimizer of $F_\lambda-V$. It implies that there exists an even integer number $2\le j\le 2k$, such that the jth derivative of $F_\lambda-V$ at x^* is positive and the ith derivative is equal to zero for each $i<j$. Indeed, if there exists no such even integer j, one can see that the $(2k+1)$th derivative is also equal to zero because x^* is assumed the global minimum. Consequently, the above formula does not hold. In the following, we define

$$\mathrm{Osc}_{I_i}F=\max_{x,x'\in I_i}|F(x)-F(x')|.$$

By choosing sufficiently large integer N, the numbers $d = \frac{\pi}{N}$ and $\epsilon = d^{\frac{1}{p}}$ can be set arbitrarily small, where the integer $p \in \mathbb{Z}_+$ will be determined later. Let

$$x_i = \frac{2i\pi}{N}, \quad I_i = [x_i - d, x_i + d],$$

then $\bigcup_{i=0}^{N-1} I_i = \mathbb{T}$. Restricted on each interval I_i, each C^∞-function $V \in \mathfrak{V}$ is approximated by the Taylor series (module constant)

$$V_i(x) = \epsilon\Big(\sum_{j=1}^{2k+1} a_j(x - x_i)^j + O(|x - x_i|^{2k+2})\Big), \quad \forall x \in I_i. \tag{2.2}$$

Given two points $(a_1, a_2, \cdots, a_{2k+1})$ and $(a_1', a_2', \cdots, a_{2k+1}')$, we obtain two functions $V_i(x)$ and $V_i'(x)$ in the form of Taylor series as shown in (2.2). Let $\Delta V = V_i' - V_i$, $\Delta a_j = a_j' - a_j$ for $j = 1, 2, \cdots, 2k+1$. We have $\Delta V(x_i) = 0$ and

$$\begin{aligned}
&\Delta V(x_i + d) + \Delta V(x_i - d) = 2\epsilon(\Delta a_2 d + \Delta a_4 d^3 + \cdots + \Delta a_{2k} d^{2k-1})d + O(\epsilon d^{2k+2}),\\
&\Delta V(x_i + d) - \Delta V(x_i - d) = 2\epsilon(\Delta a_1 + \Delta a_3 d^2 + \cdots + \Delta a_{2k+1} d^{2k})d + O(\epsilon d^{2k+2}),\\
&\Delta V\Big(x_i \pm \frac{1}{2}d\Big) = \epsilon\Big(\pm\frac{1}{2}\Delta a_1 + \frac{1}{4}\Delta a_2 d \pm \frac{1}{8}\Delta a_3 d^2 + \cdots\\
&\qquad\qquad + \frac{1}{2^{2k}}\Delta a_{2k} d^{2k-1} \pm \frac{1}{2^{2k+1}}\Delta a_{2k+1} d^{2k}\Big)d + O(\epsilon d^{2k+2}).
\end{aligned}$$

It follows that

$$\mathrm{Osc}_{I_i}(V_i' - V_i) \geq \frac{\epsilon}{2^{2k+1}} \max\{|\Delta a_1|, |\Delta a_2|d, |\Delta a_3|d^2, \cdots, |\Delta a_{2k+1} d^{2k}|\}d. \tag{2.3}$$

Let $M_1 = 3 \cdot 2^{2k+1}M$. We construct a grid for the parameters $\{a_j\}_{j=1}^{2k+1}$ by splitting the domain equally into a family of cuboids and setting the size by

$$\Delta a_1 = M_1 d^{2k+1-\frac{1}{p}}, \quad \Delta a_2 = M_1 d^{2k-\frac{1}{p}}, \ \cdots, \ \Delta a_{2k} = M_1 d^{2-\frac{1}{p}}, \quad \Delta a_{2k+1} = M_1 d^{1-\frac{1}{p}}.$$

These cuboids are denoted by $\mathbf{c}_{ij}$ with $j \in \mathbb{J}_i = \{1, 2, \cdots\}$, the cardinality of the set of the subscripts is up to the order

$$\#(\mathbb{J}_i) = M_2[d^{-(k+1)(2k+1) + \frac{2k+1}{p}}],$$

where the integer $0 < M_2 \in \mathbb{N}$ is independent of d. If $\mathrm{Osc}_{I_i} F_\lambda(\cdot) \leq M d^{2k+2}$, we obtain from the formula (2.3) that

$$\mathrm{Osc}_{I_i}(F_\lambda(x) - V(x)) \geq 2M d^{2k+2},$$

if

$$V(x) = \epsilon(a_1(x - x_i) + a_2(x - x_i)^2 + \cdots + a_{2k+1}(x - x_i)^{2k+1} + O(|x - x_i|^{2k+2}))$$

with

$$\max\{|a_1|d^{-(2k+1)+\frac{1}{p}}, |a_2|d^{-2k+\frac{1}{p}}, \cdots, |a_{2k}|d^{-2+\frac{1}{p}}, |a_{2k+1}|d^{-1+\frac{1}{p}}\} \geq M_1.$$

The coefficients $(a_1, a_2, \cdots, a_{2k+1})$ depend on the position x_i and the parameters $(A_1, B_1, \cdots, A_{k+1}, B_{k+1})$. The gird for $(a_1, a_2, \cdots, a_{2k+1})$ induces the partition for the parameters

$(A_1, B_1, \cdots, A_{k+1}, B_{k+1})$, determined by the equation

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ \vdots \\ a_{2k} \\ a_{2k+1} \end{bmatrix} = \mathbf{C}_{(2k+1)\times(2k+2)} \begin{bmatrix} A_1 \\ B_1 \\ A_2 \\ B_2 \\ \vdots \\ A_{k+1} \\ B_{k+1} \end{bmatrix}, \tag{2.4}$$

where $\mathbf{C}_{(2k+1)\times(2k+2)}$ is a $(2k+1)\times(2k+2)$ matrix as following

$$\mathbf{C}_{(2k+1)\times(2k+2)} = [I_1, I_2, \cdots, I_{2k+1}, I_{2k+2}]$$

in which each column has $2k+1$ entries which take the form

$$I_{2j-1} = \begin{bmatrix} j\cos\left(\frac{\pi}{2} + jx_i\right) \\ j^2\cos(\pi + jx_i) \\ \vdots \\ j^{2k}\cos\left(2k\frac{\pi}{2} + jx_i\right) \\ j^{2k+1}\cos\left((2k+1)\frac{\pi}{2} + jx_i\right) \end{bmatrix},$$

$$I_{2j} = \begin{bmatrix} j\sin\left(\frac{\pi}{2} + jx_i\right) \\ j^2\sin(\pi + jx_i) \\ \vdots \\ j^{2k}\sin\left(2k\frac{\pi}{2} + jx_i\right) \\ j^{2k+1}\sin\left((2k+1)\frac{\pi}{2} + jx_i\right) \end{bmatrix},$$

where the integer j ranges from 1 to $k+1$.

The coefficient matrix $\mathbf{C}_{(2k+1)(2k+2)}$ is non-singular for each $x_i \in \mathbb{T}$. Indeed, and let $\mathbf{M}_1$ be the $(2k+1)\times(2k+1)$ matrix constituted by first $(2k+1)$ columns of $\mathbf{C}$, and let $\mathbf{M}_2$ be the $(2k+1)\times(2k+1)$ matrix constituted by first $2k$ columns and the last column of $\mathbf{C}$. We find

$$\det(\mathbf{M}_1)(x_i) = (-1)^{k-1} M_3 \sin(k+1)x_i,$$
$$\det(\mathbf{M}_2)(x_i) = (-1)^{k} M_3 \cos(k+1)x_i,$$

where the constant M_3 is not equal to zero, and only depends on the integer k:

$$M_3 = \prod_{j=2}^{k}(j^3 - j)(j^4 - j^2)\prod_{j=3}^{k}\prod_{\ell=2}^{j-1}(j^2 - \ell^2)^2((k+1)^3 - (k+1))\prod_{j=2}^{k}((k+1)^2 - j^2).$$

It induces a positive lower bound

$$\inf_{x_i\in\mathbb{T}}\{|\det(\mathbf{M}_1)(x_i)|, |\det(\mathbf{M}_2)(x_i)|\} = \frac{M_3}{2}\sqrt{2}.$$

Therefore, the grid for $\{a_j\}_{j=1}^{2k+1}$ induces a grid for $(A_1, B_1, \cdots, A_{k+1}, B_{k+1})$ which contains as many as $M_4[d^{-(k+1)(2k+1)+\frac{2k+1}{p}}]$ $(2k+2)$-dimensional strips ($M_4 > 0$ is independent of d), denoted by $\mathbf{s}_{ij}$ with $j \in \mathbb{J}_i$. Each $\mathbf{s}_{ij}$ is mapped onto $\mathbf{c}_{ij}$ by (2.4).

Given certain parameter $\lambda \in [0,1]$, if there exist Taylor coefficients $\{a_j\}_{j=1}^{2k+1}$ which determine a perturbation V such that

$$\mathrm{Osc}_{I_i}(F_\lambda(x) - V(x)) \leq Md^{2k+2},$$

then for any other Taylor coefficients $\{a'_j\}_{j=1}^{2k+1}$ satisfying the condition

$$\max\Big\{\frac{|a_1 - a'_1|}{M_1 d^{2k+1-\frac{1}{p}}}, \frac{|a_2 - a'_2|}{M_1 d^{2k-\frac{1}{p}}}, \cdots, \frac{|a_{2k} - a'_{2k}|}{M_1 d^{2-\frac{1}{p}}}, \frac{|a_{2k+1} - a'_{2k+1}|}{M_1 d^{1-\frac{1}{p}}}\Big\} \geq 1,$$

which determines another perturbation V', one obtains from the formula (2.3) that

$$\mathrm{Osc}_{I_i}(F_\lambda(x) - V'(x)) \geq 2Md^{2k+2}. \tag{2.5}$$

Under the map defined by (2.4), the inverse image of a cuboid $\mathbf{c}_j$ with the size

$$2M_1 d^{2k+1-\frac{1}{p}} \times 2M_1 d^{2k-\frac{1}{p}} \times \cdots \times 2M_1 d^{2-\frac{1}{p}} \times 2M_1 d^{1-\frac{1}{p}}$$

is a strip in the parameter space of $(A_1, B_1, A_2, B_2, \cdots, A_{k+1}, B_{k+1})$, denoted by $\mathbf{s}_j(\lambda)$, with the Lebesgue measure as small as $N_3^{-1}[d^{(k+1)(2k+1)-\frac{2k+1}{p}}]$. If the cuboid $\mathbf{c}_j$ is centered at $(a_{1j}, a_{2j}, \cdots, a_{2k+1,j})$, then for $(a'_{1j}, a'_{2j}, \cdots, a'_{2k+1,j}) \notin \mathbf{c}_j$, (2.5) holds. In other words, if $(A'_1, B'_1, A'_2, B'_2, \cdots, A'_{k+1}, B'_{k+1}) \notin \mathbf{s}_j$, (2.5) holds.

Splitting the interval $[0,1]$ equally into small sub-intervals E_ℓ with the size $|E_\ell| = M_5^{-1} d^{\frac{2k+2}{\alpha}}$, we obtain as many as $[M_5 d^{-\frac{2k+2}{\alpha}}]$ small intervals. As the function F_λ is α-Hölder continuous in λ, suitably large positive number M_5 can be chosen so that

$$\max_{x \in I_i} |F_\lambda(x) - F_{\lambda'}(x)| < \frac{1}{2} Md^{2k+2}, \quad \forall \lambda, \lambda' \in E_\ell.$$

Therefore, for $V \in \mathfrak{V}$ with $(A_1, B_1, A_2, B_2, \cdots, A_{k+1}, B_{k+1}) \notin \mathbf{s}_j$, one has

$$\mathrm{Osc}_{I_i}(F_\lambda(x) - V(x)) \geq Md^{2k+2}. \tag{2.6}$$

Picking up one parameter λ_ℓ in each small interval E_ℓ, we obtain $[M_5 d^{-\frac{2k+2}{\alpha}}]$ strips $\mathbf{s}_j(\lambda_\ell)$. By considering all small intervals I_i with $i = 0, 1, \cdots, N-1$, we find

$$\mathrm{meas}\Big(\bigcup_{j,\ell} \mathbf{s}_j(\lambda_\ell)\Big) \leq N_3^{-1} d^{(k+1)(2k+1)-\frac{2k+1}{p}} M_5 d^{-\frac{2k+1}{\alpha}} d^{-1} = M_5 N_3^{-1} d^T,$$

where

$$T = (2k+2)\Big(k - \frac{1}{\alpha}\Big) + \Big(k - \frac{2k+1}{p}\Big) > 0$$

if we choose $p = 2k+2$ and set

$$k = \Big[\frac{1}{4}\Big(\frac{2}{\alpha} + 1 + \sqrt{\Big(\frac{2}{\alpha} + 1\Big)^2 + 16}\Big)\Big] - 1.$$

Letting

$$\mathbf{S}^c = \mathbb{I}^{2k+2} \backslash \bigcup_{j,\ell} \mathbf{s}_j(\lambda_\ell),$$

we obtain the Lebesgue measure estimate

$$\mathrm{meas}(\mathbf{S}^c) \geq 1 - M_5 N_3^{-1} d^T \to 1 \quad \text{as} \ \ d \to 0.$$

Obviously, for any $(A_1, B_1, A_2, B_2, \cdots, A_{k+1}, B_{k+1}) \in \mathbf{S}^c$, $\lambda \in [0,1]$ and $i = 0, 1, 2, \cdots,$ $N-1$, (2.6) holds. It implies the density that all global minimal points of $F_\lambda(\cdot)$ satisfy the following property: There is

$$1 \leq \ell \leq k, \quad \frac{\partial^{2\ell}(F_\lambda - V)}{\partial x^{2\ell}} > 0.$$

Letting $\alpha = 1$, one immediately obtains the first part of the theorem.

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