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A Note on Heegaard Genus of Self-amalgamated 3-Manifold*

Qilong GUO[1] Ruifeng QIU[2] Yanqing ZOU[1]

Abstract Let M be a connected orientable compact irreducible 3-manifold. Suppose that ∂M consists of two homeomorphic surfaces F_1 and F_2, and both F_1 and F_2 are compressible in M. Suppose furthermore that $g(M, F_1) = g(M) + g(F_1)$, where $g(M, F_1)$ is the Heegaard genus of M relative to F_1. Let M_f be the closed orientable 3-manifold obtained by identifying F_1 and F_2 using a homeomorphism $f : F_1 \to F_2$. The authors show that if f is sufficiently complicated, then $g(M_f) = g(M, \partial M) + 1$.

Keywords Heegaard splitting, Self-amalgamated, Sufficiently complicated
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1 Introduction

All manifolds in this paper are assumed to be compact and orientable, unless otherwise stated.

Let M be a connected irreducible 3-manifold, and suppose that ∂M consists of two homeomorphic surfaces F_1 and F_2. Let M_f be the closed 3-manifold obtained by identifying F_1 and F_2 using a homeomorphism $f : F_1 \to F_2$. By the construction of self-amalgamation of Heegaard splitting, we have $g(M_f) \leq g(M, \partial M) + 1$. So a natural and interesting question is:

Question 1.1 When does $g(M_f) = g(M, \partial M) + 1$?

In [2], Du and Qiu proved that when M is sufficiently complicated, the equality holds. In [3], Guo and Zou proved that when M is irreducible and ∂-irreducible, and satisfies some conditions, and if the gluing map is sufficiently complicated, the equality holds. In this paper we consider the case that M has two compressible boundary components. Here is our result.

Theorem 1.1 *Let M be a connected orientable compact irreducible 3-manifold, and ∂M consist of two homeomorphic surfaces F_1 and F_2 which are both compressible. Suppose that $g(M, F_1) = g(M) + g(F_1)$, where $g(M, F_1)$ is the Heegaard genus of M relative to F_1. Let M_f be the closed orientable 3-manifold obtained by identifying F_1 and F_2 through a homeomorphism $f : F_1 \to F_2$. Then there is a function defined for all such M_f, such that if $d(M_f) > 2g(M)$, then $g(M_f) = g(M, \partial M) + 1$.*

Remark 1.1 $g(M, F_1)$ is defined as in Definition 2.2, and $d(M_f)$ is defined as in Definition 2.5.

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[1]School of Mathematical Sciences, Dalian University of Technology, Dalian 116023, Liaoning, China.
E-mail: guoqilong1984@hotmail.com yanqing_dut@163.com
[2]Department of Mathematics, East China Normal University, Shanghai 200241, China.
E-mail: rfqiu@math.ecnu.edu.cn
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Note that the irreducibility of M and compressibility of F_1 and F_2 imply that $g(F_1) = g(F_2) \geq 2$.

The proof of our main result is a bit similar to that in [6], but the case that we deal with is a bit harder.

2 Preliminaries

Let M be a 3-manifold. First let us review some notions about Heegaard splittings.

Definition 2.1 *Suppose that S is a properly embedded closed surface in M, which separates M into two compression bodies V and W such that $S = \partial_+ V = \partial_+ W$. Then we say that $V \bigcup_S W$ is a Heegaard splitting of M, and call S a Heegaard surface and $g(S)$ the genus of this Heegaard splitting. If $g(S)$ is minimal among all the Heegaard surfaces of M, then $g(S)$ is called the Heegaard genus of M, denoted by $g(M)$.*

Definition 2.2 *Suppose that $\partial_1 M$ (maybe empty) is a collection of components of ∂M. If $M = V \bigcup_S W$ is a Heegaard splitting such that $\partial_1 M = \partial_- V$(or ∂W_-), then we say that $M = V \bigcup_S W$ is a Heegaard splitting of M relative to $\partial_1 M$. If $g(S)$ is minimal among all Heegaard splittings of M relative to $\partial_1 M$, then $g(S)$ is called the Heegaard genus of M relative to $\partial_1 M$, denoted by $g(M, \partial_1 M)$.*

Definition 2.3 *Suppose that F is a properly embedded surface in a 3-manifold M, and F splits M into two submanifolds M_1 and M_2. We say that F is strongly irreducible if F has compressing disks on both sides, and each compressing disk in M_1 meets each compressing disk in M_2. We say that F is ∂-strongly irreducible if*

(1) *every compressing and ∂-compressing disk in M_1 meets every compressing and ∂-compressing disk in M_2, and*

(2) *there is at least one compressing or ∂-compressing disk on each side of F.*

Definition 2.4 *Let F be a connected closed surface with $g(F) > 1$. The curve complex of F is the complex whose vertices are the isotopy classes of essential simple closed curves on F, and $k+1$ vertices in this complex determine a k-simplex if they can be represented by pairwise disjoint curves. We denote the curve complex of F by $\mathcal{C}(F)$ and denote its 0-skeleton by $\mathcal{C}^{(0)}(F)$.*

For $\alpha, \beta \in \mathcal{C}^{(0)}(F)$, we define

$$d_{\mathcal{C}(F)}(\alpha,\beta) = \min\{n; \textit{there exists a sequence of essential simple closed curves } c_0, \cdots, c_n, \textit{ such that } [c_0] = \alpha, [c_n] = \beta \textit{ and } c_i \cap c_{i+1} = \varnothing \textit{ for any } 1 \leq i \leq n\}.$$

For two subsets U_1, U_2 of $\mathcal{C}^{(0)}(F)$, we define

$$d_{\mathcal{C}(F)}(U_1, U_2) = \min\{d_{\mathcal{C}(F)}(\alpha,\beta); \alpha \in U_1, \beta \in U_2\}.$$

In this paper, we do not distinguish a vertex in $\mathcal{C}(F)$ from a simple closed curve in F representing this vertex, unless otherwise stated.

Definition 2.5 *Let F be a properly embedded closed bicompressible surface in M. Define*

$$d_M(F) = \min\{d_{\mathcal{C}(F)}([\alpha],[\beta]); \alpha \textit{ bounds an essential disk on one side of } F \textit{ and } \beta \textit{ bounds an essential disk on the other side}\}.$$

Suppose that M is connected and ∂M consists of two boundary components F_1 and F_2 with $g(F_1) = g(F_2)$. Let M_f be the closed orientable 3-manifold obtained by identifying F_1 and F_2 via a homeomorphism $f : F_1 \to F_2$, and let F be the surface in M_f which is the image of F_1 and F_2 after gluing. It is often helpful to view M as a sub-manifold of M_f, i.e., $M = M_f - \text{int}(N(F))$, where $N(F) = F \times [1,2]$ is a closed small regular neighborhood of F in M_f, and F_i can be viewed as $F \times \{i\}$ $(i = 1,2)$.

Definition 2.6 *Let M, F_1, F_2, f and M_f be as above. Suppose that F_1 and F_2 are both compressible, and let $U_i = \{[\alpha] \in \mathcal{C}(F_i);\ \alpha$ bounds a disk in $M\}$. By projection $F \times [1,2]$ to F, we may view U_1 and U_2 as subsets of $\mathcal{C}(F)$. Then we define $d(M_f) = d_{\mathcal{C}(F)}(U_1, U_2)$.*

Definition 2.7 *Let P and Q be two closed bicompressible separating surfaces in an irreducible and ∂-irreducible 3-manifold M. Maximally compressing P on both sides and deleting all resulting 2-sphere components, we get*

$$M = N_1 \bigcup_{F_1^P} H_1^P \bigcup_{P} H_2^P \bigcup_{F_2^P} N_2,$$

where H_i^P is a compression body with $\partial_+ H_i^P = P$, and F_i^P is a collection (may be empty) of close surfaces of genus more than zero for $i = 1,2$. In this case, P is a Heegaard surface of the manifold $H_1^P \bigcup_P H_2^P$. Similarly we define $H_1^Q \bigcup_Q H_2^Q$. P and Q are said to be well separated in M if we can have isotopy $H_1^P \bigcup_P H_2^P$ so that it is disjoint from $H_1^Q \bigcup_Q H_2^Q$.

3 Some Lemmas

Lemma 3.1 (see [4, 7]) *If S is a Heegaard surface of a 3-manifold M, and $(Q, \partial Q) \subset (M, \partial M)$ is an essential connected surface, then $d_M(S) \leq 2 - \chi(Q)$.*

Lemma 3.2 (see [9]) *If P and Q are both strongly irreducible connected closed separating surfaces in a 3-manifold M, then one of the following holds:*

(1) *P and Q are well separated, or*

(2) *P and Q are isotopic, or*

(3) *$d_M(P) \leq 2g(Q)$.*

Lemma 3.3 (see [1]) *Let M be an irreducible 3-manifold with ∂M incompressible, if nonempty. Suppose $M = V \bigcup_S W$, where S is a strongly irreducible Heegaard surface. Suppose that M contains an incompressible closed non-boundary parallel surface Q. Then one of the following holds:*

(1) *S may be isotopied to be transverse to Q, with every component of $S - \eta(Q)$ incompressible in the respective sub-manifold of $M - \eta(Q)$;*

(2) *S may be isotopied to be transverse to Q, with every component of $S - \eta(Q)$ incompressible in the respective sub-manifold of $M - \eta(Q)$, except for exactly one strongly irreducible component;*

(3) *S may be isotopied to be almost transverse to Q (i.e., S is transverse to Q except for one saddle point), with every component of $S - \eta(Q)$ incompressible in the respective sub-manifold of $M - \eta(Q)$.*

In each case, $\eta(Q)$ is a suitable (open) regular neighborhood of F in M.

Lemma 3.4 (see [2, 5]) *Let M be an irreducible 3-manifold, and let $V \bigcup_S W$ be a Heegaard splitting of M. Suppose that Q is a properly embedded strongly irreducible surface in M and $\partial Q \neq \varnothing$. Then either $d_M(S) \leq 2 - \chi(Q)$ or Q lies in an I-bundle of one component of ∂M.*

Lemma 3.5 *Let $V \bigcup_S W$ be a Heegaard splitting of a connected 3-manifold M. If $V \bigcup_S W$ is* *the amalgamation of* $\big(V_0 \bigcup\limits_{P_0} W_0\big) \bigcup\limits_{H_1} \big(V_1 \bigcup\limits_{P_1} W_1\big)$*, then*

$$g(S) = g(P_0) + g(P_1) - |P_0| - |P_1| - g(H_1) + |H_1| + 1.$$

Generally, if $V \bigcup\limits_{S} W$ *is the amalgamation of* $\big(V_0 \bigcup\limits_{P_0} W_0\big) \bigcup\limits_{H_1} \cdots \bigcup\limits_{H_n} \big(V_n \bigcup\limits_{P_n} W_n\big)$*, then*

$$g(S) = \sum_{i=0}^{n} g(P_i) - \sum_{i=0}^{n} |P_i| - \sum_{i=1}^{n} g(H_i) + \sum_{i=1}^{n} |H_i| + 1.$$

Proof We only prove the final result and the first result is a special case of the final result. By the construction of amalgamation, we can see that

$$\partial_- V = \partial_- V_0 \cup \Big(\bigcup_{i=1}^{n} (\partial_- V_i - H_i)\Big) = \Big(\bigcup_{i=0}^{n} \partial_- V_i\Big) - \Big(\bigcup_{i=1}^{n} H_i\Big).$$

Then V is obtained by attaching 1-handles to $N(\partial_- V)$, where $N(\partial_- V)$ is a closed neighborhood of $\partial_- V$ in M if $\partial_- V \neq \varnothing$, or $N(\partial_- V)$ is a 3-ball if $\partial_- V = \varnothing$. Let N be the number of 1-handles attached to $N(\partial_- V)$, and then

$$g(S) = g(\partial_- V) + N - (|\partial_- V| - 1).$$

Note that every $V_i \bigcup\limits_{P_i} W_i$ provides $(g(P_i) - g(\partial V_i)_- + |\partial_- V_i| - |P_i|)$ 1-handles attached to $\partial_- V$, so

$$N = \sum_{i=0}^{n} (g(P_i) - g(\partial_- V_i) + |\partial_- V_i| - |P_i|).$$

Hence

$$\begin{aligned} g(S) &= \Big(\sum_{i=0}^{n} g(\partial_- V_i) - \sum_{i=1}^{n} g(H_i)\Big) + N - \Big(\Big(\sum_{i=0}^{n} |\partial_- V_i| - \sum_{i=1}^{n} |H_i|\Big) - 1\Big) \\ &= \sum_{i=0}^{n} g(P_i) - \sum_{i=0}^{n} |P_i| - \sum_{i=1}^{n} g(H_i) + \sum_{i=1}^{n} |H_i| + 1. \end{aligned}$$

As an application of the above lemma, we prove the following result which will be used in the proof of Theorem 1.1.

Lemma 3.6 *Suppose* $M = V \bigcup\limits_{S} W = N_0 \bigcup\limits_{H_1} \cdots \bigcup\limits_{H_{n-1}} N_n = \big(V_0 \bigcup\limits_{P_0} W_0\big) \bigcup\limits_{H_1} \cdots \bigcup\limits_{H_{n-1}} \big(V_n \bigcup\limits_{P_n} W_n\big)$. *Suppose that* F *is a component of* P_k *for some* k *and* F *is non-separating in* M*. Let* $M' = M - \text{int}(N(F))$*, where* $N(F)$ *is a product neighborhood of* F *in* M*. Denote two copies of* F *in* M' *by* F_1 *and* F_2*, and then* $g(M'; F_1) \leq g(S) + g(F) - 1$.

Proof Denote the component of N_k which contains F by N_k^1, and write $N_k^2 = N_k - N_k^1$. Without loss of generality, we assume that $N(F)$ is contained in $\text{int}(N_k^1)$. Then $M' = N_0 \bigcup\limits_{H_1} \cdots \bigcup\limits_{H_{k-1}} (N_k - \text{int}(N(F))) \bigcup\limits_{H_k} \cdots \bigcup\limits_{H_{n-1}} N_n$. Write $V_k^i = V_k \cap N_k^i, W_k^i = W_k \cap N_k^i$, and $P_k^i = P_k \cap N_k^i$ for $i = 1, 2$. Then $F = P_k^1$ and $N_k - \text{int}(N(F)) = (N_k^1 - \text{int}(N(F)) \sqcup N_k^2 \cong V_k^1 \sqcup W_k^1 \sqcup N_k^2$. Since V_k^1 is a compression body, $V_k^1 \cong V_k^1 \cup (\partial_+ V_k^1 \times I)$. For the same reason, $W_k^1 \cong (\partial_+ W_k^1 \times I) \cup W_k^1$. Hence

$$\begin{aligned} M' &= N_0 \bigcup_{H_1} \cdots \bigcup_{H_{k-1}} (V_k^1 \sqcup W_k^1 \sqcup N_k^2) \bigcup_{H_k} \cdots \bigcup_{H_{n-1}} N_n \\ &= \Big(V_0' \bigcup_{P_0'} W_0'\Big) \bigcup_{H_1} \cdots \bigcup_{H_{n-1}} \Big(V_n' \bigcup_{P_n'} W_n'\Big), \end{aligned}$$

where $V_i' = V_i, W_i' = W_i, P_i' = P_i$ for $i \neq k$, and $V_k' \cong V_k^1 \sqcup (\partial_+ W_k^1 \times I) \sqcup V_k^2, W_k' \cong (\partial_+ V_k^1 \times I) \sqcup W_k^1 \sqcup W_k^2$. So $g(P_i') = g(P_i)$ for $i \neq k$, and $g(P_k') = g(\partial_+ V_k^1) + g(\partial_+ W_k^1) + g(P_k^2) = g(F) + g(P_k)$.

Amalgamating $\left(V_0' \bigcup_{P_0'} W_0'\right) \bigcup_{H_1} \cdots \bigcup_{H_{n-1}} \left(V_n' \bigcup_{P_n'} W_n'\right)$, and we get a Heegaard spitting of M relative to F_1, denoted by S'. By Lemma 3.5,

$$g(S') = \sum_{i=0}^{n} g(P_i') - \sum_{i=0}^{n} |P_i'| - \sum_{i=1}^{n} g(H_i) + \sum_{i=1}^{n} |H_i| + 1.$$

Note that $P_i' = P_i$ for $i \neq k$, $g(P_k') = g(\partial_+ V_k^1) + g(\partial_+ W_k^1) + g(P_k^2) = g(F) + g(P_k)$ and $|P_k'| = |P_k| + 1$. We get $g(S') - g(S) = g(F) - 1$. So $g(M', F_1) \leq g(S') = g(S) + g(F) - 1$.

4 Proof of Theorem 1.1

Proof of Theorem 1.1 The idea is as follows. Suppose that $\widehat{V} \bigcup_{\widehat{S}} \widehat{W}$ is a minimal Heegaard splitting of M_f. We will construct a Heegaard surface S of M (relative to F_1) from $\widehat{S}$, such that if $g(\widehat{S}) < g(M) + 1$, then $g(S) < g(M) + g(F_1)$ under the assumption $d(M_f) > 2g(M)$. Now we suppose that $g(\widehat{S}) < g(M) + 1$. Since $g(M, F_1) = g(M) + g(F_1)$, and $g(M) = g(M, F_1 \cup F_2) = g(M, \partial M)$, $g(\widehat{S}) \leq g(M) = g(M, \partial M)$.

As in [8], the untelescoping of the Heegaard splitting gives a decomposition

$$M_f = \widehat{V} \bigcup_{\widehat{S}} \widehat{W} = N_0 \bigcup_{H_1} \cdots \bigcup_{H_{n-1}} N_n = \left(V_0 \bigcup_{P_0} W_0\right) \bigcup_{H_1} \cdots \bigcup_{H_{n-1}} \left(V_n \bigcup_{P_n} W_n\right),$$

where for each i, $V_i \bigcup_{P_i} W_i$ is a strongly irreducible Heegaard splitting of N_i, and H_i is incompressible in M_f. Furthermore, for each i, $g(H_i) < g(\widehat{S}),\ g(P_i) \leq g(\widehat{S})$.

Let Q_i $(i = 1, 2)$ be the surface obtained by maximally compressing F_i in M and removing all resulting 2-sphere components. Then $Q_1 \sqcup Q_2$ bounds a sub-manifold M_F in M, and F is a Heegaard surface of M_F. Write $M_F = M_{F_1} \bigcup_F M_{F_2}$, where M_{F_i} contains F_i as in Figure 1. Since F_1 and F_2 are compressible, we have that both M_{F_1} and M_{F_2} are nontrivial compression bodies.

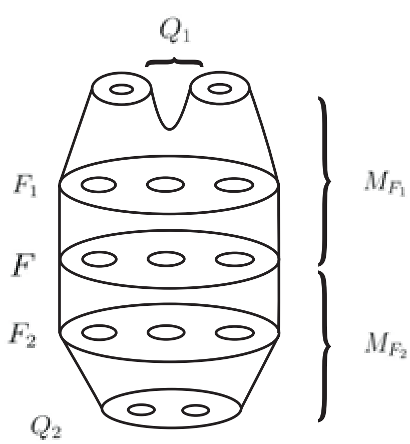

Figure 1 M_{F_1} and M_{F_2}

Recall that $U_i = \{[\alpha] \in \mathcal{C}(F_i);\ \alpha$ bounds a disk in $M\}$, and write $U_i' = \{[\alpha] \in \mathcal{C}(F); \alpha$ bounds a disk in $M_{F_i}\} \subset U_i$, so then $d_{M_F}(F) = d_{C(F)}(U_1', U_2') \geq d_{C(F)}(U_1, U_2) = d(M_f) > 2g(M)$.

Claim 4.1 Each H_i can be isotopied to be disjoint with F.

Proof If not, then H_i can not be made disjoint with M_F for some i. Without loss of generality, we assume that each component of $H_i \cap M_F$ is an essential surface of M_F. Choosing some component H' of $H_i \cap M_F$, we have $\chi(H') > \chi(H_i) = 2 - 2g(H_i) > 2 - 2g(\widehat{S})$, so $d_{M_F}(F) \leq 2-\chi(H') < 2g(\widehat{S})$ according to Lemma 3.1. But it is a contradiction since $d_{M_F}(F) > 2g(M) \geq 2g(\widehat{S})$.

By Claim 4.1, $F \subset \mathrm{int}(N_k)$ for some k. Denote the component of N_k containing F by N_k^1 and denote $N_k^2 = N_k - N_k^1$. For $i = 1, 2$, write $V_k^i = V_k \cap N_k^i$, $W_k^i = W_k \cap N_k^i$, $P_k^i = P_k \cap N_k^i$, $H_k^i = H_k \cap N_k^i$, $H_{k+1}^i = H_{k+1} \cap N_k^i$. Since $\partial N_k^1 = H_k^1 \cup H_{k+1}^1$ is incompressible in M, any compressing disk for F can be isotopied into N_k. So after isotopying, we may assume $M_F \subset N_k^1$. Then there are two possibilities for P_k^1 and F:

Case 1 P_k^1 can not be isotopied to be disjoint with F.

By Lemma 3.3, we can assume each component of $P_k^1 \cap F$ is incompressible in M_F except for at most one strongly irreducible component. Furthermore, we assume that each component is not ∂-parallel. Then by maximally ∂-compressing $P_k^1 \cap F$, we will get at least one connected surface which is either an essential surface or a strongly irreducible and ∂-strongly irreducible surface, and we choose such a component, denoted by Q. Then by Lemma 3.1 and Lemma 3.4, we have $d_{M_F}(F) \leq 2 - \chi(Q) \leq 2 - \chi(P_k) = 2g(P_k) \leq 2g(\widehat{S}) \leq 2g(M)$, which is a contradiction since $d_{M_F}(F) > 2g(M)$.

Case 2 We can have isotopy P_k^1 such that $P_k^1 \cap F = \varnothing$.

Without loss of generality, we assume that $F \subset V_k^1$, and then F must separate V_k^1, so F separates N_k^1. Since P_k^1 and F are obviously not well separated and $d_{M_F}(F) > 2g(P_k^1)$, by Lemma 3.2, P_k^1 and F are isotopic. Without loss of generality, we assume $P_k^1 = F$. Then by Lemma 3.6, we get $g(M, F_1) \leq g(\widehat{S}) + g(F_1) - 1 < g(M) + g(F_1)$, which is also a contradiction.

From the above, we show that $g(\widehat{S}) < g(M)+1$ is impossible, so $g(M_f) = g(\widehat{S}) = g(M)+1 = g(\partial M) + 1$.

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