*Chin. Ann. Math.*
36B(1), 2015, 57–66
DOI: 10.1007/s11401-014-0876-2

**Chinese Annals of Mathematics, Series B**


# Gradient Estimates for a Nonlinear Parabolic Equation with Diffusion on Complete Noncompact Manifolds*

Liang ZHAO[1] Zongwei MA[2]

**Abstract** The authors obtain some gradient estimates for positive solutions to the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} = \triangle u - b(x,t)u^{\sigma}$$

on complete noncompact manifolds with Ricci curvature bounded from below, where $0 < \sigma < 1$ is a real constant, and $b(x,t)$ is a function which is $C^2$ in the $x$-variable and $C^1$ in the $t$-variable.

**Keywords** Gradient estimates, Positive solutions, Harnack inequality
**2000 MR Subject Classification** 58J05, 58J35

## 1 Introduction

In this paper, we study the following nonlinear parabolic equation:

$$\frac{\partial u}{\partial t} = \triangle u - b(x,t)u^{\sigma} \tag{1.1}$$

on complete noncompact manifolds $M$ with Ricci curvature bounded from below, where $0 < \sigma < 1$ is a real constant, and $b(x,t)$ is a function which is $C^2$ in the $x$-variable and $C^1$ in the $t$-variable.

Gradient estimates play an important role in the study of the PDE, especially the Laplacian equation and the heat equation. Li and Yau [5] developed the fundamental gradient estimate, which is now widely called the Li-Yau estimate, for any positive solution $u(x,t)$ to the heat equation on a Riemannian manifold $M$, and showed how the classical Harnack inequality can be derived from their gradient estimate. Later, Hamilton [3] got the matrix Harnack estimate for the heat equation.

Let $(M^n, g)$ be an $n$-dimensional complete noncompact Riemannian manifold. For a smooth real-valued function $f$ on $M^n$, the drifting Laplacian is defined by

$$\triangle_f = \triangle - \nabla f \cdot \nabla.$$

Manuscript received March 16, 2013. Revised December 22, 2013.
[1]Department of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 210016, China. E-mail: zhaozongliang09@163.com
[2]College of Mathematics Physics and Information Engineering, Jiaxing University, Jiaxing 314001, Zhejiang, China. E-mail: 282277856@qq.com
*This work was supported by the Jiangsu Provincial Natural Science Foundation of China (No. BK20140804) and the Fundamental Research Funds of the Central Universities (No. NS2014076).

There is a naturally associated measure $\mathrm{d}\mu = \mathrm{e}^{-f}\mathrm{d}V$ on $M^n$ which makes the operator $\triangle_f$ self-adjoint. The $N$-Bakry-Emery Ricci tensor is defined by

$$\mathrm{Ric}_f^N = \mathrm{Ric} + \nabla^2 f - \frac{1}{N}\mathrm{d}f \otimes \mathrm{d}f$$

for $0 \leq N \leq \infty$ and $N = 0$ if and only if $f = 0$. Here $\nabla^2$ is the Hessian operator and Ric is the Ricci tensor. Huang and Li [4] considered the generalized equation

$$u_t = \triangle_f u^\alpha$$

on Riemannian manifolds and got some interesting gradient estimates. Wu [6] gave a local Li-Yau-type gradient estimate for the positive solutions to a general nonlinear parabolic equation

$$u_t = \triangle_f u - au\log u - qu$$

in $M \times [0, \tau]$, where $a \in \mathbb{R}$, $\phi$ is a $C^2$-smooth function and $q = q(x, t)$ is a function which generalizes many previous well-known results about gradient estimates. Zhang and Ma [7] considered gradient estimates on positive solutions to the following nonlinear equation:

$$\triangle_f u + cu^{-\alpha} = 0, \quad \alpha > 0 \tag{1.2}$$

on complete noncompact manifolds, and the authors got a gradient estimate for positive solutions of the above equation (1.2) when $N$ is finite and the $N$-Bakry-Emery Ricci tensor is bounded from below.

**Theorem 1.1** (see [7]) *Let $(M^n, g)$ be a complete noncompact $n$-dimensional Riemannian manifold with the $N$-Bakry-Emery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_p(2R)$ around $p \in M$. Let $u$ be a positive solution to* (1.2). *Then*

(1) *if $c > 0$, we have*

$$\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq \frac{(N+n)(N+n+2)c_1^2}{R^2} + \frac{(N+n)[(N+n-1)c_1 + c_2]}{R^2} + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + 2(N+n)K;$$

(2) *if $c < 0$, we have*

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq{}& (A+\sqrt{A})|c|\Big(\inf_{B_p(2R)} u\Big)^{-\alpha-1} + \frac{(N+n)[(N+n-1)c_1 + c_2]}{R^2} \\ &+ \frac{(N+n)c_1^2}{R^2}\Big(n + N + 2 + \frac{n+N}{2\sqrt{A}}\Big) + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} \\ &+ \Big(2 + \frac{1}{\sqrt{A}}\Big)(n+N)K,\end{aligned}$$

*where $A = (N+n)(\alpha+1)(\alpha+2)$.*

Recently, Zhu [8] investigated the nonlinear parabolic equation

$$u_t = \triangle u + \lambda(x,t)u^\alpha(x,t), \tag{1.3}$$

where $0 < \alpha < 1$, and $\lambda(x,t)$ is a function defined on $M \times (-\infty, 0]$, which is $C^1$ in the first variable and $C^0$ in the second variable. The author got a Hamilton-type estimate and a Liouville-type theorem for positive solutions to (1.3).

**Theorem 1.2** (see [8]) *Let $(M^n, g)$ be a Riemannian manifold of dimension $n \geq 2$ with* $\mathrm{Ric}(M) \geq -k$ *for some $k \geq 0$. Suppose that $u$ is a positive solution to* (1.3) *in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. Suppose also that $u \leq M$ and $\frac{|\nabla \lambda|^2}{\lambda} \leq \theta$ in $Q_{R,T}$. Then there exists a constant $C = C(\alpha, M)$, such that*

$$\frac{|\nabla u|}{u} \leq CM^{1-\alpha}\Big(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k}\Big) + C\theta^{\frac{1}{4}} M^{\frac{2}{3}(1-\alpha)}$$

*in $Q_{\frac{R}{2}, \frac{T}{2}}$.*

In this paper, we will study the interesting Li-Yau type estimate for the positive solutions to (1.1).

Motivated by the above work, we present our main results about (1.1) as follows.

**Theorem 1.3** *Let $(M^n, g)$ be a complete noncompact $n$-dimensional Riemannian manifold with Ricci curvature bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_p(2R)$ around $p \in M$. Assume that $|b| \leq \lambda(2R)$, $\triangle b \leq \theta(2R)$ and $|\nabla b| \leq \gamma(2R)$ in $B_p(2R) \times [0, T)$ for some constants $\lambda(2R), \theta(2R)$ and $\gamma(2R)$. Let $u$ be a positive solution to* (1.1) *with $u \leq M_1$. Then for any constant $0 < \beta < 1$, if $\beta < \sigma < 1$, we have*

$$\begin{aligned}\beta\frac{|\nabla u|^2}{u^2} - bu^{\sigma-1} - \frac{u_t}{u} \leq{}& \frac{n}{2\beta}\Big(\frac{(n-1)(1+\sqrt{K}R)c_1^2 + 2c_1^2}{R^2} - \lambda(\sigma-1)M_1^{\sigma-1} \\ &+ \frac{nc_1^2}{4\beta(1-\beta)R^2} + \frac{1}{t}\Big) + \Big\{\frac{3n^2}{8\beta}\Big(\frac{nM_1^{4\sigma-4}\gamma^4(1-\beta)^2}{4\epsilon\beta^4}\Big)^{\frac{1}{3}} \\ &+ \frac{n^3}{8\beta(1-\beta)^2(1-\epsilon)}\Big(\frac{M_1^{\sigma-1}\lambda(\sigma-1)(\beta-\sigma)}{2\beta} + K\Big)^2 \\ &+ \frac{n}{2\beta}[M_1^{\sigma-1}\theta + \lambda(1-\sigma)M_1^{\sigma-1}]\Big\}^{\frac{1}{2}},\end{aligned}$$

*where $c_1$ and $c_2$ are positive constants and $\epsilon \in (0,1)$.*

Let $R \to \infty$, and we can get the global Li-Yau type gradient estimates for the nonlinear parabolic equation (1.1).

**Corollary 1.1** *Let $(M^n, g)$ be a complete noncompact $n$-dimensional Riemannian manifold with Ricci curvature bounded from below by the constant $-K$, where $K > 0$. Assume that $|b| \leq \lambda(M^n)$, $\triangle b \leq \theta(M^n)$ and $|\nabla b| \leq \gamma(M^n)$ in $M^n \times [0, T)$ for some constants $\lambda$, $\theta$ and $\gamma$. Let $u$ be a positive solution to* (1.1) *with $u \leq M_1$. Then for any constant $0 < \beta < 1$, if $\beta < \sigma < 1$, we have*

$$\beta\frac{|\nabla u|^2}{u^2} - bu^{\sigma-1} - \frac{u_t}{u} \leq \frac{n}{2\beta}\Big(\lambda(\sigma-1)M_1^{\sigma-1} + \frac{1}{t}\Big) + N,$$

*where*

$$N=\Big\{\frac{3n^2}{8\beta}\Big(\frac{nM_1^{4\sigma-4}\gamma^4(1-\beta)^2}{4\epsilon\beta^4}\Big)^{\frac{1}{3}}+\frac{n^3}{8\beta(1-\beta)^2(1-\epsilon)}\Big(\frac{M_1^{\sigma-1}\lambda(\sigma-1)(\beta-\sigma)}{2\beta}+K\Big)^2$$
$$+\frac{n}{2\beta}[M_1^{\sigma-1}\theta+\lambda(1-\sigma)M_1^{\sigma-1}]\Big\}^{\frac{1}{2}},$$

*$c_1$ and $c_2$ are positive constants and $\epsilon\in(0,1)$.*

As an application, we get the following Harnack inequality.

**Theorem 1.4** *Let $(M^n,g)$ be a complete noncompact $n$-dimensional Riemannian manifold with $\mathrm{Ric}(M^n)\ge -K$, where $K>0$. Assume that $b$ is a nonpositive constant and $|b|\le\lambda(M^n)$. Let $u(x,t)$ be a positive smooth solution to the equation*

$$u_t=\triangle u-bu^\sigma$$

*on $M^n\times[0,+\infty)$. Then if $\beta<\sigma<1$, for any points $(x_1,t_1)$ and $(x_2,t_2)$ on $M^n\times[0,+\infty)$ with $0<t_1<t_2$, we have the following Harnack inequality:*

$$u(x_1,t_1)\le u(x_2,t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{n}{2\beta}}\mathrm{e}^{\phi(x_1,x_2,t_1,t_2)+\widetilde{N}(t_2-t_1)},$$

*where $\phi(x_1,x_2,t_1,t_2)=\inf\limits_{\gamma}\int_{t_1}^{t_2}\frac{1}{4\beta}|\dot{\gamma}|^2\mathrm{d}t$ and*

$$\widetilde{N}=\Big\{\frac{n^3}{8\beta(1-\beta)^2(1-\epsilon)}\Big(\frac{M_1^{\sigma-1}\lambda(\sigma-1)(\beta-\sigma)}{2\beta}+K\Big)^2$$
$$+\frac{n}{2\beta}[\lambda(1-\sigma)M_1^{\sigma-1}]\Big\}^{\frac{1}{2}}+\frac{n}{2\beta}\lambda(\sigma-1)M_1^{\sigma-1}.$$

## 2 Proof of Theorem 1.3

Let $u$ be a positive solution to (1.1). Set $w=\ln u$. Then $w$ satisfies the equation

$$w_t=\triangle w+|\nabla w|^2-b\mathrm{e}^{(\sigma-1)w}. \tag{2.1}$$

**Lemma 2.1** *Let $(M^n,g)$ be a complete noncompact $n$-dimensional Riemannian manifold with Ricci curvature bounded from below by the constant $-K=:-K(2R)$, where $R>0$ and $K>0$ in the metric ball $B_p(2R)$ around $p\in M^n$. Let $w$ be a positive solution to (2.1). Then*

$$\Big(\triangle-\frac{\partial}{\partial t}\Big)F\ge -2\nabla w\cdot\nabla F+t\Big\{\frac{2\beta}{n}(|\nabla w|^2-b\mathrm{e}^{(\sigma-1)w}-w_t)^2$$
$$+b(\sigma-1)(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2+2(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b$$
$$-\mathrm{e}^{(\sigma-1)w}\triangle b\Big\}+b(\sigma-1)\mathrm{e}^{(\sigma-1)w}F-\frac{F}{t},$$

*where*

$$F=t(\beta|\nabla w|^2-b\mathrm{e}^{(\sigma-1)w}-w_t),$$

*and $\beta$ is any constant satisfying $0<\beta<1$.*

**Proof** Define

$$F = t(\beta|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t).$$

By the Bochner formula, we have

$$\triangle|\nabla w|^2 \geq \frac{2}{n}|\triangle w|^2 + 2\nabla w\nabla(\triangle w) - 2K|\nabla w|^2. \tag{2.2}$$

Noticing that

$$\triangle w_t = (\triangle w)_t = -2\nabla w\nabla w_t + b_t\mathrm{e}^{(\sigma-1)w} + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}w_t + w_{tt} \tag{2.3}$$

and

$$\begin{aligned}\triangle w &= -|\nabla w|^2 + b\mathrm{e}^{(\sigma-1)w} + w_t\\ &= (1-\frac{1}{\beta})(b\mathrm{e}^{(\sigma-1)w} + w_t) - \frac{F}{\beta t}\\ &= (\beta-1)|\nabla w|^2 - \frac{F}{t},\end{aligned}$$

we know

$$\triangle F = t(\beta\triangle|\nabla w|^2 - \triangle(b\mathrm{e}^{(\sigma-1)w}) - \triangle w_t).$$

By (2.2)–(2.3), we obtain

$$\begin{aligned}\beta\triangle|\nabla w|^2 &\geq \frac{2\beta}{n}\Big((\beta-1)|\nabla w|^2 - \frac{F}{t}\Big)^2 + 2\beta\nabla w\nabla(\triangle w) - 2\beta K|\nabla w|^2\\ &= \frac{2\beta}{n}\Big((\beta-1)|\nabla w|^2 - \frac{F}{t}\Big)^2\\ &\quad + 2\beta\nabla w\nabla\Big[\Big(1-\frac{1}{\beta}\Big)\Big(b\mathrm{e}^{(\sigma-1)w} + w_t\Big) - \frac{F}{\beta t}\Big]\\ &\quad - 2\beta K|\nabla w|^2\\ &= \frac{2\beta}{n}\Big((\beta-1)|\nabla w|^2 - \frac{F}{t}\Big)^2 + 2(\beta-1)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b\\ &\quad + 2b(\beta-1)(\sigma-1)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 + 2(\beta-1)\nabla w\nabla w_t\\ &\quad - \frac{2}{t}\nabla w\nabla F - 2\beta K|\nabla w|^2\end{aligned}$$

and

$$\begin{aligned}\triangle(b\mathrm{e}^{(\sigma-1)w}) &= \mathrm{e}^{(\sigma-1)w}\triangle b + 2(\sigma-1)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b + b(\sigma-1)^2\mathrm{e}^{(\sigma-1)w}|\nabla w|^2\\ &\quad + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}\triangle w\\ &= \mathrm{e}^{(\sigma-1)w}\triangle b + 2(\sigma-1)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b + b(\sigma-1)^2\mathrm{e}^{(\sigma-1)w}|\nabla w|^2\\ &\quad + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}\Big[(\beta-1)|\nabla w|^2 - \frac{F}{t}\Big].\end{aligned}$$

So, we have

$$\begin{aligned}\triangle F \geq t\Big\{ & \frac{2\beta}{n}(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 + 2(\beta-1)\mathrm{e}^{(\sigma-1)w}\nabla w \nabla b \\ & + 2b(\beta-1)(\sigma-1)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 + 2(\beta-1)\nabla w \nabla w_t \\ & - \frac{2}{t}\nabla w \nabla F - 2\beta K|\nabla w|^2 - \mathrm{e}^{(\sigma-1)w}\triangle b - 2(\sigma-1)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b \\ & - b(\sigma-1)^2\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 - b(\sigma-1)\mathrm{e}^{(\sigma-1)w}\Big[(\beta-1)|\nabla w|^2 - \frac{F}{t}\Big] \\ & - (-2\nabla w \nabla w_t + b_t\mathrm{e}^{(\sigma-1)w} + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}w_t + w_{tt})\Big\}\end{aligned}$$

and

$$F_t = \frac{F}{t} + t(2\beta\nabla w\nabla w_t - b_t\mathrm{e}^{(\sigma-1)w} - b(\sigma-1)\mathrm{e}^{(\sigma-1)w}w_t - w_{tt}).$$

This implies that

$$\begin{aligned}\Big(\triangle - \frac{\partial}{\partial t}\Big)F \geq & -2\nabla w\cdot\nabla F + t\Big\{\frac{2\beta}{n}(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 \\ & + b(\sigma-1)(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 + 2(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b \\ & - \mathrm{e}^{(\sigma-1)w}\triangle b - 2\beta K|\nabla w|^2\Big\} + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}F - \frac{F}{t}.\end{aligned}$$

We complete the proof of Lemma 2.1.

**Proof of Theorem 1.3** We take a $C^2$ cut-off function $\widetilde{\varphi}$ defined on $[0,\infty)$, such that $\widetilde{\varphi}(r)=1$ for $r\in[0,1]$, $\widetilde{\varphi}(r)=0$ for $r\in[2,\infty)$, and $0\leq\widetilde{\varphi}(r)\leq 1$. Furthermore, $\widetilde{\varphi}$ satisfies

$$-\frac{\widetilde{\varphi}'(r)}{\widetilde{\varphi}^{\frac{1}{2}}(r)} \leq c_1$$

and

$$\widetilde{\varphi}''(r) \geq -c_2$$

for some absolute constants $c_1, c_2 > 0$. Denote by $r(x)$ the distance between $x$ and $p$ in $M$. Set

$$\varphi(x) = \widetilde{\varphi}\Big(\frac{r(x)}{R}\Big).$$

Using an argument of Cheng and Yau [2], we can assume $\varphi(x)\in C^2(M)$ with support in $B_p(2R)$. Direct calculation shows that on $B_p(2R)$,

$$\frac{|\nabla\varphi|^2}{\varphi} \leq \frac{c_1^2}{R^2}. \tag{2.4}$$

By the Laplacian comparison theorem in [1],

$$\triangle\varphi \geq -\frac{(n-1)(1+\sqrt{K}R)c_1^2 + c_2}{R^2}. \tag{2.5}$$

For $T\geq 0$, let $(x_0,t_0)$ be a point in $B_{2R}(p)\times[0,t]$ at which $\varphi F$ attains its maximum value $P$. We assume that $P$ is positive (otherwise the proof is trivial). At the point $(x_0,t_0)$, we have

$$\nabla(\varphi F) = 0, \quad \triangle(\varphi F)\leq 0, \quad F_t \geq 0. \tag{2.6}$$

It follows that

$$\varphi\triangle F + F\triangle\varphi - 2F\varphi^{-1}|\nabla\varphi|^2 \le 0.$$

This inequality together with the inequalities (2.4)–(2.5) yields

$$\varphi\triangle F \le HF,$$

where

$$H = \frac{(n-1)(1+\sqrt{K}R)c_1^2 + c_2 + 2c_1^2}{R^2}.$$

At $(x_0, t_0)$, by Lemma 2.1 and (2.6), we have

$$\begin{aligned}
0 &\ge \varphi\triangle F - HF \\
&\ge -HF + \varphi\Big\{-\frac{F}{t_0} + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}F \\
&\quad + \frac{2\beta t_0}{n}(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 - 2\nabla w\cdot\nabla F \\
&\quad + bt_0(\sigma-1)(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 + 2t_0(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b \\
&\quad - t_0\mathrm{e}^{(\sigma-1)w}\triangle b + t_0 b(\sigma-1)\mathrm{e}^{(\sigma-1)w} - 2\beta t_0 K|\nabla w|^2\Big\} \\
&\ge -HF - \varphi t_0^{-1}F + 2F\nabla w\nabla\varphi + \frac{2\beta t_0}{n}\varphi(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 \\
&\quad + b(\sigma-1)\mathrm{e}^{(\sigma-1)w}\varphi F + bt_0\varphi(\sigma-1)(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}|\nabla w|^2 \\
&\quad - t_0\varphi\mathrm{e}^{(\sigma-1)w}\triangle b + t_0\varphi b(\sigma-1)\mathrm{e}^{(\sigma-1)w} + 2t_0\varphi(\beta-\sigma)\mathrm{e}^{(\sigma-1)w}\nabla w\nabla b - 2\beta\varphi t_0 K|\nabla w|^2 \\
&\ge -HF - \varphi t_0^{-1}F + 2F\nabla w\nabla\varphi + \frac{2\beta t_0}{n}\varphi(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 \\
&\quad + \lambda(\sigma-1)M_1^{\sigma-1}\varphi F - \lambda t_0\varphi(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}|\nabla w|^2 \\
&\quad - t_0\varphi M_1^{\sigma-1}\theta + t_0\varphi\lambda(\sigma-1)M_1^{\sigma-1} + 2t_0\varphi(\beta-\sigma)M_1^{\sigma-1}|\nabla w|\gamma - 2\beta\varphi t_0 K|\nabla w|^2.
\end{aligned}$$

Multiplying both sides of the above inequality by $t_0\varphi$, and noting the fact that $0<\varphi<1$, we have

$$\begin{aligned}
0 &\ge -Ht_0\varphi F - \varphi F + 2t_0\varphi F\nabla w\nabla\varphi + \lambda(\sigma-1)M_1^{\sigma-1}t_0\varphi F \\
&\quad + \frac{2\beta t_0^2}{n}\varphi^2(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 - \lambda t_0^2\varphi^2(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}|\nabla w|^2 \\
&\quad - M_1^{\sigma-1}\theta t_0^2 - \lambda(1-\sigma)M_1^{\sigma-1}t_0^2 + 2t_0^2\varphi^{\frac{1}{2}}(\beta-\sigma)M_1^{\sigma-1}|\nabla w|\gamma - 2\beta\varphi^2 t_0^2 K|\nabla w|^2 \\
&\ge -Ht_0\varphi F - \varphi F - \frac{2c_1}{R}t_0\varphi F|\nabla w|\varphi^{\frac{3}{2}} + \lambda(\sigma-1)M_1^{\sigma-1}t_0\varphi F \\
&\quad + \frac{2\beta t_0^2}{n}\varphi^2\Big[(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 - \frac{n\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta}|\nabla w|^2\Big] \\
&\quad - M_1^{\sigma-1}\theta t_0^2 - \lambda(1-\sigma)M_1^{\sigma-1}t_0^2 + 2t_0^2\varphi^{\frac{1}{2}}(\beta-\sigma)M_1^{\sigma-1}|\nabla w|\gamma - 2\beta\varphi^2 t_0^2 K|\nabla w|^2.
\end{aligned}$$

Let

$$y = \varphi|\nabla w|^2, \quad z = \varphi(b\mathrm{e}^{(\sigma-1)w} + w_t).$$

It follows that

$$
\begin{aligned}
0 \geq\ & \varphi F(-Ht_0 + \lambda(\sigma-1)M_1^{\sigma-1}t_0 - 1) - \frac{2c_1}{R}t_0 F|\nabla w|\varphi^{\frac{3}{2}} \\
& + \frac{2\beta t_0^2}{n}\Big\{\varphi^2(|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t)^2 - \Big(\frac{n\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + nK\Big)\varphi^2|\nabla w|^2 \\
& + \frac{\beta-\sigma}{\beta}nM_1^{\sigma-1}|\nabla w|\gamma\varphi^{\frac{1}{2}}\Big\} - M_1^{\sigma-1}\theta t_0^2 - \lambda(1-\sigma)M_1^{\sigma-1}t_0^2 \\
\geq\ & \varphi F(-Ht_0 + \lambda(\sigma-1)M_1^{\sigma-1}t_0 - 1) \\
& + \frac{2\beta t_0^2}{n}\Big\{(y-z)^2 - \Big(\frac{n\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + nK\Big)y - \frac{n}{\beta}c_1R^{-1}y^{\frac{1}{2}}(\beta y - z) \\
& + \frac{\beta-\sigma}{\beta}nM_1^{\sigma-1}\gamma y^{\frac{1}{2}}\Big\} - M_1^{\sigma-1}\theta t_0^2 - \lambda(1-\sigma)M_1^{\sigma-1}t_0^2.
\end{aligned}
$$

Following the method in [5, pp. 161–162], we know

$$
\begin{aligned}
& (y-z)^2 - nc_1R^{-1}y^{\frac{1}{2}}(y-\alpha z) - n\widetilde{K}y - n(\alpha-1)\gamma y^{\frac{1}{2}} \\
\geq\ & \alpha^{-2}(y-\alpha z)^2 - \frac{n^2}{8}c_1^2\alpha^2(\alpha-1)^{-1}R^{-2}(y-\alpha z) \\
& - \frac{3}{4}4^{-\frac{1}{3}}n^{\frac{4}{3}}(\gamma^4(\alpha-1)^2\alpha^2\epsilon^{-1})^{\frac{1}{3}} - \frac{n^2}{4}(1-\epsilon)^{-1}(\alpha-1)^{-2}\alpha^2\widetilde{K}^2
\end{aligned}
$$

for any $0 < \epsilon < 1$.

Therefore, in our case, we have

$$
\begin{aligned}
& (y-z)^2 - \Big(\frac{n\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + nK\Big)y - \frac{n}{\beta}c_1R^{-1}y^{\frac{1}{2}}(\beta y - z) + \frac{\beta-\sigma}{\beta}nM_1^{\sigma-1}\gamma y^{\frac{1}{2}} \\
\geq\ & (y-z)^2 - nc_1R^{-1}y^{\frac{1}{2}}\Big(y - \frac{1}{\beta}z\Big) - n\Big(\frac{\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + K\Big)y - n\frac{\sigma-\beta}{\beta}M_1^{\sigma-1}\gamma y^{\frac{1}{2}} \\
\geq\ & (y-z)^2 - nc_1R^{-1}y^{\frac{1}{2}}\Big(y - \frac{1}{\beta}z\Big) - n\Big(\frac{\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + K\Big)y - n\Big(\frac{1}{\beta}-1\Big)M_1^{\sigma-1}\gamma y^{\frac{1}{2}} \\
\geq\ & \Big(\frac{1}{\beta}\Big)^{-2}\Big(y - \frac{1}{\beta}z\Big)^2 - \frac{n^2}{8}c_1^2\Big(\frac{1}{\beta}\Big)^2\Big(\frac{1}{\beta}-1\Big)^{-1}R^{-2}\Big(y - \frac{1}{\beta}z\Big) \\
& - \frac{3}{4}4^{-\frac{1}{3}}n^{\frac{4}{3}}\Big[(M_1^{\sigma-1}\gamma)^4\Big(\frac{1}{\beta}-1\Big)^2\Big(\frac{1}{\beta}\Big)^2\epsilon^{-1}\Big]^{\frac{1}{3}} \\
& - \frac{n^2}{4}(1-\epsilon)^{-1}\Big(\frac{1}{\beta}-1\Big)^{-2}\Big(\frac{1}{\beta}\Big)^2\Big(\frac{\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + K\Big)^2.
\end{aligned}
$$

Noticing that

$$
\beta y - z = \frac{\varphi F}{t},
$$

we obtain

$$
\begin{aligned}
0 \geq\ & \varphi F(-Ht_0 + \lambda(\sigma-1)M_1^{\sigma-1}t_0 - 1) + \frac{2\beta}{n}(\varphi F)^2 - \frac{nc_1^2t_0}{4\beta(1-\beta)R^2}(\varphi F) \\
& + \frac{2\beta t_0^2}{n}\Big\{ - \frac{3}{4}4^{-\frac{1}{3}}n^{\frac{4}{3}}\Big[(M_1^{\sigma-1}\gamma)^4\Big(\frac{1}{\beta}-1\Big)^2\Big(\frac{1}{\beta}\Big)^2\epsilon^{-1}\Big]^{\frac{1}{3}} \\
& - \frac{n^2}{4}(1-\epsilon)^{-1}\Big(\frac{1}{\beta}-1\Big)^{-2}\Big(\frac{1}{\beta}\Big)^2\Big(\frac{\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta} + K\Big)^2\Big\} \\
=\ & \frac{2\beta}{n}(\varphi F)^2 - \phi(\varphi F) - t_0^2\psi,
\end{aligned}
$$

where

$$\phi = Ht_0 - \lambda(\sigma-1)M_1^{\sigma-1}t_0 + \frac{nc_1^2 t_0}{4\beta(1-\beta)R^2} + 1$$

and

$$\begin{aligned}\psi = &\frac{3}{4}4^{-\frac{1}{3}}n^{\frac{4}{3}}\Big[(M_1^{\sigma-1}\gamma)^4\Big(\frac{1}{\beta}-1\Big)^2\Big(\frac{1}{\beta}\Big)^2\epsilon^{-1}\Big]^{\frac{1}{3}}\\ &+\frac{n^2}{4}(1-\epsilon)^{-1}\Big(\frac{1}{\beta}-1\Big)^{-2}\Big(\frac{1}{\beta}\Big)^2\Big(\frac{\lambda(\sigma-1)(\beta-\sigma)M_1^{\sigma-1}}{2\beta}+K\Big)^2\\ &+M_1^{\sigma-1}\theta+\lambda(1-\sigma)M_1^{\sigma-1}.\end{aligned}$$

From the inequality $Ax^2 - 2Bx \le C$, we have $x \le \frac{2B}{A} + \sqrt{\frac{C}{A}}$.

We can get

$$(\varphi F)(x_0,t_0) \le \frac{n}{2\beta}\phi + \Big(\frac{n}{2\beta}\psi\Big)^{\frac{1}{2}}.$$

Notice that for all $t \in [0,T]$,

$$\sup_{B_p(2R)} T[\beta|\nabla w|^2 - b\mathrm{e}^{(\sigma-1)w} - w_t] \le (\varphi F)(x_0,t_0).$$

We complete the proof of Theorem 1.3.

**Proof of Theorem 1.4** For any points $(x_1,t_1)$ and $(x_2,t_2)$ on $M\times[0,+\infty)$ with $0<t_1<t_2$, we take a curve $\gamma(t)$ parameterized with $\gamma(t_1)=x_1$ and $\gamma(t_2)=x_2$. One gets from Corollary 1.1 that

$$\begin{aligned}\log u(x_2,t_2)-\log u(x_1,t_1) &= \int_{t_1}^{t_2}((\log u)_t + \langle\nabla\log u,\dot\gamma\rangle)\mathrm{d}t\\ &\ge \int_{t_1}^{t_2}\Big(\beta|\nabla\log u|^2 - \frac{n}{2\beta t} - bu^{\sigma-1} - \widetilde{N} - |\nabla\log u||\dot\gamma|\Big)\mathrm{d}t\\ &\ge -\int_{t_1}^{t_2}\Big(\frac{1}{4\beta}|\dot\gamma|^2 + \frac{n}{2\beta t} + bu^{\sigma-1} + \widetilde{N}\Big)\mathrm{d}t\\ &\ge -\Big(\int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\mathrm{d}t + \log\Big(\frac{t_2}{t_1}\Big)^{\frac{n}{2\beta}} + \widetilde{N}(t_2-t_1)\Big),\end{aligned}$$

which means that

$$\log\frac{u(x_1,t_1)}{u(x_2,t_2)} \le \int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\mathrm{d}t + \log\Big(\frac{t_2}{t_1}\Big)^{\frac{n}{2\beta}} + \widetilde{N}(t_2-t_1).$$

Therefore,

$$u(x_1,t_1) \le u(x_2,t_2)\Big(\frac{t_2}{t_1}\Big)^{\frac{n}{2\beta}}\mathrm{e}^{\phi(x_1,x_2,t_1,t_2)+\widetilde{N}(t_2-t_1)},$$

where $\phi(x_1,x_2,t_1,t_2) = \inf_\gamma \int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\mathrm{d}t$ and

$$\begin{aligned}\widetilde{N} = &\Big\{\frac{n^3}{8\beta(1-\beta)^2(1-\epsilon)}\Big(\frac{M_1^{\sigma-1}\lambda(\sigma-1)(\beta-\sigma)}{2\beta}+K\Big)^2\\ &+\frac{n}{2\beta}[\lambda(1-\sigma)M_1^{\sigma-1}]\Big\}^{\frac{1}{2}} + \frac{n}{2\beta}\lambda(\sigma-1)M_1^{\sigma-1}.\end{aligned}$$

**Acknowledgements** The authors would like to thank the editors and the anonymous referees for their valuable comments and helpful suggestions that helped to improve the quality of this paper. Moreover, the authors would like to thank their supervisor Professor Kefeng Liu for his constant encouragement and help.

## References

[1] Aubin, T., Nonlinear Analysis on Manifolds, Springer-Verlag, New York, 1982.

[2] Cheng, S. Y. and Yau, S. T., Differential equations on Riemannian manifolds and their geometric applications, *Commun. Pure Appl. Math.*, **28**, 1975, 333–354.

[3] Hamilton, R., A matrix Harnack estimate for the heat equation, *Comm. Anal. Geom.*, **1**, 1993, 113–126.

[4] Huang, G. Y. and Li, H. Z., Gradient estimates and entropy formulae of porous medium and fast diffusion equations for the Witten Laplacian, 2012. arXiv: math.DG/1203.5482v1

[5] Li, P. and Yau, S. T., On the parabolic kernel of the Schrodinger operator, *Acta Math.*, **156**, 1986, 153–201.

[6] Wu, J. Y., Li-Yau type estimates for a nonlinear parabolic equation on complete manifolds, *J. Math. Anal. Appl.*, **369**, 2010, 400–407.

[7] Zhang, J. and Ma, B. Q., Gradient estimates for a nonlinear equation $\triangle_f u + cu^{-\alpha} = 0$ on complete noncompact manifolds, *Comm. Math.*, **19**, 2011, 73–84.

[8] Zhu, X. B., Gradient estimates and Liouville theorems for nonlinear parabolic equations on noncompact Riemannian manifolds, *Nonlinear Anal.*, **74**, 2011, 5141–5146.