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# A Note on Schwarz-Pick Lemma for Bounded Complex-Valued Harmonic Functions in the Unit Ball of $\mathbb{R}^n$*

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**Abstract** In this paper, the authors prove a Schwarz-Pick lemma for bounded complex-valued harmonic functions in the unit ball of $\mathbb{R}^n$.

**Keywords** Harmonic functions, Schwarz-Pick lemma, Unit ball
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## 1 Introduction

This paper is a note about Chen's paper (see [1]). Using the same method as in [1], we obtain Theorem 1.1, which extends the Schwarz-Pick lemma (see [1]) for planar harmonic mappings to bounded complex-valued harmonic functions in the unit ball of $\mathbb{R}^n$. In addition, motivated by [1] and this paper, we consider a Schwarz lemma for harmonic mappings between real unit balls in another paper. Now we introduce some denotations and the background.

Let $n$ be a positive integer greater than 1. $\mathbb{R}^n$ is the real space of dimension $n$. For $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$, let $|x| = (|x_1|^2 + \cdots + |x_n|^2)^{\frac{1}{2}}$. Let $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ be the unit ball of $\mathbb{R}^n$. The unit sphere, i.e., the boundary of $\mathbb{B}^n$ is denoted by $S$; the normalized surface-area measure on $S$ is denoted by $\sigma$ (so that $\sigma(S) = 1$). Let $S^+$ denote the northern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n > 0\}$ and $S^-$ denote the southern hemisphere $\{x = (x_1, \cdots, x_n) \in S : x_n < 0\}$. $N = (0, \cdots, 0, 1)$ denotes the north pole of $S$. $B_r = \{x \in \mathbb{R}^n : |x| < r\}$ is the open ball centered at origin of radius $r$; its closure is the closed ball $\overline{B}_r$. A twice continuously differentiable, complex-valued function $F$ defined on $\mathbb{B}^n$ is harmonic on $\mathbb{B}^n$ if and only if $\Delta F \equiv 0$, where $\Delta = D_1^2 + \cdots + D_n^2$ and $D_j^2$ denotes the second partial derivative with respect to the $j$-th coordinate variable $x_j$. By $\Omega_n$, we denote the class of all complex-valued harmonic functions $F(x)$ on $\mathbb{B}^n$ with $|F(x)| < 1$ for $x \in \mathbb{B}^n$.

Let $\mathbb{D}$ be the unit disk in the complex plane $\mathbb{C}$. Denote the disk $\{z \in \mathbb{C} : |z| < r\}$ by $D_r$; its closure is the closed disk $\overline{D}_r$.

For a holomorphic function $f$ from $\mathbb{D}$ into $\mathbb{D}$, the classical Schwarz lemma says that if $f(0) = 0$, then

$$|f(z)| \le |z| \tag{1.1}$$

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holds for $z \in \mathbb{D}$. For $0 < r < 1$, (1.1) may be written in the following form:

$$f(\overline{D}_r) \subset \overline{D}_r. \tag{1.2}$$

So the classical Schwarz lemma can be regarded as concerning the region of $f(\overline{D}_r)$. If the condition $f(0) = 0$ is relaxed, then what the region of $f(\overline{D}_r)$ is? The answer can be found in the classical Schwarz-Pick lemma. By Schwarz-Pick lemma (see [2]), it is known that

$$\frac{|f(z_1) - f(z_2)|}{|1 - \overline{f(z_2)}f(z_1)|} \leq \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|} \tag{1.3}$$

holds for $z_1, z_2 \in \mathbb{D}$. Using the notations

$$d_p(z_1, z_2) = \frac{|z_1 - z_2|}{|1 - \overline{z_2}z_1|}$$

for the pseudo-distance between $z_1, z_2 \in \mathbb{D}$, we know that

$$d_p(f(z_1), f(z_2)) \leq d_p(z_1, z_2) \tag{1.4}$$

for $z_1, z_2 \in \mathbb{D}$ by (1.3). Denote $\overline{\triangle}(z, r) = \{\zeta \in \mathbb{D} : d_p(\zeta, z) \leq r, z \in \mathbb{D}, 0 < r < 1\}$ for the closed pseudo-disk with center at $z$ and pseudo-radius $r$. Then (1.4) may be written in the following form:

$$f(\overline{\triangle}(z, r)) \subset \overline{\triangle}(f(z), r)$$

for $z \in \mathbb{D}$ and $0 < r < 1$. Note that $\overline{\triangle}(0, r) = \overline{D}_r$. So for $f$ without the assumption $f(0) = 0$, we know

$$f(\overline{D}_r) \subset \overline{\triangle}(f(0), r). \tag{1.5}$$

When $f(0) = 0$, (1.5) becomes (1.2).

For a complex-valued harmonic function $F$ on $\mathbb{D}$ such that $F(\mathbb{D}) \subset \mathbb{D}$ and $F(0) = 0$, it is known (see [3]) that

$$|F(z)| \leq \frac{4}{\pi} \arctan |z| \tag{1.6}$$

holds for $z \in \mathbb{D}$. For $0 < r < 1$, (1.6) may be written in the following form:

$$F(\overline{D}_r) \subset \overline{D}_{\frac{4}{\pi} \arctan r}. \tag{1.7}$$

If the condition $F(0) = 0$ is relaxed, then what the region of $F(\overline{D}_r)$ is? Unfortunately, the composition $f \circ F$ of a harmonic function $F$ and a holomorphic function $f$ do not need to be harmonic, so it is a serious problem to seek the estimate corresponding to (1.5) for a harmonic function $F$ without the assumption $F(0) = 0$. Fortunately, Chen resolved this problem in [1]. In [1], for any $0 < r < 1$ and $0 \leq \rho < 1$, the author constructs a closed domain $E_{r,\rho}$, which contains $\rho$ and is symmetric to the real axis, with the following properties: Let $z \in \mathbb{D}$ and $w = \rho e^{i\alpha}$ be given. For every complex-valued harmonic function $F$ with $F(\mathbb{D}) \subset \mathbb{D}$ and $F(z) = w$, the author has $F(\overline{\triangle}(z, r)) \subset e^{i\alpha} E_{r,\rho} = \{e^{i\alpha}\zeta : \zeta \in E_{r,\rho}\}$; conversely, for every $w' \in e^{i\alpha} E_{r,\rho}$, there exists a complex-valued harmonic function $F$ such that $F(\mathbb{D}) \subset \mathbb{D}$, $F(z) = w$ and $F(z') = w'$ for some $z' \in \partial\triangle(z, r)$. Obviously, by Chen's result, we know that for a complex-valued harmonic function $F$ on $\mathbb{D}$ such that $F(\mathbb{D}) \subset \mathbb{D}$ without the assumption $F(0) = 0$, if $F(0) = \rho e^{i\alpha}$, then

$$F(\overline{D}_r) \subset e^{i\alpha} E_{r,\rho}, \tag{1.8}$$

which is sharp. (1.8) is the estimate for complex-valued harmonic functions corresponding to (1.5). Note that a complex-valued harmonic function $F$ on $\mathbb{D}$ such that $F(\mathbb{D}) \subset \mathbb{D}$ can be seen as $F \in \Omega_2$. So it is natural to consider the same problem as in $\Omega_n$.

For $F \in \Omega_n$, the harmonic Schwarz lemma (see [4]) says that if $F(0) = 0$, then

$$|F(x)| \leq U(|x|N) \tag{1.9}$$

holds for $x \in \mathbb{B}^n$, where $U$ is the Poisson integral of the function that equals 1 on $S^+$ and $-1$ on $S^-$. For $0 < r < 1$, (1.9) may be written in the following form:

$$F(\overline{B}_r) \subset \overline{D}_{U(rN)}. \tag{1.10}$$

If the condition $F(0) = 0$ is relaxed, then what the region of $F(\overline{B}_r)$ is? This problem will be solved in this paper.

In this paper, by the same method as in [1], we obtain the following theorem about the region of $F(\overline{B}_r)$. The result is sharp. When $n = 2$, our result is coincident with (1.8). And when $F(0) = 0$, our result is coincident with (1.10). Note that in the following theorem, $E_{r,\rho}$ is defined as (3.1).

**Theorem 1.1** *Let $0 \leq \rho < 1$, $\alpha \in \mathbb{R}$ and $0 < r < 1$ be given. Then for every harmonic function $F$ with $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = \rho e^{i\alpha}$, we have $F(\overline{B}_r) \subset e^{i\alpha} E_{r,\rho} = \{e^{i\alpha}\zeta : \zeta \in E_{r,\rho}\}$; conversely, for every $w' \in e^{i\alpha} E_{r,\rho}$, there exists a harmonic function $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = \rho e^{i\alpha}$ and $F(rN) = w'$.*

The theorem above will be proved in three steps as follows:

Step 1 Find the extremal line of $F(\overline{B}_r)$ in the normal direction of $e^{0i}$, which is related to the value of $F(0)$.

Step 2 Find the extremal line of $F(\overline{B}_r)$ in the normal direction of a given direction. For a given direction of $e^{i\beta}$ with $-\pi \leq \beta \leq \pi$, construct a new harmonic function $F_\beta = e^{-i\beta} F$ through rotating $F(\overline{B}_r)$ by an anti-clockwise rotation of angle $\beta$. Using the result of Step 1, we will have the the extremal line of $F_\beta(\overline{B}_r)$ in the normal direction of $e^{0i}$, which is denoted by $l'_\beta$. Note that $F(\overline{B}_r)$ can be obtained from $F_\beta(\overline{B}_r)$ by a clockwise rotation of angle $\beta$. Then the extremal line of $F(\overline{B}_r)$ in the normal direction of $e^{i\beta}$, which is denoted by $l_\beta$, can be obtained from $l'_\beta$ by a clockwise rotation of angle $\beta$.

Step 3 Using the result of Step 2, we will obtain all the extremal lines of $F(\overline{B}_r)$ in every normal direction, with which we can wrap $F(\overline{B}_r)$ and obtain the region of $F(\overline{B}_r)$.

Step 1 will be solved in Section 2. Step 2 and Step 3 will be solved in Section 3.

## 2 Some Lemmas

In this section, we will introduce some lemmas, which are important for the proof of Theorem 3.1. Lemma 2.1 will be used in Lemma 2.2. Lemma 2.2 will be used in Lemma 2.3. Lemmas 2.3–2.4 will be used in Theorem 3.1.

Now we give Lemma 2.1 first. Lemma 2.1 constructs a bijection $(R, I)$ from $\mathbb{R} \times \mathbb{R}^+$ onto the upper half disk $\{(a, b) : a \in \mathbb{R}, b \in \mathbb{R}, a^2 + b^2 < 1, b > 0\}$, which will be used to construct $u_{a,b,r}$ in Lemma 2.2 for the case $b > 0$.

For $0 < r < 1$, $\mu > 0$ and a real number $\lambda$, define

$$A_{r,\lambda,\mu}(\omega) = \frac{1}{\mu}\Big(\frac{1}{|rN - \omega|^n} - \lambda\Big), \quad \omega \in S \tag{2.1}$$

and

$$R(r,\lambda,\mu)=\int_S \frac{A_{r,\lambda,\mu}(\omega)}{\sqrt{1+A^2_{r,\lambda,\mu}(\omega)}}\mathrm{d}\sigma,\quad I(r,\lambda,\mu)=\int_S \frac{1}{\sqrt{1+A^2_{r,\lambda,\mu}(\omega)}}\mathrm{d}\sigma. \tag{2.2}$$

The idea of the conformation of $A_{r,\lambda,\mu}(\omega)$, $R(r,\lambda,\mu)$ and $I(r,\lambda,\mu)$ originates from the needs of (2.16) and (2.21).

**Lemma 2.1** *Let $0<r<1$ be fixed. Then, there exists a unique pair of real functions $\lambda=\lambda(r,a,b)$ and $\mu=\mu(r,a,b)>0$, defined on the upper half disk $\{(a,b):a^2+b^2<1,\ b>0\}$ and analytic in the real sense, such that $R(r,\lambda(r,a,b),\mu(r,a,b))=a$ and $I(r,\lambda(r,a,b),\mu(r,a,b))=b$ for any point $(a,b)$ in the half disk.*

**Proof** A simple calculation gives

$$\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}=-\frac{1}{\mu}\int_S\frac{1}{(1+A^2_{r,\lambda,\mu}(\omega))^{\frac32}}\mathrm{d}\sigma, \tag{2.3}$$

$$\frac{\partial R(r,\lambda,\mu)}{\partial\mu}=-\frac{1}{\mu}\int_S\frac{A_{r,\lambda,\mu}(\omega)}{(1+A^2_{r,\lambda,\mu}(\omega))^{\frac32}}\mathrm{d}\sigma, \tag{2.4}$$

$$\frac{\partial I(r,\lambda,\mu)}{\partial\lambda}=\frac{1}{\mu}\int_S\frac{A_{r,\lambda,\mu}(\omega)}{(1+A^2_{r,\lambda,\mu}(\omega))^{\frac32}}\mathrm{d}\sigma, \tag{2.5}$$

$$\frac{\partial I(r,\lambda,\mu)}{\partial\mu}=\frac{1}{\mu}\int_S\frac{A^2_{r,\lambda,\mu}(\omega)}{(1+A^2_{r,\lambda,\mu}(\omega))^{\frac32}}\mathrm{d}\sigma. \tag{2.6}$$

It is easy to see that

(i) by (2.3), $\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}<0$ for any $\lambda$ and $\mu>0$, $R(r,\lambda,\mu)$ is strictly decreasing as a function of $\lambda$ for a fixed $\mu$;

(ii) by (2.2), for a fixed $\mu$, $R(r,\lambda,\mu)\to -1$ or 1 according to $\lambda\to+\infty$ or $\lambda\to-\infty$;

(iii) by (2.3)–(2.6) and the convexity of the square function,

$$\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}\frac{\partial I(r,\lambda,\mu)}{\partial\mu}-\frac{\partial R(r,\lambda,\mu)}{\partial\mu}\frac{\partial I(r,\lambda,\mu)}{\partial\lambda}<0$$

for any $\lambda$ and $\mu>0$;

(iiii) by (2.2), $0<I(r,\lambda,\mu)<1$ for any $\lambda$ and $\mu>0$.

By (i) and (ii), we know that for fixed $\mu$, $R(r,\lambda,\mu)$ is strictly decreasing from 1 to $-1$ as $\lambda$ increases from $-\infty$ to $+\infty$. Then for any $-1<a<1$ and fixed $\mu$, there exists a unique real number $\lambda(\mu,a)$ such that

$$R(r,\lambda,\mu)|_{\lambda=\lambda(\mu,a)}=a. \tag{2.7}$$

Further, using the implicit function theorem, we have that the function $\lambda=\lambda(\mu,a)$ defined on $\{(\mu,a):\mu>0,-1<a<1\}$ is a continuous function and

$$\frac{\partial\lambda(\mu,a)}{\partial\mu}=-\Big(\frac{\partial R(r,\lambda,\mu)}{\partial\mu}\Big/\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}\Big)\Big|_{\lambda=\lambda(\mu,a)}.$$

Next, we consider the function $I(r,\lambda(\mu,a),\mu)$ for $\mu>0$.

$$\begin{aligned}&\frac{\partial I(r,\lambda(\mu,a),\mu)}{\partial\mu}\\&=\Big(\frac{\partial I(r,\lambda,\mu)}{\partial\lambda}\frac{\partial\lambda(\mu,a)}{\partial\mu}+\frac{\partial I(r,\lambda,\mu)}{\partial\mu}\Big)\Big|_{\lambda=\lambda(\mu,a)}\\&=\Big(\Big(\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}\frac{\partial I(r,\lambda,\mu)}{\partial\mu}-\frac{\partial R(r,\lambda,\mu)}{\partial\mu}\frac{\partial I(r,\lambda,\mu)}{\partial\lambda}\Big)\Big/\frac{\partial R(r,\lambda,\mu)}{\partial\lambda}\Big)\Big|_{\lambda=\lambda(\mu,a)}.\end{aligned}$$

By (i) and (iii), we have $\frac{\partial I(r,\lambda(\mu,a),\mu)}{\partial\mu}>0$, which shows that $I(r,\lambda(\mu,a),\mu)$ is strictly increasing as a function of $\mu$ on $(0,+\infty)$ for a fixed $a$. Note (iiii). Thus, for a fixed $a$, $I(r,\lambda(\mu,a),\mu)$ has a respectively finite limit as $\mu\to0$ and $\mu\to+\infty$.

For a fixed $a$, we claim that $I(r,\lambda(\mu,a),\mu)\to0$ as $\mu\to0$, and $I(r,\lambda(\mu,a),\mu)\to\sqrt{1-a^2}$ as $\mu\to+\infty$.

As $\mu\to0$, there exists a subsequence $\mu_k\to0$ such that $\lambda(\mu_k,a)$ has a finite limit $t$ or tends to $\infty$. We only need to prove that $I(r,\lambda(\mu_k,a),\mu_k)\to0$ as $k\to\infty$. Since $I(r,\lambda(\mu_k,a),\mu_k)=\int_S\frac{1}{\sqrt{1+A^2_{r,\lambda(\mu_k,a),\mu_k}(\omega)}}\mathrm{d}\sigma$, we only need to prove that $|A_{r,\lambda(\mu_k,a),\mu_k}(\omega)|\to+\infty$ almost everywhere on $S$. Note that

$$|A_{r,\lambda(\mu_k,a),\mu_k}(\omega)|=\frac{1}{\mu_k}\Big|\frac{1}{|rN-\omega|^n}-\lambda(\mu_k,a)\Big|$$

and

$$\frac{1}{(1+r)^n}\le\frac{1}{|rN-\omega|^n}\le\frac{1}{(1-r)^n}.$$

If $\lambda(\mu_k,a)\to t$ as $k\to\infty$, then $\frac{1}{|rN-\omega|^n}-\lambda(\mu_k,a)$ is bounded and $\frac{1}{|rN-\omega|^n}-\lambda(\mu_k,a)\neq0$ almost everywhere on $S$. Thus $|A_{r,\lambda(\mu_k,a),\mu_k}(\omega)|\to+\infty$ almost everywhere on $S$. If $\lambda(\mu_k,a)\to\infty$ as $k\to\infty$, then it is obvious that $|A_{r,\lambda(\mu_k,a),\mu_k}(\omega)|\to+\infty$ uniformly for $\omega\in S$. The first claim is proved.

As $\mu\to+\infty$, $\frac{1}{\mu}\frac{1}{|rN-\omega|^n}\to0$ uniformly for $\omega\in S$. If there exists a subsequence $\mu_k\to+\infty$ such that $\frac{\lambda(\mu_k,a)}{\mu_k}\to\infty$, then $|A_{r,\lambda(\mu_k,a),\mu_k}(\omega)|\to+\infty$ uniformly for $\omega\in S$, and $I(r,\lambda(\mu_k,a),\mu_k)\to0$, a contradiction. This shows that $\frac{\lambda(\mu,a)}{\mu}$ is bounded as $\mu\to+\infty$. Thus there exists a subsequence $\mu_k\to+\infty$ such that $-\frac{\lambda(\mu_k,a)}{\mu_k}$ tends to a finite limit $t$. That is

$$\lim_{k\to\infty}-\frac{\lambda(\mu_k,a)}{\mu_k}=t. \tag{2.8}$$

We only need to prove that $I(r,\lambda(\mu_k,a),\mu_k)\to\sqrt{1-a^2}$ as $k\to\infty$. Let

$$(A(\omega))_k=A_{r,\lambda(\mu_k,a),\mu_k}(\omega).$$

By (2.1), (2.8) and $\mu_k\to+\infty$, we obtain

$$\begin{aligned}\lim_{k\to\infty}\frac{(A(\omega))_k}{\sqrt{1+((A(\omega))_k)^2}}&=\lim_{k\to\infty}\frac{\frac{1}{\mu}{}_k\Big(\frac{1}{|rN-\omega|^n}-\lambda(\mu_k,a)\Big)}{\sqrt{1+\frac{1}{\mu_k^2}\Big(\frac{1}{|rN-\omega|^n}-\lambda(\mu_k,a)\Big)^2}}\\&=\lim_{k\to\infty}\frac{-\frac{\lambda(\mu_k,a)}{\mu_k}}{\sqrt{1+\Big(\frac{\lambda(\mu_k,a)}{\mu_k}\Big)^2}}=\frac{t}{\sqrt{1+t^2}}\end{aligned} \tag{2.9}$$

uniformly for $\omega \in S$, and

$$\lim_{k\to\infty} \frac{1}{\sqrt{1+((A(\omega))_k)^2}} = \lim_{k\to\infty} \frac{1}{\sqrt{1+\frac{1}{\mu_k^2}\Big(\frac{1}{|rN-\omega|^n} - \lambda(\mu_k,a)\Big)^2}}$$
$$= \lim_{k\to\infty} \frac{1}{\sqrt{1+\Big(\frac{\lambda(\mu_k,a)}{\mu_k}\Big)^2}} = \frac{1}{\sqrt{1+t^2}} \tag{2.10}$$

uniformly for $\omega \in S$. By the Lebesgue's dominated convergence theorem, (2.2) and (2.9)–(2.10), we have

$$\lim_{k\to\infty} R(r,\lambda(\mu_k,a),\mu_k) = \lim_{k\to\infty} \int_S \frac{(A(\omega))_k}{\sqrt{1+((A(\omega))_k)^2}} \mathrm{d}\sigma$$
$$= \int_S \lim_{k\to\infty} \frac{(A(\omega))_k}{\sqrt{1+((A(\omega))_k)^2}} \mathrm{d}\sigma$$
$$= \frac{t}{\sqrt{1+t^2}} \tag{2.11}$$

and

$$\lim_{k\to\infty} I(r,\lambda(\mu_k,a),\mu_k) = \lim_{k\to\infty} \int_S \frac{1}{\sqrt{1+((A(\omega))_k)^2}} \mathrm{d}\sigma$$
$$= \int_S \lim_{k\to\infty} \frac{1}{\sqrt{1+((A(\omega))_k)^2}} \mathrm{d}\sigma$$
$$= \frac{1}{\sqrt{1+t^2}}. \tag{2.12}$$

Note that $R(r,\lambda(\mu_k,a),\mu_k) \equiv a$ by (2.7), and $\big(\frac{t}{\sqrt{1+t^2}}\big)^2 + \big(\frac{1}{\sqrt{1+t^2}}\big)^2 = 1$. Then by (2.11) we obtain that $\frac{t}{\sqrt{1+t^2}} = a$ and $\frac{1}{\sqrt{1+t^2}} = \sqrt{1-a^2}$. Consequently by (2.12),

$$\lim_{k\to\infty} I(r,\lambda(\mu_k,a),\mu_k) = \sqrt{1-a^2}.$$

The second claim is proved.

It is proved that $I(r,\lambda(\mu,a),\mu)$ is continuous and strictly increasing from 0 to $\sqrt{1-a^2}$ as $\mu$ increases from 0 to $+\infty$. Thus, for any $0<b<\sqrt{1-a^2}$ and $-1<a<1$, there exists a unique real number $\mu(a,b)$ such that

$$I(r,\lambda(\mu(a,b),a),\mu(a,b)) = b. \tag{2.13}$$

Further, using the implicit function theorem, we have that the function $\mu(a,b)$ defined on $\{(a,b) : a^2+b^2<1, b>0\}$ is a continuous function.

Denote $\lambda(\mu(a,b),a)$ by $\lambda(r,a,b)$. Denote $\mu(a,b)$ by $\mu(r,a,b)$. We have proved that there exists a unique pair of functions $\lambda=\lambda(r,a,b)$ and $\mu=\mu(r,a,b)$ such that

$$R(r,\lambda(r,a,b),\mu(r,a,b)) = a, \quad I(r,\lambda(r,a,b),\mu(r,a,b)) = b$$

on the upper half disk. The real analyticity of $\lambda=\lambda(r,a,b)$ and $\mu=\mu(r,a,b)$ is asserted by the implicit function theorem. The lemma is proved.

Let $a$ and $b$ be two numbers such that $0 \le b < 1$, $-1 < a < 1$ and $a^2 + b^2 < 1$. Let $\mathcal{U}_{a,b}$ denote the class of real-valued functions $u \in L^\infty(S)$ satisfying the following conditions:

$$\|u\|_\infty \le 1, \quad \int_S u(\omega)\mathrm{d}\sigma = a, \quad \int_S \sqrt{1-u^2(\omega)}\mathrm{d}\sigma \ge b. \tag{2.14}$$

Every function $u \in L^\infty(S)$ defines a harmonic function

$$U(x) = \int_S \frac{1-|x|^2}{|x-\omega|^n} u(\omega)\mathrm{d}\sigma \quad \text{for } x \in \mathbb{B}^n.$$

Let $0 < r < 1$ and define a functional $L_r$ on $L^\infty(S)$ by

$$L_r(u) = U(rN) = \int_S \frac{1-r^2}{|rN-\omega|^n} u(\omega)\mathrm{d}\sigma. \tag{2.15}$$

Obviously, $\mathcal{U}_{a,b}$ is a closed set, and $L_r$ is a continuous functional on $\mathcal{U}_{a,b}$. Then there exists an extremal function such that $L_r$ attains its maximum on $\mathcal{U}_{a,b}$ at the extremal function. We will claim in the following lemma that the extremal function is unique. In the proof of the following lemma, we will construct a function $u_0$ first and then prove that $u_0$ is the unique extremal function, which will be denoted by $u_{a,b,r}$.

**Lemma 2.2** *For any $a$, $b$ and $r$ satisfying the above conditions, there exists a unique extremal function $u_{a,b,r} \in \mathcal{U}_{a,b}$ such that $L_r$ attains its maximum on $\mathcal{U}_{a,b}$ at $u_{a,b,r}$.*

**Proof** Let $a$, $b$ and $r$ be fixed. First assume that $b > 0$. From Lemma 2.1, we have $\lambda = \lambda(r,a,b)$ and $\mu = \mu(r,a,b) > 0$ such that $R(r,\lambda,\mu) = a$ and $I(r,\lambda,\mu) = b$. For the need of (2.21), let

$$u_0(\omega) = \frac{A_{r,\lambda,\mu}(\omega)}{\sqrt{1+A^2_{r,\lambda,\mu}(\omega)}}, \tag{2.16}$$

where $A_{r,\lambda,\mu}(\omega)$ is defined as in (2.1). Then $\|u_0\|_\infty < 1$ and by (2.2), and we know

$$\int_S u_0(\omega)\mathrm{d}\sigma = R(r,\lambda,\mu) = a, \quad \int_S \sqrt{1-u_0^2(\omega)}\mathrm{d}\sigma = I(r,\lambda,\mu) = b. \tag{2.17}$$

This means that $u_0 \in \mathcal{U}_{a,b}$.

Let $u \in \mathcal{U}_{a,b}$. By (2.14) and (2.17), we have

$$\lambda \int_S (u_0(\omega) - u(\omega))\mathrm{d}\sigma = 0, \tag{2.18}$$

$$\mu \int_S \Big(\sqrt{1-u_0^2(\omega)} - \sqrt{1-u^2(\omega)}\Big)\mathrm{d}\sigma \le 0. \tag{2.19}$$

By the Taylor formula of the function $\sqrt{1-x^2}$, we have

$$\begin{aligned}\sqrt{1-u^2(\omega)} - \sqrt{1-u_0^2(\omega)} &= \frac{u_0(\omega)(u_0(\omega)-u(\omega))}{\sqrt{1-u_0^2(\omega)}} - \frac{(u_0(\omega)-u(\omega))^2}{2(1-\xi^2)^{\frac{3}{2}}} \\ &\le \frac{u_0(\omega)(u_0(\omega)-u(\omega))}{\sqrt{1-u_0^2(\omega)}},\end{aligned} \tag{2.20}$$

where $\xi$ is a real number between $u_0(\omega)$ and $u(\omega)$. By (2.1) and (2.16), we have

$$\frac{1}{|rN-\omega|^n}-\lambda-\frac{\mu u_0(\omega)}{\sqrt{1-u_0^2(\omega)}}=0. \tag{2.21}$$

Then by (2.15) and (2.18)–(2.21), we obtain that

$$\begin{aligned}
&\frac{L_r(u_0)-L_r(u)}{1-r^2}=\int_S\frac{u_0(\omega)-u(\omega)}{|rN-\omega|^n}\mathrm{d}\sigma\\
&\geq\int_S\frac{u_0(\omega)-u(\omega)}{|rN-\omega|^n}\mathrm{d}\sigma-\lambda\int_S(u_0(\omega)-u(\omega))\mathrm{d}\sigma-\mu\int_S\big(\sqrt{1-u^2(\omega)}-\sqrt{1-u_0^2(\omega)}\big)\mathrm{d}\sigma\\
&=\int_S\frac{u_0(\omega)-u(\omega)}{|rN-\omega|^n}\mathrm{d}\sigma-\lambda\int_S(u_0(\omega)-u(\omega))\mathrm{d}\sigma\\
&\quad-\mu\int_S\frac{u_0(\omega)(u_0(\omega)-u(\omega))}{\sqrt{1-u_0^2(\omega)}}\mathrm{d}\sigma+\mu\int_S\frac{(u_0(\omega)-u(\omega))^2}{2(1-\xi^2)^{\frac{3}{2}}}\mathrm{d}\sigma\\
&\geq\int_S\frac{u_0(\omega)-u(\omega)}{|rN-\omega|^n}\mathrm{d}\sigma-\lambda\int_S(u_0(\omega)-u(\omega))\mathrm{d}\sigma-\mu\int_S\frac{u_0(\omega)(u_0(\omega)-u(\omega))}{\sqrt{1-u_0^2(\omega)}}\mathrm{d}\sigma\\
&=\int_S(u_0(\omega)-u(\omega))\Big(\frac{1}{|rN-\omega|^n}-\lambda-\frac{\mu u_0(\omega)}{\sqrt{1-u_0^2(\omega)}}\Big)\mathrm{d}\sigma\\
&=0.
\end{aligned}$$

Thus $L_r(u_0)\geq L_r(u)$ with equality if and only if $\mu\int_S\frac{(u_0(\omega)-u(\omega))^2}{2(1-\xi^2)^{\frac{3}{2}}}\mathrm{d}\sigma=0$. Therefore $L_r(u_0)\geq L_r(u)$ with equality if and only if $u(\omega)=u_0(\omega)$ almost everywhere. This shows that $u_0(\omega)$ is the unique extremal function, which will be denoted by $u_{a,b,r}(\omega)$.

Next we consider the case $b=0$. For a real number $d$, let

$$S_d=\{x\in S:|N-x|=d\}, \tag{2.22}$$
$$S_d^+=\{x\in S:|N-x|<d\}, \tag{2.23}$$
$$S_d^-=\{x\in S:|N-x|>d\}. \tag{2.24}$$

For a fixed real number $a$ such that $-1<a<1$, there exists a unique real number $d_a$ such that $\sigma(S_{d_a}^+)=\frac{1+a}{2}$ and $\sigma(S_{d_a}^-)=\frac{1-a}{2}$. Let

$$u_0(\omega)=\begin{cases}1, & \omega\in S_{d_a}^+,\\ 0, & \omega\in S_{d_a},\\ -1, & \omega\in S_{d_a}^-.\end{cases} \tag{2.25}$$

We want to prove that $u_0$ is just the unique extremal function, which will be denoted by $u_{a,0,r}(\omega)$.

It is obvious that $u_0\in\mathcal{U}_{a,0}$. Let $u\in\mathcal{U}_{a,0}$. By (2.14) and (2.25), we have

$$\int_S(u_0(\omega)-u(\omega))\mathrm{d}\sigma=0, \tag{2.26}$$
$$u_0(\omega)-u(\omega)\geq 0\quad\text{for }\omega\in S_{d_a}^+, \tag{2.27}$$
$$u_0(\omega)-u(\omega)\leq 0\quad\text{for }\omega\in S_{d_a}^-. \tag{2.28}$$

Let

$$J_a=|rN-x_0|,\quad\text{where }x_0\in S_{d_a}. \tag{2.29}$$

Note that

$$|rN-\omega|<J_a \quad \text{for } \omega\in S^+_{d_a}, \tag{2.30}$$
$$|rN-\omega|>J_a \quad \text{for } \omega\in S^-_{d_a}. \tag{2.31}$$

Then by (2.15) and (2.26)–(2.31), we obtain that

$$\begin{aligned}
&\frac{L_r(u_0)-L_r(u)}{1-r^2}\\
&=\int_S \frac{u_0(\omega)-u(\omega)}{|rN-\omega|^n}\mathrm{d}\sigma\\
&=\int_S \Big(\frac{1}{|rN-\omega|^n}-\frac{1}{J_a^n}\Big)(u_0(\omega)-u(\omega))\mathrm{d}\sigma\\
&=\int_{S^+_{d_a}} \Big(\frac{1}{|rN-\omega|^n}-\frac{1}{J_a^n}\Big)(u_0(\omega)-u(\omega))\mathrm{d}\sigma+\int_{S^-_{d_a}} \Big(\frac{1}{|rN-\omega|^n}-\frac{1}{J_a^n}\Big)(u_0(\omega)-u(\omega))\mathrm{d}\sigma\\
&\ge 0.
\end{aligned}$$

Thus $L_r(u_0)\ge L_r(u)$ with equality if and only if $u(\omega)=u_0(\omega)$ almost everywhere. The lemma is proved.

Let $a$ and $b$ be two real numbers with $a^2+b^2<1$ and $0<r<1$. If $b\ge 0$, $u_{a,b,r}$ has been defined in Lemma 2.2. Now, define

$$v_{a,b,r}(\omega)=\sqrt{1-u^2_{a,b,r}(\omega)} \quad \text{for } \omega\in S \tag{2.32}$$

and

$$U_{a,b,r}(x)=\int_S \frac{1-|x|^2}{|x-\omega|^n}u_{a,b,r}(\omega)\mathrm{d}\sigma, \tag{2.33}$$

$$V_{a,b,r}(x)=\int_S \frac{1-|x|^2}{|x-\omega|^n}v_{a,b,r}(\omega)\mathrm{d}\sigma. \tag{2.34}$$

For $b<0$, let

$$U_{a,b,r}(x)=U_{a,-b,r}(x), \quad V_{a,b,r}(x)=-V_{a,-b,r}(x). \tag{2.35}$$

Then for any $a\in\mathbb{R}$, $b\in\mathbb{R}$ and $a^2+b^2<1$, let

$$F_{a,b,r}(x)=U_{a,b,r}(x)+\mathrm{i}V_{a,b,r}(x) \quad \text{for } x\in\mathbb{B}^n. \tag{2.36}$$

The harmonic function $F_{a,b,r}(x)=U_{a,b,r}(x)+\mathrm{i}V_{a,b,r}(x)$ satisfies $F_{a,b,r}(0)=a+b\mathrm{i}$ and $F_{a,b,r}(\mathbb{B}^n)\subset\mathbb{D}$, since we will show that $|U_{a,b,r}(x)|^2+|V_{a,b,r}(x)|^2<1$. By the convexity of the square function,

$$|U_{a,b,r}(x)|^2+|V_{a,b,r}(x)|^2\le\int_S \frac{1-|x|^2}{|x-\omega|^n}(u^2_{a,b,r}(\omega)+v^2_{a,b,r}(\omega))\mathrm{d}\sigma=1$$

with equality if and only if $u_{a,b,r}(\omega)$ and $v_{a,b,r}(\omega)$ are constants almost everywhere on $S$. However $u_{a,b,r}(\omega)$ is not possible to be a constant almost everywhere on $S$. Thus $|U_{a,b,r}(x)|^2+|V_{a,b,r}(x)|^2<1$.

The functions $F_{a,b,r}$ are the extremal functions in the following lemma.

**Lemma 2.3** *Let* $F(x) = U(x) + \mathrm{i}V(x)$ *be a harmonic function such that* $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = a + b\mathrm{i}$. *Then, for* $0 < r < 1$ *and* $\omega \in S$,

$$U(r\omega) \le U_{a,b,r}(rN)$$

*with equality at some point* $r\omega$ *if and only if* $F(x) = F_{a,b,r}(xA)$, *where* $A$ *is an orthogonal matrix such that* $r\omega A = rN$, $U_{a,b,r}$ *is defined as in* (2.33) *and* (2.35), *and* $F_{a,b,r}$ *is defined as in* (2.36). *Further,* $U(x) < U_{a,b,r}(rN)$ *for* $|x| < r$.

**Proof** Step 1 First, the case $r\omega = rN$ will be proved. Let $0 < \widetilde{r} < 1$ be fixed. Construct a function

$$G(x) = F(\widetilde{r}x) \quad \text{for } x \in \overline{\mathbb{B}}^n.$$

$G(x)$ is harmonic on $\overline{\mathbb{B}}^n$ and $G(0) = a + b\mathrm{i}$. Let $G(x) = u(x) + \mathrm{i}v(x)$. Then

$$\|u\|_\infty \le 1, \quad \int_S u(\omega)\mathrm{d}\sigma = a, \quad \int_S \sqrt{1 - u^2(\omega)}\mathrm{d}\sigma \ge \int_S |v(\omega)|\mathrm{d}\sigma \ge \Big| \int_S v(\omega)\mathrm{d}\sigma \Big| = |b|. \tag{2.37}$$

So by (2.14) we know that $u \in \mathcal{U}_{a,|b|}$ and by Lemma 2.2, we have $u(rN) \le U_{a,b,r}(rN)$ with equality if and only if $u(\omega) = u_{a,|b|,r}(\omega)$ almost everywhere on $S$. For $u_{a,|b|,r}(\omega)$, by (2.17) and (2.25), we have

$$\int_S \sqrt{1 - u^2_{a,|b|,r}(\omega)}\mathrm{d}\sigma = |b|. \tag{2.38}$$

If $u(\omega) = u_{a,|b|,r}(\omega)$ almost everywhere on $S$, then by (2.33) and (2.35), we have

$$u(x) = U_{a,|b|,r}(x) = U_{a,b,r}(x) \ \text{ for } x \in \mathbb{B}^n;$$

and by (2.32), we have

$$v_{a,|b|,r}(\omega) = \sqrt{1 - u^2_{a,|b|,r}(\omega)} = \sqrt{1 - u^2(\omega)}. \tag{2.39}$$

Note that by (2.37)–(2.39), we have

$$|b| = \int_S v_{a,|b|,r}(\omega)\mathrm{d}\sigma \ge \int_S |v(\omega)|\mathrm{d}\sigma \ge \Big| \int_S v(\omega)\mathrm{d}\sigma \Big| = |b|.$$

Then

$$v(\omega) = v_{a,|b|,r}(\omega) \quad \text{almost everywhere on } S \text{ when } b \ge 0,$$
$$v(\omega) = -v_{a,|b|,r}(\omega) \quad \text{almost everywhere on } S \text{ when } b < 0.$$

So

$$v(x) = V_{a,b,r}(x) \quad \text{for } x \in \mathbb{B}^n.$$

For $G(x) = u(x) + \mathrm{i}v(x)$, it is proved that $u(rN) \le U_{a,b,r}(rN)$ with equality if and only if $G(x) = F_{a,b,r}(x)$. Now let $\widetilde{r} \to 1$. Note that

$$\lim_{\widetilde{r}\to 1} G(x) = \lim_{\widetilde{r}\to 1} F(\widetilde{r}x) = F(x), \quad \lim_{\widetilde{r}\to 1} u(rN) = U(rN).$$

Then by the result for $G(x)$, we have $U(rN) \le U_{a,b,r}(rN)$ with equality if and only if $F(x) = F_{a,b,r}(x)$.

Step 2 Now we prove the case $r\omega \neq rN$. Construct a function

$$\widetilde{F}(x) = F(xA^{-1}) \quad \text{for } x \in \mathbb{B}^n,$$

where $A$ is an orthogonal matrix such that $r\omega A = rN$ and $A^{-1}$ is the inverse matrix of $A$. By [4], we know that $\widetilde{F}(x)$ is also a harmonic function. Let $\widetilde{F}(x) = \widetilde{U}(x) + \mathrm{i}\widetilde{V}(x)$. Note that $\widetilde{F}(0) = a + b\mathrm{i}$. Then by the result of Step 1, we have $\widetilde{U}(rN) \leq U_{a,b,r}(rN)$ with equality if and only if $\widetilde{F}(x) = F_{a,b,r}(x)$. Note that $\widetilde{U}(rN) = U(rNA^{-1}) = U(r\omega)$ and $\widetilde{F}(x) = F(xA^{-1})$. Thus $U(r\omega) \leq U_{a,b,r}(rN)$ with equality if and only if $F(xA^{-1}) = F_{a,b,r}(x)$. It is just that $U(r\omega) \leq U_{a,b,r}(rN)$ with equality if and only if $F(x) = F_{a,b,r}(xA)$.

Step 3 We will show that $U(x) < U_{a,b,r}(rN)$ for $|x| < r$. By the result of Step 2 and the maximum principle, we have $U(x) \leq U_{a,b,r}(rN)$ for $|x| \leq r$. If the equality holds for some $x_0$ with $|x_0| < r$, then $U(x)$ must be equal to $U_{a,b,r}(rN)$ identically for $|x| \leq r$. Note that if $U(rN) = U_{a,b,r}(rN)$, then by the result of Step 1, we have $U(x) = U_{a,b,r}(x)$. Thus $U_{a,b,r}(x) \equiv U_{a,b,r}(rN)$ for $|x| \leq r$. However, it is impossible since $U_{a,b,r}$ is not a constant. The proof of the lemma is complete.

**Lemma 2.4** *For fixed $0 < r < 1$ and $x \in \mathbb{B}^n$, $F_{a,b,r}(x)$ is defined as in (2.36). Then $F_{a,b,r}(x)$, as a function of variables $a$ and $b$, is analytic in the real sense on the open half disk $\{(a,b) : b > 0, a^2 + b^2 < 1\}$ and is continuous to the real diameter.*

**Proof** Let $0 < r < 1$ and $x \in \mathbb{B}^n$ be fixed. It is obvious that $F_{a,b,r}(x)$ is analytic in the real sense on the open half disk, since it is determined there by the functions $\lambda(r,a,b)$ and $\mu(r,a,b)$ formulated in Lemma 2.1, which are analytic in the real sense on the open half disk $\{(a,b) : b > 0, a^2 + b^2 < 1\}$. We only need to prove that $F_{a,b,r}(x)$ is continuous at the points of the real diameter. Note (2.36). Then we only need to prove that $U_{a,b,r}(x)$ and $V_{a,b,r}(x)$ are continuous at the points of the real diameter.

Let $-1 < a_0 < 1$ be given. We want to prove that $U_{a,b,r}(x)$ and $V_{a,b,r}(x)$ are continuous at $(a_0, 0)$. It is just to prove that $U_{a,b,r}(x) \to U_{a_0,0,r}(x)$ and $V_{a,b,r}(x) \to V_{a_0,0,r}(x)$ as $(a,b) \to (a_0, 0)$.

Step 1 For the case $(a,b) \to (a_0, 0)$ with $b = 0$, by (2.33)–(2.34), we only need to prove $u_{a,0,r}(\omega) \to u_{a_0,0,r}(\omega)$ almost everywhere on $S$ as $(a,0) \to (a_0,0)$. Recall that

$$u_{a,0,r}(\omega) = \begin{cases} 1, & \omega \in S_{d_a}^+, \\ 0, & \omega \in S_{d_a}, \\ -1, & \omega \in S_{d_a}^-, \end{cases}$$

where $S_{d_a}^+$, $S_{d_a}$ and $S_{d_a}^-$ are defined as in (2.22)–(2.24). This shows that $u_{a,0,r}(\omega) \to u_{a_0,0,r}(\omega)$ almost everywhere on $S$ as $(a,0) \to (a_0,0)$.

Step 2 For the case $(a,b) \to (a_0,0)$ with $b > 0$, by (2.33)–(2.34), we only need to prove $u_{a,b,r}(\omega) \to u_{a_0,0,r}(\omega)$ for any $\omega \in S$ as $(a,b) \to (a_0,0)$ with $b > 0$.

First we want to prove that $\mu(r,a,b) \to 0$ as $(a,b) \to (a_0,0)$ with $b > 0$, where $\mu(r,a,b)$ is defined as $\mu(a,b)$ in (2.13). Assume that $\mu(r,a,b) \nrightarrow 0$ as $(a,b) \to (a_0,0)$ with $b > 0$. Then there exists a sequence $(a_k,b_k) \to (a_0,0)$ with $b_k > 0$ such that $\mu_k = \mu(r,a_k,b_k)$ has a positive lower bound since $\mu(r,a,b) > 0$. Then by (2.2) and (2.13), we have

$$\int_S \Big(1 + \frac{1}{\mu_k^2}\Big(\frac{1}{|rN - \omega|^n} - \lambda_k\Big)^2\Big)^{-\frac{1}{2}} \mathrm{d}\sigma = I(r, \lambda_k, \mu_k) = b_k \to 0,$$

where $\lambda_k = \lambda(r, a_k, b_k)$, $\lambda(r, a, b)$ is defined as $\lambda(\mu(a, b), a)$ in (2.13). Thus $\lambda_k \to \infty$. Assume that $\lambda_k \to +\infty$. Then by (2.1) and (2.16)–(2.17), we obtain

$$u_{a_k,b_k,r}(\omega) = \frac{\dfrac{1}{\mu_k}\Big(\dfrac{1}{|rN-\omega|^n} - \lambda_k\Big)}{\Big(1 + \dfrac{1}{\mu_k^2}\Big(\dfrac{1}{|rN-\omega|^n} - \lambda_k\Big)^2\Big)^{\frac{1}{2}}} \to -1$$

uniformly for $\omega \in S$, and $a_k \to -1$, a contradiction.

Now, we want to prove that

$$\lambda(r, a, b) \to \lambda_0 = \frac{1}{J_{a_0}^n}$$

as $(a, b) \to (a_0, 0)$ with $b > 0$, where $J_{a_0}^n$ is defined as in (2.29). On the contrary, assume that $\lambda(r, a, b) \nrightarrow \lambda_0$ as $(a, b) \to (a_0, 0)$ with $b > 0$. Then there is a sequence $(a_k, b_k) \to (a_0, 0)$ with $b_k > 0$ such that $\lambda_k = \lambda(r, a_k, b_k) \to \lambda' \neq \lambda_0$. If $\lambda' = \infty$, then, as above, $|a_k| \to 1$, a contradiction. In the case that $\lambda'$ is finite, by (2.1) and (2.16)–(2.17), we have

$$u_{a_k,b_k,r}(\omega) = \frac{\dfrac{1}{|rN-\omega|^n} - \lambda_k}{\Big(\mu_k^2\Big(\dfrac{1}{|rN-\omega|^n} - \lambda_k\Big)^2\Big)^{\frac{1}{2}}} \to \mathrm{sgn}\Big\{\frac{1}{|rN-\omega|^n} - \lambda'\Big\},$$

$$\begin{aligned} a_k &= \int_S u_{a_k,b_k,r}(\omega)\mathrm{d}\sigma \to \int_S \mathrm{sgn}\Big\{\frac{1}{|rN-\omega|^n} - \lambda'\Big\}\mathrm{d}\sigma \\ &= \begin{cases} -1, & \lambda' \geq \dfrac{1}{(1-r)^n}, \\ 1, & \lambda' \leq \dfrac{1}{(1+r)^n}, \\ a', & \lambda' = \dfrac{1}{J_{a'}^n},\ -1 < a' < 1,\ a' \neq a_0. \end{cases} \end{aligned}$$

This contradicts $a_k \to a_0$.

It is proved that $\mu(r, a, b) \to 0$ and $\lambda(r, a, b) \to \lambda_0$ as $(a, b) \to (a_0, 0)$ with $b > 0$. Thus,

$$u_{a,b,r}(\omega) \to \mathrm{sgn}\Big\{\frac{1}{|rN-\omega|^n} - \lambda_0\Big\} = u_{a_0,0,r}(\omega).$$

Step 3 For the case that $(a, b) \to (a_0, 0)$ with $b < 0$, by the result of Step 2, we know that $U_{a,-b,r}(x) \to U_{a_0,0,r}(x)$ and $V_{a,-b,r}(x) \to V_{a_0,0,r}(x)$ as $(a, -b) \to (a_0, 0)$ with $-b > 0$. Note that $U_{a,-b,r}(x) = U_{a,b,r}(x)$, $V_{a,-b,r}(x) = -V_{a,b,r}(x)$ and $V_{a_0,0,r}(x) \equiv 0$. Then we have $U_{a,b,r}(x) \to U_{a_0,0,r}(x)$ and $V_{a,b,r}(x) \to V_{a_0,0,r}(x) = 0$ as $(a, b) \to (a_0, 0)$ with $b < 0$.

It is proved that $U_{a,b,r}(x)$ and $V_{a,b,r}(x)$ are continuous at $(a_0, 0)$. The lemma is proved.

## 3 Main Results

For $-\pi \leq \beta \leq \pi$ and a real number $\delta$, denote the straight line $l(\beta, \delta)$ and the closed half plane $P(\beta, \delta)$ by

$$l(\beta, \delta) = \{w = u + \mathrm{i}v : \mathrm{Re}\{w\mathrm{e}^{-\mathrm{i}\beta}\} = u\cos\beta + v\sin\beta = \delta\}$$

and

$$P(\beta, \delta) = \{w = u + \mathrm{i}v : \mathrm{Re}\{w\mathrm{e}^{-\mathrm{i}\beta}\} = u\cos\beta + v\sin\beta \leq \delta\},$$

respectively.

**Theorem 3.1** *Let $0 < r < 1$ and $0 \le \rho < 1$. Denote*

$$P_\beta = P(\beta, U_{\rho\cos\beta,-\rho\sin\beta,r}(rN)), \quad l_\beta = l(\beta, U_{\rho\cos\beta,-\rho\sin\beta,r}(rN))),$$

*and define*

$$E_{r,\rho} = \bigcap_{-\pi\le\beta\le\pi} P_\beta, \tag{3.1}$$
$$\Gamma_{r,\rho} = \{w : w = f_{r,\rho}(\beta) = \mathrm{e}^{\mathrm{i}\beta} F_{\rho\cos\beta,-\rho\sin\beta,r}(rN), -\pi \le \beta \le \pi\},$$

*where $U_{\rho\cos\beta,-\rho\sin\beta,r}$ is defined as in* (2.33) *and* (2.35)*, and $F_{\rho\cos\beta,-\rho\sin\beta,r}$ is defined as in* (2.36)*. Then*

(1) *for any harmonic function $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = \rho$, we have $F(\overline{B}_r) \subset E_{r,\rho}$;*

(2) *$E_{r,\rho}$ is a closed convex domain and is symmetrical with respect to the real axis, and $\rho$ is an interior point of $E_{r,\rho}$;*

(3) *$\Gamma_{r,\rho}$ is a convex Jordan closed curve and $\partial E_{r,\rho} = \Gamma_{r,\rho}$;*

(4) *for any $w' \in E_{r,\rho}$, there is a harmonic function $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = \rho$ and $F(rN) = w'$.*

**Proof** (1) Denote

$$P'_\beta = P(0, U_{\rho\cos\beta,-\rho\sin\beta,r}(rN)), \quad l'_\beta = l(0, U_{\rho\cos\beta,-\rho\sin\beta,r}(rN)).$$

$P_\beta$ and $l_\beta$ are obtained from $P'_\beta$ and $l'_\beta$ by an anti-clockwise rotation of angle $\beta$, respectively.

Let $F$ be a harmonic function such that $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = \rho$. For $-\pi \le \beta \le \pi$, let $F_\beta = \mathrm{e}^{-\mathrm{i}\beta} F$. Then, $F_\beta(\mathbb{B}^n) \subset \mathbb{D}$ and $F_\beta(0) = \rho(\cos\beta - \mathrm{i}\sin\beta)$. Using Lemma 2.3 to the harmonic function $F_\beta$, we have $F_\beta(\overline{B}_r) \subset P'_\beta$ and consequently, $F(\overline{B}_r) \subset P_\beta$. This shows (1).

(2) It is obvious that $E_{r,\rho}$ is a closed convex set and is symmetrical with respect to the real axis. We only need to prove that $\rho$ is an interior point of $E_{r,\rho}$.

First we want to prove that $f_{r,\rho}(\beta) \in \partial E_{r,\rho}$ for $-\pi \le \beta \le \pi$. $f_{r,\rho}(\beta) \in l_\beta$ since $F_{\rho\cos\beta,-\rho\sin\beta,r}(rN) \in l'_\beta$. Let $G(x) = \mathrm{e}^{\mathrm{i}\beta} F_{\rho\cos\beta,-\rho\sin\beta,r}(x)$. The harmonic function $G$ satisfies the conditions $G(\mathbb{B}^n) \subset \mathbb{D}$ and $G(0) = \rho$. By (1), $f_{r,\rho}(\beta) = G(rN) \in E_{r,\rho}$. Note that $E_{r,\rho} \subset P_\beta$, $l_\beta = \partial P_\beta$ and $f_{r,\rho}(\beta) \in l_\beta$ which was proved above. Then we have $f_{r,\rho}(\beta) \in \partial E_{r,\rho}$.

For $f_{r,\rho}(0), f_{r,\rho}(\pi), f_{r,\rho}\big(\frac{\pi}{2}\big)$ and $f_{r,\rho}\big(-\frac{\pi}{2}\big)$, by Lemma 2.3, we have

$$f_{r,\rho}(0) = F_{\rho,0,r}(rN) = U_{\rho,0,r}(rN) > U_{\rho,0,r}(0) = \rho, \tag{3.2}$$
$$f_{r,\rho}(\pi) = -F_{-\rho,0,r}(rN) = -U_{-\rho,0,r}(rN) < -U_{-\rho,0,r}(0) = \rho, \tag{3.3}$$
$$\mathrm{Im} f_{r,\rho}\Big(\frac{\pi}{2}\Big) = U_{0,-\rho,r}(rN) = U_{0,\rho,r}(rN) > U_{0,\rho,r}(0) = 0,$$
$$\mathrm{Im} f_{r,\rho}\Big(-\frac{\pi}{2}\Big) = -U_{0,\rho,r}(rN) < -U_{0,\rho,r}(0) = 0.$$

Then $\rho$ is an interior point of $E_{r,\rho}$ since $E_{r,\rho}$ is a convex set.

(3) First we want to prove that $\Gamma_{r,\rho}$ is a Jordan closed curve. $\Gamma_{r,\rho}$ is close and continuous by Lemma 2.4. Assume that there exist $0 < \beta_1 < \beta_2 < \pi$ such that $w_0 = f_{r,\rho}(\beta_1) = f_{r,\rho}(\beta_2)$. Then $\beta_2 - \beta_1 < \pi$ and $w_0$ is the vertex of the angular domain $P_{\beta_1} \cap P_{\beta_2}$. Further, it is easy to see that $f_{r,\rho}(\beta) = w_0$ for $\beta_1 < \beta < \beta_2$, since $l_\beta \cap \partial E_{r,\rho} = w_0$ and $f_{r,\rho}(\beta) \in l_\beta \cap \partial E_{r,\rho}$. $f_{r,\rho}(\beta)$ is analytic on $(0,\pi)$ in the real sense by Lemma 2.4. Then we have $f_{r,\rho}(\beta) = w_0$ for $0 < \beta < \pi$ and by the continuity, $f_{r,\rho}(0) = f_{r,\rho}(\pi) = w_0$. This is a contraction, since $f_{r,\rho}(0) > f_{r,\rho}(\pi)$ by (3.2)–(3.3). This shows that $\Gamma^+_{r,\rho} = \{w = f_{r,\rho}(\beta) : 0 \le \beta \le \pi\}$ is a Jordan curve. For the same

reason, $\Gamma^-_{r,\rho} = \{w = f_{r,\rho}(\beta) : -\pi \le \beta \le 0\}$ is also a Jordan curve. Then $\Gamma_{r,\rho}$ is a Jordan closed curve.

For $-\pi \le \beta \le \pi$, it is proved in (2) that $f_{r,\rho}(\beta) \in \partial E_{r,\rho}$. Then $\Gamma_{r,\rho} \subset \partial E_{r,\rho}$. Note that $\partial E_{r,\rho}$ must be a convex Jordan closed curve. Thus $\partial E_{r,\rho} = \Gamma_{r,\rho}$.

(4) For $w' \in E_{r,\rho}$, draw a straight line $l$ passing through $w'$ and intersect $\partial E_{r,\rho}$ at $w_1$ and $w_2$. Let $w' = k_1 w_1 + k_2 w_2$ with $k_1, k_2 \ge 0$ and $k_1 + k_2 = 1$. There are two real numbers $\beta_1$ and $\beta_2$ such that $f_{r,\rho}(\beta_1) = w_1$ and $f_{r,\rho}(\beta_2) = w_2$. Then the harmonic function $F = k_1 e^{i\beta_1} F_{\rho\cos\beta_1, -\rho\sin\beta_1, r} + k_2 e^{i\beta_2} F_{\rho\cos\beta_2, -\rho\sin\beta_2, r}$ satisfies $F(\mathbb{B}^n) \subset \mathbb{D}$, $F(0) = \rho$ and $F(rN) = w'$. The theorem is proved.

When $\rho = 0$, we have a corollary as follows, which is coincident with (1.10).

**Corollary 3.1** *Let $0 < r < 1$. For any harmonic mapping $F$ such that $F(\mathbb{B}^n) \subset \mathbb{D}$ and $F(0) = 0$, we have*

$$F(\overline{B}_r) \subset \overline{D}_{U(rN)},$$

*where $U$ is the Poisson integral of the function that equals $1$ on $S^+$ and $-1$ on $S^-$.*

**Proof** By Theorem 3.1, we only need to prove that $E_{r,0} = \overline{D}_{U(rN)}$. Further, by the definition of $E_{r,\rho}$ in Theorem 3.1, we only need to prove that $U_{0,0,r}(rN) = U(rN)$. Note that by (2.25),

$$u_{0,0,r}(\omega) = \begin{cases} 1, & \omega \in S^+, \\ 0, & \omega \in S, \\ -1, & \omega \in S^-. \end{cases}$$

Then by (2.33) we know that $U_{0,0,r}(rN) = U(rN)$. The corollary is proved.

From Theorem 3.1, we obtain Theorem 1.1, which is the general version of the above Theorem 3.1.

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