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Dynamics of a Function Related to the Primes*

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Abstract Let $n = p_1p_2\cdots p_k$, where p_i $(1 \le i \le k)$ are primes in the descending order and are not all equal. Let $\Omega_k(n) = P(p_1+p_2)P(p_2+p_3)\cdots P(p_{k-1}+p_k)P(p_k+p_1)$, where $P(n)$ is the largest prime factor of n. Define $w^0(n) = n$ and $w^i(n) = w(w^{i-1}(n))$ for all integers $i \ge 1$. The smallest integer s for which there exists a positive integer t such that $\Omega_k^s(n) = \Omega_k^{s+t}(n)$ is called the index of periodicity of n. The authors investigate the index of periodicity of n.

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1 Introduction

Let $\mathcal{A}_3$ be the set of all positive integers pqr, where p, q, r are primes and are not all equal. For any integer $n = pqr \in \mathcal{A}_3$, we define a function w by $w(n) = P(p+q)P(p+r)P(q+r)$, where $P(m)$ is the largest prime factor of m. Define $w^0(n) = n$ and $w^i(n) = w(w^{i-1}(n))$ for all integers $i \ge 1$.

In 2006, Goldring [4] proved that any element $n \in \mathcal{A}_3$ is w-periodic, i.e., there exists an integer $i \ge 0$, such that $w^i(n) = 20$. Denote the smallest such integer i by $\mathrm{ind}_3(n)$. Goldring [4] proved that $\mathrm{ind}_3(n) \le 4(\pi(P(n)) - 3)$, where $\pi(x)$ denotes the number of primes not exceeding x. Later, Chen and Shi [1] improved Goldring's result and proved that $\mathrm{ind}_3(n) \ll (\log P(n))^2$ for all $n \in \mathcal{A}_3$.

Let $\mathcal{P}$ be the set of all positive primes. An integer m is called a parent of n if $w(m) = n$. Write

$$\mathcal{B}_3 = \{p_1^2p_2 \mid p_1 \ne p_2, p_1, p_2 \in \mathcal{P}\},$$
$$\mathcal{C}_3 = \{p_1p_2p_3 \mid p_1, p_2, p_3 \in \mathcal{P} \text{ and are pairwise distinct}\}.$$

Chen and Shi [2] proved that for any positive integer k, there are infinitely many elements of $\mathcal{B}_3$ which have at least k parents in $\mathcal{B}_3$, and that there exist infinitely many elements of $\mathcal{B}_3$ which have no parents in $\mathcal{B}_3$.

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Later, Jia [5] studied parents of $p_1p_2p_3 \in \mathcal{B}_3 \cup \mathcal{C}_3$ and obtained some interesting results.

Recently, Chen, Shi and Wu [3] proved that there exist infinitely many $n \in \mathcal{B}_3$ which have at least $n^{1.1886}$ parents in $\mathcal{B}_3$.

For an integer $n = p_1p_2 \cdots p_k$, where p_i $(1 \le i \le k)$ are primes in the descending order and are not all equal, define

$$\Omega_k(n) = P(p_1+p_2)P(p_2+p_3)\cdots P(p_{k-1}+p_k)P(p_k+p_1).$$

Clearly, $\Omega_3(n) = w(n)$. In [6], Shi generalized Goldring's w function to the function Ω_k. Define $\Omega_k^0(n) = n$ and $\Omega_k^i(n) = \Omega_k(\Omega_k^{i-1}(n))$ for all integers $i \ge 1$. We call n a simple integer for Ω_k if there exists an integer $i \ge 0$, such that $\Omega_k^i(n)$ is a prime power. For integers $k \ge 3$, let

$$\mathcal{A}_k = \{n \in \mathbb{Z}^+ \mid \Omega(n) = k,\ n \text{ is not a simple integer for } \ \Omega_k\},$$

where $\Omega(n)$ is the total number of prime factors of n. If $2 \nmid k$, $\Omega(n) = k$ and n is not a prime power, then $\Omega_k(n)$ is also not a prime power. Otherwise, we have $P(p_1+p_2) = P(p_2+p_3) = \cdots = P(p_k+p_1) = p$, where $p_1p_2\cdots p_k = n$ and p, p_i $(1 \le i \le k)$ are primes. It follows that $p_1 \equiv -p_2 \equiv p_3 \equiv \cdots \equiv p_k \equiv -p_1 \pmod p$, and then $p = 2$ or $p = p_1$. If $p = 2$, then, by $2p_1 = (p_1+p_2) - (p_2+p_3) + \cdots + (p_k+p_1)$, it follows that $2 \mid p_1$, and then $p_1 = p$. Hence, $n = p^k$, a contradiction. Therefore, the definition of $\mathcal{A}_k$ is consistent with that of the previous set $\mathcal{A}_3$.

An element n of $\mathcal{A}_k$ is Ω_k-periodic if there exists a nonnegative integer s and a positive integer t, such that $\Omega_k^s(n) = \Omega_k^{s+t}(n)$. The smallest such integer s is called the index of periodicity of n, denoted by $\mathrm{ind}_k(n)$. The array $\overline{b_1, b_2, \cdots, b_t}$ is called a circular array of $\mathcal{A}_k$ if t elements $b_1, b_2, \cdots, b_t \in \mathcal{A}_k$ satisfy $\Omega_k(b_s) = b_{s+1}$ $(s = 1, 2, \cdots, t-1)$, $\Omega_k(b_t) = b_1$. In general, we regard all arrays such as $\overline{b_i, b_{i+1}, \cdots, b_t, b_1, \cdots, b_{i-1}}$ $(i = 1, 2, \cdots, t)$ as an equal array, denoted by $b_i^{\Omega_k}$, where b_i is an element in this circular array. An element n of $\mathcal{A}_k$ is said to lie in the circular array $b_i^{\Omega_k}$ ultimately, if there exists an integer $j \ge 0$, such that $\Omega_k^j(n) \in b_i^{\Omega_k}$. The whole circular array in $\mathcal{A}_k$ is denoted by $\mathcal{A}_k^{\Omega_k}$.

In [6], Shi proved the following theorem.

Theorem A *Every element of $\mathcal{A}_k$ is periodic and each lies in some circular array ultimately. When $k \ge 5$, $\mathcal{A}_k^{\Omega_k} = \{(2^a3^b5^c)^{\Omega_k} \mid\ a+b+c = k,\ a \ge 1,\ b \ge 2,\ c \ge 1,\ a, b, c \in \mathbb{Z}\}$. In addition, $\mathcal{A}_4^{\Omega_4} = \{60^{\Omega_4}, 90^{\Omega_4}\}$.*

In this paper, based on the method in [1], we prove the following result.

Theorem 1.1 *Let k be an odd integer with $k \ge 4$. For any integer n with k prime factors not all equal, we have*

$$\mathrm{ind}_k(n) = O_k(\log P(n))^2.$$

Remark 1.1 If k is even, then the iteration of the arithmetic function Ω_k may stop. For example, $k = 4$, $\Omega_4(3 \times 7 \times 13 \times 17) = 5^4$. Therefore, we only consider the odd case.

2 Preliminary Lemmas

Lemma 2.1 *Let $X \geq 3$ be an integer and α be a real number with $0 < \alpha < 1$. For any integer n with k ($k \geq 4$) prime factors not all equal, let $n = p_1 p_2 \cdots p_k$, where p_i ($1 \leq i \leq k$) are primes and $p_1 \geq p_2 \geq \cdots \geq p_k$. If $p_1 \leq X$ and $p_2 \leq \alpha X$, then there exists an integer i with $1 \leq i \leq 3$, such that*

$$P(\Omega_k^i(n)) \leq \frac{3+\alpha}{4}X + 4.$$

Proof If $p_1 \leq \alpha X$, then $P(\Omega_k(n)) \leq \alpha X + 2 \leq \frac{(3+\alpha)X}{4} + 4$. If $p_1 = 3 > \alpha X$, i.e., $n = 3 \cdot 2^{k-1}$, then $P(\Omega_k(n)) = 5 \leq \frac{(3+\alpha)X}{4} + 4$. If $p_k > 2$, then $P(\Omega_k(n)) \leq \frac{(1+\alpha)X}{2} \leq \frac{(3+\alpha)X}{4} + 4$. Thus we may assume that $p_1 \geq 5$, $p_1 > \alpha X$ and $p_k = 2$.

Now we consider the following two cases.

Case 1 $p_1 + 2$ is composite.

By the definition of $\Omega_k(n)$ and $p_k = 2$, we have

$$\Omega_k(n) = P(p_1 + p_2) P(p_2 + p_3) \cdots P(p_{k-1} + 2) P(2 + p_1).$$

If $p_2 = 2$, then $P(p_1 + p_2) \leq \frac{X+2}{3}$; if $p_2 \geq 3$, then $P(p_1 + p_2) \leq \frac{(1+\alpha)X}{2}$. We also have $P(p_1 + p_k) \leq \frac{X+2}{3}$.

For $i = 2, 3, \cdots, k-1$, if $p_{i+1} > 2$, then $P(p_i + p_{i+1}) \leq \alpha X$; if $p_{i+1} = 2$, then $P(p_i + p_{i+1}) \leq \alpha X + 2$.

Hence, we obtain

$$P(\Omega_k(n)) \leq \frac{3+\alpha}{4}X + 4.$$

Case 2 $p_1 + 2$ is prime.

Subcase 2.1 $n = p_1 \cdot 2^{k-1}$. It follows that

$$\Omega_k(n) = (p_1 + 2)^2 \cdot 2^{k-2}, \quad \Omega_k^2(n) = (p_1 + 2) P^2(p_1 + 4) 2^{k-3}.$$

Since $p_1 > 3$, and p_1, $p_1 + 2$ are both primes, we have $3 \mid p_1 + 4$. Hence, $P(p_1 + 4) < p_1 + 2$ and

$$\Omega_k^3(n) = P(p_1 + 2 + P(p_1 + 4)) \cdot P(P(p_1 + 4) + 2) \cdot P^2(p_1 + 4) \cdot 2^{k-4}.$$

Clearly, we have

$$P(p_1 + 4) \leq \frac{X+4}{3}, \quad P(P(p_1 + 4) + 2) \leq \frac{X+4}{3} + 2$$

and

$$P(p_1 + 2 + P(p_1 + 4)) \leq \frac{p_1 + 2 + \frac{p_1 + 4}{3}}{2} \leq \frac{2X + 5}{3}.$$

Therefore,

$$P(\Omega_k^3(n)) \leq \frac{3+\alpha}{4}X + 4.$$

Subcase 2.2 $n = p_1 p_2 \cdots p_{i-1} 2^{k-i+1}$, where $3 \leq i \leq k$ and $p_{i-1} \geq 3$. Then

$$\Omega_k(n) = (p_1 + 2) \cdot P(p_1 + p_2) \cdots P(p_{i-2} + p_{i-1}) \cdot P(p_{i-1} + 2) \cdot 2^{k-i}.$$

Let $\Omega_k(n) = q_1 q_2 \cdots q_k$, where q_i $(1 \le i \le k)$ are primes and $q_1 \ge q_2 \ge \cdots \ge q_k$. Clearly, $q_1 = p_1 + 2 \le X + 2$ and for $j = 2, 3, \cdots, k$,

$$q_j \le \max\left\{\frac{1+\alpha}{2}X,\ \alpha X + 2\right\} \le \frac{1+\alpha}{2}X + 2.$$

Since $p_1 > 3$, and p_1, $p_1 + 2$ are both primes, we have $3 \mid q_1 + 2$. Thus, for $j = 2, k$, if $q_j = 2$, then $P(q_1 + q_j) \le \frac{X+4}{3}$; if $q_j \ge 3$, then $P(q_1 + q_j) \le \frac{q_1+q_j}{2} \le \frac{(3+\alpha)X}{4} + 2$. We also have $P(q_i + q_{i+1}) \le q_2 + 2 \le \frac{(1+\alpha)X}{2} + 4$ for $i = 2, 3, \cdots, k-1$.

By $\Omega_k^2(n) = P(q_1 + q_2) \cdots P(q_{k-1} + q_k) P(q_k + q_1)$, it follows that

$$P(\Omega_k^2(n)) \le \frac{3+\alpha}{4}X + 4.$$

Therefore, by all the cases above, there exists an integer i with $1 \le i \le 3$, such that

$$P(\Omega_k^i(n)) \le \frac{3+\alpha}{4}X + 4.$$

This completes the proof of Lemma 2.1.

Lemma 2.2 *Let $X \ge 3$, $k \ge 4$ be integers and $\alpha < 1$ be a positive real number. Let $n = p_1 p_2 \cdots p_k$, where p_i $(1 \le i \le k)$ are primes in the descending order and are not all equal. If $p_1 \le X$ and $p_j \le \alpha X$ for some integer j with $2 \le j \le k$, then there exists a positive integer i with $1 \le i \le 4j - 3$, such that*

$$P(\Omega_k^i(n)) \le \frac{2^{3j-4} - 1 + \alpha}{2^{3j-4}}X + 6j - 8.$$

Proof If $p_1 = 3$, then $n = 3^s \cdot 2^{k-s}$ for some integer s with $1 \le s \le k-1$. By $X \ge 3$ and $j \ge 2$, we have

$$P(\Omega_k(n)) = 5 < \frac{2^{3j-4} - 1 + \alpha}{2^{3j-4}}X + 6j - 8.$$

Thus we may assume that $X \ge p_1 \ge 5$. We shall prove it by induction on j.

By Lemma 2.1, the result is true for $j = 2$. Now we suppose that it is true for $j = l - 1$, where $2 \le l - 1 < k$. That is, if $p_1 \le X$ and $p_{l-1} \le \alpha X$, then there exists an integer i with $1 \le i \le 4(l-1) - 3 = 4l - 7$, such that

$$P(\Omega_k^i(n)) \le \frac{2^{3(l-1)-4} - 1 + \alpha}{2^{3(l-1)-4}}X + 6(l-1) - 8 = \frac{2^{3l-7} - 1 + \alpha}{2^{3l-7}}X + 6l - 14.$$

Now we assume that $p_1 \le X$ and $p_l \le \alpha X$. We consider the following cases.

Case 1 $p_k \ge 3$.

For $1 \le s \le l-2$, we have $P(p_s + p_{s+1}) \le X$. For $l-1 \le s \le k-1$, we have $P(p_s + p_{s+1}) \le \frac{(1+\alpha)X}{2}$. We also have $P(p_1 + p_k) \le \frac{(1+\alpha)X}{2}$.

Let $\Omega_k(n) = q_1 q_2 \cdots q_k$, where q_i $(1 \le i \le k)$ are primes and $q_1 \ge q_2 \ge \cdots \ge q_k$. Then $q_1 \le X$ and $q_{l-1} \le \frac{(1+\alpha)X}{2}$.

By the induction hypothesis, there exists an integer i with $1 \le i \le 4l - 7$, such that

$$P(\Omega_k^i(\Omega_k(n))) \le \frac{2^{3l-7} - 1 + \frac{1+\alpha}{2}}{2^{3l-7}}X + 6l - 14 = \frac{2^{3l-6} - 1 + \alpha}{2^{3l-6}}X + 6l - 14.$$

Hence, there exists an integer i with $1 \le i \le 4l-6$, such that

$$P(\Omega_k^i(n)) \le \frac{2^{3l-6}-1+\alpha}{2^{3l-6}}X + 6l - 14 \le \frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X + 6l - 8.$$

Case 2 $p_k = 2$ and $p_l \ge 3$.

If $1 \le s \le l-2$, then $P(p_s + p_{s+1}) \le X+2$. If $l-1 \le s \le k-1$, then $P(p_s+p_{s+1}) \le \max\{\frac{X+\alpha X}{2}, \alpha X + 2\} \le \frac{(1+\alpha)(X+2)}{2}$. We also have $P(p_1+p_k) \le X+2$. Hence $q_1 \le X+2$.

Let $\Omega_k(n) = q_1 q_2 \cdots q_k$, where q_i $(1 \le i \le k)$ are primes and $q_1 \ge q_2 \ge \cdots \ge q_k$. Suppose that $q_{l-1} \le \frac{(1+\alpha)(X+2)}{2}$. By the induction hypothesis, there exists an integer i with $1 \le i \le 4l-7$, such that

$$P(\Omega_k^i(\Omega_k(n))) \le \frac{2^{3l-7}-1+\frac{1+\alpha}{2}}{2^{3l-7}}(X+2) + 6l - 14 \le \frac{2^{3l-6}-1+\alpha}{2^{3l-6}}X + 6l - 12.$$

Hence, there exists an integer i with $1 \le i \le 4l-6$, such that

$$P(\Omega_k^i(n)) \le \frac{2^{3l-6}-1+\alpha}{2^{3l-6}}X + 6l - 12 \le \frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X + 6l - 8.$$

Subcase 2.1 p_1+2 is composite.

It follows that $P(p_1+p_k) = P(p_1+2) \le \frac{X+2}{3} \le \frac{(1+\alpha)(X+2)}{2}$. Hence $q_{l-1} \le \frac{(1+\alpha)(X+2)}{2}$, and we are done with the proof.

Subcase 2.2 p_1+2 is prime.

If $q_{l-1} \le \frac{(1+\alpha)(X+2)}{2}$, then we are done with the proof.

Now we assume that $q_{l-1} > \frac{(1+\alpha)(X+2)}{2}$.

Since $p_k = 2$ and $p_l \ge 3$, there exists an integer t with $l \le t \le k-1$, such that $p_t \ge 3$ and $p_{t+1} = 2$. Noting that $P(p_t + p_{t+1}) \le \frac{(1+\alpha)(X+2)}{2} < q_{l-1}$ and $q_1 \ge q_2 \ge \cdots \ge q_k$, we have $q_l \ge P(p_t+p_{t+1}) = P(p_t+2) \ge 3$. By $p_1 > 3$, and since p_1 and p_1+2 are both primes, we have that $q_1 + 2 = p_1 + 4$ is composite. Now we go back to Case 1 if $q_k \ge 3$ and Subcase 2.1 if $q_k = 2$. The maximal upper bound in these two cases appears in Subcase 2.1. Hence, there exists an integer i with $1 \le i \le 4l-5$, such that

$$P(\Omega_k^i(n)) \le \frac{2^{3l-6}-1+\frac{1+\alpha}{2}}{2^{3l-6}}(X+2) + 6l - 12 \le \frac{2^{3l-5}-1+\alpha}{2^{3l-5}}X + 6l - 10.$$

Case 3 $p_k = 2$ and $p_l = 2$.

In this case, we have $\Omega_k(n) = P(p_1+p_2)\cdots P(p_{l-1}+2)P(p_1+2)2^{k-l}$.

Subcase 3.1 At least one of $p_{l-1}+2$ and p_1+2 is composite.

Let $p+2$ be composite, where $p = p_1$ or p_{l-1}. Then $P(p+2) \ge 3$ and $P(p+2) \le \frac{X+2}{3} \le \frac{(1+\alpha)(X+2)}{2}$. Let $\Omega_k(n) = q_1 q_2 \cdots q_k$, where q_i $(1 \le i \le k)$ are primes and $q_1 \ge q_2 \ge \cdots \ge q_k$. If $q_{l-1} \le \frac{(1+\alpha)(X+2)}{2}$, then we use the induction hypothesis. Now suppose that $q_{l-1} > \frac{(1+\alpha)(X+2)}{2}$. It follows that $q_{l+1} = 2 < P(p+2) \le \frac{(1+\alpha)(X+2)}{2} < q_{l-1}$. Hence $3 \le P(p+2) = q_l \le \frac{(1+\alpha)(X+2)}{2}$. If $l = k$, then $q_k \ge 3$ and we go back to Case 1. If $l < k$, then $q_k = 2$ and we go back to Case 2. The maximal upper bound in these two cases and the induction hypothesis appear in Subcase 2.2. Hence, there exists an integer i with $1 \le i \le 4l-4$, such that

$$P(\Omega_k^i(n)) \le \frac{2^{3l-5}-1+\frac{1+\alpha}{2}}{2^{3l-5}}(X+2) + 6l - 10 \le \frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X + 6l - 8.$$

Subcase 3.2 Both $p_{l-1}+2$ and p_1+2 are primes.

It follows that $p_{l-1}\ge 3$, and then $2\mid p_i+p_{i+1}$ for $i=1,2,\cdots,l-2$.

Subcase 3.2.1 $\frac{p_1+p_2}{2},\frac{p_2+p_3}{2},\cdots,\frac{p_{l-2}+p_{l-1}}{2}$ are not all primes.

We assume that $\frac{p_j+p_{j+1}}{2}$ is composite, where $1\le j\le l-2$. Then

$$P\Big(\frac{p_j+p_{j+1}}{2}\Big)\le\frac{p_j+p_{j+1}}{4}\le\frac{X}{2}.$$

Now $\Omega_k(n)=(p_1+2)(p_{l-1}+2)P\big(\frac{p_1+p_2}{2}\big)\cdots P\big(\frac{p_{l-2}+p_{l-1}}{2}\big)2^{k-l}$. Let $\Omega_k(n)=q_1q_2\cdots q_k$, where q_i $(1\le i\le k)$ are primes and $q_1\ge q_2\ge\cdots\ge q_k$. Clearly, $q_{l+1}=2$ and $q_1=p_1+2\le X+2$.

Suppose that $P\big(\frac{p_j+p_{j+1}}{2}\big)\ge 3$. If $q_l\ne P(\frac{p_j+p_{j+1}}{2})$, then $q_{l-1}\le P\big(\frac{p_j+p_{j+1}}{2}\big)\le\frac{X}{2}$, and we use the induction hypothesis. If $q_l=P\big(\frac{p_j+p_{j+1}}{2}\big)$ and $l=k$, then $q_k\ge 3$ and we go back to Case 1. If $q_l=P(\frac{p_j+p_{j+1}}{2})$ and $l<k$, then $q_k=2$ and we go back to Case 2. The maximal upper bound in this two cases and the induction hypothesis appear in Subcase 2.2. Hence, there exists an integer i with $1\le i\le 4l-4$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3l-5}-1+\frac{1}{2}}{2^{3l-5}}(X+2)+6l-10\le\frac{2^{3l-4}-1}{2^{3l-4}}X+6l-8.$$

Now suppose that $P\big(\frac{p_j+p_{j+1}}{2}\big)=2$. Let $\Omega_k(n)=q_1q_2\cdots q_{l-1}2^{k-l+1}$, where q_i $(1\le i\le l-1)$ are primes and $q_1\ge q_2\ge\cdots\ge q_{l-1}\ge 2$. By $p_1>3$, and since p_1 and p_1+2 are primes, we have $3\mid vp_1+4$. That is, $3\mid q_1+2$. Noting that $X\ge 3$, we have $q_l=2\le\frac{(1+\alpha)(X+2)}{2}$. Hence, by Subcase 3.1, there exists an integer i with $1\le i\le 4l-3$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3l-5}-1+\frac{1+\alpha}{2}}{2^{3l-5}}(X+2)+6l-10\le\frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X+6l-8.$$

Subcase 3.2.2 $\frac{p_1+p_2}{2},\frac{p_2+p_3}{2},\cdots,\frac{p_{l-2}+p_{l-1}}{2}$ are all primes.

(1) $p_{l-1}=3$.

If $\frac{3}{5}\le\alpha<1$, then, by $5\le p_1\le X$, we have $p_{l-1}=3\le\alpha X$. Thus, by the induction hypothesis, the result is true. Now we assume that $0<\alpha<\frac{3}{5}$. Noting that $2\le\alpha X$, we have $p_{l-1}=3\le\frac{3\alpha X}{2}<\frac{9X}{10}<X$. By the induction hypothesis and $\alpha<\frac{3}{5}$, there exists an integer i with $1\le i\le 4l-7$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3l-7}-1+\frac{3}{2}\alpha}{2^{3l-7}}X+6l-14<\frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X+6l-8.$$

(2) $p_{l-1}>3$.

Since $p_1>3$, and p_1 and p_1+2 are primes, we have $p_1\equiv 2\pmod 3$. Noting that $\frac{p_1+p_2}{2}$ is a prime greater than 3 and $p_2>3$, we have $p_2\equiv 2\pmod 3$. Otherwise, if $p_2\equiv 1\pmod 3$, then $3\mid p_1+p_2$, a contradiction. Similarly, we have $p_1\equiv p_2\equiv\cdots\equiv p_{l-1}\equiv 2\pmod 3$. It follows that $p_1+2\equiv 1\pmod 3$, $p_{l-1}+2\equiv 1\pmod 3$ and $\frac{p_i+p_{i+1}}{2}\equiv 2\pmod 3$ for $i=1,2,\cdots,l-2$. Now we consider

$$\Omega_k(n)=(p_1+2)\frac{p_1+p_2}{2}\cdots\frac{p_{l-2}+p_{l-1}}{2}(p_{l-1}+2)2^{k-l}.$$

For all i,j with $1\le i,j\le l-2$, we have

$$\frac{p_i+p_{i+1}}{2}+p_1+2\equiv\frac{p_j+p_{j+1}}{2}+p_{l-1}+2\equiv 0\pmod 6.$$

Hence, primes

$$P\Big(\frac{p_i+p_{i+1}}{2}+p_1+2\Big),\quad P\Big(\frac{p_j+p_{j+1}}{2}+p_{l-1}+2\Big),\quad P(p_1+4)$$

are all odd, and none of them is more than $\frac{X+4}{3}\le\frac{(1+\alpha)(X+2)}{2}$.

Let $\Omega_k^2(n)=q_1q_2\cdots q_k$, where q_i $(1\le i\le k)$ are primes and $q_1\ge q_2\ge\cdots\ge q_k$. Then $q_{l+2}=2$ (if it exists) and there exist integers r,s with $1\le r<s\le l+1$, such that $3\le q_r\le q_s\le\frac{(1+\alpha)(X+2)}{2}$. It follows that $q_l\le\frac{(1+\alpha)(X+2)}{2}$. If $q_l=2$, then $q_{l-1}\le q_r\le\frac{(1+\alpha)(X+2)}{2}$, and we use the induction hypothesis. If $q_l\ge 3$, we go back to Case 1 when $q_k\ge 3$ and Case 2 when $q_k=2$. Hence, there exists an integer i with $1\le i\le 4l-3$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3l-5}-1+\frac{1+\alpha}{2}}{2^{3l-5}}(X+2)+6l-10\le\frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X+6l-8.$$

By all the cases above, Lemma 2.2 is true for $j=l$. That is, if $p_l\le\alpha X$ and $p_1\le X$, then there exists an integer i with $1\le i\le 4l-3$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3l-4}-1+\alpha}{2^{3l-4}}X+6l-8.$$

This completes the proof of Lemma 2.2.

Lemma 2.3 *Let k be an odd integer with $k\ge 4$. Then for any integer n with k prime factors not all equal, there exists an integer i with $1\le i\le 2\log P(n)+4k-2$, such that*

$$P(\Omega_k^i(n))\le\Big(1-\frac{1}{2^{3k-3}}\Big)P(n)+6k-8.$$

Proof Suppose that $n=p_1p_2\cdots p_k$, where p_i $(1\le i\le k)$ are primes in the descending order and are not all equal. Then

$$\Omega_k(n)=P(p_1+p_2)\cdot P(p_2+p_3)\cdots P(p_{k-1}+p_k)\cdot P(p_k+p_1).$$

Now, we discuss the following cases.

Case 1 $p_1\ge p_2\ge\cdots\ge p_k\ge 3$.

Let

$$\overline{P}:=\Big\{\frac{p_1+p_2}{2},\ \frac{p_2+p_3}{2},\ \cdots,\ \frac{p_{k-1}+p_k}{2},\ \frac{p_1+p_k}{2}\Big\}.$$

Subcase 1.1 At least one element of $\overline{P}$ is composite.

Since the largest prime factor of this composite element of $\overline{P}$ is less than $\frac{p_1}{2}$, by Lemma 2.2, there exists an integer i with $1\le i\le 4k-2$, such that

$$P(\Omega_k^i(n))\le\frac{2^{3k-4}-1+\frac{1}{2}}{2^{3k-4}}p_1+6k-8=\Big(1-\frac{1}{2^{3k-3}}\Big)p_1+6k-8.$$

Subcase 1.2 All elements of $\overline{P}$ are primes.

In this case, it is clear that every element of $\overline{P}$ is an odd prime. Now we arrange these k odd primes in the descending order and denote them by $\overline{p}_{11}\ge\overline{p}_{12}\ge\cdots\ge\overline{p}_{1k}$. Then we consider

$$\frac{\overline{p}_{11}+\overline{p}_{12}}{2},\ \frac{\overline{p}_{12}+\overline{p}_{13}}{2},\ \cdots,\frac{\overline{p}_{1k-1}+\overline{p}_{1k}}{2},\ \frac{\overline{p}_{1k}+\overline{p}_{11}}{2}.$$

If all these numbers are odd primes, then we arrange them in the descending order and denote them by $\overline{p}_{21} \geq \overline{p}_{22} \geq \cdots \geq \overline{p}_{2k}$. Continue this process until they are not all primes. Suppose that for the $(t+1)$th time, there exists an integer s with $1 \leq s \leq k$, such that $\overline{p}_{t+1,s}$ is composite.

Since $\overline{p}_{ti}$ are odd primes for $i = 1, 2, \cdots, k$, we have

$$\overline{p}_{t1} \equiv \overline{p}_{t2} \equiv \cdots \equiv \overline{p}_{tk} \pmod 2.$$

That is,

$$\frac{\overline{p}_{t-1,1} + \overline{p}_{t-1,2}}{2} \equiv \frac{\overline{p}_{t-1,2} + \overline{p}_{t-1,3}}{2} \equiv \cdots \equiv \frac{\overline{p}_{t-1,k} + \overline{p}_{t-1,1}}{2} \pmod 2.$$

By $2 \nmid k$, it follows that

$$\overline{p}_{t-1,1} \equiv \overline{p}_{t-1,3} \equiv \cdots \equiv \overline{p}_{t-1,k} \equiv \overline{p}_{t-1,2} \equiv \overline{p}_{t-1,4} \equiv \cdots \equiv \overline{p}_{t-1,k-1} \pmod{2^2}.$$

Thus

$$\overline{p}_{t-1,1} \equiv \overline{p}_{t-1,2} \equiv \cdots \equiv \overline{p}_{t-1,k} \pmod{2^2}.$$

Continuing this argument, for all integers j with $1 \leq j \leq t$, we have

$$\overline{p}_{j1} \equiv \overline{p}_{j2} \equiv \cdots \equiv \overline{p}_{jk} \pmod{2^{t+1-j}}.$$

Hence

$$p_1 \equiv p_2 \equiv \cdots \equiv p_k \pmod{2^{t+1}}.$$

If $2^{t+1} > p_1$, then by $p_1 \geq \cdots \geq p_k \geq 3$, we have $p_1 = p_2 = \cdots = p_k$, a contradiction. So $2^{t+1} \leq p_1$, and then $t < 2\log p_1$. Since $\overline{p}_{t+1,s}$ is composite and $\overline{p}_{t+1,s} \leq \frac{p_1}{2}$, by Lemma 2.2, there exists an integer i with $1 \leq i \leq 2\log P(n) + 4k - 2$, such that

$$P(\Omega_k^i(n)) \leq \frac{2^{3k-4} - 1 + \frac{1}{2}}{2^{3k-4}} p_1 + 6k - 8 = \Big(1 - \frac{1}{2^{3k-3}}\Big) p_1 + 6k - 8.$$

Case 2 $p_k = 2$.

If $p_1 = 3$, then $P(\Omega_k^i(n)) = 5$ for $i = 1$ and the result is obviously true. Now we assume that $p_1 \geq 5$. Then $p_k \leq \frac{2p_1}{5}$. By Lemma 2.2, there exists an integer i with $1 \leq i \leq 4k - 3$, such that

$$P(\Omega_k^i(n)) \leq \frac{2^{3k-4} - 1 + \frac{2}{5}}{2^{3k-4}} p_1 + 6k - 8 \leq \Big(1 - \frac{1}{2^{3k-3}}\Big) p_1 + 6k - 8.$$

By all the cases above, for any integer n with k prime factors not all equal, there exists an integer i with $1 \leq i \leq 2\log P(n) + 4k - 2$, such that

$$P(\Omega_k^i(n)) \leq \Big(1 - \frac{1}{2^{3k-3}}\Big) P(n) + 6k - 8.$$

This completes the proof of Lemma 2.3.

3 Proof of Theorem 1.1

For any integer n with k prime factors not all equal, let $n = p_1 p_2 \cdots p_k$, where p_i $(1 \le i \le k)$ are primes in the descending order and are not all equal. Take $i_0 = 0$, $P(n) = p_1$ and $\Omega_k^{i_0}(n) = n$. By Lemma 2.3, there exist positive integers $i_1 < i_2 < \cdots$, such that, for all integers $t \ge 1$,

$$P(\Omega_k^{i_t}(n)) \le c_k \cdot P(\Omega_k^{i_{t-1}}(n)) + 6k - 8 \tag{3.1}$$

and

$$i_t - i_{t-1} \le 2\log(P(\Omega_k^{i_{t-1}}(n))) + 4k - 2, \tag{3.2}$$

where

$$c_k = 1 - \frac{1}{2^{3k-3}}.$$

By (3.1), we have

$$\begin{aligned} P(\Omega_k^{i_t}(n)) &\le c_k \cdot P(\Omega_k^{i_{t-1}}(n)) + 6k - 8 \\ &\le c_k^2 \cdot P(\Omega_k^{i_{t-2}}(n)) + (6k-8) \cdot c_k + 6k - 8 \\ &\le \cdots \\ &\le c_k^t \cdot P(\Omega_k^{i_0}(n)) + (6k-8) \cdot c_k^{t-1} + \cdots + (6k-8) \cdot c_k + 6k - 8 \\ &< c_k^t p_1 + (6k-8) \cdot 2^{3k-3}. \end{aligned} \tag{3.3}$$

If $p_1 \le 7 \cdot 11^{k-1} - (6k-8) \cdot 2^{3k-3}$, then by Theorem A, $\mathrm{ind}_k(n)$ is bounded and the result is true. Now we suppose that $p_1 > 7 \cdot 11^{k-1} - (6k-8) \cdot 2^{3k-3}$. Take a positive integer t_0, such that

$$t_0 - 1 < \frac{\log(7 \cdot 11^{k-1} - (6k-8) \cdot 2^{3k-3}) - \log p_1}{\log c_k} \le t_0. \tag{3.4}$$

Then

$$P(\Omega_k^{i_{t_0}}(n)) < c_k^{t_0} p_1 + (6k-8) \cdot 2^{3k-3} \le 7 \cdot 11^{k-1}.$$

By Theorem A, for every element $n \in \mathcal{A}_k$, there exists a positive integer i_n such that $\Omega_k^{i_n}(n)$ lies in some circular array $(2^a 3^b 5^c)^{\Omega_k}$ ultimately.

Let

$$c_0 = \max\{i_n \,|\, n \in \mathcal{A}_k,\ P(n) \le 7 \cdot 11^{k-1}\}.$$

Then there exists an integer j with $1 \le j \le c_0$, such that $\Omega^{i_{t_0}+j}(n)$ lies in some circular array $(2^a 3^b 5^c)^{\Omega_k}$ ultimately.

By (3.2)–(3.3), we have

$$i_{t_0} \le t_0 \cdot (2\log(p_1 + (6k-8)2^{3k-3}) + 4k - 2). \tag{3.5}$$

Thus, by (3.4)–(3.5), we have

$$\mathrm{ind}_k(n) \le i_{t_0} + c_0 \le c_1(\log^2 p_1),$$

where the constant c_1 depends only on k.

Therefore,

$$\mathrm{ind}_k(n) = O_k(\log P(n))^2.$$

This completes the proof of Theorem 1.1.

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References

[1] Chen, Y. G. and Shi, Y., Dynamics of the w function and the Green-Tao theorem on arithmetic progressions in the primes, *Proc. Amer. Math. Soc.,* **136**, 2008, 2351–2357.

[2] Chen, Y. G. and Shi, Y., Distribution of primes and dynamics of the w function, *J. Number Theory,* **128**, 2008, 2085–2090.

[3] Chen, Y. G., Shi, Y. and Wu, J., Dynamics of Golding's w-function, *J. Number Theory,* **132**, 2012, 390–409.

[4] Goldring, W., Dynamics of the w function and primes, *J. Number Theory,* **119**, 2006, 86–98.

[5] Jia, C. H., On the inverse problem relative to dynamics of the w function, *Sci. in China Ser. A,* **52**, 2009, 849–856.

[6] Shi, Y., Dynamics of the arithmetic function Ω_k, *J. Math. Res. Exposition,* **28**, 2008, 886–890.