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Bifurcation Analysis of the Multiple Flips Homoclinic Orbit*

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Abstract A high-codimension homoclinic bifurcation is considered with one orbit flip and two inclination flips accompanied by resonant principal eigenvalues. A local active coordinate system in a small neighborhood of homoclinic orbit is introduced. By analysis of the bifurcation equation, the authors obtain the conditions when the original flip homoclinic orbit is kept or broken. The existence and the existence regions of several double periodic orbits and one triple periodic orbit bifurcations are proved. Moreover, the complicated homoclinic-doubling bifurcations are found and expressed approximately.

Keywords Orbit flip, Inclination flips, Homoclinic orbit, Resonance
2000 MR Subject Classification 37C29, 34C23, 34C37

1 Introduction and Hypothesis

Homoclinic bifurcation is one of the origins of chaotic behaviors and has important applications in many fields. Amongst them, the flips cases began to catch attention in the last decade. As is well-known, homoclinic orbits generically occur as a codimension-one phenomenon if the genericity conditions are all kept. Otherwise, a higher codimension instance may take place, such as the case of resonant eigenvalues, which was considered in [1]; orbit flips, that is, the non-principal phenomenon, was treated in [2–3]; and the case of inclination flips or critically twisted homoclinic orbits, for which one can refer to [4–5], etc, where the homoclinic-doubling bifurcation, a codimension-two transition from an n-homoclinic to a $2n$-homoclinic orbit, is found to exist (see [6–10]). One may refer to the model for electro-chemical oscillators, or the FitzHugh-Nagumo nerve-axon equations (see [11]), a Shimitzu-Morioka equation for convection instabilities (see [12]), and a Hodgkin-Huxley model of thermally sensitive neurons (see [13]) for some applications.

Recently, the flip of heterodimensional cycles or accompanied by transcritical bifurcation was studied (see [14–16]). The double and triple periodic orbit bifurcations were proved to exist, and also some coexistence conditions for the homoclinic orbit and the periodic orbit were given. But the research is not concerned with multiple flips or homoclinic-doubling bifurcations.

This paper produces mainly a study of the homoclinic bifurcation with multiple flips, concretely one orbit flip and two inclination flips, which takes place at least in a four-dimensional

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system. Compared with the above work mentioned, the subject of multiple flips, especially with resonance, is very challenging and difficult, because the codimension is higher and the relationship of eigenvalues is more subtle. All these lead to the stronger degeneracy of bifurcation equations. So it is extremely hard to solve the bifurcation equations. To well handle the problem, we choose the method initially established in [17], that is, first construct a local specifical active coordinate system which reflects sufficiently the geometric structure of the corresponding invariant manifolds in a small tubular neighborhood of the homoclinic orbit, then establish a returning map, the Poincaré map, and get the associated successor function. By a delicate analysis of the bifurcation equation, we get the existence of 1-periodic orbit, 1-homoclinic orbit, some double and triple periodic orbits, and particularly the 2^n-homoclinic orbit and their corresponding bifurcation surfaces.

The system to be considered is C^r as

$$\dot{z} = f(z) + g(z, \mu), \tag{1.1}$$

and its unperturbed system is

$$\dot{z} = f(z), \tag{1.2}$$

where $r \geq 6$, $z \in \mathbb{R}^4$, $\mu \in \mathbb{R}^l$, $l \geq 4$, $0 <| \mu |\ll 1$, $f(0) = 0$, and $g(0, \mu) = g(z, 0) = 0$.

Suppose that (1.2) has an orbit $\Gamma = \{z = \gamma(t) : t \in \mathbb{R}, \gamma(\pm\infty) = 0\}$ homoclinic to the hyperbolic equilibrium $z = 0$, which has two negative and two positive eigenvalues $\lambda_1, \lambda_2, -\rho_1, -\rho_2$ satisfying $\lambda_2 > \lambda_1 > 0 > -\rho_1 > -\rho_2$. Set W^s (resp. W^{ss}) and W^u (resp. W^{uu}) to be the stable (resp. strong stable) manifold and unstable (resp. strong unstable) manifold of the equilibrium $z = 0$, respectively. We further make three assumptions:

(H1) (Resonance) $\lambda_1(\mu) \equiv \rho_1(\mu)$ for $|\mu| \ll 1$, where $\lambda_1(0) = \lambda_1$ and $\rho_1(0) = \rho_1$.

(H2) (Orbit Flip) Define $e^+ = \lim\limits_{t\to-\infty} \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$, $e_s^- = \lim\limits_{t\to+\infty} \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$. Then $e^+ \in T_0W^u$ and $e_s^- \in T_0W^{ss}$ are unit eigenvectors corresponding to λ_1 and $-\rho_2$ respectively, where T_0W^u (resp. T_0W^{ss}) is the tangent space of the corresponding manifold W^u (resp. W^{ss}) at the saddle $z = 0$.

(H3) (Inclination Flips) Denote by e_u^+ and e^- the unit eigenvectors corresponding to λ_2 and $-\rho_1$ respectively, and let

$$T_{\gamma(t)}W^u \to \operatorname{span}\{e_s^-, e^+\} \ \text{ as } t \to +\infty, \quad T_{\gamma(t)}W^s \to \operatorname{span}\{e^-, e^+\} \ \text{ as } t \to -\infty.$$

Hypothesis (H3) means that in two directions when $t \to \pm\infty$, the homoclinic orbit Γ comes through two inclination flips. Under (H1)–(H3), the genericity conditions are all broken, and therefore a codimension-4 homoclinic bifurcation occurs.

2 Normal Forms and Successor Functions

To well construct the Poincaré map and get the associated successor functions, we first need to transform (1.1) into a normal form in some neighborhood U of the origin O. In fact, with hypotheses (H1)–(H3) and the normal form theory, there must be a C^{r-4} system as follows after four successive C^r to C^{r-3} transformations in U (see [10] for the details),

$$\begin{aligned}
\dot{x} &= [\lambda_1(\mu) + a(\mu)xy + o(|xy|)]x + O(u)[O(x^2y) + O(v)],\\
\dot{y} &= [-\rho_1(\mu) + b(\mu)xy + o(|xy|)]y + O(v)[O(xy^2) + O(u)],\\
\dot{u} &= [\lambda_2(\mu) + c(\mu)xy + o(|xy|)]u + x^2H_1(x, y, v),\\
\dot{v} &= [-\rho_2(\mu) + d(\mu)xy + o(|xy|)]v + y^2H_2(x, y, u),
\end{aligned} \tag{2.1}$$

with the assumption

(H4) $H_1(x, 0, 0) = 0$, $H_2(0, y, 0) = 0$.

Indeed we have

$$\begin{aligned}
x^2H_1(x, y, v) &= a_1x^{2+k_1}y^{k_2} + a_2x^{2+k_3}v^{k_4} + a_3x^{2+k_5}y^{k_6}v^{k_7} + \text{h.o.t.},\\
y^2H_2(x, y, u) &= b_1y^{2+l_1}x^{l_2} + b_2y^{2+l_3}u^{l_4} + b_3y^{2+l_5}x^{l_6}u^{l_7} + \text{h.o.t.},
\end{aligned}$$

where $2 + k_i - \frac{\lambda_2}{\lambda_1} > \max\{\frac{\rho_2}{\lambda_1}, 2\}$, $i = 1, 3, 5$, $k_2 > \max\{\frac{\rho_2}{\lambda_1}, 2\}$, $k_4 > \max\{\frac{2\lambda_1}{\rho_2}, 1\}$ and $k_6 + k_7\frac{\rho_2}{\rho_1} > \max\{\frac{\rho_2}{\rho_1}, \frac{2\lambda_1}{\rho_1}\}$; $2 + l_i > \frac{\rho_2}{\lambda_1}$, $i = 1, 3, 5$, $l_2, l_4 > 0$ and $l_6 + l_7\frac{\lambda_2}{\lambda_1} > 0$; $\lambda_1(0) = \lambda_1$, $\lambda_2(0) = \lambda_2$, $\rho_1(0) = \rho_1$ and $\rho_2(0) = \rho_2$; $a(\mu)$, $b(\mu)$, $c(\mu)$ and $d(\mu)$ are parameters depending on μ. Equivalently, in U, we have

$$\begin{aligned}
&W^u_{\rm loc} = \{y = v = 0\}, && W^s_{\rm loc} = \{x = u = 0\},\\
&W^{u+}_{\rm loc} = \Gamma \cap W^u_{\rm loc} = \{y = u = v = 0\}, && W^{s-}_{\rm loc} = \{x = u = 0, v = v(y)\},\\
&W^{uu}_{\rm loc} = \{x = y = v = 0\}, && W^{ss}_{\rm loc} = \Gamma \cap W^s_{\rm loc} = \{x = y = u = 0\},
\end{aligned}$$

where $z = (x, y, u, v) \in \mathbb{R}^4$, $W^{u+}_{\rm loc}$ (resp. $W^{s-}_{\rm loc}$) is the local weak unstable (resp. weak stable) manifold which is tangent to e^+ (resp. e^-) at $z = 0$ and $v(y)$ satisfies $v(0) = v'(0) = 0$. Namely, we have straightened the corresponding invariant manifolds. It is possible to choose some moment T, such that $\gamma(-T) = \{\delta, 0, 0, 0\}$ and $\gamma(T) = \{0, 0, 0, \delta\}$, where δ is small enough and $\{(x, y, u, v) : |x|, |y|, |u|, |v| < 2\delta\} \subset U$.

Now we turn to consider the linear variational system of (1.2) and its adjoint system

$$\dot{z} = Df(\gamma(t))z, \tag{2.2}$$

$$\dot{z} = -(Df(\gamma(t)))^*z. \tag{2.3}$$

From the matrix theory, we immediately have the following lemma.

Lemma 2.1 *There exists a fundamental solution matrix $Z(t) = (z_1(t), z_2(t), z_3(t), z_4(t))$ of (2.2) satisfying*

$$Z(-T) = \begin{pmatrix} w_{11} & w_{21} & 0 & w_{41}\\ 0 & 0 & 0 & w_{42}\\ w_{13} & 0 & 1 & w_{43}\\ w_{14} & 0 & 0 & w_{44}\end{pmatrix} \quad \textit{and} \quad Z(T) = \begin{pmatrix} 0 & 0 & w_{31} & 0\\ w_{12} & 0 & w_{32} & 1\\ 1 & 0 & w_{33} & 0\\ 0 & 1 & w_{34} & 0\end{pmatrix},$$

where $z_1(t) \in (T_{\gamma(t)}W^u)^c \cap (T_{\gamma(t)}W^s)^c$, $z_2(t) = -\frac{\dot{\gamma}(t)}{|\dot{\gamma}(T)|} \in T_{\gamma(t)}W^u \cap T_{\gamma(t)}W^s$, $z_3(t) \in T_{\gamma(t)}W^u$ and $z_4(t) \in T_{\gamma(t)}W^s$, and $w_{14}w_{21}w_{31}w_{42} \neq 0$, $w_{21} < 0$.

Proof Notice that the tangent subspace $T_{\gamma(-T)}W^u$ is invariant and $W^{uu}_{\rm loc} \cap U$ is straightened to be u axis. It is possible to choose $z_3(-T) = (0, 0, 1, 0)$ since $z_3(t) \in T_{\gamma(t)}W^u$. While for

$w_{31} \neq 0$, it is because $\lim_{t\to+\infty} T_{\gamma(t)}W^u = \mathrm{span}\{e^+, e_s^-\}$ and $z_3(T) \in T_{\gamma(T)}W^u$ points to the x axis.

As to $z_i(-T)$ or $z_i(T)$, $i = 1, 2, 4$, one may refer to [9–10] for the similar proof, and we omit the details here.

Remark 2.1 The matrix $(Z^{-1}(t))^*$ is a fundamental solution matrix of (2.3), denoted by $\Phi(t) = (\phi_1(t), \phi_2(t), \phi_3(t), \phi_4(t)) = (Z^{-1}(t))^*$. Then $\phi_1(t) \in (T_{\gamma(t)}W^u)^c \cap (T_{\gamma(t)}W^s)^c$ is bounded and tends to zero exponentially as $| t |\to +\infty$.

Take a new coordinate $N = N(t) = (n_1(t), 0, n_3(t), n_4(t))$ and define

$$s(t) \triangleq \gamma(t) + Z(t)N^* = \gamma(t) + z_1(t)n_1 + z_3(t)n_3 + z_4(t)n_4. \tag{2.4}$$

Substitute it into (1.1),

$$\dot{z(t)} = \dot{\gamma(t)} + \dot{Z(t)}N^* + Z(t)\dot{N}^* = f(\gamma(t) + Z(t)N^*) + g(\gamma(t) + Z(t)N^*, \mu).$$

A simple calculation gives the system in a new coordinates,

$$\dot{n}_i = \phi_i^*(t)g_\mu(\gamma(t), 0)\mu + \text{h.o.t.}, \quad i = 1, 3, 4.$$

Now we want to construct a Poincaré map. Firstly choose two cross sections of Γ (see Figure 1),

$$S_0 = \{z = s(T) : \ |x|, |y|, |u|, |v| < 2\delta\} \subset U,$$
$$S_1 = \{z = s(-T) : \ |x|, |y|, |u|, |v| < 2\delta\} \subset U.$$

Then integrating both sides from $-T$ to T of the above equation, we further achieve

$$n_i(T) = n_i(-T) + M_i\mu + \text{h.o.t.}, \quad i = 1, 3, 4, \tag{2.5}$$

which means a regular map is well defined (see Figure 1(1)):

$$F_1 : S_1 \to S_0, \quad N(-T) \mapsto N(T),$$

where $M_1 = \int_{-\infty}^{+\infty} \phi_1^*(t)g_\mu(\gamma(t), 0)\mathrm{d}t$ is Melnikov vector (see [9–10, 17]) and $M_i = \int_{-T}^{T}\phi_i^*(t)$ $g_\mu(\gamma(t), 0)\mathrm{d}t$, $i = 3, 4$.

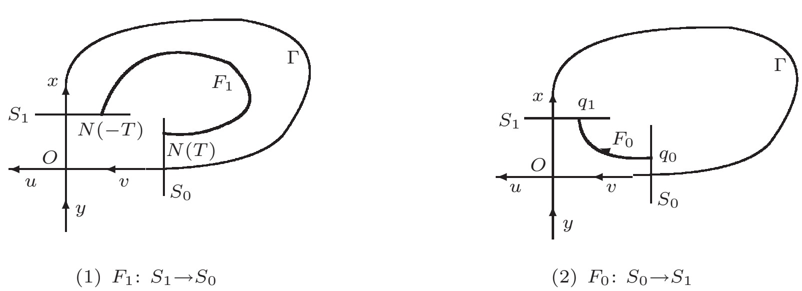


Figure 1 Transition Maps

Set points $s(T)=q_{2j}(x_{2j},y_{2j},u_{2j},v_{2j})\in S_0$, $s(-T)=q_{2j+1}(x_{2j+1},\, y_{2j+1},\, u_{2j+1},\, v_{2j+1})\in S_1$ and $N_{2j}(T)=(n_{2j,1},0,n_{2j,3},n_{2j,4})$, $N_{2j+1}(-T)=(n_{2j+1,1},0,n_{2j+1,3},\ n_{2j+1,4})$, $j=0,1,2,\cdots$. Then take $t=-T$ and T respectively in (2.4), and it is easy to get the following relationship between two coordinate systems:

$$\begin{aligned}&n_{2j,1}=u_{2j}-w_{33}w_{31}^{-1}x_{2j},\\&n_{2j,3}=w_{31}^{-1}x_{2j},\\&n_{2j,4}=y_{2j}-w_{12}u_{2j}+(w_{12}w_{33}-w_{32})w_{31}^{-1}x_{2j},\end{aligned}\tag{2.6}$$

$$\begin{aligned}&n_{2j+1,1}=w_{14}^{-1}v_{2j+1}-w_{44}w_{14}^{-1}w_{42}^{-1}y_{2j+1},\\&n_{2j+1,3}=u_{2j+1}-w_{13}w_{14}^{-1}v_{2j+1}+(w_{13}w_{44}w_{14}^{-1}-w_{43})w_{42}^{-1}y_{2j+1},\\&n_{2j+1,4}=w_{42}^{-1}y_{2j+1},\end{aligned}\tag{2.7}$$

and

$$x_{2j+1}\approx\delta,\quad v_{2j}\approx\delta.\tag{2.8}$$

Next, we set up a singular map $F_0:\ S_0\to S_1,\ q_0(x_0,y_0,u_0,v_0)\mapsto q_1(x_1,y_1,u_1,v_1)$ (see Figure 1(2)) induced by the solutions of system (2.1) in the neighborhood U. Firstly there is

$$\begin{aligned}x(t)&=\mathrm{e}^{\lambda_1(\mu)(t-T-\tau)}\Big\{x_1+\int_{T+\tau}^{t}a(\mu)\mathrm{e}^{-\lambda_1(\mu)(s-T-\tau)}x^2y\,\mathrm{d}s\Big\}+\text{h.o.t.}\\&=\mathrm{e}^{\lambda_1(\mu)(t-T-\tau)}x_1+a(\mu)\int_{T+\tau}^{t}\mathrm{e}^{\lambda_1(\mu)(t-s)}\mathrm{e}^{2\lambda_1(\mu)(s-T-\tau)}x_1^2\mathrm{e}^{-\rho_1(\mu)(s-T)}y_0\,\mathrm{d}s+\text{h.o.t.},\end{aligned}$$

where τ is the time going from $q_0\in S_0$ to $q_1\in S_1$. Denote the Silnikov time $s=\mathrm{e}^{-\lambda_1(\mu)\tau}$. Then there is $x_0\triangleq x(T)=sx_1+O(x_1^2y_0s^2\ln s)$. By the same approach, one can also get the values y_1, u_0 and v_1 as below

$$\begin{aligned}&y_1=y(T+\tau)=s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+O(x_1y_0^2s^2\ln s),\\&u_0=u(T)=s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}}u_1+O(x_1y_0u_1s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}+1}\ln s),\\&v_1=v(T+\tau)=s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}}v_0+O(x_1y_0v_0s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}+1}\ln s).\end{aligned}\tag{2.9}$$

It is exactly the definition of the map F_0.

With all the equations from (2.5)–(2.9), the Poincaré map $F\triangleq F_1\circ F_0$ is well defined as

$$\begin{aligned}n_{21}&=n_{11}+M_1\mu+\text{h.o.t.}\\&=w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}}-w_{44}w_{14}^{-1}w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+M_1\mu+\text{h.o.t.},\\n_{23}&=n_{13}+M_3\mu+\text{h.o.t.}\\&=u_1-w_{13}w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}}+(w_{13}w_{44}w_{14}^{-1}-w_{43})w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+M_3\mu+\text{h.o.t.},\\n_{24}&=n_{14}+M_4\mu+\text{h.o.t.}\\&=w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+M_4\mu+\text{h.o.t.}\end{aligned}$$

Then we can get the associated successor function $G(s,\,u_1,\,y_0)=(G_1,\,G_3,\,G_4)=F(q_0)-q_0$

$$\begin{aligned}&G_1=w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}}-u_1s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}}+w_{33}w_{31}^{-1}\delta s-w_{44}w_{14}^{-1}w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+M_1\mu+\text{h.o.t.},\\&G_3=u_1-w_{13}w_{14}^{-1}\delta s^{\frac{\rho_2(\mu)}{\lambda_1(\mu)}}-w_{31}^{-1}\delta s+(w_{13}w_{44}w_{14}^{-1}-w_{43})w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0+M_3\mu+\text{h.o.t.},\\&G_4=w_{42}^{-1}s^{\frac{\rho_1(\mu)}{\lambda_1(\mu)}}y_0-y_0+w_{12}s^{\frac{\lambda_2(\mu)}{\lambda_1(\mu)}}u_1+(w_{32}-w_{12}w_{33})w_{31}^{-1}\delta s+M_4\mu+\text{h.o.t.}\end{aligned}\tag{2.10}$$

3 Bifurcation Analysis

Notice that if the flying time τ of an orbit starting at a point in S_0 to S_1 is finite, then a periodic orbit of (1.1) exists, and accordingly the Silnikov time $s = \mathrm{e}^{-\lambda_1(\mu)\tau} > 0$; if τ is infinite, then a homoclinic orbit exists, and accordingly $s = 0$. So from the establishment of the associated successor function, it is enough to look for the nonnegative solutions s of (2.10) to study the bifurcations.

First of all, $G_3 = 0$ and $G_4 = 0$ reveal that

$$\begin{aligned} u_1 &= w_{31}^{-1}\delta s - M_3\mu + \text{h.o.t.}, \\ y_0 &= (w_{32} - w_{33}w_{12})w_{31}^{-1}\delta s + M_4\mu + \text{h.o.t.} \end{aligned}$$

Putting them into $G_1 = 0$, we get immediately the bifurcation equation

$$\begin{aligned} F(s,\mu) \equiv& -w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s - w_{32}w_{44}(w_{14}w_{31}w_{42})^{-1}\delta s^2 + w_{33}w_{31}^{-1}\delta s \\ &+ w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s^{\frac{\lambda_2}{\lambda_1}+1} + M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0. \end{aligned} \tag{3.1}$$

Here for concision, we have omitted the parameter μ in $\lambda_i(\mu)$ and $\rho_i(\mu)$, and replaced the exponent $\frac{\rho_1}{\lambda_1}$ by one owing to (H1). Setting $Q = (s, u_1, y_0)$ and $\widetilde{G} = \frac{\partial(G_1,G_3,G_4)}{\partial Q}$, we find that, when $w_{33} \neq 0$,

$$\det\widetilde{G}\Big|_{\substack{Q=0\\ \mu=0}} = \begin{vmatrix} w_{33}w_{31}^{-1}\delta & 0 & 0 \\ -w_{31}^{-1}\delta & 1 & 0 \\ (w_{32}-w_{33}w_{12})w_{31}^{-1}\delta & 0 & -1 \end{vmatrix} \neq 0.$$

Therefore the implicit function theorem reveals that $G = 0$ has a unique solution

$$s = s(\mu), \quad u_1 = u_1(\mu), \quad y_0 = y_0(\mu)$$

satisfying $s(0) = 0$, $u_1(0) = 0$ and $y_0(0) = 0$. So (1.1) has a unique periodic orbit as $s > 0$ or a unique homoclinic orbit as $s = 0$, and they do not coexist. Furthermore, $F(s,\mu) = 0$ has explicitly a sufficiently small positive solution $s = -\delta^{-1}w_{33}^{-1}w_{31}M_1\mu + \text{h.o.t.}$ if $w_{31}w_{33}M_1\mu < 0$. On the other hand, it has a solution $s = 0$ when $\mu \in H^1 \triangleq \{\mu \mid M_1\mu + \text{h.o.t.} = 0\}$. So we have the following theorem.

Theorem 3.1 *Suppose that* $M_1 \neq 0$ *and* $w_{33} \neq 0$ *hold. Then* (1.1) *has at most one* 1-*periodic orbit or one* 1-*homoclinic orbit in the neighborhood of* Γ. *Moreover, a* 1-*periodic orbit exists* (*resp. does not exist*) *as* μ *is in the region defined by* $w_{31}w_{33}M_1\mu < 0$ (*resp.* > 0) *and a* 1-*homoclinic orbit exists as* $\mu \in H^1$, *but they do not coexist* (*see Figure* 2(1–2)).

From Theorem 3.1, we know that (1.1) may have a codimension-1 orbit near Γ homoclinic to the equilibrium O along x and y axes when $\mu \in H^1$, while a codimension-2 orbit flip homoclinic orbit could exist if $y_0 = M_4\mu + \text{h.o.t.} = 0$, where y_0 is given by $G_4 = 0$ (see Figure 2(3)). So the following corollary is true.

Corollary 3.1 *Assume that the hypotheses of Theorem* 3.1 *are still valid. Then* (1.1) *has exactly a codimension*-2 *orbit flip homoclinic orbit near* Γ *as* $\mu \in \{\mu \mid F(0,\mu) = M_1\mu + \text{h.o.t.} = 0,\ y_0 = M_4\mu + \text{h.o.t.} = 0\}$.

To well develop our study, we define two functions

$$H_1(s,\mu) = w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s + \text{h.o.t.},$$

$$H_2(s,\mu) = w_{14}\delta^{-1}M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + s^{\frac{\rho_2}{\lambda_1}} + w_{14}\delta^{-1}M_1\mu + \text{h.o.t.},$$

where $W = H_1(s,\mu)$ is a line and $W = H_2(s,\mu)$ is a curve according to the variable s. Clearly $F(s,\mu) = -w_{14}^{-1}\delta(H_1(s,\mu) - H_2(s,\mu))$.

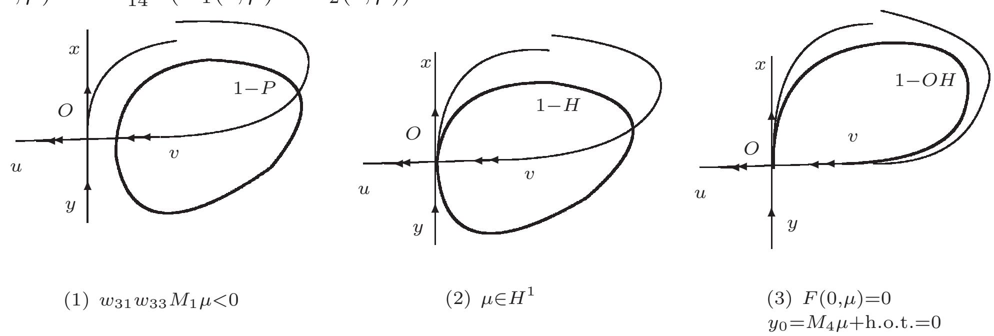


Figure 2 Existence of the 1-periodic orbit (1-P) and the 1-homoclinic orbit (1-H)

Theorem 3.2 *Suppose that* $\text{Rank}(M_1, M_3) = 2$, $\lambda_2 < \rho_2 < 2\lambda_1$ *and* $w_{33} = 0$ *are valid. Then*

(1) (1.1) *has a unique* (*resp. not any*) 1-*periodic orbit for* $M_1\mu M_3\mu < 0$ (*resp.* $M_1\mu M_3\mu > 0$ *and* $w_{14}w_{42}w_{44}M_1\mu M_4\mu < 0$) *and* $|M_1\mu| \ll |M_3\mu|^{\frac{\rho_2}{\rho_2-\lambda_2}}$.

(2) (1.1) *has a unique* (*resp. not any*) 1-*periodic orbit for* $w_{14}M_1\mu < 0$ (*resp.* $w_{14}M_1\mu > 0$ *and* $w_{42}w_{44}M_4\mu < 0$) *and* $|M_3\mu| \ll |M_1\mu|^{\frac{\rho_2-\lambda_2}{\rho_2}}$.

Proof Denote $\overline{s} = \big(-\frac{M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$ as $w_{14}M_1\mu < 0$ and $w_{14}M_3\mu > 0$. Then for $|M_1\mu| \ll |M_3\mu|^{\frac{\rho_2}{\rho_2-\lambda_2}}$, there is $|w_{14}\delta^{-1}M_3\mu\overline{s}^{\frac{\lambda_2}{\lambda_1}}| \gg |\overline{s}^{\frac{\rho_2}{\lambda_1}}|$, so that $H_2(s,\mu) = 0$ has a small positive solution $\overline{s}$. That means the curve $W = H_2(s,\mu)$ is monotonously increasing through the line $W = 0$. Moreover, take $s^* = \big(-\frac{2M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$, $H_2(s^*,\mu) = w_{14}\delta^{-1}M_3\mu\big(-\frac{2M_1\mu}{M_3\mu}\big) + w_{14}\delta^{-1}M_1\mu + \text{h.o.t.} = -w_{14}\delta^{-1}M_1\mu + \text{h.o.t.} \gg w_{44}w_{42}^{-1}\delta^{-1}M_4\mu\big(-\frac{2M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.} = H_1(s^*,\mu)$. So $H_1(s,\mu) = H_2(s,\mu)$ must have a solution $\tilde{s} \in (\overline{s}, s^*)$ (resp. $\tilde{s} \in (0,\overline{s})$) as $w_{42}w_{44}M_4\mu > 0$ (resp. $w_{42}w_{44}M_4\mu < 0$) (see Figure 3(1) (resp. Figure 3(2))), or equivalently $F(s,\mu) = 0$ has a positive solution $\tilde{s}$, which means the existence of a 1-periodic orbit. For the case $w_{14}M_3\mu < 0$, $w_{14}M_1\mu > 0$, the proof is similar (see Figure 3(3)). Thus (1) is true.

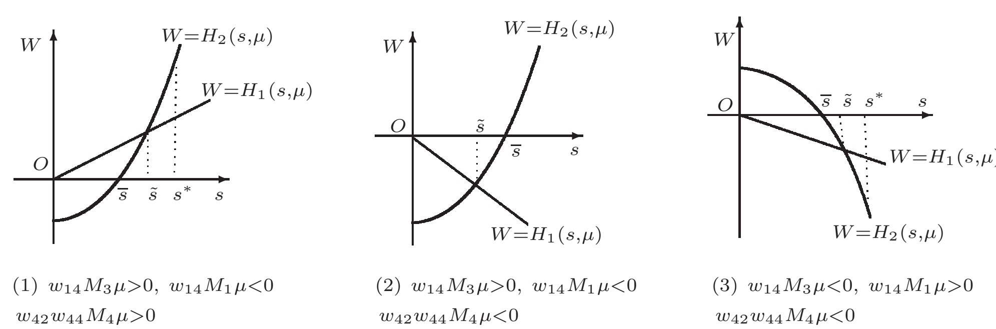


Figure 3 Existence of 1-periodic orbit

Now if $|M_3\mu| \ll |M_1\mu|^{\frac{\rho_2-\lambda_2}{\rho_2}}$, we can verify that $|w_{14}\delta^{-1}M_3\mu\overline{s}^{\frac{\lambda_2}{\lambda_1}}| \ll |\overline{s}^{\frac{\rho_2}{\lambda_1}}|$, where $\overline{s} = (-w_{14}\delta^{-1}M_1\mu)^{\frac{\lambda_1}{\rho_2}} + \text{h.o.t.}$ for $w_{14}M_1\mu < 0$, that is, the term $s^{\frac{\rho_2}{\lambda_1}}$ has a lower order in $H_2(s,\mu)$, so $\overline{s}$ is a positive solution of $H_2(s,\mu) = 0$. With the similar study of the solution of $H_1(s,\mu) = H_2(s,\mu)$, it is easy to get the result (2). This completes the proof.

Next we begin to look for the double 1-periodic orbit bifurcation. Redefine two functions

$$\begin{aligned}P(s,\mu) &= w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s - w_{14}\delta^{-1}M_1\mu + \text{h.o.t.},\\ Q(s,\mu) &= s^{\frac{\rho_2}{\lambda_1}} + w_{14}\delta^{-1}M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + \text{h.o.t.}\end{aligned}$$

We know that a double 1-periodic orbit bifurcation surface exists if and only if $P(s,\mu) = Q(s,\mu)$, $P'(s,\mu) = Q'(s,\mu)$ and $P''(s,\mu) \neq Q''(s,\mu)$ hold, that is,

$$\begin{aligned}&w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s - M_1\mu = w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}} + M_3\mu s^{\frac{\lambda_2}{\lambda_1}} + \text{h.o.t.},\\ &w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu = \rho_2\lambda_1^{-1}w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}-1} + \lambda_2\lambda_1^{-1}M_3\mu s^{\frac{\lambda_2}{\lambda_1}-1} + \text{h.o.t.},\\ &0 \neq \rho_2(\rho_2-\lambda_1)w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}-2} + \lambda_2(\lambda_2-\lambda_1)M_3\mu s^{\frac{\lambda_2}{\lambda_1}-2} + \text{h.o.t.}\end{aligned} \tag{3.2}$$

In the region $|M_4\mu| \ll |M_3\mu|^{\frac{\rho_2-\lambda_1}{\rho_2-\lambda_2}}$, the second equation of (3.2) yields a solution

$$s_* = \Big(\frac{\lambda_1 w_{44}M_4\mu}{\lambda_2 w_{14}w_{42}M_3\mu}\Big)^{\frac{\lambda_1}{\lambda_2-\lambda_1}} + \text{h.o.t.}$$

as $w_{42}w_{44}M_4\mu > 0$ and $w_{14}M_3\mu > 0$ or $w_{42}w_{44}M_4\mu < 0$ and $w_{14}M_3\mu < 0$. Combining with the value of s_*, the first equation of (3.2) gives a tangency condition, which corresponds to the existence of the double periodic orbit bifurcation surface

$$SN_1^1:\ M_1\mu = \frac{\lambda_2-\lambda_1}{\lambda_1}M_3\mu\Big(\frac{\lambda_1 w_{44}M_4\mu}{\lambda_2 w_{14}w_{42}M_3\mu}\Big)^{\frac{\lambda_2}{\lambda_2-\lambda_1}} + \text{h.o.t.}$$

for $w_{14}M_1\mu > 0$, $w_{14}M_3\mu > 0$ and $w_{42}w_{44}M_4\mu > 0$ or $w_{14}M_1\mu < 0$, $w_{14}M_3\mu < 0$ and $w_{42}w_{44}M_4\mu < 0$.

Notice that, when the tangency takes place, the line $W = P(s,\mu)$ lies under the curve $W = Q(s,\mu)$. So if $-w_{14}M_1\mu$ increases (resp. decreases), the line must intersect the curve at two (resp. no) sufficiently small positive points. Namely, $F(s,\mu) = 0$ has two small positive solutions, or equivalently, two 1-periodic orbits appear on the side of SN_1^1 which points to $(\operatorname{sgn} M_3\mu)M_1$.

If $|M_3\mu| \ll |M_4\mu|^{\frac{\rho_2-\lambda_2}{\rho_2-\lambda_1}}$, the leading term of $Q(s,\mu)$ becomes $s^{\frac{\rho_2}{\lambda_1}}$, and therefore the tangent point is

$$s_* = \Big(\frac{\lambda_1 w_{44}M_4\mu}{\rho_2\delta w_{42}}\Big)^{\frac{\lambda_1}{\rho_2-\lambda_1}} + \text{h.o.t.}$$

as $w_{42}w_{44}M_4\mu > 0$. Another double 1-periodic orbit bifurcation surface SN_2^1 exists with the expression

$$w_{14}M_1\mu = \frac{\rho_2-\lambda_1}{\lambda_1}\delta\Big(\frac{\lambda_1 w_{44}M_4\mu}{\rho_2\delta w_{42}}\Big)^{\frac{\rho_2}{\rho_2-\lambda_1}} + \text{h.o.t.}$$

confined to $w_{42}w_{44}M_4\mu > 0$ and $w_{14}M_1\mu > 0$, which has the normal vector M_1 at $\mu = 0$.

Furthermore, when $2\lambda_1 > \rho_2 > \lambda_2$, $P''(s,\mu) = Q''(s,\mu)$ is solvable with the solution

$$s^* = \Big(-\frac{\lambda_2(\lambda_2-\lambda_1)w_{14}M_3\mu}{\rho_2(\rho_2-\rho_1)\delta}\Big)^{\frac{\lambda_1}{\rho_2-\lambda_2}} + \text{h.o.t.}$$

for $w_{14}M_3\mu < 0$, which is a triple small positive solution of $P(s,\mu) = Q(s,\mu)$. Multiplying both sides of the second equation of (3.2) by s^* and substituting it into the first one, we get easily the triple 1-periodic orbit bifurcation surface

$$SN^2: \quad w_{14}M_1\mu = \frac{(\lambda_1-\rho_2)(\rho_2-\lambda_2)\delta}{\lambda_1\lambda_2}\Big(-\frac{\lambda_2(\lambda_2-\lambda_1)w_{14}M_3\mu}{\rho_2(\rho_2-\rho_1)\delta}\Big)^{\frac{\rho_2}{\rho_2-\lambda_2}} + \text{h.o.t.},$$
$$M_4\mu = \frac{\rho_2(\lambda_2-\rho_2)w_{42}\delta}{\lambda_1(\lambda_2-\lambda_1)w_{44}}\Big(-\frac{\lambda_2(\lambda_2-\lambda_1)w_{14}M_3\mu}{\rho_2(\rho_2-\rho_1)\delta}\Big)^{\frac{\rho_2-\lambda_1}{\rho_2-\lambda_2}} + \text{h.o.t.}$$

for $w_{14}M_1\mu < 0$, $w_{14}M_3\mu < 0$ and $w_{42}w_{44}M_4\mu < 0$.

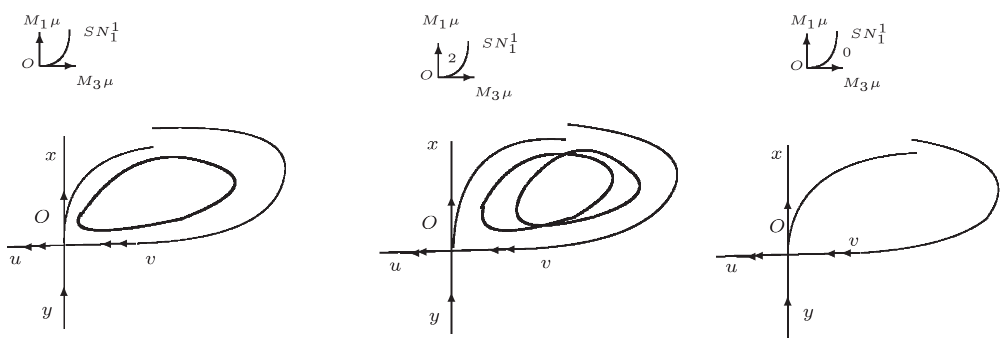
Obviously, the hypersurface has a normal plane $\text{span}\{M_1, M_4\}$ at $\mu = 0$. To sum up, there are the following results.

Theorem 3.3 *Suppose that* $\text{Rank}(M_1, M_3, M_4) = 3$, $2\lambda_1 > \rho_2 > \lambda_2$ *and* $w_{33} = 0$ *hold. Then in the small neighborhood of the orgin of* μ *space,*

(1) *For* $w_{14}M_1\mu > 0$, $w_{14}M_3\mu > 0$ *and* $w_{42}w_{44}M_4\mu > 0$ (*or* $w_{14}M_1\mu < 0$, $w_{14}M_3\mu < 0$ *and* $w_{42}w_{44}M_4\mu < 0$)*, there exists a double* 1*-periodic orbit bifurcation surface* SN_1^1 *as* $|M_4\mu| \ll |M_3\mu|^{\frac{\rho_2-\lambda_1}{\rho_2-\lambda_2}}$ *or a double* 1*-periodic orbit bifurcation surface* SN_2^1 *as* $|M_3\mu| \ll |M_4\mu|^{\frac{\rho_2-\lambda_2}{\rho_2-\lambda_1}}$.

Moreover, they both have the normal vector M_1 *at* $\mu = 0$ *and bifurcate two* 1*-periodic orbits on their side pointing to the direction* $(\text{sgn}\, M_3\mu)M_1$ *or* $(\text{sgn}\, w_{14})M_1$ *respectively, and have no* 1*-periodic orbits on the other side* (*see Figure* 4).

(2) *For* $w_{14}M_1\mu < 0$, $w_{14}M_3\mu < 0$ *and* $w_{42}w_{44}M_4\mu < 0$, *there exists a triple* 1*-periodic orbit bifurcation surface* SN^2 *with the normal plane* $\text{span}\{M_1, M_4\}$ *at* $\mu = 0$, *such that* (1.1) *has exactly a triple* 1*-periodic orbit when* $\mu \in SN^2$.



(1) $\mu \in SN_1^1$ a double 1-periodic orbit

(2) $\mu \in SN_1^1$ pointing to $(\text{sgn}\, M_3\mu)M_1$, a two 1-periodic orbit

(3) $\mu \in SN_1^1$ pointing against $(\text{sgn}\, M_3\mu)M_1$, no 1-periodic orbit

Figure 4 $w_{14}M_1\mu > 0$, $w_{14}M_3\mu > 0$, $w_{42}w_{44}M_4\mu > 0$

Remark 3.1 There does not exist an n-multiple 1-periodic orbit bifurcation surface in any case for $n > 3$.

In the case of $M_i\mu = 0$ for $i = 1, 3, 4$, some fruitful bifurcation results are obtained.

Theorem 3.4 *Suppose that $2\lambda_1 > \rho_2 > \lambda_2$ and $w_{33} = 0$ hold. Then there are the following results:*

(1) *If $M_1 = 0$, (1.1) has exactly a 1-homoclinic orbit, and moreover,*

(a) (1.1) *has a unique (resp. not any) 1-periodic orbit for $w_{42}w_{44}M_4\mu > 0$ (resp. $w_{42}w_{44}M_4\mu < 0$ and $w_{14}M_3\mu > 0$);*

(b) *for $\mathrm{Rank}(M_3, M_4) = 2$, $w_{14}M_3\mu < 0$ and $w_{42}w_{44}M_4\mu < 0$, there exists a double 1-periodic orbit bifurcation surface*

$$SN_3^1 : \ M_4\mu = -\frac{(\rho_2 - \lambda_2)\delta w_{42}}{(\lambda_2 - \lambda_1)w_{44}}\Big(- \frac{(\lambda_2 - \lambda_1)w_{14}M_3\mu}{(\rho_2 - \lambda_1)\delta}\Big)^{\frac{\rho_2-\lambda_1}{\rho_2-\lambda_2}} + \text{h.o.t.}$$

with a normal vector M_4 at $\mu = 0$, which undergoes two 1-periodic orbits when μ lies on the side of SN_3^1 pointing to the direction $(\mathrm{sgn}\, w_{42}w_{44})M_4$ and no undergoes 1-periodic orbit in the opposite direction.

(2) *If $M_3 = 0$,*

(a) (1.1) *has exactly a (resp. not any) 1-periodic orbit for $w_{14}M_1\mu < 0$ (resp. $w_{14}M_1\mu > 0$ and $w_{42}w_{44}M_4\mu < 0$);*

(b) *for $\mathrm{Rank}(M_1, M_4) = 2$, $w_{14}M_1\mu > 0$ and $w_{42}w_{44}M_4\mu > 0$, there exists a double 1-periodic orbit bifurcation surface*

$$SN_4^1 : \ w_{14}M_1\mu = \frac{\rho_2 - \lambda_1}{\lambda_1\delta}\Big(\frac{\lambda_1 w_{44}M_4\mu}{\rho_2\delta w_{42}}\Big)^{\frac{\rho_2}{\rho_2-\lambda_1}} + \text{h.o.t.}$$

with a normal vector M_1 at $\mu = 0$, which bifurcates two 1-periodic orbits when μ locates on the side of SN_4^1 directed by $-(\mathrm{sgn}\, w_{14})M_1$ and bifurcates no 1-periodic orbit on the opposite side.

(3) *If $M_4 = 0$,*

(a) (1.1) *has exactly a (resp. not any) 1-periodic orbit for $w_{14}M_1\mu < 0$ (resp. $w_{14}M_1\mu > 0$ and $w_{14}M_3\mu > 0$);*

(b) *for $\mathrm{Rank}(M_1, M_3) = 2$, $w_{14}M_1\mu > 0$ and $w_{14}M_3\mu < 0$, there exists a double 1-periodic orbit bifurcation surface*

$$SN_5^1 : \ w_{14}M_1\mu = \frac{\rho_2 - \lambda_2}{\lambda_2}\delta\Big(- \frac{\lambda_2 w_{14}M_3\mu}{\rho_2\delta}\Big)^{\frac{\rho_2}{\rho_2-\lambda_2}} + \text{h.o.t.}$$

with a normal vector M_1 at $\mu = 0$, which bifurcates two 1-periodic orbits when μ locates on the side of SN_5^1 directed by $-(\mathrm{sgn}\, w_{14})M_1$ and bifurcates no 1-periodic orbit on the opposite side.

(4) *If $M_1^2 + M_3^2 = 0$ or $M_1^2 + M_4^2 = 0$, (1.1) has exactly a 1-homoclinic orbit and a 1-periodic orbit for $w_{42}w_{44}M_4\mu > 0$ or $w_{14}M_3\mu < 0$ respectively.*

(5) *If $M_3^2 + M_4^2 = 0$, (1.1) has only a 1-periodic orbit for $w_{14}M_3\mu < 0$.*

(6) *If $M_1^2 + M_3^2 + M_4^2 = 0$, (1.1) has only a 1-homoclinic orbit.*

Proof Recall that for $M_1 = 0$, (3.1) becomes

$$F(s, \mu) \equiv s(-w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu + M_3\mu s^{\frac{\lambda_2}{\lambda_1}-1} + w_{14}^{-1}\delta s^{\frac{\rho_2}{\lambda_1}-1} + \text{h.o.t.}) = 0.$$

Obviously, it has a zero solution that corresponds to a 1-homoclinic orbit. Set $s^{\frac{\lambda_2-\lambda_1}{\lambda_1}}=h$, and

$$P_1(h,\mu)=\delta^{-1}w_{44}w_{42}^{-1}M_4\mu-\delta^{-1}w_{14}M_3\mu h+\text{h.o.t.},$$
$$Q_1(h,\mu)=h^{\frac{\rho_2-\lambda_1}{\lambda_2-\lambda_1}}+\text{h.o.t.}$$

Similar to the proof of Thereom 3.2, it is easy to get a 1-periodic orbit as $w_{42}w_{44}M_4\mu>0$ owing to the relative position of the line $W=P_1(h,\mu)$ and the curve $W=Q_1(h,\mu)$. Thereby the claim (1)(a) holds.

Further, equations $P_1(h,\mu)=Q_1(h,\mu)$, $P_1'(h,\mu)=Q_1'(h,\mu)$ and $P_1''(h,\mu)\neq Q_1''(h,\mu)$ exactly determine the double 1-periodic orbit bifurcation surface SN_3^1 for $w_{14}M_3\mu<0$ and $w_{42}w_{44}M_4\mu<0$ with the corresponding tangent point $\overline{h}=\left(-\frac{(\lambda_2-\lambda_1)w_{14}M_3\mu}{(\rho_2-\lambda_1)\delta}\right)^{\frac{\lambda_2-\lambda_1}{\rho_2-\lambda_2}}+\text{h.o.t.}$ Without difficulty, on the side of SN_3^1 which points to $(\operatorname{sgn} w_{42}w_{44})M_4$, there appear two 1-periodic orbits. Now (1) is complete.

For the cases (2) and (3), the proofs are very close to those in Theorems 3.2–3.3 or as above. We omit the details here and only give the tangent points of the corresponding double 1-periodic orbit bifurcation surface. When $M_3=0$, set

$$P_2(s,\mu)=w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s-w_{14}\delta^{-1}M_1\mu+\text{h.o.t.},$$
$$Q_2(s,\mu)=s^{\frac{\rho_2}{\lambda_1}}+\text{h.o.t.}$$

So $P_2'(s,\mu)=Q_2'(s,\mu)$ gives a small positive solution $s_*=\left(\frac{\lambda_1w_{44}M_4\mu}{\rho_2w_{42}\delta}\right)^{\frac{\lambda_1}{\rho_2-\lambda_1}}+\text{h.o.t.}$ for $w_{42}w_{44}M_4\mu>0$. $P_2(s_*,\mu)=Q_2(s_*,\mu)$ subsequently defines SN_4^1.

In the case $M_4=0$, one may set another kind of curves

$$P_3(h,\mu)=-\delta^{-1}w_{14}M_3\mu h-\delta^{-1}w_{14}M_1\mu+\text{h.o.t.},$$
$$Q_3(h,\mu)=h^{\frac{\rho_2}{\lambda_2}}+\text{h.o.t.},$$

where $s^{\frac{\lambda_2}{\lambda_1}}=h$, and the relevant tangent point is $h_*=\left(-\frac{\lambda_2w_{14}M_3\mu}{\rho_2\delta}\right)^{\frac{\lambda_2}{\rho_2-\lambda_2}}+\text{h.o.t.}$ for $w_{14}M_3\mu<0$.

As to the case (4), if $M_1^2+M_3^2=0$, $F(s,\mu)=s(-w_{44}w_{42}^{-1}w_{14}^{-1}M_4\mu+w_{14}^{-1}\delta s^{\frac{\rho_2-\lambda_1}{\lambda_1}}+\text{h.o.t.})=0$ has two solutions $s_1=0$ and $s_2=\left(\frac{w_{44}M_4\mu}{w_{42}\delta}\right)^{\frac{\lambda_1}{\rho_2-\lambda_1}}+\text{h.o.t.}$ for $w_{42}w_{44}M_4\mu>0$.

If $M_1^2+M_4^2=0$, $F(s,\mu)=s^{\frac{\lambda_2}{\lambda_1}}(w_{14}^{-1}\delta s^{\frac{\rho_2-\lambda_2}{\lambda_1}}+M_3\mu+\text{h.o.t.})=0$ gives $s_1=0$ and $s_3=(-w_{14}\delta^{-1}M_3\mu)^{\frac{\lambda_1}{\rho_2-\lambda_2}}+\text{h.o.t.}$ for $w_{14}M_3\mu<0$, corresponding to a 1-homoclinic orbit and a 1-periodic orbit respectively.

The last claim is very clear. We finish the proof here.

From now on, we try to study the homoclinic doubling bifurcation. To begin with, we need to get the second returning successor function $F\circ F(q_0)-q_0$. Reset τ_1 and τ_2 to be the time going from $q_0(x_0,y_0,u_0,v_0)\in S_0$ to $q_1(x_1,y_1,u_1,v_1)\in S_1$ and from $q_2(x_2,y_2,u_2,v_2)\in S_0$ to $q_3(x_3,y_3,u_3,v_3)\in S_1$ respectively, $s_1=\mathrm{e}^{-\lambda_1\tau_1}$, $s_2=\mathrm{e}^{-\lambda_1\tau_2}$, and $F_0(q_0)=q_1$, $F_1(q_1)=q_2$, $F_0(q_2)=q_3$, $F_1(q_3)=q_4=q_0$. Repeat the process of the establishment of (2.10), $F\circ F(q_0)-q_0$ can be expressed as $G^2(s_1,s_2,u_1,u_3,y_0,y_2)=(G_1^1,G_1^3,G_1^4,G_2^1,G_2^3,G_2^4)$:

$$G_1^1=w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}}-w_{44}w_{14}^{-1}w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0-u_3s_2^{\frac{\lambda_2}{\lambda_1}}+w_{33}w_{31}^{-1}\delta s_2+M_1\mu+\text{h.o.t.},$$

$$G_1^3 = u_1 - w_{13}w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s_2 + (w_{13}w_{44}w_{14}^{-1} - w_{43})w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0 + M_3\mu + \text{h.o.t.},$$
$$G_1^4 = w_{42}^{-1}s_1^{\frac{\rho_1}{\lambda_1}}y_0 - y_2 + w_{12}s_2^{\frac{\lambda_2}{\lambda_1}}u_3 + (w_{32} - w_{12}w_{33})w_{31}^{-1}\delta s_2 + M_4\mu + \text{h.o.t.},$$
$$G_2^1 = w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 - u_1 s_1^{\frac{\lambda_2}{\lambda_1}} + w_{33}w_{31}^{-1}\delta s_1 + M_1\mu + \text{h.o.t.},$$
$$G_2^3 = u_3 - w_{13}w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{31}^{-1}\delta s_1 + (w_{13}w_{44}w_{14}^{-1} - w_{43})w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 + M_3\mu + \text{h.o.t.},$$
$$G_2^4 = w_{42}^{-1}s_2^{\frac{\rho_1}{\lambda_1}}y_2 - y_0 + w_{12}s_1^{\frac{\lambda_2}{\lambda_1}}u_1 + (w_{32} - w_{12}w_{33})w_{31}^{-1}\delta s_1 + M_4\mu + \text{h.o.t.}$$

When $w_{33} = 0$ and $2\lambda_1 > \rho_2 > \lambda_2$, eliminating y_0, u_1, y_2 and u_3 from $G_i^j = 0$, $i = 1, 2$ and $j = 3, 4$, and putting them into $G_1^1 = 0$ and $G_2^1 = 0$, we obtain

$$w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_1 - w_{31}^{-1}\delta s_1 s_2^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0, \tag{3.3}$$
$$w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_2 - w_{31}^{-1}\delta s_2 s_1^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_1^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0. \tag{3.4}$$

Notice that a 2-homoclinic orbit Γ^2 means that the orbit returns twice near the singular point in finite and infinite time respectively, which corresponds to the solution $s_1 = 0$ and $s_2 > 0$ or $s_1 > 0$ and $s_2 = 0$ of (3.3) and (3.4). So it is sufficient to seek the small solutions of $s_1 = 0$ and $s_2 > 0$ by the symmetry of G^2. Therefore

$$M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0, \tag{3.5}$$
$$s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{42}^{-1}\delta^{-1}M_4\mu s_2 + \delta^{-1}w_{14}M_1\mu + \text{h.o.t.} = 0. \tag{3.6}$$

(3.5) has a solution $s_2 = \big(-\frac{M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$ for $M_1\mu M_3\mu < 0$ and $\frac{|M_1\mu|}{|M_3\mu|}$ sufficiently small. With this, (3.6) determines a 2-homoclinic orbit bifurcation surface in the region $\{\mu \mid\ |M_1\mu| \ll |M_3\mu|^{\frac{\rho_2}{\rho_2-\lambda_2}}\}$

$$H^2:\ w_{14}M_1\mu = -\delta\Big(-\frac{M_1\mu}{M_3\mu}\Big)^{\frac{\rho_2}{\lambda_2}} + w_{44}w_{42}^{-1}M_4\mu\Big(-\frac{M_1\mu}{M_3\mu}\Big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$$

for $M_1\mu M_3\mu < 0$ and $(w_{14}M_1\mu)w_{42}w_{44}M_4\mu > 0$, which has a normal vector M_1 at $\mu = 0$.

Going through the same procedure of finding H^2, one can still find a 4-homoclinic orbit bifurcation surface H^4 (see [9–10]). Concretely, we first have, parallel to (3.3) and (3.4),

$$\begin{aligned}
&w_{14}^{-1}\delta s_1^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_1 - w_{31}^{-1}\delta s_1 s_2^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0,\\
&w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_2 - w_{31}^{-1}\delta s_2 s_3^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_3^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0,\\
&w_{14}^{-1}\delta s_3^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_3 - w_{31}^{-1}\delta s_3 s_4^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_4^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0,\\
&w_{14}^{-1}\delta s_4^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_4 - w_{31}^{-1}\delta s_4 s_1^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_1^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0.
\end{aligned} \tag{3.7}$$

Without loss of generality, it is enough to consider the solution $s_1 = 0$ and $s_i > 0$, $i = 2, 3, 4$. Then (3.7) is

$$M_3\mu s_2^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0, \tag{3.8}$$

$$w_{14}^{-1}\delta s_2^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_2 - w_{31}^{-1}\delta s_2 s_3^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_3^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0, \tag{3.9}$$

$$w_{14}^{-1}\delta s_3^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_3 - w_{31}^{-1}\delta s_3 s_4^{\frac{\lambda_2}{\lambda_1}} + M_3\mu s_4^{\frac{\lambda_2}{\lambda_1}} + M_1\mu + \text{h.o.t.} = 0, \tag{3.10}$$

$$w_{14}^{-1}\delta s_4^{\frac{\rho_2}{\lambda_1}} - w_{44}w_{14}^{-1}w_{42}^{-1}M_4\mu s_4 + M_1\mu + \text{h.o.t.} = 0. \tag{3.11}$$

Evidently (3.8) gives $s_2 = \big(-\frac{M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$ for $M_1\mu M_3\mu < 0$. And $s_4 = \frac{w_{14}w_{42}M_1\mu}{w_{44}M_4\mu} + \text{h.o.t.}$ from (3.11) for $w_{14}w_{42}w_{44}M_1\mu M_4\mu > 0$. (3.10) permits

$$s_3 = \frac{w_{14}w_{42}M_1\mu}{w_{44}M_4\mu} - \frac{w_{14}w_{42}\delta_u}{w_{44}M_4\mu}\Big(\frac{w_{14}w_{42}M_1\mu}{w_{44}M_4\mu}\Big)^{\frac{\lambda_2}{\lambda_1}} + \text{h.o.t.}$$

With all these values of s_i for $i = 2, 3, 4$, (3.9) finally defines the 4-homoclinic orbit bifurcation surface with the same principal part of H^2:

$$H^4: \ w_{14}M_1\mu = -\delta\Big(-\frac{M_1\mu}{M_3\mu}\Big)^{\frac{\rho_2}{\lambda_2}} + w_{44}w_{42}^{-1}M_4\mu\Big(-\frac{M_1\mu}{M_3\mu}\Big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$$

for $M_1\mu M_3\mu < 0$ and $(w_{14}M_1\mu)w_{42}w_{44}M_4\mu > 0$, which has a vector M_1 at $\mu = 0$.

If we repeat the process, the 2^n-homoclinic orbit bifurcation surface H^{2^n} can be achieved in the approximate form $w_{14}M_1\mu = w_{44}w_{42}^{-1}M_4\mu\big(-\frac{M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$ To sum up, we conclude the following.

Theorem 3.5 *Suppose that* $\text{Rank}(M_1, M_3, M_4) = 3$, $2\lambda_1 > \rho_2 > \lambda_2$ *and* $w_{33} = 0$ *hold. Then in the neighborhood of the origin of* μ *space, there exists a* 2^n*-homoclinic orbit bifurcation surface* H^{2^n} : $w_{14}M_1\mu = w_{44}w_{42}^{-1}M_4\mu\big(-\frac{M_1\mu}{M_3\mu}\big)^{\frac{\lambda_1}{\lambda_2}} + \text{h.o.t.}$ *for* $|M_1\mu| \ll |M_3\mu|^{\frac{\rho_2}{\rho_2-\lambda_2}}$, $(w_{14}M_1\mu)w_{42}w_{44}M_4\mu > 0$ *and* $M_1\mu M_3\mu < 0$*, which has the normal vector* M_1 *at* $\mu = 0$.

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