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Global Existence, Uniqueness and Pathwise Property of Solutions to a Stochastic Rössler-Lorentz System*

Song JIANG[1] Junping YIN[2]

Abstract The authors integrate two well-known systems, the Rössler and Lorentz systems, to introduce a new chaotic system, called the Lorentz-Rössler system. Then, taking into account the effect of environmental noise, the authors incorporate white noise in both Rössler and Lorentz systems to have a corresponding stochastic system. By deriving the uniform a priori estimates for an approximate system and then taking them to the limit, the authors prove the global existence, uniqueness and the pathwise property of solutions to the Lorentz-Rössler system. Moreover, the authors carried out a number of numerical experiments, and the numerical results demonstrate their theoretic analysis and show some new qualitative properties of solutions which reveal that the Lorentz-Rössler system could be used to design more complex and more secure nonlinear hop-frequence time series.

Keywords Stochastic differential equations, Rössler-Lorentz systems, Existence, Pathwise property
2000 MR Subject Classification 60J65, 34K40

1 Introduction

The original Rössler system reads as

$$\begin{aligned}\dot{x}_1 &= x_2 - x_3,\\ \dot{x}_2 &= x_1 + ax_2,\\ \dot{x}_3 &= b + x_3(x_1 - c),\end{aligned}\tag{1.1}$$

which contains only one quadratic nonlinear term x_1x_3 and was introduced by Rössler in 1976 (see [18]). In the recent years, this model has received increasing attention due to its theoretical challenges and great potential applications in secure communications (see [2, 5, 9, 15]), chemical reaction, biological systems and so on (see [1]). The Lorentz system, introduced by Lorentz [12] in 1963,

$$\begin{aligned}\dot{x}_1 &= \sigma(x_2 - x_1),\\ \dot{x}_2 &= rx_1 - x_2 - x_1x_3,\\ \dot{x}_3 &= x_1x_2 - \beta x_3,\end{aligned}\tag{1.2}$$

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[1]Institute of Applied Physics and Computational Mathematics, Beijing 100088, China.
[2]Institute of Applied Physics and Computational Mathematics, Beijing 100088, China. Beijing Center for Mathematics and Information Interdisciplinary Sciences, China. E-mail: yinjp829829@126.com
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is a well-known model, which has found a lot of applications in different fields, and for example, we refer the reader to [16] on chaos, [7] on instabilities and lasers, [6] on thermospheres, [8] on brushless DC motors, [4] on electric circuits, and [17] on chemical reactions.

Due to their wide applications, the systems (1.1)–(1.2), and in particular, the related nonlinear stochastic systems, have been intensively studied in the last decades in the literature (see [3, 10–11, 13, 19–20]).

In this paper, we consider a more general model which includes the Rössler and Lorentz systems and study the well-poseness of its corresponding stochastic system:

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) - \gamma(t)(x_2 - x_3), \\ \dot{x}_2 &= r x_1 - x_2 - \alpha_1(t) x_1 x_3 + x_1 + a x_2, \\ \dot{x}_3 &= \alpha_2(t) x_1 x_2 - \beta x_3 + b + \alpha_3(t) x_1 (x_3 - c). \end{aligned} \tag{1.3}$$

In fact, if we suitably take the coefficients in (1.3), then the system (1.3) reduces to (1.1) or (1.2). Therefore, the main properties of the Rössler and Lorentz systems can be included in this model. The following Figure 1 shows the attractor and the time series of (1.3) with the initial data $X_0 = (8, 5, 30)$, where we have taken $\alpha_3(t) = 0,\ b = 0,\ \alpha_1(t) = \alpha_2(t) = 1,\ a = -1$, such that (1.3) reduces to the Lorentz system (1.2).

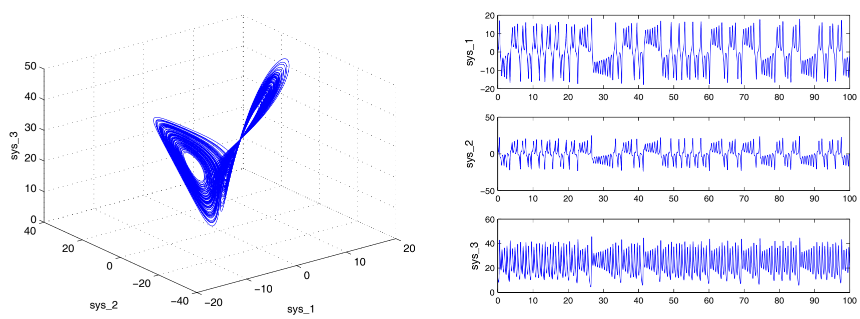

Figure 1 The attractor and the time series of the Lorentz system.

Let $\gamma(t) = 1,\ \alpha_1(t) = \alpha_2(t) = 0,\ \sigma = 0$, and appropriately choose other coefficients in (1.3). Then one can obtain the attractor of the Rössler system and the time series of it are shown in Figure 2.

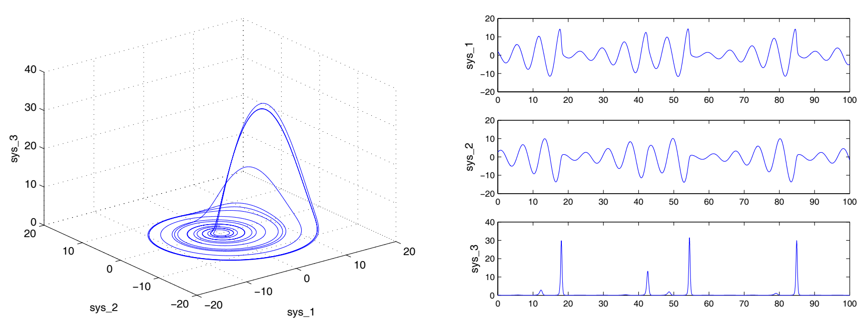

Figure 2 The behavior similar to the Rössler system (the initial value $X_0 = (3, -4, 2)$).

If we take $\sigma = 9,\ \gamma(t) = -1,\ r = 27,\ a = b = c = 0,\ \alpha_1(t) = \alpha_2(t) = 2,\ \alpha_3(t) = 1,\ \beta = \frac{8}{3}$ in (1.3), then we obtain a Lorentz-Rössler system, the solution behavior of which is presented in Figure 3 below.

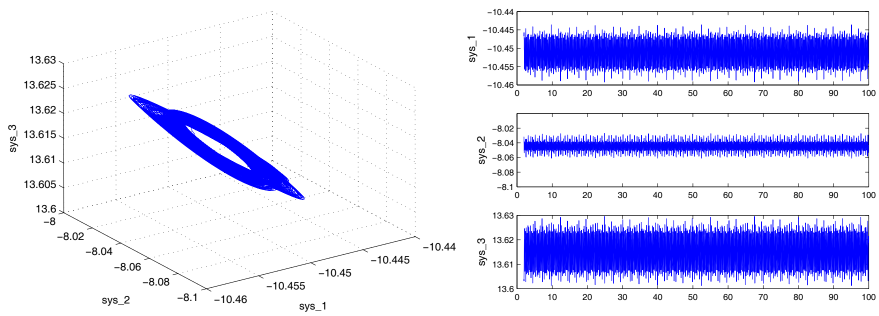

Figure 3 A Rössler-Lorentz system with initial value $X_0 = (1, 0.1, 1)$.

We can also consider the case of time-dependent coefficients in (1.3), and for example, if we take $\sigma = 10,\ \gamma(t) = \frac{1}{t^2+10000},\ r = 27,\ a = b = 0,\ c = 1,\ \alpha_1(t) = 2,\ \alpha_2(t) = \frac{2}{(\cos(t^2)+10)} + 10,\ \alpha_3(t) = 1$, then the behavior of the solution is shown in Figure 4.

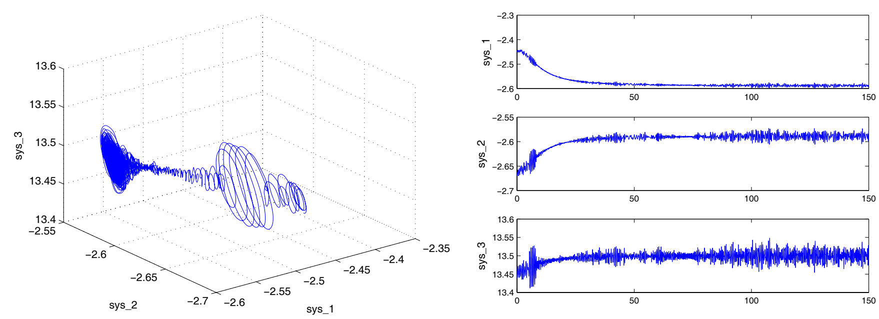

Figure 4 The Rössler-Lorentz with time-dependent coefficients.

Remark 1.1 From Figures 1–4, we clearly see that upon different choices of the coefficients, there are many diversified attractors given by the system (1.3). Since the structure of solutions to our new system (1.3) is more complex than that of both Lorentz and Rössler systems, shown for example in Figure 3, which have many good properties including nonlinear complexity, non-periodicity, well-proportioned character and so on, the system (1.3) can be used in more secure applications to design a more complex and more secure hop-frequency time series. If we use these nonlinear time series to control the frequency hopping pattern, the communications would become much more difficult to be disturbed.

In order to make the model (1.3) more widely applicable, we should take into account the effect of environment noise, particularly, in secure outer communications (complex electric circumstances), convulsed circuits communications, multi-level chemical reactions and so on.

Thus we incorporate white noise in each equation of the system (1.3):

$$
\begin{aligned}
\mathrm{d}x_1 &= [\sigma(x_2 - x_1) - \gamma(t)(x_2 - x_3)]\mathrm{d}t + \sum_{i=1}^{3} u_{1i}(x_1, x_2, x_3)\mathrm{d}B_i(t),\\
\mathrm{d}x_2 &= [rx_1 - x_2 - \alpha_1(t)x_1x_3 + x_1 + ax_2]\mathrm{d}t + \sum_{i=1}^{3} u_{2i}(x_1, x_2, x_3)\mathrm{d}B_i(t),\\
\mathrm{d}x_3 &= [\alpha_2(t)x_1x_2 - \beta x_3 + b + \alpha_3(t)x_1(x_3 - c)]\mathrm{d}t + \sum_{i=1}^{3} u_{3i}(x_1, x_2, x_3)\mathrm{d}B_i(t),
\end{aligned}
\tag{1.4}
$$

where u_{ij} represents the intensity of the noise at time t and $B_i(t)$ is a standard white noise, namely, $B_i(t)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. If we consider the case of Figures 1–4 with environment noise, then the corresponding stochastic system (1.4) can be illustrated by Figures 5–8, respectively.

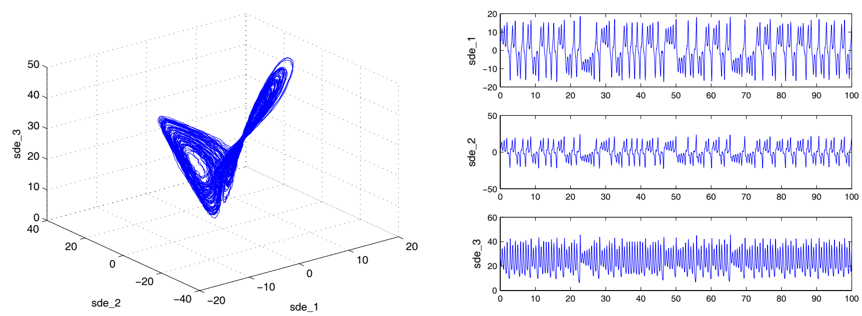

Figure 5 The attractor and time series of the Stochastic Lorentz system with a standard irrelated white noise ($u_{ii} = 1,\ u_{ij,i\neq j} = 0,\ i, j = 1, 2, 3$).

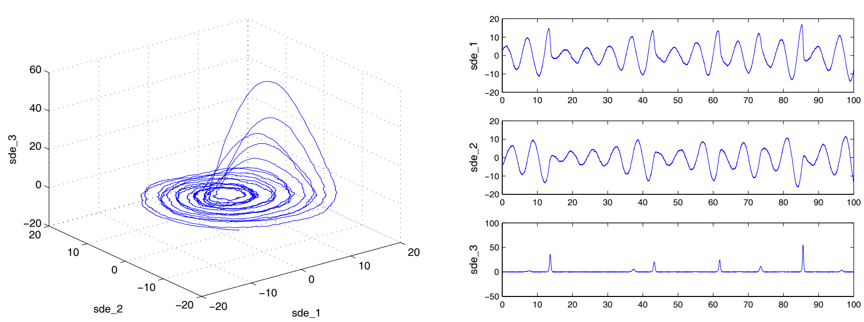

Figure 6 The stochastic Rössler system, and the intensity of the noise: $u_{ii} = 0.5,\ u_{ij,i\neq j} = 0,\ i, j = 1, 2, 3.$

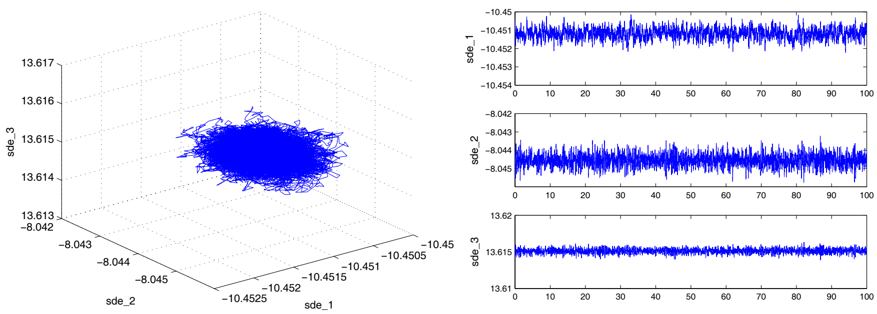

Figure 7 The stochastic Rössler-Lorentz system with the noise $u_{ii} = 0.001$, $u_{ij,i\neq j} = 0$, $i, j = 1, 2, 3$.

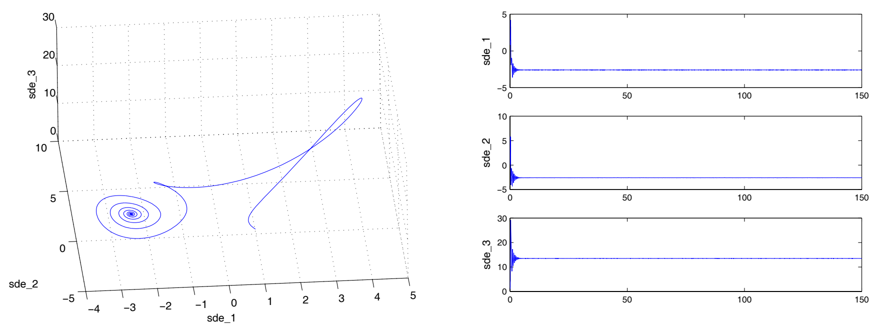

Figure 8 The time-dependent stochastic Rössler-Lorentz system with the noise $u_{ii} = 0.01$, $u_{ij,i\neq j} = 0$, $i, j = 1, 2, 3$.

In the current paper, we shall mainly investigate the Rössler-Lorentz system (1.4), and consider the case that $\gamma(t)$ is not a constant but depends on the third variable and the coefficient of the third interactive term $\alpha_3(t)$, namely $\gamma(t) = \alpha_3(t)x_3(t)$.

Throughout this article, unless otherwise specified, we assume $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ to be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). $B(t) := (B_1(t), B_2(t), B_3(t))^{\mathrm{T}}$ denotes a three-dimensional Brownian motion defined on this probability space.

The rest of this paper is arranged as follows. In Section 2, we describe some fundamental conditions and notations, while in Section 3, the global existence and uniqueness for the stochastic system (1.4) are established. Moreover, the pathwise property of solutions is obtained. In Section 4, some numerical examples with inner random perturbations are presented, which demonstrate the results of our theoretical analysis, and exhibit the various behaviors with different inner perturbations. By the end of Section 4, we give some conclusions and discussions.

2 Assumptions and Notations

We firstly split the system (1.4) into different parts and give some fundamental conditions. In this paper, we consider the generalized system (1.4) only forward in time $t \in [0, \infty)$. Let

$X=(x_1,x_2,x_3)\in\mathbb{R}^3$. The Rössler-Lorentz system can be rewritten as

$$\mathrm{d}X=-[AX+C(X)-F]\mathrm{d}t+U(X,t)\mathrm{d}B_t,\quad 0\le t<+\infty, \tag{2.1}$$

$$X(0)=X_0, \tag{2.2}$$

where the initial datum $X_0=(x_{10},x_{20},x_{30})^{\mathrm{T}}$ is a fixed point independent of $\mathcal{F}_t$ for all $t>0$. The four parts of the drift for (2.1) are given by

$$\begin{aligned} &A=\begin{pmatrix}\sigma & -\sigma & 0\\ -(r+1) & 1-a & 0\\ -\alpha_3(t)c & 0 & \beta\end{pmatrix},\quad C(X)=\begin{pmatrix}-\alpha_3(t)x_3x_2+\alpha_3(t)x_3^2\\ \alpha_1(t)x_1x_3\\ -\alpha_2(t)x_1x_2-\alpha_3(t)x_1x_3\end{pmatrix},\\ &F=\begin{pmatrix}0\\0\\b\end{pmatrix},\quad U(X,t)=\begin{pmatrix}u_{11}(t) & u_{12}(t) & u_{13}(t)\\ u_{21}(t) & u_{22}(t) & u_{23}(t)\\ u_{31}(t) & u_{32}(t) & u_{33}(t)\end{pmatrix},\end{aligned} \tag{2.3}$$

where $U(X,t):\mathbb{R}^3\times[0,\infty)\to$ space of 3×3-matrices is a noise term. For the system (2.1), we assume:

(A1) The matrix A satisfies $(AX,X)\ge\lambda\|X\|_2^2$ for some constant $\lambda>0$.

(A2) The constant b and the coefficients $\alpha_1(t),\alpha_2(t),\alpha_3(t)$ are bounded. In addition, the coefficients of the interactive terms satisfy $\alpha_1(t)=\alpha_2(t)+\alpha_3(t)$.

(A3) The noise term $U(X,t)$ satisfies a Lipschitz condition and a linear growth condition, i.e.,

$$\|U(X,t)\|_2^2:=\mathrm{trace}(U(X,t)U^{\mathrm{T}}(X,t))\le C_1(1+\|X\|_2^2).$$

(A4) (The alternative condition) Either $C_1<\frac{\lambda}{p-1}$ or $\|U(X,t)\|_2^2\le C_1$.

Remark 2.1 If $U(X,t)\equiv 0$, then the Lorentz-Rössler system becomes a deterministic one. As for deterministic equations, there are a number of methods to be used in the study of the well-poseness. And in this paper, we mainly consider the stochastic system. It is important that the system should keep a small variety in a weak noise environment, for example, outdoor-move communications, airplane oscillating communications, and complex electromagnetic environment communications.

At the end of this section, we give some notations which will be frequently used throughout the paper. For any real matrix $R=[r_{ij}]\in\mathbb{R}^{d\times m}$, we define

$$\|R\|_2^2=\mathrm{trace}(R^{\mathrm{T}}R)=\mathrm{trace}(RR^{\mathrm{T}})=\sum_{i,j}r_{ij}^2.$$

For any $p\in\mathbb{N}$ even and $X\in\mathbb{R}^3$, we denote $\|X\|_2^p=(x_1^2+x_2^2+x_3^2)^{\frac{p}{2}}$. For any two variables $X,Y\in\mathbb{R}^3$, (X,Y) stands for the usual inner product. C_i $(i=1,2,\cdots)$ will denote generic constants which vary from line to line and depend on some parameters.

We remark that the condition (A3) can be easily satisfied. For example,

$$U_i(X,t)=x_i(t)\sum_{j=1}^{3}\sigma_{ij}(t)\mathrm{d}B_j(t)\quad\text{or}\quad U_i(X,t)=\sum_{j=1}^{3}\sigma_{ij}(t)\mathrm{d}B_j(t)$$

satisfies (A3) when all the $\sigma_{ij}(t)$ are bounded on $\mathbb{R}_+$.

3 Global Existence and Pathwise Property

In this section, we shall show a global existence result and a pathwise property for the stochastic system (1.4) with initial data $X(0) = X_0 = (x_{10}, x_{20}, x_{30})^{\mathrm{T}} \in \mathbb{R}^3$ and $U(X,t) \neq 0$. In order to guarantee the existence of a unique global solution for any given initial data, we require in general that the terms of the system (1.4) satisfy the linear growth and uniform Lipschitz continuity conditions. For the system (2.1), the terms $-A(t)X$, F, $U(X,t)$ satisfy these two conditions. However, the term $C(X,t)$ does not satisfy both the linear growth and uniform Lipschitz continuity conditions. To circumvent this difficulty, we first introduce a modified system, which is solvable, by truncating $C(X,t)$ appropriately. Then, uniform priori estimates for the modified system enable us to show that the modified system converges to the original one as the truncation level goes to infinity, and thus we obtain a solution to (1.4).

Our modified system is given in the following lemma.

Lemma 3.1 *Let $\chi_N \in C^1(\mathbb{R}^3, \mathbb{R})$ with $\chi_N(X) = 1$ for $\|X\|_2 \le N$ and $\chi_N(X) = 0$ for $\|X\|_2 \ge N+1$. Define $C_N(X) := \chi_N(X)C(X)$, and consider the modified system:*

$$\mathrm{d}X_N = -(AX_N + C_N(X_N) - F)\mathrm{d}t + U(X_N)\mathrm{d}B_t, \quad t \in [0,\infty),\ X_N(0) = X_0, \tag{3.1}$$

where $X_N = (x_{N1}, x_{N2}, x_{N3})$. Assume that the initial data is independent of $\{\mathcal{F}_t\}_{t>0}$ and satisfies $E\|X_0\|_2^2 < \infty$. Then for any fixed $N > 0$, the modified system (3.1) possesses a continuous almost sure unique and global solution that is $\{\mathcal{F}_t\}$ measurable.

Proof It is easy to see that $C_N(X_N)$ is bounded and satisfies a linear growth condition. To show that $C_N(X_N)$ is uniformly Lipschitz continuous for any fixed $N > 0$, we observe for $X_N, \overline{X}_N \in \mathbb{R}^3$ that

$$\begin{aligned}&\|C_N(X_N) - C_N(\overline{X}_N)\|_2^2 \\ =&\left\| \begin{pmatrix} \alpha_3(t)(-\chi_N(X_N)X_{N3}X_{N2} + \chi_N(\overline{X}_N)\overline{X}_{N3}\overline{X}_{N2} + \chi_N(X_N)X_{N3}^2 - \chi_N(\overline{X}_N)\overline{X}_{N3}^2) \\ \alpha_1(t)(\chi_N(X_N)X_{N1}X_{N3} - \chi_N(\overline{X}_N)\overline{X}_{N1}\overline{X}_{N3}) \\ \alpha_2(t)(-\chi_N(X_N)X_{N1}X_{N2} + \chi_N(\overline{X}_N)\overline{X}_{N1}\overline{X}_{N2}) - \alpha_3(t)(\chi_N(X_N)X_{N1}X_{N3} \\ +\chi_N(\overline{X}_N)\overline{X}_{N1}\overline{X}_{N3}) \end{pmatrix} \right\|_2^2.\end{aligned}$$

We consider three cases for the right-hand side of the above identity.

Case 1 If both $\|X_N\|_2 \ge N+1$ and $\|\overline{X}_N\|_2 \ge N+1$, then obviously, the uniform Lipschitz continuity of $C_N(X_N)$ can be ensured.

Case 2 If either $\|X_N\|_2 \ge N+1$ and $\|\overline{X}\|_2 < N+1$, or $\|X_N\|_2 < N+1$ and $\|\overline{X}\|_2 \ge N+1$, we have (without loss of generality, let $\|X_N\|_2 \ge N+1$ and $\|\overline{X}_N\|_2 < N+1$, which implies $\|X_N - \overline{X}_N\|_2^2 \neq 0$):

$$\begin{aligned}&\|C_N(X_N) - C_N(\overline{X}_N\|_2^2 \\ =&\left\| \begin{pmatrix} \alpha_3(t)\chi_N(X_N)(-X_{N3}X_{N2} + X_{N3}^2) \\ \chi_N(X_N)\alpha_1(t)X_{N1}X_{N3} \\ -\chi_N(X_N)(\alpha_2(t)X_{N1}X_{N2} + \alpha_3(t)X_{N1}X_{N3}) \end{pmatrix} \right\|_2^2 \\ \le& \left(4N \max_i \sup\{\alpha_1(t), \alpha_2(t), \alpha_3(t)\}(N+1)\right)^2 \|X_N\|_2^2 \le C_\alpha \|X_N - \overline{X}_N\|_2^2,\end{aligned}$$

where C_α is dependent on N, the boundary of $\alpha_1(t), \alpha_2(t), \alpha_3(t)$ and the assumption of Case 2.

Case 3 If $\|X_N\|_2 \le N+1$ and $\|\overline{X}\|_2 \le N+1$, we first find that

$$\begin{aligned}
&\alpha_3(t)(\chi_N(\overline{X}_N)\overline{X}_{N3}\overline{X}_{N2} - \chi_N(X_N)X_{N3}X_{N2})\\
\le& \sup_{t\in[0,\infty)} |\alpha_3(t)|(|\chi_N(\overline{X}_N)||\overline{X}_{N2} - X_{N2}| + |(\chi_N(\overline{X}_N)\overline{X}_{N3} - \chi_N(X_N)X_{N3})X_{N2}|)\\
\le& \sup_{t\in[0,\infty)} |\alpha_3(t)|(|\chi_N(\overline{X}_N)||\overline{X}_{N2} - X_{N2}| + |\chi_N(\overline{X}_N) - \chi_N(X_N)|X_{N2}\overline{X}_{N3}\\
&+ \chi_N(X_N)X_{N2}|\overline{X}_{N3} - X_{N3}|)\\
\le& C_3\|\overline{X}_N - X_N\|_2,
\end{aligned}$$

where C_3 is dependent on $\sup\limits_{t\in[0,\infty)} \alpha_3(t)$, N and the continuous coefficients of the function χ. We can employ a similar argument to deal with all the other terms of $C_N(X_N) - C_N(\overline{X}_N)$, and thus derive that

$$\|C_N(X_N) - C_N(\overline{X}_N)\|_2^2 \le C_5\|X_N - \overline{X}_N\|_2^2.$$

On the other hand, all other coefficients in (1.4) obviously satisfy a linear growth as well as the uniform Lipschitz continuity condition. Thanks to the truncation function $\chi_N \in C^1(\mathbb{R}^3, \mathbb{R})$, the modified nonlinear term $C_N(X_N)$ remains differentiable, and its derivatives are continuous and have compact supports. Thus, Lemma 3.1 follows from the usual existence and uniqueness theorem.

To get the uniform a priori estimates of the solution to the system (3.1), we first deal with the Itô derivatives of the Lyapunov functions.

Lemma 3.2 *Let the assumptions* (A1) *and* (A2) *hold. Then we have*

$$\begin{aligned}
\mathrm{d}\|X_N\|_2^p =& -\frac{p\lambda}{2}\|X_N\|_2^p\mathrm{d}t + \|X_N\|_2^{p-2}\frac{pb^2}{2\lambda}\mathrm{d}t + \frac{p}{2}(p-1)\|X_N\|_2^{p-2}\|U(X_N)\|_2^2\mathrm{d}t\\
&+ p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}U(X_N)\mathrm{d}B_t + \phi(t)\mathrm{d}t, \qquad (3.2)
\end{aligned}$$

where $\phi(t) \le 0$ is an adapted process.

Proof Denoting the Lyapunov function

$$V(X) = (x_1^2 + x_2^2 + x_3^2)^{\frac{p}{2}} = \|X\|_2^p,$$

we use the Itô formula with respect to $V(X_N)$ for $p \in \mathbb{N}$ even to evaluate the Itô derivatives of the Lyapunov function of the solution to the modified system (3.1) as follows:

$$\begin{aligned}
\mathrm{d}U(X_N) =& \sum_{i=1}^3 \frac{p}{2}(x_{N1}^2 + x_{N2}^2 + x_{N3}^2)^{\frac{p}{2}-1}2x_{Ni}\mathrm{d}x_{Ni}\\
&+ \frac{1}{2}\sum_{i,j=1}^3 \frac{p}{2}\Big(\frac{p}{2}-1\Big)(x_{N1}^2 + x_{N2}^2 + x_{N3}^2)^{\frac{p}{2}-2}2x_{Ni}2x_{Nj}\mathrm{d}x_{Ni}\mathrm{d}x_{Nj}\\
&+ \frac{1}{2}\sum_{i=1}^3 \frac{p}{2}(x_{N1}^2 + x_{N2}^2 + x_{N3}^2)^{\frac{p}{2}-1}2\mathrm{d}x_{Ni}\mathrm{d}x_{Ni}\\
=& p\|X_N\|_2^{p-2}(\mathrm{d}X_N, X_N) + p\Big(\frac{p}{2}-1\Big)\|X_N\|_2^{p-4}\sum_{i,j=1}^3 x_{Ni}\mathrm{d}x_{Ni}x_{Nj}\mathrm{d}x_{Nj}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{p}{2}\|X_N\|_2^{p-2}\sum_{i=1}^{3}\mathrm{d}x_{Ni}\mathrm{d}x_{Ni}\\
&= p\|X_N\|_2^{p-2}\{-[(AX_N,X_N)+(\chi_N(X_N)C(X_N),X_N)-(F,X_N)]\}\mathrm{d}t\\
&\quad+p\Big(\frac{p}{2}-1\Big)\|X_N\|_2^{p-4}\mathrm{trace}(X_NX_N^{\mathrm{T}}U(X_N)U^{\mathrm{T}}(X_N))\mathrm{d}t\\
&\quad+\frac{p}{2}\|X_N\|_2^{p-2}\mathrm{trace}(U(X_N)U^{\mathrm{T}}(X_N))\mathrm{d}t+p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}U(X_N)\mathrm{d}B_t. \qquad (3.3)
\end{aligned}
$$

We have to control every term on the right-hand side of (3.3). To this end, we first need the following estimates.

Lemma 3.3 *For any real matrix $R\in\mathbb{R}^{m\times n}$, $Q\in\mathbb{R}^{n\times m}$, the following inequalities hold:*

$$\|RQ\|_2\le\|R\|_2\|Q\|_2,\quad |\mathrm{trace}(RQ)|\le\|R\|_2\|Q\|_2.$$

In fact, recalling the definition of the trace operator and Hölder's inequality, we easily see that Lemma 3.3 follows from the following inequalities:

$$\|RQ\|_2^2=\sum_{ij}\Big(\sum_k r_{ik}q_{kj}\Big)^2\le\sum_{i,j}\Big(\sum_k r_{ik}^2\sum_l q_{lj}^2\Big)=\sum_{i,j,k,l}(r_{ik}^2q_{lj}^2)=\sum_{i,k}r_{ik}^2\sum_{j,l}q_{lj}^2=\|R\|_2^2\|Q\|_2^2,$$

$$|\mathrm{trace}(RQ)|^2=\Big(\sum_{i,j}r_{ij}q_{ji}\Big)^2\le\sum_{i,j}r_{ij}^2\sum_{i,j}q_{ji}^2=\|R\|_2^2\|Q\|_2^2.$$

Hence, by Lemma 3.3, we have that

$$\mathrm{trace}(X_NX_N^{\mathrm{T}}U(X_N)U^{\mathrm{T}}(X_N))\le\|X_NX_N^{\mathrm{T}}\|_2\|U(X_N)U^{\mathrm{T}}(X_N)\|_2\le\|X_N\|_2^2\|U(X)\|_2^2. \qquad (3.4)$$

Recalling the assumption (A1) and Hölder's inequality, we deduce that

$$-(AX_N,X_N)+(F,X_N)\le-\frac{\lambda}{2}\|X_N\|_2^2+\frac{b^2}{2\lambda}, \qquad (3.5)$$

while from (A2) we get

$$
\begin{aligned}
(\chi_N(X_N)C(X_N),X_N)&=\left(\chi_N(X_N)\begin{pmatrix}-\alpha_3(t)x_{N3}x_{N2}+\alpha_3(t)x_{N3}^2\\ \alpha_1(t)x_{N1}x_{N3}\\ -\alpha_2(t)x_{N1}x_{N2}-\alpha_3(t)x_{N1}x_{N3}\end{pmatrix},\begin{pmatrix}x_{N1}\\x_{N2}\\x_{N3}\end{pmatrix}\right)\\
&=\chi_N(X_N)(-\alpha_3(t)x_{N3}x_{N2}x_{N1}+\alpha_3(t)x_{N3}^2x_{N1}\\
&\quad+\alpha_1(t)x_{N1}x_{N3}x_{N2}-\alpha_2(t)x_{N1}x_{N2}x_{N3}-\alpha_3(t)x_{N1}x_{N3}^2)\\
&\equiv 0. \qquad (3.6)
\end{aligned}
$$

Using Lemma 3.3 again, and putting together the estimates (3.4)–(3.6), we conclude that

$$
\begin{aligned}
\mathrm{d}\|X_N\|_2^p&=-\frac{p\lambda}{2}\|X_N\|_2^p\mathrm{d}t+p\|X_N\|_2^{p-2}\frac{b^2}{2\lambda}\mathrm{d}t+0+p\Big(\frac{p}{2}-1\Big)\|X_N\|_2^{p-2}\|U(X_N)\|_2^2\mathrm{d}t\\
&\quad+\frac{p}{2}\|X_N\|_2^{p-2}\|U(X_N)\|_2^2\mathrm{d}t+p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}U(X_N)\mathrm{d}B_t+\phi(t)\mathrm{d}t,
\end{aligned}
$$

where $\phi(t)\le 0$ is an adapted process, which compensates all the above computations. Thus, the proof of Lemma 3.2 is complete.

Next, we derive uniform a priori estimates for $X_N(t)$. We begin with the following lemma.

Lemma 3.4 *Assume that the assumptions* (A1)–(A4) *hold, and let* $p \in \mathbb{N}$ *be even and fixed, and the initial expectation* $E\|X_0\|_2^p < \infty$. *Then,*

$$\sup_{t\in[0,\infty)} E\|X_N(t)\|_2^p \le C_p, \tag{3.7}$$

where the constant $C_p > 0$ *depends only on* $E\|X_0\|_2^p, p, \lambda, C_1, b$, *but not on* N.

Proof First, we introduce the stopping time. For any $D \in \mathbb{N}$, let

$$\tau_D = \inf\{t \in [0,\infty) : \|X_N(t)\|_2 \ge D\}. \tag{3.8}$$

In view of the assumption (A4), we consider the following two cases, respectively.

Case 1 If $\|U(X_N)\|_2^2 \le C_1$, then (3.2) becomes

$$\begin{aligned} \mathrm{d}\|X_N\|_2^p \le &-\frac{p\lambda}{2}\|X_N\|_2^p \mathrm{d}t + \|X_N\|_2^{p-2}\frac{pb^2}{2\lambda}\mathrm{d}t + \frac{p}{2}(p-1)C_1\|X_N\|_2^{p-2}\mathrm{d}t \\ &+ p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}t + p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}U(X_N)\mathrm{d}B_t. \end{aligned}$$

Integrating the above inequality from 0 to $t \wedge \tau_D$ and taking the expectation, we obtain

$$\begin{aligned} E\|X_N(t\wedge\tau_D)\|_2^p \le &E(\|X_0\|_2^p) + E\int_0^{t\wedge\tau_D}\Big(-\frac{p\lambda}{2}\Big)E\|X_N(s)\|_2^p \mathrm{d}s \\ &+ \int_0^{t\wedge\tau_D}\Big(\frac{pb^2}{2\lambda} + \frac{C_1 p(p-1)}{2}\Big)E\|X_N(s)\|_2^{p-2}\mathrm{d}s + 0. \end{aligned}$$

In particular, we take $p = 2$ and use the Gronwall inequality to infer that

$$\begin{aligned} &E\|X_N(t\wedge\tau_D)\|_2^2 \\ &\le E\|X_0\|_2^2 + \Big(\frac{pb^2}{2\lambda} + \frac{C_1 p(p-1)}{2}\Big)(t\wedge\tau_D) \\ &\quad - \frac{p\lambda}{2}\int_0^{t\wedge\tau_D}\exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D - s)\Big]\Big[E\|X_0\|_2^2 + \Big(\frac{pb^2}{2\lambda} + \frac{C_1 p(p-1)}{2}\Big)s\Big]\mathrm{d}s \\ &= E\|X_0\|_2^2\exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D)\Big] + \Big[\frac{pb^2}{2\lambda} + \frac{C_1p(p-1)}{2}\Big]\frac{2}{p\lambda}\Big\{1 - \exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D)\Big]\Big\} \\ &\le C_2^{(1)} := E\|X_0\|_2^2 + \frac{b^2}{\lambda_{\min}^2} + \frac{C_1(p-1)}{\lambda}, \end{aligned}$$

where $C_2^{(1)}$ is independent of t. Computing recursively, we obtain for any finite $p \in \mathbb{N}$ even that

$$\begin{aligned} &E\|X_N(t\wedge\tau_D)\|_2^p \\ &\le E\|X_0\|_2^p + \int_0^{t\wedge\tau_D}\Big(-\frac{p\lambda}{2}\Big)E\|X_N(s)\|_2^p\mathrm{d}s + \int_0^{t\wedge\tau_D}\Big[\frac{pb^2}{2\lambda} + \frac{C_1p(p-1)}{2}\Big]C_{p-2}^{(1)}\mathrm{d}s \\ &= E\|X_0\|_2^p + \Big[\frac{pb^2}{2\lambda} + \frac{C_1p(p-1)}{2}\Big]C_{p-2}^{(1)}(t\wedge\tau_D)\int_0^{t\wedge\tau_D}(-\frac{p\lambda}{2})E\|X_N(s)\|_2^p\mathrm{d}s. \end{aligned}$$

If we apply the Gronwall inequality again, we get

$$\begin{aligned}
&E\|X_N(t\wedge\tau_D)\|_2^p\\
&\le E\|X_0\|_2^p+\Big[\frac{pb^2}{2\lambda}+\frac{C_1p(p-1)}{2}\Big]C_{p-2}^{(1)}(t\wedge\tau_D)\\
&\quad+\int_0^{t\wedge\tau_D}\Big(-\frac{p\lambda}{2}\Big)\exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D-s)\Big]\Big\{E\|X_0\|_2^p+\Big[\frac{pb^2}{2\lambda}+\frac{C_1p(p-1)}{2}\Big]C_{p-2}^{(1)}s\Big\}\mathrm{d}s\\
&=E\|X_0\|_2^p\exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D)\Big]+\Big[\frac{pb^2}{2\lambda}+\frac{C_1p(p-1)}{2}\Big]C_{p-2}^{(1)}\frac{2}{p\lambda}\Big\{1-\exp\Big[-\frac{p\lambda}{2}(t\wedge\tau_D)\Big]\Big\}\\
&\le C_p^{(1)}:=E\|X_0\|_2^p+\Big[\frac{b^2}{\lambda_{\min}^2}+\frac{C_1(p-1)}{\lambda}\Big]C_{p-2}^{(1)},
\end{aligned}$$

where $C_p^{(1)}, C_{p-2}^{(1)}, \cdots, C_2^{(1)}$ are independent of $t\wedge\tau_D$.

Therefore, for any finite $p\in\mathbb{N}$ even, we have

$$\sup_{t\in[0,\infty)} E\|X_N(t\wedge\tau_D)\|_2^p\le C_p^{(1)}. \tag{3.9}$$

Case 2 If $C_1<\frac{\lambda}{p-1}$, then (3.2) becomes

$$\begin{aligned}
\mathrm{d}\|X_N\|_2^p\le{}&\Big(\frac{C_1p(p-1)}{2}-\frac{p\lambda}{2}\Big)\|X_N\|_2^p\mathrm{d}t+\Big(\frac{pb^2}{2\lambda}+\frac{C_1p(p-1)}{2}\Big)\|X_N\|_2^{p-2}\mathrm{d}t\\
&+p\|X_N\|_2^{p-2}X_N^{\mathrm{T}}U(X_N)\mathrm{d}B_t.
\end{aligned}\tag{3.10}$$

Keeping in mind that $C_1<\frac{\lambda}{p-1}$ and (3.10), we derive that

$$\sup_{t\in[0,\infty)} E\|X_N(t\wedge\tau_D)\|_2^p\le C_2^{(2)},$$

where $C_2^{(2)}=E\|X_0\|_2^2+\frac{\frac{b^2}{\lambda}+C_1(p-1)}{\lambda-C_1(p-1)}$.

We can argue similarly to (3.9) to obtain

$$E\|X_N(t\wedge\tau_D)\|_2^p\le C_p^{(2)}, \tag{3.11}$$

where $C_p^{(2)}=E\|X_0\|_2^p+\frac{\frac{b^2}{\lambda}+C_1(p-1)}{\lambda-C_1(p-1)}$.

Combining (3.9) with (3.11), we conclude

$$\sup_{t\in[0,\infty)} E\|X_N(t\wedge\tau_D)\|_2^p\le C_p:=C_p^{(1)}+C_p^{(2)},$$

where $C_p^{(1)}$ and $C_p^{(2)}$ are independent of N and t, and depend on the constants $E\|X_0\|_2^p,\lambda,p,b,C_1$ only.

It is obvious that the stopping time satisfies $\tau_D\to\infty$ as $D\to\infty$. By the continuity of the solution $X_N(t)$ in t, we see that for any fixed $t\in[0,\infty)$, $\|X_N(t\wedge\tau_D)\|_2^p$ is bounded, $t\wedge\tau_D\to t$ as $D\to\infty$, and

$$\|X_N(t\wedge\tau_D)\|_2^p\longrightarrow\|X_N(t)\|_2^p\ \text{ a.s.}\quad(\text{as }\ D\to\infty). \tag{3.12}$$

Putting all the above uniform estimates for $E\|X_N(t\wedge\tau_D)\|_2^p$ together, using (3.12) and the Fatou lemma, we obtain

$$E\|X_N(t)\|_2^p = E\lim_{D\to\infty}\|X_N(t\wedge\tau_D)\|_2^p \le \lim_{D\to\infty}\inf E\|X_N(t\wedge\tau_D)\|_2^p \le C_p,$$

where C_p is independent of t. Hence,

$$\sup_{t\in[0,\infty)} E\|X_N(t)\|_2^p \le C_p,$$

which completes the proof.

Under a weaker condition on the initial expectation, we still have Lemma 3.4, namely, the following lemma.

Lemma 3.5 *Let* $E\|X_0\|_2^4<\infty$ *and* $T>0$ *be arbitrary but fixed. Assume that the conditions* (A1)–(A4) *hold. Then there exists a constant* $\widetilde{C}_4$*, such that*

$$E\sup_{t\in[0,T]}\|X_N(t)\|_2^2 \le \widetilde{C}_4, \tag{3.13}$$

where $\widetilde{C}_4$ *is independent of* N*, and depends possibly on* $T, C_2, C_4, C_1, \lambda, b$ *only.*

Proof In view of the uniform boundedness Lemma 3.4, we see that it is not necessary to use the stopping time in the following discussion. If we integrate (3.2) from 0 to t ($t\in[0,T]$), we obtain

$$\begin{aligned}
&\sup_{t\in[0,T]}\|X_N(t)\|_2^2\\
&=\|X_0\|_2^2+\sup_{t\in[0,T]}\int_0^t\frac{b^2}{\lambda}\mathrm{d}s-\lambda\sup_{t\in[0,T]}\int_0^t\|X_N(s)\|_2^2\mathrm{d}s+\sup_{t\in[0,T]}\int_0^t\|U(X_N(s))\|_2^2\mathrm{d}s\\
&\quad+2\sup_{t\in[0,T]}\int_0^t X_N^{\mathrm{T}}(s)U(X(s))\mathrm{d}B_s+\sup_{t\in[0,T]}\int_0^t\phi(s)\mathrm{d}s.
\end{aligned}$$

Taking the expectation to the above identity, and using the Burholder-Davis-Gundy inequality to control the stochastic integral, we infer that

$$\begin{aligned}
E\sup_{t\in[0,T]}\|X_N(t)\|_2^2 &\le E\|X_0\|_2^2+\frac{b^2}{\lambda}T-\lambda\int_0^T E\|X_N(s)\|_2^2\mathrm{d}s+\int_0^T E\|U(X_N(s))\|_2^2\mathrm{d}s\\
&\quad+2E\sup_{t\in[0,T]}\int_0^t X_N^{\mathrm{T}}(s)U(X_N(s))\mathrm{d}B_s\\
&\le E\|X_0\|_2^2+\frac{b^2}{\lambda}T+\int_0^T C_1(1+E\|X_N(s)\|_2^2)\mathrm{d}s\\
&\quad+8E\Big|\int_0^T\|X_N^{\mathrm{T}}(s)U(X_N(s))\|_2^2\mathrm{d}s\Big|^{\frac12}\\
&\le E\|X_0\|_2^2+\Big(\frac{b^2}{\lambda}+C_1+C_1C_2\Big)T+8+8E\int_0^T\|X_NT(s)\|_2^2\|U(X_N(s))\|_2^2\mathrm{d}s\\
&\le E\|X_0\|_2^2+\Big(\frac{b^2}{\lambda}+C_1+9C_1C_2+8C_1C_4\Big)T+8:=\widetilde{C}_4,
\end{aligned}$$

which implies the boundedness given in the lemma.

Making use of the above lemmas, we are able to show the main results of this paper, i.e., Theorems 3.1–3.2 below.

Theorem 3.1 *Suppose that the conditions* (A1)–(A4) *are satisfied. Then, the Rössler-Lorentz systems* (2.1)–(2.2) *with* $E\|X_0\|_2^4 < \infty$ *possesses a global unique almost sure continuous solution process, which has the following property:*

If, in addition, $E\|X_0\|_2^p < \infty$ *for a fixed* $p \in \mathbb{N}$ *even, then*

$$\sup_{t\in[0,\infty)} E\|X(t)\|_2^p \le C_p. \tag{3.14}$$

Proof First, we show the existence. By virtue of Lemma 3.1, the truncated system (3.1) has a continuous solution $X_N(t)$ ($t \in [0,\infty)$). To prove the existence, we show that the sequence $X_N(t)$ is convergent $X_N(t) \to X(t)$ as $N \to \infty$ in some sense. Let τ_D denote the stopping time introduced in (3.8) for an $N \in \mathbb{N}$. From Lemma 3.5 and the Chebyshev inequality we get

$$\begin{aligned}\mathbb{P}\{\tau_N(\omega) < N\} &\le \mathbb{P}\Big\{\sup_{t\in[0,N]} \|X_N(t)\|_2 \ge N\Big\} \le \frac{E\sup_{t\in[0,N]}\|X_N(t)\|_2^2}{N^2}\\ &\le \frac{E\|X_0\|_2^2 + (\frac{b^2}{\lambda} + C_1 + 9C_1C_2 + 8C_1C_4)N + 8}{N^2} \to 0, \quad \text{as } N \to \infty,\end{aligned} \tag{3.15}$$

whence

$$\mathbb{P}\{\tau_\infty(\omega) < \infty\} = 0$$

and

$$\mathbb{P}\{\tau_\infty(\omega) = \infty\} = 1.$$

Thus for almost every $\omega \in \Omega$, there exists an $N_0(\omega)$, such that $\tau_{N_0(\omega)}$ is large enough. Moreover, one has

$$C_{N'}(X) = C_N(X) = C(X), \quad N' \ge N > 0 \;\text{ for all } \|X\|_2 \le N. \tag{3.16}$$

Hence,

$$\tau_{N'} \ge \tau_N \;\text{ and }\; X_{N'}^{X_0}(\cdot,\omega) = X_N^{X_0}(\cdot,\omega) \quad \text{(almost sure) on } [0,\tau_N] \;\text{ for all } N' \ge N. \tag{3.17}$$

By virtue of (3.17), if $\tau_N \to \infty$, then $\tau_{N'} \to \infty$ for all $N' \ge N$ ($N \to \infty$). Therefore, the set $\{\omega : \tau_N(\omega) \to \infty\}$ is monotonously increasing and converges to Ω as $N \to \infty$.

Moreover, because for any $N \in \mathbb{N}$, $X_N(t)$ is continuous in t and converges uniformly in t to $X(t)$, $X(t)$ is also continuous in t (in fact, note that if $\tau_{N_0(\omega)}$ is sufficiently large, then we can express for almost all $\omega \in \Omega$ the limit function as $X(\cdot,\omega) := X_{N'}(\cdot,\omega)$ for all $N' \ge N_0(\omega)$).

Next, we have to show that the limit function $X(t)$ is indeed a solution of the original Rössler-Lorentz system. When $t = 0$, it is obvious that $X_N(0) = X(0) = X_0$ for all $N \in \mathbb{N}$, while for $t > 0$ we show that $X(t)$ solves (2.1)–(2.2) by taking it to the limitation in (3.1) as $N \to \infty$. To this end, recalling the definition of (3.8), we have

$$C_N(X_N(t \wedge \tau_N)) = C(X(t \wedge \tau_N))$$

and

$$X_N(t\wedge\tau_N)=X(t\wedge\tau_N)\quad \text{for all}\ \ t<\infty.$$

Moreover, the almost sure convergence of $\tau_N \overset{(N\to\infty)}{\longrightarrow} \infty$ implies that

$$\begin{aligned}&\mathbb{P}\Big\{\sup_{t\in[0,T]}\Big\|\int_{t_0}^{t}[A(X_N(s)-X(s))+(F-F)+C_N(X_N(s))-C(X(s))]\mathrm{d}s\\&+\int_{t_0}^{t}[U(X_N(s))-U(X(s))]\mathrm{d}B_s\Big\|_2>0\Big\}\le \mathbb{P}\{\tau_N<t\}\overset{(N\to\infty)}{\longrightarrow}0,\end{aligned}\tag{3.18}$$

where we have used the fact that if $\tau_N\ge t$, then $\tau_N\wedge t=t$, and consequently, $X_N(t)=X(t)$, $C_N(X_N(t))=C(X(t))$ and

$$\begin{aligned}&\sup_{t\in[0,T]}\Big\|\int_{t_0}^{t}[A(X_N(s)-X(s))+(F-F)+C_N(X_N(s))-C(X(s))]\mathrm{d}s\\&+\int_{t_0}^{t}[U(X_N(s))-U(X(s))]\mathrm{d}B_s\Big\|_2=0.\end{aligned}$$

Therefore, $X(\cdot)$ is a solution of the stochastic Rössler-Lorentz system on $[0,\infty)$. Meanwhile, the boundary of the moments (3.14) can be obtained by the uniform-in-t convergence of $X_N(t)$ to $X(t)$ and Lemma 3.4. Finally, for any fixed $N>0$, the coefficients of the truncated system (3.1) satisfy the uniform Lipschitz continuity condition, so the uniqueness of the the system (3.1) can be ensured. Combining (3.18) with the uniform-in-t convergence of $X_N(t)$ to $X(t)$, we obtain the uniqueness of solutions to the Rössler-Lorentz system by an almost sure means.

In Theorem 3.1 we have discussed the existence, uniqueness and the moment properties of solutions to the systems (2.1)–(2.2). Now, we study the pathwise property.

Theorem 3.2 *Let the conditions in Theorem* 3.1 *hold. Then for any initial data* $X_0\in\Omega$*, the solution* $X(t)$ *of* (2.1) *established in Theorem* 3.1 *satisfies*

$$\limsup_{t\to\infty}\frac{\log(\|X(t)\|_2)}{\log t}\le 1,\quad a.s.\tag{3.19}$$

Proof We use the same notations as in the proof of Theorem 3.1. Following a procedure similar to that used for (3.2), we obtain

$$\begin{aligned}\mathrm{d}\|X\|_2^p=&-\frac{p\lambda}{2}\|X\|_2^p\mathrm{d}t+\|X\|_2^{p-2}\frac{pb^2}{2\lambda}\mathrm{d}t+\frac{p}{2}(p-1)\|X\|_2^{p-2}\|U(X)\|_2^2\mathrm{d}t\\&+p\|X_N\|_2^{p-2}X^{\mathrm{T}}U(X)\mathrm{d}B_t+\psi(t)\mathrm{d}t,\end{aligned}\tag{3.20}$$

where $\psi(t)\le 0$ is an adapted process.

First, we discuss the behavior of the solution in the time interval $[t,t+1]$. Let $p=2$. For any $t>0$, we integrate (3.20) over (s,t), and then take the supremum and expectation to get

$$\begin{aligned}&E\sup_{t\le s\le t+1}\|X(s)\|_2^2\\&\le E\|X_0\|_2^2+\frac{b^2}{\lambda}+E\sup_{t\le s\le t+1}\int_t^s\|U(X(u))\|_2^2\mathrm{d}u+E\sup_{t\le s\le t+1}\int_t^s X^{\mathrm{T}}(u)U(X(u))\mathrm{d}B_u\\&=E\|X(t)\|_2^2+\frac{b^2}{\lambda}+\int_t^{t+1}E\|U(X(u))\|_2^2\mathrm{d}u+E\sup_{t\le s\le t+1}\int_t^s X^{\mathrm{T}}(u)U(X(u))\mathrm{d}B_u,\end{aligned}\tag{3.21}$$

which, together with the Burkholder-Davis-Gaudy inequality and the condition (A3), gives

$$
\begin{aligned}
E\sup_{t\le s\le t+1}\|X(s)\|_2^2 &\le E\|X(t)\|_2^2+\frac{b^2}{\lambda}+C_1+\int_t^{t+1}E\|X(u)\|_2^2\mathrm{d}u\\
&\quad+\sqrt{32}E\Big(\int_t^{t+1}\|X^{\mathrm{T}}(u)U(X(u))\|_2^2\mathrm{d}u\Big)^{\frac12}\\
&\le 2C_2+\frac{b^2}{\lambda}+\sqrt{32}+C_1+\sqrt{32}C_1C_2+\sqrt{32}C_1C_4:=\widetilde{C}, \qquad (3.22)
\end{aligned}
$$

where $\widetilde{C}$ depends on C_1, C_2, C_4, λ and b^2 only, but not on t. Thus, the inequality (3.22) implies that the constant $\widetilde{C}$ satisfies

$$E\sup_{k\le s\le k+1}\|X(s)\|_2^2\le\widetilde{C},\quad k=1,2,\cdots.$$

Let ε be arbitrary. It follows from the Chebyshev inequality that

$$\mathbb{P}\Big\{\sup_{k\le s\le k+1}\|X(s)\|_2>k^{1+\varepsilon}\Big\}\le\frac{E\sup\limits_{k\le s\le k+1}\|X(s)\|_2^2}{k^{2+2\varepsilon}}\le\frac{\widetilde{C}}{k^{2+2\varepsilon}},\quad k=1,2,\cdots.$$

Applying the well-known Borel-Cantelli Lemma (see [14]), we find that for almost all $\omega\in\Omega$,

$$\sup_{k\le s\le k+1}\|X(s)\|_2\le k^{1+\varepsilon}\quad\text{for all but finitely many } k. \qquad (3.23)$$

Hence, there exists a $k_0(\omega)$, such that for almost all $\omega\in\Omega$, (3.23) holds whenever $k\ge k_0$. Consequently, for almost all $\omega\in\Omega$, if $k\ge k_0$ and $k\le t\le k+1$, one has

$$\frac{\log(\|X(t)\|_2)}{\log(t)}\le\frac{\log(k^{1+\varepsilon})}{\log(k)}=1+\varepsilon.$$

Taking $t\to\infty$, we conclude

$$\limsup_{t\to\infty}\frac{\log(\|X(t)\|_2)}{\log(t)}\le 1+\varepsilon,$$

which, by letting $\varepsilon\to 0$, gives the pathwise property (3.19). In other words, the solution would not grow faster than $t^{1+\varepsilon}$, with probability one.

4 Numerical Results

In this section, we present some numerical results with different parameters which show qualitative differences between the stochastic and deterministic Lorentz-Rössler systems, and illustrate our theoretic analysis. We use the stochastic and deterministic Runge-Kutta schemes to carry out our numerical tests, for all of which the initial datum is taken to be $X_0=(1,0.1,1)$ and all coefficients in the systems are chosen so that the conditions (A1)–(A4) are satisfied. For simplicity, we only consider the independent noise ($u_{ij,i\ne j}=0,\ i,j=1,2,3$). First we consider some examples to exhibit the moment estimates of the Rössler-Lorentz system.

Example 4.1 Let the coefficients of the linear terms be $\sigma=2,\ r=0,\ a=-7,\ b=1,\ \beta=2,\ c=0$, and nonlinear parameters be $\alpha_1(t)=\frac{1}{\sin(t^2)+2}+\frac{1}{\cos(t^2)+2},\ \alpha_2(t)=\frac{1}{\cos(t^2)+2},\ \alpha_3(t)=\frac{1}{\sin(t^2)+2}$. With these choices, it is easy to see that the conditions (A1)–(A4) hold. The stochastic and deterministic solutions to (1.4) are shown in the following Figures 9–10, respectively.

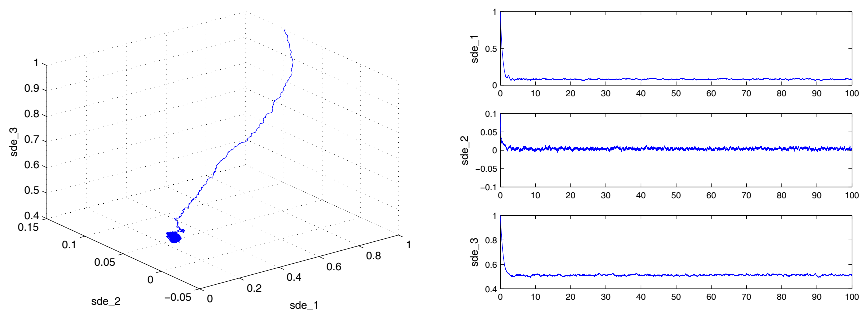

Figure 9 The attractor and the time series of the stochastic Rössler-Lorentz system with a white noise $u_{ii} = 0.01,\ i = 1, 2, 3.$

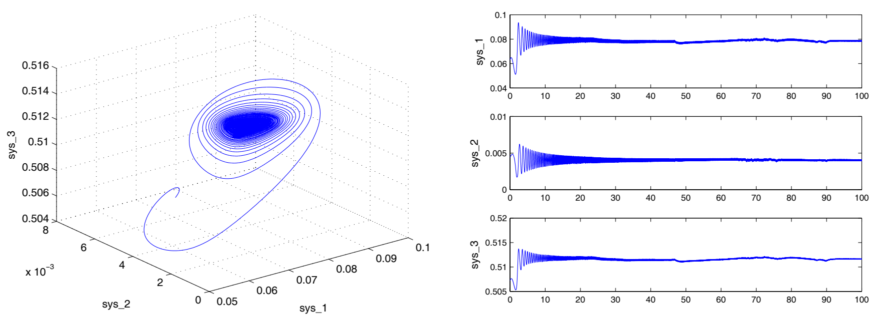

Figure 10 The attractor and the time series of the deterministic Rössler-Lorentz system.

If we take the nonlinear terms to be small transformations:

$$\alpha_1(t) = \frac{1}{\sin(t)+2} + \frac{1}{\cos(t^2)+2}, \quad \alpha_2(t) = \frac{1}{\cos(t^2)+2}, \quad \alpha_3(t) = \frac{1}{\sin(t)+2},$$

then we find that the stochastic and deterministic solutions possess very different trajectories (see Figures 11–12).

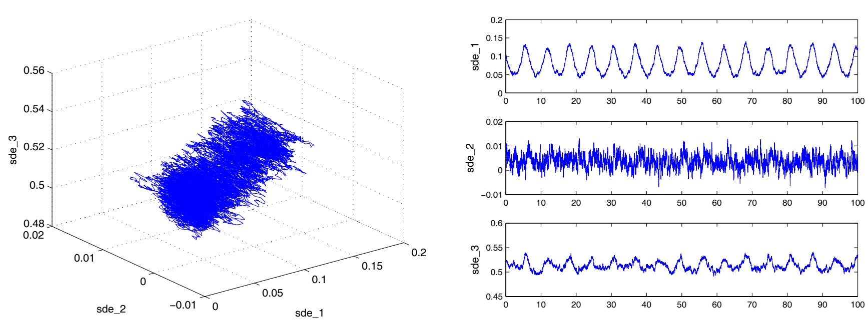

Figure 11 The attractor and the time series of the stochastic Rössler-Lorentz system with a white noise $u_{ii} = 0.01,\ i = 1, 2, 3.$

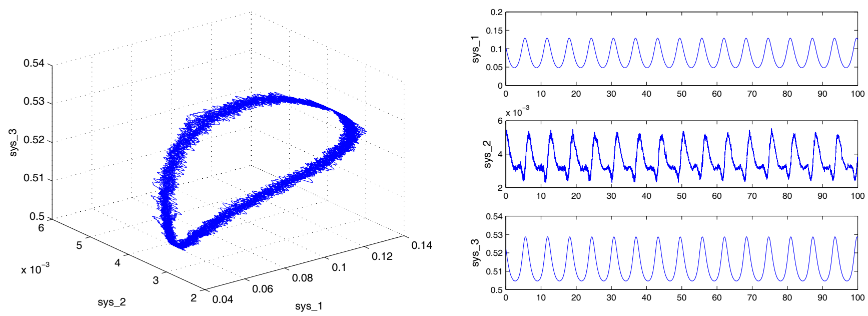

Figure 12 The attractor and the time series of the deterministic Rössler-Lorentz system.

Example 4.2 Let $c = 0$. We take $\beta > 0$ and $a < 1$ to satisfy the condition (A1). In this example, we first consider the case with $\sigma = 12,\ r = -10,\ a = -7,\ b = 1,\ \beta = 2,\ c = 0$, and

$$\alpha_1(t) = \frac{1}{\sin(t) + 2} + \frac{5}{\sin(t^2) + 2}, \quad \alpha_2(t) = \frac{5}{\sin(t^2) + 2}, \quad \alpha_3(t) = \frac{1}{\sin(t) + 2}.$$

The corresponding numerical solutions are illustrated in Figures 13–14.

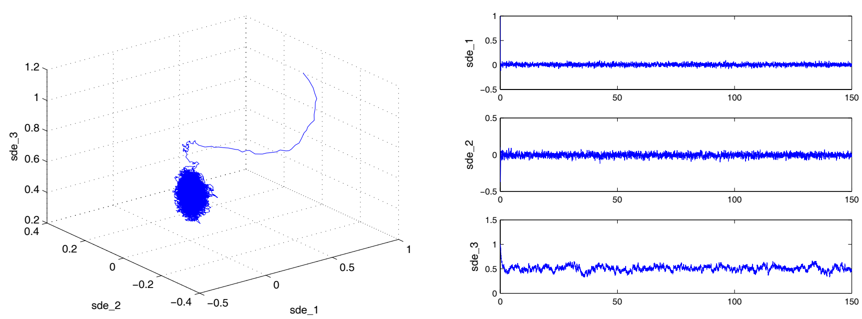

Figure 13 Stochastic system with a white noise $u_{ii} = 0.1,\ i = 1, 2, 3$.

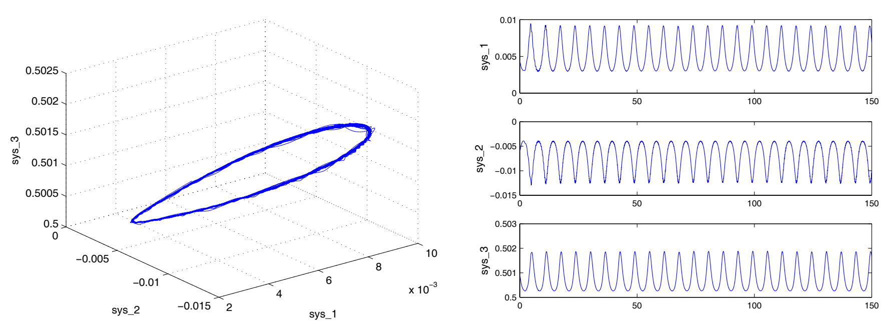

Figure 14 Deterministic system.

Then, we consider the effect of matrix A. Change the linear terms and assume that $\sigma = 9,\ r = -10,\ a = -18,\ b = 1,\ \beta = 2,\ c = 0$, and the corresponding numerical solutions are shown in Figures 15–16.

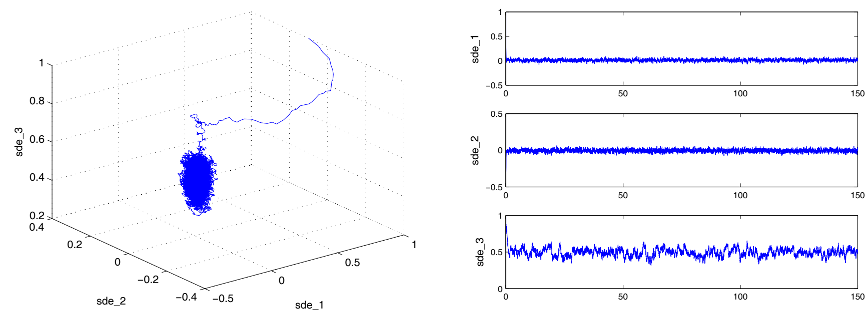

Figure 15 Transformed linear terms system with a white noise $u_{ii} = 0.1,\ i = 1, 2, 3.$

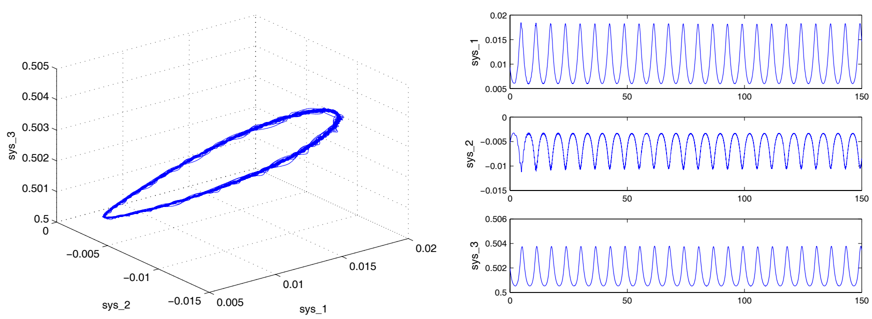

Figure 16 Deterministic system with different linear terms.

Example 4.3 In this example, we mainly test the effect of noises. Take $\sigma = 2,\ r = 0,\ a = -18,\ b = 1,\ \beta = 8(\cos t + 1),\ c = 0$, and

$$\alpha_1(t) = \frac{t}{2t+2} - 2\sin(t) + 3 + \frac{1}{\cos(3t)+2}, \quad \alpha_3(t) = \frac{t}{2t+2}, \quad \alpha_2(t) = -2\sin(t) + 3 + \frac{1}{\cos(3t)+2}.$$

Thus, the corresponding stochastic and deterministic numerical solutions are presented in Figures 17–20.

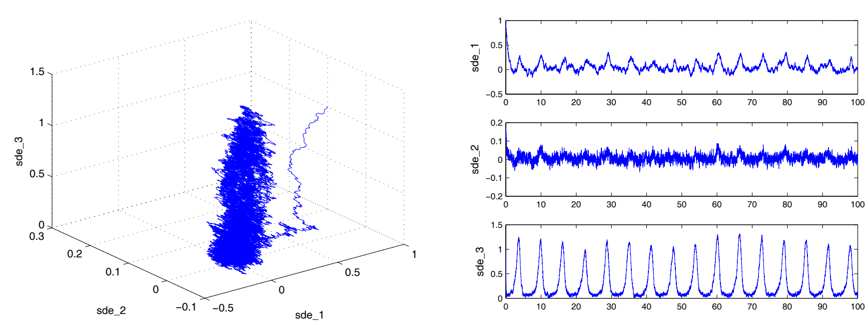

Figure 17 Stochastic system with a white noise $u_{ii} = 0.1,\ i = 1, 2, 3.$

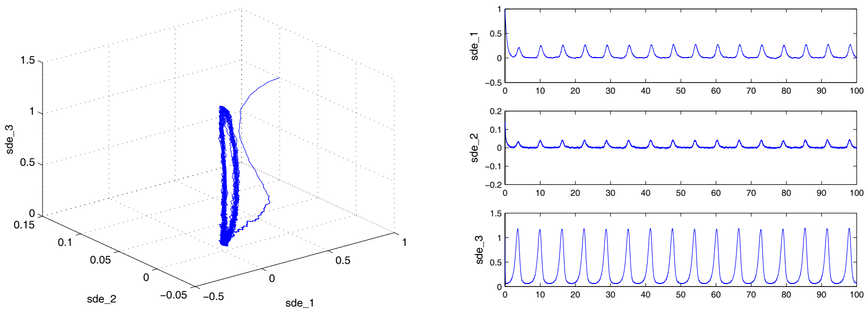

Figure 18 Stochastic system with a white noise $u_{ii} = 0.01,\ i = 1, 2, 3.$

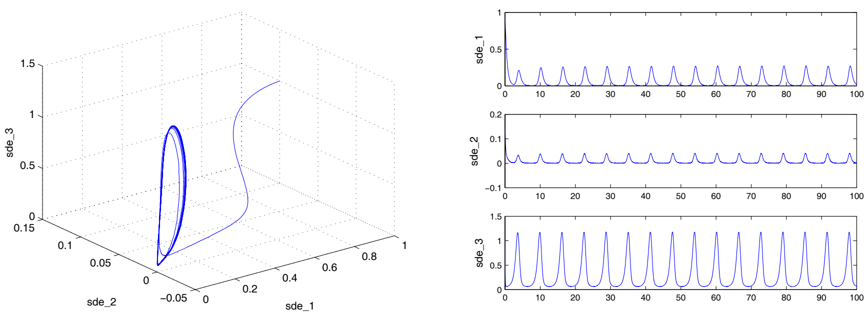

Figure 19 Stochastic system with a white noise $u_{ii} = 0.001,\ i = 1, 2, 3.$

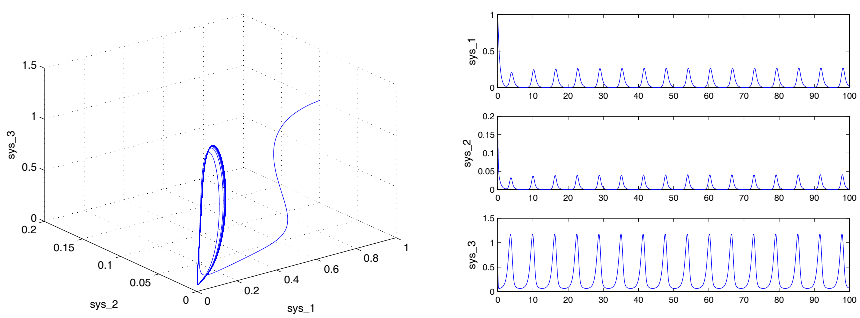

Figure 20 Deterministic system with a white noise $u_{ii} = 0,\ i = 1, 2, 3.$

In this paper, we have established a sufficient condition under which the stochastic system (1.3) has a unique solution. Moreover, we have carried out a number of numerical experiments, which show some interesting qualitative behaviors of solutions, summarized as follows.

(1) From Examples 1–3 we obviously see that for any time T, Theorem 3.1 can be ensured, and the boundary of the moments of the solutions can be obtained.

(2) Examples 1–2 clearly show the dynamical behavior of the system (1.4), which mainly depends on the nonlinear terms. In particular, Example 1 gives small transforms on nonlinear terms, but the respective paths have large differences.

(3) For a more general Lorentz-Rössler system, if the uniqueness can be ensured, we can show, for Example 3 in particular, that the stochastic system converges toward the corresponding deterministic system when the intensity of the noise goes to zero.

(4) All the numerical results on the system (1.3) give us very abundant expressions, including the behavior of the well-known Lorentz and Rössler systems. Furthermore, parts of the systems could be used to make a more complex and more secure hop-frequency time series.

(5) In view of the numerical results given in Figure 4, we have clearly found that the combined stochastic Lorentz-Rössler system possesses better properties than the corresponding deterministic system.

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References

[1] Chen, G. and Dong, X., From Chaos to Order Methodologies Perspectives and Applications, World Scientific, Singapore, 1994.

[2] Chen, S. H. and Lü, J. H., Synchronization of an uncertain unified chaotic systems using backstepping design, *Chaos, Solitions and Fractals*, **14**, 2002, 643–647.

[3] Cheng, S. R., Stochastic population systems, *Stochastic Analysis and Applications*, **27**, 2009, 854–874.

[4] Cuomo, K. M., Circuit implementation of synchronized chaos with application to communications, *Phys. Rev. Lett.*, **71**, 1993, 65–68.

[5] Fang, J. Q., Hong, Y. and Chen, G., Switching manifold approach to chaos synchronization, *Phys. Rev. E*, **59**, 1999, 2523–2526.

[6] Gorman, M., Widmann, P. J. and Robbins, K. A., Nonlinear dynamics of a convection loop: A quantitative comparison of experiment with theory, *Phys. D*, **19**, 1986, 255–267.

[7] Haken, H., Analogy between higher instabilities in fluids and lasers, *Phys. Lett. A*, **53**, 1975, 77–78.

[8] Hemnti, N., Strange attractors inbrushless DC motors, *IEEE Transactions on Circuits and Systems I: Fundamental Theory and Applications*, **41**, 1994, 40–45.

[9] Itoh, M., Yang, T. and Chua, L. O., Conditions for impulsive synchronization of chaotic and hyperchaotic systems, *Int. J. Bifur. Chaos*, **11**, 2001, 551–560.

[10] Knobloch, E., Chaos in the segmented disc dynamics, *Phys. Lett. A*, **82**(9), 1981, 439–440.

[11] Liu, M. and Wang, K., Persistence, extinction and global asymptotical stability of a non-autonomous predator-prey model with random perturbation, *Appl. Math. Modelling*, **36**, 2012, 5344–5353.

[12] Lorentz, E. N., Determinstic nonperiodic flows, *J. Atmospheric Sci.*, **12**, 1963, 139–163.

[13] Mao, X., Exponential Stability of Stochastic Differential Equations, Marcel Dekker Inc., New York, 1994.

[14] Mao, X., Stochastic Differential Equations and Applications, Horwood Publishing, Chichester, 1997.

[15] Peifer, M., Schelter, B., Winterhalder, M., et al., Mixing properties of the Rössler system and consequences for coherence and synchronization analysis, *Phys. Rev. E.*, **72**, 2005, 1–7.

[16] Peitgen, H. O., Jürgens, H. and Saupe, D., Chaos and Fractals, Springer-Verlag, Berlin, Heidelberg, 1992.

[17] Poland, D., Cooperative catalysis and chemical chaos: a chemical model for Lorentz equation, *Phys. D*, **65**, 1993, 86–99.

[18] Rössler, O. E., An equation for continuous chaos, *Phys. Lett. A*, **57**, 1976, 397–398.

[19] Xing, Z. W. and Peng, J. G., Boundedness, persistence and extinction of a stochastic non-autonomous logistic system with time delays, *Appl. Math. Modelling*, **36**, 2012, 3379–3386.

[20] Yang, Q. S. and Jiang, D. Q., A note on asymptotic behaviors of stochastic population model with Allee effect, *Appl. Math. Modelling*, **35**, 2011, 4611–4619.