

Bochner-Kodaira Techniques on Kähler Finsler Manifolds*

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Abstract A horizontal Hodge Laplacian operator $\square_{\mathcal{H}}$ is defined for Hermitian holomorphic vector bundles over PTM on Kähler Finsler manifold, and the expression of $\square_{\mathcal{H}}$ is obtained explicitly in terms of horizontal covariant derivatives of the Chern-Finsler connection. The vanishing theorem is obtained by using the $\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}$ -method on Kähler Finsler manifolds.

Keywords Kähler Finsler manifold, Horizontal Hodge Laplacian operator, Bochner-Kodaira technique

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1 Introduction

Bochner [1–3] initiated a method, i.e., the well-known “Bochner technique”, which used the Laplace operator and the general maximum principle of Hopf to deal with the relation between vector or tensor fields and the curvature of manifolds, and got the global properties of manifolds. From then on, the Bochner technique became a very useful method in geometrical study. Both in Riemannian and Kählerian manifolds, the Bochner technique was discussed in details in [4–8]. The Bochner technique is used to integrate the Laplacian of the pointwise square norm of a harmonic form over a compact Riemannian manifolds, yielding thereby two terms. One is the global square norm of the covariant derivatives of the harmonic form. The other involves the curvature tensor. Under the suitable condition of the curvature tensor, it can be obtained that the harmonic form must be zero or parallel. It was applied to $(0, q)$ -forms on a Kähler manifold with values in Hermitian holomorphic line bundles, due to Kodaira [9]. Later, the technique was called as the Bochner-Kodaira technique. The Bochner-Kodaira technique is the important method in differential geometry and is variated as the $\partial\bar{\partial}$ Bochner-Kodaira technique due to Siu [10–11].

Recently, under the initiation of S. S. Chern, the global differential geometry of real and complex Finsler manifolds gained a great development (see [12–17]), Abate and Pateizio [16] set up a Cartan-Finsler connection in real Finsler manifolds and a Chern-Finsler connection in complex Finsler manifolds. The main purpose of this paper is to generalize the Bochner-Kodaira

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techniques from Kähler manifolds to Kähler Finsler manifolds, and the vanishing theorem is obtained by using the $\partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}$ Bochner-Kodaira technique.

2 Complex Finsler Manifolds

Let M be a compact complex manifold of dimension n , and $\pi : T^{1,0}M \rightarrow M$, where $T^{1,0}M$ is the holomorphic tangent bundle of M . We denote by $o : M \rightarrow T^{1,0}M$ the zero section of $T^{1,0}M$, and set $\widetilde{M} = T^{1,0}M \setminus o(M)$, which means the holomorphic tangent bundle minus its zero section. Let $z = (z^1, \dots, z^n)$ and $(z, v) = (z^1, \dots, z^n, v^1, \dots, v^n)$ be the local coordinates on M and the induced complex coordinates on $T^{1,0}M$, respectively. For simplicity, we denote

$$\partial_{\mu} = \frac{\partial}{\partial z^{\mu}}, \quad \partial_{\alpha} = \frac{\partial}{\partial v^{\alpha}}, \quad 1 \leq \mu, \alpha \leq n,$$

which give a local holomorphic frame field of $T^{1,0}\widetilde{M}$.

Since there is a natural $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acting on TM by scalar multiplication, the projective tangent bundle PTM can be defined by $\text{PTM} = \{(z, [v]) \mid (z, [v]) = (z, \lambda v), \forall (z, v) \in \widetilde{M}, \lambda \in \mathbb{C}^*\}$. The local coordinate system (z, v) for TM may also be considered as a local coordinate system for PTM as long as v is considered as a homogeneous coordinate system. The reason for working on PTM rather than TM is that PTM is compact, if M is compact.

Let F be a strongly pseudoconvex complex Finsler metric defined on $T^{1,0}M$, that is $F : T^{1,0}M \rightarrow \mathbb{R}^+$ is a continuous function satisfying the following conditions (see [16]):

- (i) $G = F^2$ is smooth on \widetilde{M} ;
- (ii) $F(v) > 0$ for all $v \in \widetilde{M}$;
- (iii) $F(\lambda v) = |\lambda|F(v)$ for all $v \in T^{1,0}M$ and $\lambda \in \mathbb{C}$;
- (iv) The Hermitian matrix $(G_{\alpha\bar{\beta}})$ is positive definite on \widetilde{M} , where

$$G_{\alpha\bar{\beta}} = \frac{\partial^2 G}{\partial v^{\alpha} \partial \bar{v}^{\beta}},$$

and the derivatives with respect to the z -coordinates will be denoted by indexes after a semi-colon, for instance,

$$G_{;\mu\nu} = \frac{\partial^2 G}{\partial z^{\mu} \partial z^{\nu}} \quad \text{or} \quad G_{\alpha;\bar{\nu}} = \frac{\partial^2 G}{\partial \bar{z}^{\nu} \partial v^{\alpha}}.$$

A manifold M endowed with a strongly pseudoconvex complex Finsler metric will be called a strongly pseudoconvex complex Finsler manifold.

Let $T_{\mathbb{C}}\widetilde{M}$ be the complex tangent bundle of the real tangent bundle $T_{\mathbb{R}}\widetilde{M}$ of \widetilde{M} , and $T_{\mathbb{C}}M$ be the complex tangent bundle of the real tangent bundle $T_{\mathbb{R}}M$ of M . Then the differential $d\pi : T_{\mathbb{C}}\widetilde{M} \rightarrow T_{\mathbb{C}}M$ of $\pi : \widetilde{M} \rightarrow M$ defines the vertical bundle \mathcal{V} over \widetilde{M} by

$$\mathcal{V} = \text{Ker} d\pi \cap T^{1,0}\widetilde{M} \subset T^{1,0}\widetilde{M},$$

which is a holomorphic vector bundle of rank n over \widetilde{M} . A local frame field of \mathcal{V} is given by $\{\dot{\partial}_1, \dots, \dot{\partial}_n\}$ and there is a well-defined Hermitian metric on \mathcal{V} induced by F given by

$$\langle V_1, V_2 \rangle_v = G_{\alpha\bar{\beta}}(z, v) V_1^{\alpha} \bar{V}_2^{\beta}, \quad (2.1)$$

where $(z, v) \in \widetilde{M}$, $V_1, V_2 \in \mathcal{V}_v$ with $V_j = V_j^{\alpha} \dot{\partial}_{\alpha}$ ($j = 1, 2$). Then there is a unique Chern-Finsler connection D associated to the Hermitian structure induced by F . Then the Chern

Finsler connection is the Hermitian connection of the holomorphic vector bundle $(\mathcal{V}, \langle, \rangle)$. Let $\mathcal{H} \subset T^{1,0}\widetilde{M}$ be the complex horizontal bundle associated to the Chern Finsler connection, and the natural local frame $\{\delta_1, \dots, \delta_n\}$ for \mathcal{H} is given by

$$\delta_\mu = \partial_\mu - \Gamma_{;\mu}^\alpha \dot{\partial}_\alpha,$$

where $\Gamma_{;\mu}^\alpha = G^{\bar{\tau}\alpha} G_{\bar{\tau};\mu}$ are called the nonlinear connection coefficients associated to (M, F) . In this paper, we shall only use the adapted frame $\{\delta_\mu, \dot{\partial}_\alpha\}$ for $T^{1,0}\widetilde{M}$ and its dual frame $\{dz^\mu, \delta v^\alpha\}$, where $\delta v^\alpha = dv^\alpha + \Gamma_{;\mu}^\alpha dz^\mu$, because they have a simple rule of transformation under the change of coordinates.

Using the complex horizontal map $\Theta : \mathcal{V} \rightarrow \mathcal{H}$, the Hermitian metric \langle, \rangle on \mathcal{V} can be transferred on \mathcal{H} by setting $\langle H_1, H_2 \rangle_v = \langle \Theta^{-1}(H_1), \Theta^{-1}(H_2) \rangle_v$ for $v \in \widetilde{M}$ and $H_1, H_2 \in \mathcal{H}_v$. Note that we shall use $v \in \widetilde{M}$ rather than $(z, v) \in \widetilde{M}$ for simplicity, when there is no chance of confusion. The Hermitian metric \langle, \rangle on $T^{1,0}\widetilde{M}$ is defined by requiring \mathcal{H} to be orthogonal to \mathcal{V} and the Chern Finsler connection extends to the complex linear connection still called the Chern Finsler connection on \widetilde{M} , which is compatible with the Hermitian metric \langle, \rangle on $T^{1,0}\widetilde{M}$, but is not torsion free in general. It has \mathcal{H} -valued $(2,0)$ -torsion θ and \mathcal{V} -valued $(1,1)$ -torsion τ , and θ relates to the Kählerianity of the Chern Finsler connection D . More precisely, a strongly pseudoconvex complex Finsler metric F is called strongly Kähler if and only if

$$\Gamma_{\mu;\nu}^\alpha - \Gamma_{\nu;\mu}^\alpha = 0, \quad (2.2)$$

called Kähler if and only if

$$(\Gamma_{\mu;\nu}^\alpha - \Gamma_{\nu;\mu}^\alpha)v^\mu = 0, \quad (2.3)$$

and called weakly Kähler if and only if

$$G_\alpha(\Gamma_{\mu;\nu}^\alpha - \Gamma_{\nu;\mu}^\alpha)v^\mu = 0. \quad (2.4)$$

Recently, Chen and Shen [18] showed that a Kähler-Finsler metric must be a strongly Kähler-Finsler metric. Then, it is necessary to consider the Kählerian case in this paper.

By defining $D(\bar{X}) = \overline{DX}$ and the complex linearity, the Chern Finsler connection D can be extended to the whole complex vector bundle $T_{\mathbb{C}}\widetilde{M}$ and its dual complex vector bundle $T_{\mathbb{C}}^*\widetilde{M}$ by requiring $D\varphi(X) + \varphi(DX) = d\varphi(X)$ for every $\varphi \in \chi(T_{\mathbb{C}}^*\widetilde{M})$ and $X \in \chi(T_{\mathbb{C}}\widetilde{M})$. Thus the Chern Finsler connection can also be extended to the complex linear connection $D : \chi(T_{\mathbb{C}}^{r,s}\widetilde{M}) \rightarrow \chi(T_{\mathbb{C}}^{r,s}\widetilde{M} \otimes T_{\mathbb{C}}^{r,s}\widetilde{M})$ in the usual way. All the extended connections are still called the Chern Finsler connection with the conjugation and preserving the type. Let ∇ be the covariant differentiation defined by D . Since the complex Finsler fundamental tensor $G_{\alpha\bar{\beta}}$ is both \mathcal{H} -metrical and \mathcal{V} -metrical, i.e.,

$$\nabla_{\delta_\gamma} G_{\alpha\bar{\beta}} = 0, \quad \nabla_{\dot{\partial}_\gamma} G_{\alpha\bar{\beta}} = 0, \quad (2.5)$$

$G^{\alpha\bar{\beta}}$ are also both \mathcal{H} -metrical and \mathcal{V} -metrical.

3 Bochner-Kodaira Techniques for the Pull-Back Bundles of Holomorphic Vector Bundles

The principal step in the Bochner-Kodaira technique is the computation of the Laplacian. There are some results about the Laplacian and their applications for horizontal (p, q) -forms

on the base manifold or the tangent bundle (see [19–23]). In preparation, we will give simple statements for the Laplacian for the horizontal (p, q) -forms on PTM, and omit the complicated computation. In this section, we focus on the horizontal Laplacian $\square_{\mathcal{H}}$ of the horizontal (p, q) -form with the value in the Hermitian holomorphic vector bundles on PTM. Then, we will give the expression of $\square_{\mathcal{H}}$ explicitly in terms of the horizontal covariant derivatives of the Chern Finsler connection, which was called the Bochner-Kodaira technique.

Let (M, F) be a compact Kähler Finsler manifold. It is known that F induces naturally a non-degenerated Hermitian metric on the total space PTM,

$$\tilde{G} = G_{\alpha\bar{\beta}} dz^\alpha \otimes d\bar{z}^\beta + (\ln G)_{\alpha\bar{\beta}} \delta v^\alpha \otimes \delta \bar{v}^\beta.$$

Denote

$$\omega_{\mathcal{V}} = \sqrt{-1}(\ln G)_{\alpha\bar{\beta}} \delta v^\alpha \wedge \delta \bar{v}^\beta, \quad \omega_{\mathcal{H}} = \sqrt{-1}G_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta.$$

Then the invariant volume form of PTM is given by

$$dv = \frac{\omega_{\mathcal{H}}^n}{n!} \wedge \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!}.$$

If we denote by $d\sigma$ the pure vertical form of the volume form of PTM, thus

$$d\sigma = \frac{\omega_{\mathcal{V}}^{n-1}}{(n-1)!},$$

and then

$$dv = \frac{\omega_{\mathcal{H}}^n}{n!} \wedge d\sigma.$$

Let $\mathcal{A}^{p,q}$ be the space of horizontal (p, q) -forms on PTM, that is, those coefficients of every $\varphi \in \mathcal{A}^{p,q}$ are zero homogeneous with respect to fibre coordinates, and the elements of $\mathcal{A}^{p,q}$ in local coordinates are

$$\begin{aligned} \varphi &= \frac{1}{p!q!} \sum \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\beta_1} \wedge \dots \wedge d\bar{z}^{\beta_q}, \\ \psi &= \frac{1}{p!q!} \sum \psi_{c_1 \dots c_p \bar{d}_1 \dots \bar{d}_q} dz^{c_1} \wedge \dots \wedge dz^{c_p} \wedge d\bar{z}^{d_1} \wedge \dots \wedge d\bar{z}^{d_q}. \end{aligned}$$

Then the pointwise inner product is given by

$$\langle \varphi, \psi \rangle = \frac{1}{p!q!} \sum \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} \overline{\psi_{c_1 \dots c_p \bar{d}_1 \dots \bar{d}_q}} G^{\bar{c}_1 \alpha_1} \dots G^{\bar{c}_p \alpha_p} G^{\bar{\beta}_1 d_1} \dots G^{\bar{\beta}_q d_q}. \quad (3.1)$$

If we denote

$$\begin{aligned} A_p &= (\alpha_1, \dots, \alpha_p), \quad \alpha_1 < \alpha_2 < \dots < \alpha_p, \quad 1 \leq \alpha_i \leq n, \\ A_{n-p} &= (\alpha_{p+1}, \dots, \alpha_n), \quad \alpha_{p+1} < \dots < \alpha_n, \quad 1 \leq \alpha_i \leq n, \end{aligned}$$

where $(\alpha_1, \dots, \alpha_p, \alpha_{p+1}, \dots, \alpha_n)$ is a permutation of $(1, 2, \dots, n)$. Similarly, for $B_q = (\beta_1, \dots, \beta_q)$, $B_{n-q} = (\beta_{q+1}, \dots, \beta_n)$; $C_p = (c_1, \dots, c_p)$, $C_{n-p} = (c_{p+1}, \dots, c_n)$; $D_q = (d_1, \dots, d_q)$, $D_{n-q} = (d_{q+1}, \dots, d_n)$, and

$$G^{\bar{C}_p A_p} = G^{\bar{c}_1 \alpha_1} \dots G^{\bar{c}_p \alpha_p}, \quad G^{\bar{B}_q D_q} = G^{\bar{\beta}_1 d_1} \dots G^{\bar{\beta}_q d_q},$$

then

$$\langle \varphi, \psi \rangle = \varphi_{A_p \bar{B}_q} \overline{\psi_{C_p \bar{D}_q}} G^{\bar{C}_p A_p} G^{\bar{B}_q D_q} = \varphi_{A_p \bar{B}_q} \overline{\psi_{\bar{A}_p B_q}}, \quad (3.2)$$

where $\psi_{\bar{A}_p B_q} = \psi_{C_p \bar{D}_q} G^{C_p \bar{A}_p} G^{B_q \bar{D}_q}$.

Notice that there is a global inner product in $\mathcal{A}^{p,q}$ given by

$$(\varphi, \psi)_{\text{PTM}} = \int_{\text{PTM}} \langle \varphi, \psi \rangle dv. \quad (3.3)$$

Then we can define the operator $*$: $\mathcal{A}^{p,q} \rightarrow \mathcal{A}^{n-q,n-p}$ by the relation

$$\int_{\text{PTM}} \varphi \wedge * \bar{\psi} \wedge d\sigma = (\varphi, \psi)_{\text{PTM}}. \quad (3.4)$$

It is easy to obtain that the operator $*$ has the following properties:

- (i) $*\bar{\psi} = *\bar{\psi}$;
- (ii) $**\psi = (-1)^{p+q}\psi$.

Under the local coordinate, if $\psi = \sum \psi_{A_p \bar{B}_q} dz^{A_p} \wedge dz^{\bar{B}_q}$,

$$*\psi = (i)^n (-1)^{\frac{1}{2}n(n-1)+pn} \sum G_{A_q A_{n-q} \bar{B}_p \bar{B}_{n-p}} G^{\bar{C}_q A_q} G^{\bar{B}_p D_p} \psi_{D_p \bar{C}_q} dz^{A_{n-q}} \wedge dz^{\bar{B}_{n-p}}, \quad (3.5)$$

where $G_{A_q A_{n-q} \bar{B}_p \bar{B}_{n-p}} = G_{\alpha_1 \dots \alpha_q \alpha_{q+1} \dots \alpha_n \bar{\beta}_1 \dots \bar{\beta}_p \bar{\beta}_{p+1} \dots \bar{\beta}_n} = \det(G_{\alpha_i \bar{\beta}_k})$.

If (M, F) is a Kähler Finsler manifold, then by the symmetry of the horizontal connection coefficients: $\Gamma_{\mu;\nu}^\alpha = \Gamma_{\nu;\mu}^\alpha$, the horizontal derivatives can be replaced by the horizontal covariant derivatives, that is,

$$\partial_{\mathcal{H}} \varphi = \frac{1}{p!q!} \sum \nabla_{\delta_\mu} \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} dz^\mu \wedge dz^{\alpha_1} \wedge \dots \wedge dz^{\bar{\beta}_q}, \quad (3.6)$$

$$\bar{\partial}_{\mathcal{H}} \varphi = \frac{1}{p!q!} \sum \nabla_{\delta_{\bar{\mu}}} \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} dz^{\bar{\mu}} \wedge dz^{\alpha_1} \wedge \dots \wedge dz^{\bar{\beta}_q}. \quad (3.7)$$

Let $\partial_{\mathcal{H}}^*$ and $\bar{\partial}_{\mathcal{H}}^*$ be the adjoint operators of $\partial_{\mathcal{H}}$ and $\bar{\partial}_{\mathcal{H}}$ with respect to the global inner product in $\mathcal{A}^{p,q}$, respectively, that is,

$$(\partial_{\mathcal{H}} \psi, \varphi) = (\psi, \partial_{\mathcal{H}}^* \varphi), \quad (\bar{\partial}_{\mathcal{H}} \psi, \varphi) = (\psi, \bar{\partial}_{\mathcal{H}}^* \varphi),$$

which satisfy $\partial_{\mathcal{H}}^* = - * (\bar{\partial}_{\mathcal{H}} *)$ and $\bar{\partial}_{\mathcal{H}}^* = - * (\partial_{\mathcal{H}} *)$. Then by solving $(\bar{\partial}_{\mathcal{H}} \psi, \varphi) = (\psi, \bar{\partial}_{\mathcal{H}}^* \varphi)$, one can get

$$(\bar{\partial}_{\mathcal{H}}^* \varphi)_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} = -(-1)^p \sum G^{\bar{\mu} \nu} \nabla_{\delta_{\bar{\beta}}} \varphi_{\alpha_1 \dots \alpha_p \bar{\mu} \bar{\beta}_1 \dots \bar{\beta}_q}. \quad (3.8)$$

By defining the horizontal Laplacian operator

$$\square_{\mathcal{H}} = \bar{\partial}_{\mathcal{H}} \bar{\partial}_{\mathcal{H}}^* + \bar{\partial}_{\mathcal{H}}^* \bar{\partial}_{\mathcal{H}}, \quad (3.9)$$

we can get its expression explicitly in terms of the horizontal covariant derivatives of the Chern Finsler connection.

Theorem 3.1 (see [21]) *For any $\varphi \in \mathcal{A}^{p,q}$,*

$$\begin{aligned} & (\square_{\mathcal{H}} \varphi)_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} \\ &= -G^{\bar{\mu} \nu} \nabla_{\delta_{\bar{\nu}}} \nabla_{\delta_{\bar{\beta}}} \varphi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q} + \sum_{\nu, \mu} \sum_{i=1}^q (-1)^{i-1} G^{\bar{\mu} \nu} [\nabla_{\delta_{\bar{\nu}}}, \nabla_{\delta_{\bar{\beta}_i}}] \varphi_{\alpha_1 \dots \alpha_p \bar{\mu} \bar{\beta}_1 \dots \widehat{\bar{\beta}_i} \dots \bar{\beta}_q}. \end{aligned} \quad (3.10)$$

In the following, we emphasize on deriving the horizontal Laplacian operator for the horizontal (p, q) -form with value in the Hermitian holomorphic vector bundle on PTM by extending the operators $\partial_{\mathcal{H}}, \bar{\partial}_{\mathcal{H}}, *, \bar{\partial}_{\mathcal{H}}^*, \partial_{\mathcal{H}}^*$, without confusion, and we continue to use all the symbols, respectively.

Let \tilde{E} be a Hermitian holomorphic vector bundle over PTM with the Hermitian metric $L_{\alpha\bar{\beta}}$. In local coordinates, the curvature operator of metric $L_{\alpha\bar{\beta}}$ is given by (see [6, p. 117] and [16, p. 93])

$$\Omega_{\beta}^{\alpha} = \bar{\partial}(L^{\alpha\bar{\gamma}}\partial L_{\beta\bar{\gamma}}),$$

where

$$\begin{aligned}\bar{\partial} &= \delta_{\bar{\nu}} d\bar{z}^{\nu} + \dot{\partial}_{\bar{\alpha}} \bar{\psi}^{\alpha}, & \partial &= \delta_{\nu} dz^{\nu} + \dot{\partial}_{\alpha} \psi^{\alpha}, \\ \delta_{\nu} &= \partial_{\nu} - \Gamma_{\nu}^{\alpha} \partial_{\alpha}, & \psi^{\alpha} &= dv^{\alpha} + \Gamma_{\mu}^{\alpha} dz^{\mu}.\end{aligned}$$

The coefficients of the horizontal part of Ω_{β}^{α} are

$$\Omega_{\alpha\bar{\beta};\mu\bar{\nu}} = \delta_{\mu}\delta_{\bar{\nu}}L_{\alpha\bar{\beta}} + (\delta_{\mu}\Gamma_{\bar{\nu}}^{\bar{\lambda}})(\dot{\partial}_{\bar{\lambda}}L_{\alpha\bar{\beta}}) - L^{\sigma\bar{\tau}}(\delta_{\mu}L_{\alpha\bar{\tau}})(\delta_{\bar{\nu}}L_{\sigma\bar{\beta}}), \quad (3.11)$$

so we can define a horizontal curvature form of \tilde{E} by

$$\Theta_{\alpha\bar{\beta}} = -\sqrt{-1}\Omega_{\alpha\bar{\beta};\mu\bar{\nu}}dz^{\mu} \wedge d\bar{z}^{\nu}. \quad (3.12)$$

If the strongly pseudoconvex complex Finsler manifold is a Kähler manifold, then the nonlinear connection $\Gamma_{\nu}^{\alpha} = 0$, and the horizontal curvature form (3.12) reduces to an ordinary curvature form of the Kähler manifold (see [10]). Furthermore, for a Kähler Finsler manifold, there exists a normal coordinate system (see [24–25]), for which one also has the nonlinear connection $\Gamma_{\nu}^{\alpha} = 0$, and in this case, the horizontal curvature form (3.12) takes the form of a Kähler manifold, that is, in the normal coordinate system, the Kähler Finsler manifold is very similar to a Kähler manifold, so one often uses the normal coordinate system to simplify calculations (see [5]).

Let $\mathcal{A}^{p,q}(\tilde{E})$ be the space of complex horizontal (p, q) -forms on PTM with value in \tilde{E} . If $\{e_{\sigma}\}_{\sigma=1}^r$ is a local holomorphic frame of \tilde{E} , then $\phi, \psi \in \mathcal{A}^{p,q}(\tilde{E})$ are

$$\phi = \sum_{\sigma=1}^r e_{\sigma} \phi^{\sigma}, \quad \psi = \sum_{\mu=1}^r e_{\mu} \psi^{\mu},$$

respectively, where $\phi^{\sigma}, \psi^{\mu} \in \mathcal{A}^{p,q}$, and we have

$$\bar{\partial}_{\mathcal{H}}\phi = \sum_{\sigma=1}^r e_{\sigma} \bar{\partial}_{\mathcal{H}}\phi^{\sigma}. \quad (3.13)$$

Then we define

$$\tilde{\nabla}_{\delta_{\alpha}}\phi = e_{\mu}(L^{\bar{\lambda}\mu}\delta_{\alpha}L_{\bar{\lambda}\bar{\nu}}\phi^{\nu} + \nabla_{\delta_{\alpha}}\phi^{\mu}), \quad \tilde{\nabla}_{\delta_{\bar{\beta}}}\phi = e_{\mu}\nabla_{\delta_{\bar{\beta}}}\phi^{\mu}, \quad (3.14)$$

where $\nabla_{\delta_{\alpha}}, \nabla_{\delta_{\bar{\beta}}}$ are the horizontal covariant derivatives of the Chern Finsler connection.

We can define the pointwise inner product on $\mathcal{A}^{p,q}(\tilde{E})$ by

$$\langle \phi, \psi \rangle = \sum_{\sigma,\mu=1}^r \langle e_{\sigma}, e_{\mu} \rangle \langle \phi^{\sigma}, \psi^{\mu} \rangle = \sum_{\sigma,\mu=1}^r L_{\sigma\bar{\mu}} \langle \phi^{\sigma}, \psi^{\mu} \rangle, \quad (3.15)$$

where $\langle \phi^\sigma, \psi^\mu \rangle$ is defined by (3.1).

The global inner product can be defined by

$$\begin{aligned} (\phi, \psi) &= \int_{\text{PTM}} \langle \phi, \psi \rangle dv = \int_{\text{PTM}} \sum_{\lambda, \mu=1}^r L_{\lambda\bar{\mu}} \langle \phi^\lambda, \psi^\mu \rangle dv \\ &= \int_{\text{PTM}} \sum_{\lambda, \mu=1}^r L_{\lambda\bar{\mu}} \phi^\lambda \wedge * \bar{\psi}^\mu \wedge d\sigma, \end{aligned} \quad (3.16)$$

and we define $\|\psi\|^2 = (\phi, \phi)$, as usual.

The adjoint $\bar{\partial}_{\mathcal{H}}^*$ of $\bar{\partial}_{\mathcal{H}}$ on $\mathcal{A}^{p,q}(\tilde{E})$ can be defined by solving

$$(\bar{\partial}_{\mathcal{H}} \phi, \psi) = (\phi, \bar{\partial}_{\mathcal{H}}^* \psi) \quad (3.17)$$

for $\bar{\partial}_{\mathcal{H}}^*$. Let $\psi \in \mathcal{A}^{p,q+1}(\tilde{E})$, $\phi \in \mathcal{A}^{p,q}(\tilde{E})$ and $\tau = L_{\lambda\bar{\mu}} \phi^\lambda \wedge * \bar{\psi}^\mu \wedge d\sigma$, and then

$$\begin{aligned} 0 &= \int_{\text{PTM}} d\tau = \int_{\text{PTM}} d_{\mathcal{H}} \tau = \int_{\text{PTM}} \bar{\partial}_{\mathcal{H}} \tau \\ &= \int_{\text{PTM}} L_{\lambda\bar{\mu}} \bar{\partial}_{\mathcal{H}} \phi^\lambda \wedge * \bar{\psi}^\mu \wedge d\sigma + (-1)^{p+q} \int_{\text{PTM}} \phi^\lambda \wedge \bar{\partial}_{\mathcal{H}} (L_{\lambda\bar{\mu}} * \bar{\psi}^\mu) \wedge d\sigma, \end{aligned}$$

but

$$\int_{\text{PTM}} L_{\lambda\bar{\mu}} \bar{\partial}_{\mathcal{H}} \phi^\lambda \wedge * \bar{\psi}^\mu \wedge d\sigma = \int_{\text{PTM}} L_{\lambda\bar{\mu}} \phi^\lambda \wedge * (\bar{\partial}_{\mathcal{H}}^* \bar{\psi}^\mu) \wedge d\sigma.$$

Thus

$$L_{\bar{\lambda}\mu} * (\bar{\partial}_{\mathcal{H}}^* \psi)^\mu = -(-1)^{p+q} \partial_{\mathcal{H}} (L_{\nu\bar{\lambda}} * \psi^\nu).$$

Then

$$\begin{aligned} * (\bar{\partial}_{\mathcal{H}}^* \psi)^\mu &= -(-1)^{p+q} L^{\bar{\lambda}\mu} \partial_{\mathcal{H}} (L_{\nu\bar{\lambda}} * \psi^\nu), \\ (\bar{\partial}_{\mathcal{H}}^* \psi)^\mu &= - * (\partial_{\mathcal{H}} * \psi^\mu) - L^{\bar{\lambda}\mu} \delta_\alpha L_{\nu\bar{\lambda}} * (dz^\alpha \wedge * \psi^\nu) \\ &= \bar{\partial}_{\mathcal{H}}^* \psi^\mu - L^{\bar{\lambda}\mu} \delta_\alpha L_{\nu\bar{\lambda}} * (dz^\alpha \wedge * \psi^\nu). \end{aligned} \quad (3.18)$$

In order to obtain the expression of $(\bar{\partial}_{\mathcal{H}}^* \psi)^\mu$ explicitly, we need to compute $*(dz^\alpha \wedge * \psi^\nu)$. For $\psi^\nu \in \mathcal{A}^{p,q}$, we can write

$$\psi^\nu = \psi_{\alpha_1 \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_q}^\nu dz^{\alpha_1} \wedge \dots \wedge dz^{\alpha_p} \wedge d\bar{z}^{\bar{\beta}_1} \wedge \dots \wedge d\bar{z}^{\bar{\beta}_q} = \psi_{A_p \bar{B}_q}^\nu dz^{A_p} \wedge d\bar{z}^{B_q}$$

and

$$* \psi^\nu = i^n (-1)^{\frac{1}{2}n(n-1)+pn} G_{A_q A_{n-q} \bar{B}_p \bar{B}_{n-p}} G^{\bar{J}_q A_q} G^{\bar{B}_p I_p} \psi_{I_p \bar{J}_q}^\nu dz^{A_{n-q}} \wedge d\bar{z}^{B_{n-p}}.$$

If we set

$$\psi_{A_{n-q} \bar{B}_{n-p}}^\nu = i^n (-1)^{\frac{1}{2}n(n-1)+pn} G_{A_q A_{n-q} \bar{B}_p \bar{B}_{n-p}} G^{\bar{J}_q A_q} G^{\bar{B}_p I_p} \psi_{I_p \bar{J}_q}^\nu, \quad (3.19)$$

then

$$\begin{aligned} * \psi^\nu &= \psi_{A_{n-q} \bar{B}_{n-p}}^\nu dz^{A_{n-q}} \wedge d\bar{z}^{B_{n-p}} \\ &= \text{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} \psi_{B_{n-q} \bar{A}_{n-p}}^\nu dz^{B_{n-q}} \wedge d\bar{z}^{A_{n-p}} \end{aligned}$$

and

$$\begin{aligned}
*(dz^\alpha \wedge *\psi^\nu) &= * \left(\operatorname{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} \psi_{B_{n-q} \overline{A}_{n-p}}^\nu dz^\alpha \wedge dz^{B_{n-q}} \wedge d\overline{z}^{A_{n-p}} \right) \\
&= (i)^n (-1)^{\frac{1}{2}n(n-1)+(n-q+1)n} \operatorname{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} G_{B_{n-p} B_p \overline{\beta} \overline{A}_{n-q} \overline{E}_{q-1}} \\
&\quad \cdot G_{\overline{C}_{n-p} B_{n-p}} G_{\overline{A}_{n-q} D_{n-q}} G_{\overline{\beta} \alpha} \psi_{D_{n-q} \overline{C}_{n-p}}^\nu dz^{B_p} \wedge d\overline{z}^{E_{q-1}},
\end{aligned}$$

where E_{q-1} is the increasing set of numbers complementary to the set $\beta A_{n-q} \subset (1, \dots, n)$. Let χ_{q-1} be the increasing set of numbers complementary to the set $\beta B_{n-q} \subset (1, \dots, n)$, and set

$$\eta = i^n (-1)^{\frac{1}{2}n(n-1)+(n-q+1)n} \operatorname{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} \operatorname{sgn} \begin{pmatrix} A_{n-p} & A_p \\ B_{n-q} & B_q \end{pmatrix}.$$

Then

$$\begin{aligned}
*(dz^\alpha \wedge *\psi^\nu) &= \eta G_{A_{n-p} A_p \overline{\beta} \overline{B}_{n-q} \overline{\chi}_{q-1}} G_{\overline{C}_{n-p} A_{n-p}} G_{\overline{B}_{n-q} D_{n-q}} G_{\overline{\beta} \alpha} \psi_{D_{n-q} \overline{C}_{n-p}}^\nu dz^{A_p} \wedge d\overline{z}^{\chi_{q-1}} \\
&= \eta \operatorname{sgn} \begin{pmatrix} B_{n-q} & B_q \\ \beta B_{n-q} & \chi_{q-1} \end{pmatrix} G_{A_{n-p} A_p \overline{B}_{n-q} \overline{B}_q} G_{\overline{C}_{n-p} A_{n-p}} G_{\overline{B}_{n-q} D_{n-q}} G_{\overline{\beta} \alpha} \\
&\quad \cdot \psi_{D_{n-q} \overline{C}_{n-p}}^\nu dz^{A_p} \wedge d\overline{z}^{\chi_{q-1}}
\end{aligned}$$

and

$$\begin{aligned}
&i^n (-1)^{\frac{1}{2}n(n-1)+pn} \eta \operatorname{sgn} \begin{pmatrix} B_{n-q} & B_q \\ \beta B_{n-q} & \chi_{q-1} \end{pmatrix} \\
&= (-1)^{n^2+pn-qn} \operatorname{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} \operatorname{sgn} \begin{pmatrix} A_{n-p} & A_p \\ B_{n-q} & B_q \end{pmatrix} \operatorname{sgn} \begin{pmatrix} B_{n-q} & B_q \\ \beta B_{n-q} & \chi_{q-1} \end{pmatrix} \\
&= (-1)^{n^2+pn-qn+(n-p)p+(n-q)(q-1)} \operatorname{sgn} \begin{pmatrix} B_q & B_{n-q} \\ A_p & A_{n-p} \end{pmatrix} \operatorname{sgn} \begin{pmatrix} A_p & A_{n-p} \\ \beta \chi_{q-1} & B_{n-q} \end{pmatrix} \\
&= (-1)^p \operatorname{sgn} \begin{pmatrix} B_q \\ \beta \chi_{q-1} \end{pmatrix},
\end{aligned}$$

by which, and (3.18), we get

$$*(dz^\alpha \wedge *\psi^\nu) = (-1)^p \operatorname{sgn} \begin{pmatrix} B_q \\ \beta \chi_{q-1} \end{pmatrix} G_{\overline{\beta} \alpha} \psi_{A_p \overline{B}_q}^\nu dz^{A_p} \wedge d\overline{z}^{\chi_{q-1}}.$$

Hence, we have the following result.

Proposition 3.1 For any $\psi \in \mathcal{A}^{p,q}(\tilde{E})$,

$$(\overline{\partial}_{\mathcal{H}}^* \psi)_{A_p \overline{B}_{q-1}}^\mu = (\overline{\partial}_{\mathcal{H}}^* \psi^\mu)_{A_p \overline{B}_{q-1}} - (-1)^p \sum L^{\bar{\lambda} \mu} \delta_\alpha L_{\nu \bar{\lambda}} G_{\overline{\beta} \alpha} \psi_{A_p \overline{\beta} \overline{B}_{q-1}}^\nu. \quad (3.20)$$

By defining the horizontal Laplacian operator $\square_{\mathcal{H}}$ for the holomorphic vector bundle on PTM, we have

$$\square_{\mathcal{H}} = \overline{\partial}_{\mathcal{H}} \overline{\partial}_{\mathcal{H}}^* + \overline{\partial}_{\mathcal{H}}^* \overline{\partial}_{\mathcal{H}}$$

and $\square_{\mathcal{H}} \varphi = 0$ for $\varphi \in \mathcal{A}^{p,q}(\tilde{E})$, if and only if φ is harmonic horizontal (p, q) -form with value in \tilde{E} .

Theorem 3.2 *If (M, F) is a Kähler Finsler manifold, for any $\psi \in \mathcal{A}^{p,q}(\tilde{E})$, we have*

$$\begin{aligned} (\square_{\mathcal{H}}\psi)_{A_p\overline{B}_q}^\mu &= -G^{\overline{\beta}\alpha}\tilde{\nabla}_{\delta_\alpha}\tilde{\nabla}_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu + \sum_{k=1}^q G^{\overline{\beta}\alpha}[\nabla_{\delta_\alpha}, \nabla_{\delta_{\overline{\beta}_k}}]\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \\ &\quad - \sum_{k=1}^q (\Omega_{\nu;\overline{\beta}_k}^\mu - \Gamma_{\nu;\overline{\beta}_k}^\mu)\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu, \end{aligned} \quad (3.21)$$

where $\Omega_{\nu;\overline{\beta}_k}^\mu = \Omega_{\nu;\overline{\beta}_k\alpha}^\mu G^{\overline{\beta}\alpha}$, $\Gamma_{\nu;\overline{\beta}_k}^\mu = G^{\overline{\beta}\alpha}L^{\overline{\lambda}\mu}\dot{\partial}_\gamma(L_{\nu\overline{\lambda}})\delta_{\overline{\beta}_k}(\Gamma_\alpha^\gamma)$.

Proof From (3.3) and (3.13),

$$(\overline{\partial}_{\mathcal{H}}\psi)_{A_p\overline{\beta}\overline{B}_q}^\mu = (\overline{\partial}_{\mathcal{H}}\psi^\mu)_{A_p\overline{\beta}\overline{B}_q} = (-1)^p \left(\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu - \sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}}\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\mu \right).$$

Then

$$\begin{aligned} (\overline{\partial}_{\mathcal{H}}^*\overline{\partial}_{\mathcal{H}}\psi)_{A_p\overline{B}_q}^\mu &= (\overline{\partial}_{\mathcal{H}}^*\overline{\partial}_{\mathcal{H}}\psi^\mu)_{A_p\overline{B}_q} - \sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha} \left(\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu \right. \\ &\quad \left. - \sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}}\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\mu \right) \\ &= (\overline{\partial}_{\mathcal{H}}^*\overline{\partial}_{\mathcal{H}}\psi^\mu)_{A_p\overline{B}_q} - \sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu \\ &\quad + \sum L^{\overline{\lambda}\mu}\partial_\alpha L_{\overline{\lambda}\nu}G^{\overline{\beta}\alpha}\sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}}\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \end{aligned}$$

and

$$\begin{aligned} (\overline{\partial}_{\mathcal{H}}\overline{\partial}_{\mathcal{H}}^*\psi)_{A_p\overline{B}_q}^\mu &= (\overline{\partial}_{\mathcal{H}}\overline{\partial}_{\mathcal{H}}^*\psi^\mu)_{A_p\overline{B}_q} - \sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}} \left(\sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha}\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \right) \\ &= (\overline{\partial}_{\mathcal{H}}\overline{\partial}_{\mathcal{H}}^*\psi^\mu)_{A_p\overline{B}_q} - \sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}} \left(\sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha} \right) \psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \\ &\quad - \sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha} \left(\sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}}\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \right). \end{aligned}$$

Thus

$$\begin{aligned} (\square_{\mathcal{H}}\psi)_{A_p\overline{B}_q}^\mu &= (\square_{\mathcal{H}}\psi^\mu)_{A_p\overline{B}_q} - \sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\nu \\ &\quad - \sum_{k=1}^q \nabla_{\delta_{\overline{\beta}_k}} \left(\sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}G^{\overline{\beta}\alpha} \right) \psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \\ &= -G^{\overline{\beta}\alpha}\nabla_{\delta_\alpha}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu + \sum_{k=1}^q G^{\overline{\beta}\alpha}[\nabla_{\delta_\alpha}, \nabla_{\delta_{\overline{\beta}_k}}]\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\mu \\ &\quad - \sum G^{\overline{\beta}\alpha}L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\nu - \sum_{k=1}^q (\Omega_{\nu;\overline{\beta}_k}^\mu - \Gamma_{\nu;\overline{\beta}_k}^\mu)\psi_{A_p\overline{\beta}_1\cdots\overline{\beta}_{k-1}\overline{\beta}_{k+1}\cdots\overline{\beta}_q}^\nu \\ &= -G^{\overline{\beta}\alpha} \left(\nabla_{\delta_\alpha}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\mu + \sum L^{\overline{\lambda}\mu}\delta_\alpha L_{\nu\overline{\lambda}}\nabla_{\delta_{\overline{\beta}}}\psi_{A_p\overline{B}_q}^\nu \right) + \sum_{k=1}^q G^{\overline{\beta}\alpha} \end{aligned}$$

and

$$\begin{aligned}
& [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}_k}}] \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu - \sum_{k=1}^q (\Omega_{\nu; \bar{\beta}_k}^\mu - \Gamma_{\nu; \bar{\beta}_k}^\mu) \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\nu \\
&= -G^{\bar{\beta}\alpha} \tilde{\nabla}_{\delta_\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \psi_{A_p \bar{B}_q}^\mu + \sum_{k=1}^q G^{\bar{\beta}\alpha} [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}_k}}] \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu \\
&\quad - \sum_{k=1}^q (\Omega_{\nu; \bar{\beta}_k}^\mu - \Gamma_{\nu; \bar{\beta}_k}^\mu) \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\nu,
\end{aligned}$$

where $\Omega_{\nu; \bar{\beta}_k}^\mu = \Omega_{\nu; \bar{\beta}_k}^\mu G^{\bar{\beta}\alpha}$, $\Gamma_{\nu; \bar{\beta}_k}^\mu = G^{\bar{\beta}\alpha} L^{\bar{\lambda}\mu} \dot{\partial}_\gamma (L_{\nu \bar{\lambda}}) \delta_{\bar{\beta}_k}^\gamma (\Gamma_\alpha^\gamma)$.

Lemma 3.1 (see [16]) *Let $D : \chi(T^{1,0}\widetilde{M}) \longrightarrow \chi(T_C^* \widetilde{M} \otimes T_C^{1,0}\widetilde{M})$ be the complex linear connection on \widetilde{M} induced by the Chern-Finsler connection. Then for any $V_\alpha, V_\beta \in \chi(T^{1,0}\widetilde{M})$, we have*

$$D_{V_\alpha} D_{V_\beta} - D_{V_\beta} D_{V_\alpha} = D_{[V_\alpha, V_\beta]} + \Omega(V_\alpha, V_\beta), \quad (3.22)$$

$$D_{V_\alpha} D_{\bar{V}_\beta} - D_{\bar{V}_\beta} D_{V_\alpha} = D_{[V_\alpha, \bar{V}_\beta]} + \Omega(V_\alpha, \bar{V}_\beta), \quad (3.23)$$

$$D_{\bar{V}_\alpha} D_{\bar{V}_\beta} - D_{\bar{V}_\beta} D_{\bar{V}_\alpha} = D_{[\bar{V}_\alpha, \bar{V}_\beta]}, \quad (3.24)$$

where Ω is the curvature operator of the Chern Finsler connection D . In local coordinates, the curvature operator is given by

$$\Omega = \Omega_\beta^\alpha \otimes [dz^\beta \otimes \delta_\alpha + \delta v^\beta \otimes \dot{\partial}_\alpha]$$

and

$$\Omega_\beta^\alpha = R_{\beta; \mu \bar{\nu}}^\alpha dz^\mu \wedge d\bar{z}^\nu + R_{\beta \delta; \bar{\nu}}^\alpha \delta v^\delta \wedge d\bar{z}^\nu + R_{\beta \bar{\gamma}; \mu}^\alpha dz^\mu \wedge \delta \bar{v}^\gamma + R_{\beta \delta \bar{\gamma}}^\alpha \delta v^\delta \wedge \delta \bar{v}^\gamma.$$

Let us calculate the second term on the right-hand side of (3.21). For a form φ_α of type $(1, 0)$,

$$[\nabla_{\delta_\mu}, \nabla_{\delta_{\bar{\nu}}}] \varphi_\alpha = \nabla_{[\delta_\mu, \delta_{\bar{\nu}}]} \varphi_\alpha + \Omega(\delta_\mu, \delta_{\bar{\nu}}) \varphi_\alpha = -\Gamma_{\alpha\sigma}^\tau \delta_{\bar{\nu}}(\Gamma_{;\mu}^\sigma) \varphi_\tau + R_{\alpha; \mu \bar{\nu}}^\tau \varphi_\tau.$$

We denote $\mathcal{T}_{\alpha\mu\bar{\nu}}^\tau = -\Gamma_{\alpha\sigma}^\tau \delta_{\bar{\nu}}(\Gamma_{;\mu}^\sigma)$, $\mathcal{T}_{\alpha\bar{\nu}}^\tau = G^{\bar{\beta}\mu} \mathcal{T}_{\alpha\mu\bar{\nu}}^\tau$, $\mathcal{T}_{\alpha}^\tau = G^{\bar{\nu}\mu} \mathcal{T}_{\alpha\mu\bar{\nu}}^\tau$, $R_{\alpha; \bar{\nu}}^\tau = G^{\bar{\beta}\mu} R_{\alpha; \mu \bar{\nu}}^\tau$ and $R_\alpha^\tau = G^{\bar{\nu}\mu} R_{\alpha; \mu \bar{\nu}}^\tau$. Obviously, we have $\overline{\mathcal{T}_{\alpha\mu\bar{\nu}}^\tau} = \mathcal{T}_{\alpha\mu\nu}^\tau$ and $\overline{R_{\alpha; \mu \bar{\nu}}^\tau} = R_{\alpha; \bar{\mu} \nu}^\tau$. Then

$$G^{\bar{\beta}\mu} [\nabla_{\delta_\mu}, \nabla_{\delta_{\bar{\nu}}}] \varphi_\alpha = \mathcal{T}_{\alpha\bar{\nu}}^\tau \varphi_\tau + R_{\alpha; \bar{\nu}}^\tau \varphi_\tau.$$

For a form $\varphi_{\bar{\beta}}$ of type $(0, 1)$,

$$G^{\bar{\gamma}\mu} [\nabla_{\delta_\mu}, \nabla_{\delta_{\bar{\nu}}}] \varphi_{\bar{\beta}} = -G^{\bar{\gamma}\mu} \overline{[\nabla_{\delta_\mu}, \nabla_{\delta_{\bar{\nu}}}] \varphi_{\bar{\beta}}} = -(\mathcal{T}_{\bar{\beta}\bar{\nu}}^{\bar{\gamma}} \varphi_{\bar{\tau}} + R_{\bar{\beta}; \bar{\nu}}^{\bar{\gamma}} \varphi_{\bar{\tau}}).$$

Similarly, we see

$$\begin{aligned}
& G^{\bar{\beta}\alpha} [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}_k}}] \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu \\
&= \sum_{i=1}^p (\mathcal{T}_{\alpha_i \bar{\beta}_k}^\tau + R_{\alpha_i; \bar{\beta}_k}^\tau) \psi_{\alpha_1 \dots \alpha_{i-1} \tau \alpha_{i+1} \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu \\
&\quad - (\mathcal{T}_{\bar{\beta}_k}^{\bar{\tau}} + R_{\bar{\beta}_k}^{\bar{\tau}}) \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\tau} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu.
\end{aligned} \quad (3.25)$$

Then, (3.21) can be rewritten as

$$\begin{aligned}
(\square_{\mathcal{H}}\psi)_{A_p\bar{B}_q}^\mu &= -G^{\bar{\beta}\alpha}\tilde{\nabla}_{\delta_\alpha}\tilde{\nabla}_{\delta_{\bar{\beta}}}\psi_{A_p\bar{B}_q}^\mu \\
&+ \sum_{k=1}^q \sum_{i=1}^p (\mathcal{T}_{\alpha_i}^\tau \bar{\beta}_k + R_{\alpha_i;\bar{\beta}_k}^\tau) \psi_{\alpha_1\cdots\alpha_{i-1}\tau\alpha_{i+1}\cdots\alpha_p\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}^\mu \\
&- \sum_{k=1}^q (\mathcal{T}_{\bar{\beta}_k}^\tau + R_{\bar{\beta}_k}^\tau) \psi_{A_p\bar{\beta}_1\cdots\bar{\beta}_{k-1}\tau\bar{\beta}_{k+1}\cdots\bar{\beta}_q}^\mu \\
&- \sum_{k=1}^q (\Omega_{\nu;\bar{\beta}_k}^\mu - \Gamma_{\nu;\bar{\beta}_k}^\mu) \psi_{A_p\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}^\nu. \tag{3.26}
\end{aligned}$$

Let $\beta = \langle e_\mu \nabla_{\delta_{\bar{\beta}}} \psi^\mu, \psi \rangle d\bar{z}^\beta$, where $\langle \cdot, \cdot \rangle$ is understood in the sense of (3.15). When M is a compact Kähler Finsler manifold, by Stokes' theorem, $\int_{\text{PTM}} (\bar{\partial}_{\mathcal{H}}^* \beta) dv = 0$, so

$$\begin{aligned}
0 &= \int_{\text{PTM}} G^{\bar{\beta}\alpha} \nabla_{\delta_\alpha} (L_{\mu\bar{\nu}} \nabla_{\delta_{\bar{\beta}}} \psi_{A_p\bar{B}_q}^\mu \overline{\psi_{\nu\bar{A}_p\bar{B}_q}^\nu}) dv \\
&= \int_{\text{PTM}} G^{\bar{\beta}\alpha} \nabla_{\delta_\alpha} (L_{\mu\bar{\nu}} \nabla_{\delta_{\bar{\beta}}} \psi_{A_p\bar{B}_q}^\mu) \overline{\psi_{\nu\bar{A}_p\bar{B}_q}^\nu} dv + \int_{\text{PTM}} G^{\bar{\beta}\alpha} L_{\mu\bar{\nu}} \nabla_{\delta_{\bar{\beta}}} \psi_{A_p\bar{B}_q}^\mu \overline{\nabla_{\delta_{\bar{\alpha}}} \psi_{\nu\bar{A}_p\bar{B}_q}^\nu} dv \\
&= \int_{\text{PTM}} G^{\bar{\beta}\alpha} L_{\mu\bar{\nu}} \tilde{\nabla}_{\delta_\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \psi_{A_p\bar{B}_q}^\mu \overline{\psi_{\nu\bar{A}_p\bar{B}_q}^\nu} dv + \|\tilde{\nabla}_{\mathcal{H}}\psi\|^2.
\end{aligned}$$

Thus, $\|\tilde{\nabla}_{\mathcal{H}}\psi\|^2 = -(e_\mu (G^{\bar{\beta}\alpha} \tilde{\nabla}_{\delta_\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \psi^\mu), \psi)$.

When M is a compact Kähler Finsler manifold, by contracting (3.21) with $\frac{1}{p!q!}\psi \in \mathcal{A}^{p,q}(\tilde{E})$ and integrating over PTM, we obtain

$$\begin{aligned}
&\|\bar{\partial}_{\mathcal{H}}^* \psi\|^2 + \|\bar{\partial}_{\mathcal{H}} \psi\|^2 \\
&= \|\tilde{\nabla}_{\mathcal{H}}\psi\|^2 + \frac{1}{(p-1)!(q-1)!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} (\mathcal{T}_{\alpha}^\tau \bar{\nu} + R_{\alpha;\bar{\nu}}^\tau) \psi_{\tau A_{p-1}\bar{\beta}\bar{B}_{q-1}}^\mu \overline{\psi_{\lambda\bar{\alpha}\bar{A}_{p-1}\bar{\nu}\bar{B}_{q-1}}^\nu} dv \\
&- \frac{1}{p!(q-1)!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} ((\mathcal{T}_{\bar{x}}^\tau + R_{\bar{x}}^\tau) \psi_{A_p\bar{\beta}\bar{\chi}_{q-1}}^\mu \\
&+ (\Omega_{\nu;\bar{\beta}_k}^\mu - \Gamma_{\nu;\bar{\beta}_k}^\mu) \psi_{A_p\bar{\beta}\bar{\chi}_{q-1}}^\nu) \overline{\psi_{\lambda\bar{A}_p\bar{x}\bar{\chi}_{q-1}}^\nu} dv, \tag{3.27}
\end{aligned}$$

where $\|\cdot\|$ denotes the global L^2 norm over PTM, and $\tilde{\nabla}_{\mathcal{H}}\psi$ denotes the \tilde{E} -valued tensor with components $\nabla_{\delta_{\bar{\beta}}} \psi^\nu$. (3.27) is the $\tilde{\nabla}_{\mathcal{H}}$ Bochner-Kodaira technique.

Since

$$[\tilde{\nabla}_{\delta_\alpha}, \tilde{\nabla}_{\delta_{\bar{\beta}}}] \psi^\mu = \tilde{\nabla}_{\delta_\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \psi^\mu - \tilde{\nabla}_{\delta_{\bar{\beta}}} \tilde{\nabla}_{\delta_\alpha} \psi^\mu = [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}}}] \psi^\mu - \Omega_{\nu;\bar{\beta}\alpha}^\mu \psi^\nu,$$

that is,

$$-\tilde{\nabla}_{\delta_\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \psi^\mu = -\tilde{\nabla}_{\delta_{\bar{\beta}}} \tilde{\nabla}_{\delta_\alpha} \psi^\mu - [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}}}] \psi^\mu + \Omega_{\nu;\bar{\beta}\alpha}^\mu \psi^\nu, \tag{3.28}$$

by applying the commutation formula for $[\tilde{\nabla}_{\delta_\alpha}, \tilde{\nabla}_{\delta_{\bar{\beta}}}] \psi^\mu$ to (3.21), with (3.25), we obtain

$$\begin{aligned}
(\square_{\mathcal{H}}\psi)_{A_p\bar{B}_q}^\mu &= -G^{\bar{\beta}\alpha} \tilde{\nabla}_{\delta_{\bar{\beta}}} \tilde{\nabla}_{\delta_\alpha} \psi_{A_p\bar{\beta}B_q}^\mu - G^{\bar{\beta}\alpha} [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}}}] \psi_{A_p\bar{B}_q}^\mu + \Omega_{\nu;\bar{\beta}\alpha}^\mu \psi_{A_p\bar{B}_q}^\nu \\
&+ \sum_{k=1}^q G^{\bar{\beta}\alpha} [\nabla_{\delta_\alpha}, \nabla_{\delta_{\bar{\beta}_k}}] \psi_{A_p\bar{\beta}_1\cdots\bar{\beta}_{k-1}\bar{\beta}\bar{\beta}_{k+1}\cdots\bar{\beta}_q}^\nu
\end{aligned}$$

$$\begin{aligned}
& - \sum_{k=1}^q (\Omega_{\nu; \bar{\beta}_k}^\mu - \Gamma_{\nu; \bar{\beta}_k}^\mu) \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\nu \\
& = - G^{\bar{\beta} \alpha} \tilde{\nabla}_{\bar{\beta}} \tilde{\nabla}_{\delta_\alpha} \psi_{A_p \bar{B}_q}^\mu - \sum_{k=1}^q (\Omega_{\nu; \bar{\beta}_k}^\mu - \Gamma_{\nu; \bar{\beta}_k}^\mu) \psi_{A_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\nu \\
& + \sum_{k=1}^q \sum_{i=1}^p (\mathcal{T}_{\alpha_i}^\tau \bar{\beta}_k + R_{\alpha_i; \bar{\beta}_k}^\tau) \psi_{\alpha_1 \dots \alpha_{i-1} \tau \alpha_{i+1} \dots \alpha_p \bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu \\
& - \sum_{i=1}^p (\mathcal{T}_{\alpha_i}^\tau + R_{\alpha_i}^\tau) \psi_{\alpha_1 \dots \alpha_{i-1} \tau \alpha_{i+1} \dots \alpha_p \bar{B}_q}^\mu. \tag{3.29}
\end{aligned}$$

When M is a compact Kähler Finsler manifold, and by contracting (3.29) with $\frac{1}{p!q!} \psi \in \mathcal{A}^{p,q}(\tilde{E})$ and integrating over PTM, we obtain

$$\begin{aligned}
& \|\bar{\partial}_{\mathcal{H}}^* \psi\|^2 + \|\bar{\partial}_{\mathcal{H}} \psi\|^2 \\
& = \|\tilde{\nabla}_{\mathcal{H}} \psi\|^2 - \frac{1}{p!q!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} \Omega_{\nu}^\mu \psi_{A_p \bar{B}_q}^\nu \overline{\psi^{\lambda \bar{A}_p B_q}} dv \\
& - \frac{1}{p!(q-1)!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} (\Omega_{\nu; \bar{x}}^\mu - \Gamma_{\nu; \bar{x}}^\mu) \psi_{A_p \bar{\beta}_{q-1}}^\nu \overline{\psi^{\lambda \bar{A}_p x \chi_{q-1}}} dv \\
& - \frac{1}{(p-1)!q!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} (\mathcal{T}_{\nu}^\tau + R_{\nu}^\tau) \psi_{\tau A_{p-1} \bar{B}_q}^\mu \overline{\psi^{\lambda \bar{\alpha} \bar{A}_{p-1} B_q}} dv \\
& + \frac{1}{(p-1)!(q-1)!} \int_{\text{PTM}} L_{\mu\bar{\lambda}} (\mathcal{T}_{\alpha}^\tau \bar{\nu} + R_{\alpha; \bar{\nu}}^\tau) \psi_{\tau A_{p-1} \bar{\beta}_{q-1}}^\mu \overline{\psi^{\lambda \bar{\alpha} \bar{A}_{p-1} \nu B_{q-1}}} dv, \tag{3.30}
\end{aligned}$$

where $\|\cdot\|$ denotes the global L^2 norm over PTM, and $\tilde{\nabla}_{\mathcal{H}} \psi$ denotes the \tilde{E} -valued tensor with components $\tilde{\nabla}_{\delta_\alpha} \psi^\mu$. (3.30) is the $\tilde{\nabla}_{\mathcal{H}}$ Bochner-Kodaira technique.

Remark 3.1 If the Kähler Finsler manifold is a Kähler manifold, then (3.27) and (3.30) coincide with (1.3.3) and (1.3.5) in [11].

4 $\partial_{\mathcal{H}} \bar{\partial}_{\mathcal{H}}$ Bochner-Kodaira Technique

$\partial \bar{\partial}$ Bochner-Kodaira technique was initiated by Siu [10] and named in [11], and the method is a modification of the classical Bochner-Kodaira technique by replacing the operator \square with $\partial \bar{\partial}$ and exploits the bigraded structure of the differential forms on a Kähler manifold in a more serious way than usual in the computations with \square . In this section, we study the $\partial_{\mathcal{H}} \bar{\partial}_{\mathcal{H}}$ Bochner-Kodaira technique on Kähler Finsler manifolds and get the vanishing theorem for the Hermitian holomorphic bundle on PTM on Kähler Finsler manifolds.

Let \tilde{E} be the Hermitian holomorphic vector bundle of rank r on PTM is the same bundle as stated in Section 3. We know that there exist normal coordinates of M and normal fiber coordinates of \tilde{E} , which can be found with detailed information in [24–25]. This greatly simplifies our calculations in local coordinates, and then, in all the following computations, we will use the normal coordinates of M and the normal fiber coordinates of \tilde{E} .

Letting $\varphi = \sum e_\mu \varphi^\mu \in \mathcal{A}^{0,q}(\tilde{E})$, and under the normal coordinates of M and the normal fiber coordinates of \tilde{E} , we have $\tilde{\nabla}_{\mathcal{H}} \varphi = \sum e_\mu \nabla_{\mathcal{H}} \varphi^\mu$, where $\nabla_{\mathcal{H}}$ is the horizontal covariant differentiation of Chern Finsler connection. If M is a Kähler Finsler manifold, and $\nabla_{\mathcal{H}} \varphi^\mu = \partial_{\mathcal{H}} \varphi^\mu$, letting

$$\Phi = \partial_{\mathcal{H}} \bar{\partial}_{\mathcal{H}} (L_{\mu\bar{\lambda}} \varphi^\mu \wedge \overline{\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1}), \tag{4.1}$$

where $\omega_{\mathcal{H}} = \sqrt{-1}G_{\alpha\bar{\beta}}dz^{\alpha} \wedge d\bar{z}^{\beta}$, and $d_{\mathcal{H}}\omega_{\mathcal{H}} = 0$ on Kähler Finsler manifolds, then

$$\begin{aligned}\Phi &= \partial_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}(L_{\mu\bar{\lambda}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-p-1}) \\ &= \sqrt{-1}\Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} + L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \\ &\quad + (-1)^{q+1}L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} + (-1)^q L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\nabla_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \\ &\quad + L_{\mu\bar{\lambda}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1}.\end{aligned}\tag{4.2}$$

When M is a compact Kähler Finsler manifold, and by integrating over PTM, we obtain

$$\begin{aligned}0 &= \int_{\text{PTM}} \Phi \wedge d\sigma \\ &= \sqrt{-1} \int_{\text{PTM}} \Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + \int_{\text{PTM}} L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + (-1)^{q+1} \int_{\text{PTM}} L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + (-1)^q \int_{\text{PTM}} L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\nabla_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + \int_{\text{PTM}} L_{\mu\bar{\lambda}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma.\end{aligned}\tag{4.3}$$

By applying integration by parts to the second term and the last term, from

$$\begin{aligned}&d(L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma) \\ &= L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + (-1)^{q+1}L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma,\end{aligned}$$

it follows that

$$\int_{\text{PTM}} L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma = (-1)^q \int_{\text{PTM}} L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma.$$

Likewise,

$$\int_{\text{PTM}} L_{\mu\bar{\lambda}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma = (-1)^q \int_{\text{PTM}} L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma.$$

Hence

$$\begin{aligned}&\sqrt{-1} \int_{\text{PTM}} \Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge \bar{\varphi}^{\bar{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + (-1)^q \int_{\text{PTM}} L_{\mu\bar{\lambda}}\bar{\partial}_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\partial_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma \\ &\quad + (-1)^q \int_{\text{PTM}} L_{\mu\bar{\lambda}}\nabla_{\mathcal{H}}\varphi^{\mu} \wedge \overline{\nabla_{\mathcal{H}}\varphi^{\lambda}} \wedge \omega_{\mathcal{H}}^{n-q-1} \wedge d\sigma = 0.\end{aligned}\tag{4.4}$$

We are going to transform, by using the exterior algebra of Hermitian vector spaces, each term in (4.4) to a corresponding term obtained from the $\tilde{\nabla}_{\mathcal{H}}$ Bochner-Kodaira technique.

We will list the formulae we need concerning exterior algebras of Hermitian vector spaces, which were collected together by Siu [11]. Let L be the operator of taking a wedge product with $\omega_{\mathcal{H}} = \sqrt{-1}G_{\alpha\bar{\beta}}dz^{\alpha} \wedge d\bar{z}^{\beta}$, and \wedge be the adjoint operator of L with the pointwise inner product \langle, \rangle on $\mathcal{A}^{p,q}$ defined by (3.1). A k -form $\psi \in \mathcal{A}^{p,q}$ is called primitive if $\wedge\psi = 0$.

For any primitive k -form ψ and $s \leq r$,

$$\wedge^s L^r \psi = \left(\prod_{i=0}^{s-1} (r-i) \right) \left(\prod_{j=1}^s (n-k-r+j) \right) L^{r-s} \psi. \quad (4.5)$$

Let $\varepsilon_{p,q} = (-1)^{\frac{1}{2}(p+q)(p+q+1)}(\sqrt{-1})^{p-q}$. For any primitive (p,q) -form ψ with $p+q = k$,

$$*L^l \psi = \varepsilon_{p,q} \frac{l!}{(n-k-l)!} L^{n-k-l} \psi \quad (4.6)$$

for $0 \leq l \leq n-k$. One has $*L^l \psi = 0$ if $l > n-k$.

Every k -form v can be uniquely written as

$$v = \sum_r L^r v_r,$$

where each v_r is primitive, and r runs from $\max\{0, k-n\}$ to the largest integer $\lfloor \frac{k}{2} \rfloor$, not exceeding $\frac{k}{2}$.

Lemma 4.1 (see [11]) *For any $(1, q)$ -form η ,*

$$\overline{\epsilon_{1,q}} \eta \wedge \bar{\eta} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} = (\langle \eta, \eta \rangle - \langle \wedge \eta, \wedge \eta \rangle) \frac{\omega_{\mathcal{H}}^n}{n!}. \quad (4.7)$$

For detailed information of these formulae, see [6, p.69].

Lemma 4.2

$$\begin{aligned} & -\overline{\epsilon_{0,q}} \Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge \overline{\varphi^{\lambda}} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} \wedge d\sigma \\ &= \left(\frac{1}{q!} \Omega_{\mu\bar{\lambda}} \varphi_{\bar{B}_q}^{\mu} \overline{\varphi^{\lambda B_q}} - \frac{1}{(q-1)!} \Omega_{\mu\bar{\lambda}; \bar{x}} \overline{\varphi_{\bar{B}_{q-1}}^{\mu}} \overline{\varphi^{\lambda x B_{q-1}}} \right) dv, \end{aligned} \quad (4.8)$$

where as in (3.11), $\Theta_{\mu\bar{\lambda}} = -\sqrt{-1} \Omega_{\mu\bar{\lambda}; \alpha\bar{x}} dz^{\alpha} \wedge d\bar{z}^x$ and $\Omega_{\mu\bar{\lambda}} = G^{\alpha\bar{x}} \Omega_{\mu\bar{\lambda}; \alpha\bar{x}}$.

Proof Since φ^{λ} is $(0, q)$ -form, it is primitive. By (4.6),

$$\begin{aligned} -\overline{\epsilon_{0,q}} \Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge \overline{\varphi^{\lambda}} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{n!} \wedge d\sigma &= -\Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu} \wedge *L\overline{\varphi^{\lambda}} \wedge d\sigma \\ &= \langle -\Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu}, L\overline{\varphi^{\lambda}} \rangle dv \\ &= \langle \wedge(-\Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu}), \varphi^{\lambda} \rangle dv. \end{aligned}$$

Using local coordinates, we have

$$-(\Theta_{\mu\bar{\lambda}} \wedge \varphi^{\mu})_{\alpha\bar{\beta}B_q} = \sqrt{-1} \left(\Omega_{\mu\bar{\lambda}; \alpha\bar{\beta}} \varphi_{B_q}^{\mu} - \sum_{k=1}^q \Omega_{\mu\bar{\lambda}; \alpha\bar{\beta}_k} \varphi_{\bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^{\mu} \right).$$

Contracting both sides with $G^{\alpha\bar{\beta}}$, we obtain

$$-(\wedge(\Theta_{\mu\bar{\lambda}} \wedge \varphi^\mu))_{\bar{B}_q} = \Omega_{\mu\bar{\lambda}} \varphi_{\bar{B}_q}^\mu - \sum_{k=1}^q \Omega_{\mu\bar{\lambda}; \bar{\beta}_k} \varphi_{\bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu.$$

Taking the inner product with φ^λ , we obtain the desired equation.

By applying (4.6) to the case $\eta = \bar{\partial}_{\mathcal{H}}\varphi$ and $l = 0$, we get

$$\begin{aligned} \overline{\epsilon_{0,q+1}} L_{\mu\bar{\lambda}} \bar{\partial}_{\mathcal{H}}\varphi^\mu \wedge \overline{\bar{\partial}_{\mathcal{H}}\varphi^\lambda} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} \wedge d\sigma &= \sum L_{\mu\bar{\lambda}} \bar{\partial}_{\mathcal{H}}\varphi^\mu \wedge * \overline{\bar{\partial}_{\mathcal{H}}\varphi^\lambda} \wedge d\sigma \\ &= \langle \bar{\partial}_{\mathcal{H}}\varphi, \bar{\partial}_{\mathcal{H}}\varphi \rangle dv. \end{aligned} \quad (4.9)$$

By using Lemma 4.1 in the case $\eta = \partial_{\mathcal{H}}\varphi^\mu = \nabla_{\mathcal{H}}\varphi^\mu$, we have

$$\overline{\epsilon_{1,q}} L_{\mu\bar{\lambda}} \partial_{\mathcal{H}}\varphi^\mu \wedge \overline{\partial_{\mathcal{H}}\varphi^\lambda} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} \wedge d\sigma = (\langle \tilde{\nabla}_{\mathcal{H}}\varphi, \tilde{\nabla}_{\mathcal{H}}\varphi \rangle - \langle \bar{\partial}_{\mathcal{H}}^*\varphi, \bar{\partial}_{\mathcal{H}}^*\varphi \rangle) dv. \quad (4.10)$$

Using $\epsilon_{1,q} = -\epsilon_{0,q+1} = (-1)^{q+1} \sqrt{-1} \epsilon_{0,q}$, we multiply (4.4) by $(-1)^q \frac{\overline{\epsilon_{1,q}}}{(n-q-1)!} = \sqrt{-1} \frac{\overline{\epsilon_{0,q}}}{(n-q-1)!}$, and we obtain

$$\begin{aligned} \int_{\text{PTM}} \left(\frac{1}{q!} \Omega_{\mu\bar{\lambda}} \varphi_{\bar{B}_q}^\mu - \frac{1}{(q-1)!} \sum_{k=1}^q \Omega_{\mu\bar{\lambda}; \bar{\beta}_k} \varphi_{\bar{\beta}_1 \dots \bar{\beta}_{k-1} \bar{\beta}_{k+1} \dots \bar{\beta}_q}^\mu \right) \overline{\varphi^{\lambda \bar{B}_q}} dv \\ - \|\bar{\partial}_{\mathcal{H}}\varphi\|^2 + \|\tilde{\nabla}_{\mathcal{H}}\varphi\|^2 - \|\bar{\partial}_{\mathcal{H}}^*\varphi\|^2 = 0, \end{aligned} \quad (4.11)$$

which is the same as (3.30) obtained by $\tilde{\nabla}_{\mathcal{H}}$ Bochner-Kodaira technique under the normal coordinates of M and the normal fiber coordinates of \tilde{E} .

Then combining (4.4) with (4.9)–(4.10), we have

$$\begin{aligned} \overline{\epsilon_{0,q}} \int_{\text{PTM}} \Theta_{\alpha\bar{\beta}} \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega_H^{n-q-1}}{(n-q-1)!} \wedge d\sigma \\ = -\|\bar{\partial}_{\mathcal{H}}\varphi\|^2 - \|\bar{\partial}_{\mathcal{H}}^*\varphi\|^2 + \|\tilde{\nabla}_{\mathcal{H}}\varphi\|^2. \end{aligned} \quad (4.12)$$

Then, we have the following result.

Theorem 4.1 (Vanishing Theorem) *Let M be an n -dimensional compact Kähler Finsler manifold, $\varphi \in \mathcal{A}^{0,q}(\tilde{E})$, $\varphi = \sum e_\alpha \varphi^\alpha$. If the horizontal curvature form $\Theta_{\alpha\bar{\beta}}$ of \tilde{E} satisfies*

$$\overline{\epsilon_{0,q}} \int_{\text{PTM}} \Theta_{\alpha\bar{\beta}} \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} \wedge d\sigma < 0, \quad (4.13)$$

then there is no nonzero horizontal harmonic $(0,q)$ -form over PTM with valued in \tilde{E} , for all $0 < q \leq n$.

Proof Since M is compact, φ is horizontal harmonic if and only if $\bar{\partial}_{\mathcal{H}}\varphi = \bar{\partial}_{\mathcal{H}}^*\varphi = 0$. From (4.12), we have

$$\overline{\epsilon_{0,q}} \int_{\text{PTM}} \Theta_{\alpha\bar{\beta}} \varphi^\alpha \wedge \overline{\varphi^\beta} \wedge \frac{\omega_{\mathcal{H}}^{n-q-1}}{(n-q-1)!} \wedge d\sigma \geq 0,$$

which contradicts (4.13) when φ is not identically zero. Hence $\varphi \equiv 0$.

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