Chin. Ann. Math.
36B(1), 2015, 147–160
DOI: 10.1007/s11401-014-0869-1

Chinese Annals of Mathematics, Series B

The Cocycle Property of Stochastic Differential Equations Driven by G-Brownian Motion*

Huijie QIAO[1]

Abstract In this paper, solutions of the following non-Lipschitz stochastic differential equations driven by G-Brownian motion:

$$X_t = x + \int_0^t b(s,\omega,X_s)\mathrm{d}s + \int_0^t h(s,\omega,X_s)\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(s,\omega,X_s)\mathrm{d}B_s$$

are constructed. It is shown that they have the cocycle property. Moreover, under some special non-Lipschitz conditions, they are bi-continuous with respect to t, x.

Keywords Cocycle property, Non-Lipschitz condition, SDEs driven by G-Brownian motion
2000 MR Subject Classification 60H05, 60H10, 60J65

1 Introduction

Let Ω denote the space of all $\mathbb{R}^d$-valued continuous paths $\omega : [0,+\infty) \ni t \mapsto \omega_t \in \mathbb{R}^d$, with $\omega_0 = 0$, equipped with a uniform convergence topology. If a linear expectation E_P, which is induced by the Wiener measure P, is given, the canonical process $B_t(\omega)$ is a d-dimensional Brownian motion on the probability space $(\Omega, \mathcal{F}, P)$. Here $\mathcal{F}$ stands for a Borel σ-field of Ω. Now, if a sublinear expectation $\mathbb{E}$ is given, what is the canonical process $B_t(\omega)$ on some constructed probability space? It is a G-Brownian motion introduced in [7]. The stochastic calculus with respect to the G-Brownian motion has been established (see [7–9]). Relative to the Lévy theorem of the Brownian motion, a martingale characterization of G-Brownian motion has been shown in [10]. The BDG inequality for G-stochastic integrals was also established (see [6]). Moreover, the G-Itô formula in [8] was obtained and later Gao in [6] extended it by the localization method.

Consider the following stochastic differential equation (SDE, for short) driven by G-Brownian motion:

$$X_t = x + \int_0^t b(s,\omega,X_s)\mathrm{d}s + \int_0^t h(s,\omega,X_s)\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(s,\omega,X_s)\mathrm{d}B_s, \tag{1.1}$$

Manuscript received February 23, 2012. Revised October 28, 2013.
Department of Mathematics, Southeast University, Nanjing 211189, China.
E-mail: hjqiaogean@seu.edu.cn
*This work was supported by the National Natural Science Foundation of China (No. 11001051).

where $x \in \mathbb{R}^n$, $b : \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^n$, $\sigma : \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d}$, $h : \mathbb{R}_+ \times \Omega \times \mathbb{R}^n \mapsto \mathbb{R}^{n\times d^2}$ with the following form:

$$h = \begin{pmatrix} h_{11}^{(1)} & \cdots & h_{1d}^{(1)} & h_{21}^{(1)} & \cdots & h_{2d}^{(1)} & \cdots & h_{d1}^{(1)} & \cdots & h_{dd}^{(1)} \\ h_{11}^{(2)} & \cdots & h_{1d}^{(2)} & h_{21}^{(2)} & \cdots & h_{2d}^{(2)} & \cdots & h_{d1}^{(2)} & \cdots & h_{dd}^{(2)} \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ h_{11}^{(n)} & \cdots & h_{1d}^{(n)} & h_{21}^{(n)} & \cdots & h_{2d}^{(n)} & \cdots & h_{d1}^{(n)} & \cdots & h_{dd}^{(n)} \end{pmatrix},$$

and

$$\langle B\rangle = (\langle B^1, B^1\rangle, \langle B^1, B^2\rangle, \cdots, \langle B^1, B^d\rangle, \cdots, \langle B^d, B^1\rangle, \langle B^d, B^2\rangle, \cdots, \langle B^d, B^d\rangle)^{\mathrm{T}}.$$

Here A^{T} stands for the transposition of matrix A. The second and third integral on the right side of (1.1) will be introduced in Section 2. If b, h and σ satisfy Lipschitz conditions, Peng [8] showed the existence and uniqueness of the solution to Equation (1.1) in the space $M_G^2(0,T)$ (see the definition in Section 2) by the contracting mapping theorem. Under the same conditions, Gao defined the Picard iterative approximation sequence and obtained the unique solution to (1.1) (see [6]). However, many coefficients do not satisfy the Lipschitz condition. Therefore the extension to non-Lipschitz conditions is necessary. Here we do this. The unique solution to (1.1) is constructed through successive approximation. It is worthwhile to mention that non-Lipschitz conditions have been studied in [2]. But our assumption is more general than theirs.

Moreover, we study a property of (1.1). Because the property is similar to the cocycle property of SDEs driven by Brownian motion, we also call it the cocycle property. As we know, it is the first time to mention the cocycle property of SDEs driven by G-Brownian motion.

Using the cocycle property, under some special non-Lipschitz conditions, we get a bi-continuous modification of the solution with respect to t, x.

This paper is arranged as follows. In Section 2 we prepare some preliminaries to the readers' convenience. In Section 3, the solution to (1.1) is constructed and its cocycle property is proved. We consider its bi-continuity under some special non-Lipschitz conditions in Section 4.

The following conventions will be used throughout the paper: C with or without indices will denote different positive constants (depending on the indices) whose values may change from one place to another.

2 Preliminaries

First of all, we introduce G-expectation (see [5–6]).

$\mathbb{S}^d$ denotes the space of $d \times d$ symmetric matrices. Γ is a given nonempty, bounded and closed subset of $\mathbb{R}^{d\times d}$ which is the space of all $d \times d$ matrices. $\mathrm{lip}(\mathbb{R}^d)$ is the set of bounded Lipschitz continuous functions on $\mathbb{R}^d$. $|\cdot|$ denotes the length of a vector in $\mathbb{R}^n$. $\|\cdot\|$ stands for the Hilbert-Schmidt norm of a matrix.

For $A \in \mathbb{S}^d$, set

$$G(A) = \frac{1}{2}\sup_{\gamma\in\Gamma} \mathrm{tr}[\gamma\gamma^{\mathrm{T}} A].$$

For each $\varphi \in \mathrm{lip}(\mathbb{R}^d)$, define

$$\mathbb{E}(\varphi) := u(1, 0),$$

where $u(t, x)$ is the viscosity solution to the following G-heat equation:

$$\begin{cases} \dfrac{\partial u}{\partial t} - G(D^2 u) = 0, \quad (t, x) \in [0, \infty) \times \mathbb{R}^d, \\ u(0, x) = \varphi(x), \end{cases} \tag{2.1}$$

and $D^2 u = (\partial^2_{x_i x_j} u)^d_{i,j=1}$ (the existence and uniqueness of (2.1) in the sense of the viscosity solution can be found in [4]). Then $\mathbb{E} : \mathrm{lip}(\mathbb{R}^d) \mapsto \mathbb{R}$ is a sublinear expectation, i.e.,

(1) $X \geqslant Y,\ \mathbb{E}[X] \geqslant \mathbb{E}[Y]$,

(2) $\mathbb{E}[X + Y] \leqslant \mathbb{E}[X] + \mathbb{E}[Y]$,

(3) for all $\lambda \geqslant 0,\ \mathbb{E}[\lambda X] = \lambda \mathbb{E}[X]$,

(4) for all $c \in \mathbb{R},\ \mathbb{E}[X + c] = \mathbb{E}[X] + c$.

This sublinear expectation is also called a G-normal distribution on $\mathbb{R}^d$ and is denoted by $N(0, \Sigma)$, where $\Sigma := \{\gamma\gamma^{\mathrm{T}}, \gamma \in \Gamma\}$.

To well understand the sublinear expectation, we introduce another concept. Let $\mathcal{F}_t := \sigma(\omega_s, s \leqslant t)$ and $\mathcal{A}^{\Gamma}_{0,\infty}$ be the collection of all Γ-valued $\{\mathcal{F}_t, t \geqslant 0\}$-adapted processes on the interval $[0, \infty)$. For each $\theta \in \mathcal{A}^{\Gamma}_{0,\infty}$, we denote

$$B^{0,\theta}_t := \int_0^t \theta_s \mathrm{d}\omega_s.$$

P_θ denotes the law of the process $B^{0,\theta}_t$ under P. Define

$$\overline{C}(A) := \sup_{\theta \in \mathcal{A}^{\Gamma}_{0,\infty}} P_\theta(A), \quad A \in \mathcal{F}.$$

Then $\overline{C}(\cdot)$ is a Choquet capacity (see [5]). A set A is polar if $\overline{C}(A) = 0$ and a property holds "quasi-surely" (q.s., for short) if it holds outside a polar set. For each $X \in \mathcal{F}$ such that $E_{P_\theta} X$ exists for each $\theta \in \mathcal{A}^{\Gamma}_{0,\infty}$. Set

$$\overline{\mathbb{E}} X := \sup_{\theta \in \mathcal{A}^{\Gamma}_{0,\infty}} E_{P_\theta} X$$

and then for all $X \in L^1_G(\mathcal{F})$ (introduced in the sequel)

$$\mathbb{E} X = \overline{\mathbb{E}} X.$$

For its proof, refer to [5].

For each $t > 0$, set

$$\begin{aligned} \mathrm{Lip}(\mathcal{F}_t) := \{ & f(\omega_{t_1}, \omega_{t_2}, \cdots, \omega_{t_n}) : n \geqslant 1, \\ & t_1, \cdots, t_n \in [0, t],\ f \in \mathrm{lip}(\mathbb{R}^{d \times n})\}, \\ \mathrm{Lip}(\mathcal{F}) := & \bigcup_{n=1}^{\infty} \mathrm{Lip}(\mathcal{F}_n) \subset C_b(\Omega). \end{aligned}$$

Let $\mathcal{H}$ be a vector lattice of real functions defined on Ω such that $\mathrm{Lip}(\mathcal{F}) \subset \mathcal{H}$ and if that $X_1, \cdots, X_n \in \mathcal{H}$, then $g(X_1, \cdots, X_n) \in \mathcal{H}$ for each $g \in \mathrm{lip}(\mathbb{R}^n)$.

Let $\mathbb{E} : \mathcal{H} \mapsto \mathbb{R}$ be a sublinear expectation on $\mathcal{H}$. A d-dimensional random vector X with each component in $\mathcal{H}$ is said to be G-normal distributed under the sublinear expectation $\mathbb{E}[\cdot]$ if for each $\varphi \in \mathrm{lip}(\mathbb{R}^d)$,

$$u(t,x) := \mathbb{E}(\varphi(x + \sqrt{t}X)), \quad t \geqslant 0,\ x \in \mathbb{R}^d$$

is the viscosity solution of the G-heat equation (2.1).

$\mathbb{E}$ is called to be a G-expectation if the d-dimensional canonical process $\{B_t(\omega), t \geqslant 0\}$ is a G-Brownian motion under the sublinear expectation, that is,

(i) $B_0 = 0$;

(ii) for any $s, t \geqslant 0, B_t \sim B_{t+s} - B_s \sim N(0, t\Sigma)$;

(iii) for any $m \geqslant 1, 0 = t_0 < t_1 < \cdots < t_m < \infty$, the increment $B_{t_m} - B_{t_{m-1}}$ is independent of $B_{t_1}, \cdots, B_{t_{m-1}}$, i.e., for each $\varphi \in \mathrm{lip}(\mathbb{R}^{d\times m})$,

$$\mathbb{E}(\varphi(B_{t_1}, \cdots, B_{t_{m-1}}, B_{t_m} - B_{t_{m-1}})) = \mathbb{E}(\psi(B_{t_1}, \cdots, B_{t_{m-1}})),$$

where $\psi(x_1, \cdots, x_{m-1}) = \mathbb{E}(\varphi(x_1, \cdots, x_{m-1}, B_{t_m} - B_{t_{m-1}}))$. In particular, for any $\mathbf{a} \in \mathbb{R}^d, B_t^{\mathbf{a}} := (\mathbf{a}, B_t) := \sum\limits_{i=1}^{d} \mathbf{a}^i B_t^i$ is a one-dimensional G-Brownian motion.

Next, we only introduce a stochastic integral about $B_t^{\mathbf{a}}$ for convenience of statement (see [5–6]).

To the G-expectation $\mathbb{E}$, the topological completion of $\mathrm{Lip}(\mathcal{F}_t)$ (resp. $\mathrm{Lip}(\mathcal{F})$) under the Banach norm $\mathbb{E}[|\cdot|]$ is denoted by $L_G^1(\mathcal{F}_t)$ (resp. $L_G^1(\mathcal{F})$). $\mathbb{E}[\cdot]$ can be extended uniquely to a sublinear expectation on $L_G^1(\mathcal{F})$. The extension is also denoted by $\mathbb{E}$.

For $T \in \mathbb{R}_+,\ 0 = t_0 < t_1 < \cdots < t_N = T$. Let $p \geqslant 1$ be fixed. Define

$$M_G^{p,0}(0,T) := \Big\{\eta_t(\omega) = \sum_{j=1}^{N} \xi_{j-1}(\omega) 1_{[t_{j-1}, t_j)}(t); \xi_{j-1}(\omega) \in L_G^p(\mathcal{F}_{t_{j-1}})\Big\},$$

where $L_G^p(\mathcal{F}_t) = \{\xi \in L_G^1(\mathcal{F}_t); \mathbb{E}|\xi|^p < \infty\}$. For $\eta_t(\omega) \in M_G^{p,0}(0,T)$,

$$\frac{1}{T}\int_0^T \mathbb{E}(\eta_t)\mathrm{d}t = \frac{1}{T}\sum_{j=1}^{N} \mathbb{E}(\xi_{j-1})(t_j - t_{j-1}).$$

For each $p \geqslant 1$, $M_G^p(0,T)$ denotes the completion of $M_G^{p,0}(0,T)$ under the norm

$$\|\eta\|_{M_G^p(0,T)} = \frac{1}{T}\Big(\int_0^T \mathbb{E}|\eta_t|^p \mathrm{d}t\Big)^{\frac{1}{p}}.$$

For each $\eta \in M_G^{2,0}(0,T)$, define

$$I(\eta) = \int_0^T \eta(s)\mathrm{d}B_s^{\mathbf{a}} := \sum_{j=1}^{N} \xi_{j-1}(B_{t_j}^{\mathbf{a}} - B_{t_{j-1}}^{\mathbf{a}}),$$

and then we get a stochastic integral with respect to G-Brownian motion. Besides, the mapping $I: M_G^{2,0}(0,T) \mapsto L_G^2(\mathcal{F}_T)$ can be continuously extended to $I: M_G^2(0,T) \mapsto L_G^2(\mathcal{F}_T)$. For each $\eta \in M_G^2(0,T)$, the stochastic integral is defined by

$$\int_0^T \eta(s)\mathrm{d}B_s^{\mathbf{a}} =: I(\eta).$$

Let $\pi_t^N = \{t_0^N, t_1^N, \cdots, t_N^N\},\ N = 1, 2, \cdots$, be a sequence of partitions of $[0,t]$ and

$$\mu(\pi_t^N) = \max_{1 \leqslant i \leqslant N} |t_i^N - t_{i-1}^N|.$$

The quadratic variation process $\langle B^{\mathbf{a}}\rangle_t$ of the process $B_t^{\mathbf{a}}$ is defined by

$$\langle B^{\mathbf{a}}\rangle_t = \lim_{\mu(\pi_t^N)\to 0} \sum_{k=1}^N (B^{\mathbf{a}}_{t_k^N} - B^{\mathbf{a}}_{t_{k-1}^N})^2 = (B_t^{\mathbf{a}})^2 - 2\int_0^t B_s^{\mathbf{a}}\mathrm{d}B_s^{\mathbf{a}}.$$

For each fixed $s \geqslant 0$,

$$\langle B^{\mathbf{a}}\rangle_{t+s} - \langle B^{\mathbf{a}}\rangle_s = \langle (B^s)^{\mathbf{a}}\rangle_t,$$

where $B_t^s = B_{t+s} - B_s$ and $(B^s)_t^{\mathbf{a}} = (\mathbf{a}, B_t^s)$.

Define a mapping $M_G^{1,0}(0,T) \mapsto L_G^1(\mathcal{F}_T)$ as follows:

$$Q_{0,T}(\eta) = \int_0^T \eta(s)\mathrm{d}\langle B^{\mathbf{a}}\rangle_s := \sum_{j=1}^N \xi_{j-1}(\langle B^{\mathbf{a}}\rangle_{t_j} - \langle B^{\mathbf{a}}\rangle_{t_{j-1}}).$$

Then $Q_{0,T}$ can be uniquely extended to $M_G^1(0,T)$. We still denote this mapping by

$$\int_0^T \eta(s)\mathrm{d}\langle B^{\mathbf{a}}\rangle_s =: Q_{0,T}(\eta), \quad \eta \in M_G^1(0,T).$$

The following two theorems from [6, Theorems 2.1–2.2] are BDG inequalities for the G-stochastic integral.

Theorem 2.1 *For $p \geqslant 2$ and $\eta \in M_G^p(0,T)$, set $X_t = \int_0^t \eta(s)\mathrm{d}B_s^{\mathbf{a}}$. Then there exists a continuous modification $\widetilde{X}$ of X, i.e., for $\omega \in \widetilde{\Omega} \subset \Omega$ with $\overline{C}(\widetilde{\Omega}^c) = 0$, $\widetilde{X}_\cdot(\omega)$ is continuous and $\overline{C}(|X_t - \widetilde{X}_t| \neq 0) = 0$ for all $t \in [0,T]$, such that*

$$\overline{\mathbb{E}}\Big(\sup_{s\leqslant u\leqslant t} |\widetilde{X}_u - \widetilde{X}_s|^p\Big) \leqslant C(p,\mathbf{a})\mathbb{E}\Big(\Big(\int_s^t |\eta_u|^2\mathrm{d}u\Big)^{\frac{p}{2}}\Big),$$

where $0 < C(p,\mathbf{a}) < \infty$ is a constant only dependent on p and $\mathbf{a}$.

Theorem 2.2 *Let $p \geqslant 1$ and $\eta \in M_G^p(0,T)$. Then there exists a continuous modification $\widetilde{Y}_t^{\mathbf{a}}$ of $Y_t^{\mathbf{a}} := \int_0^t \eta_u\mathrm{d}\langle B^{\mathbf{a}}\rangle_u$ such that for any $0 \leqslant s < t \leqslant T$,*

$$\overline{\mathbb{E}}\Big(\sup_{s\leqslant u\leqslant t} |\widetilde{Y}_u^{\mathbf{a}} - \widetilde{Y}_s^{\mathbf{a}}|^p\Big) \leqslant C(p)(t-s)^{p-1}\int_s^t \mathbb{E}|\eta_u|^p\mathrm{d}u.$$

3 SDEs Driven by G-Brownian Motion

Theorem 3.1 *Suppose that for $p \geqslant 2$,*

(1) *there exists a function $H(t,u): \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*

(1a) *for fixed t, $H(t,u)$ is continuous nondecreasing with respect to u,*

(1b) *for $T>0$, any $0<t\leqslant T$ and $X_t \in L_G^p(\mathcal{F}_t)$,*

$$b(t,X_t), h(t,X_t), \sigma(t,X_t) \in M_G^p(0,T)$$

and

$$\mathbb{E}|b(t,X_t)|^p + \mathbb{E}\|h(t,X_t)\|^p + \mathbb{E}\|\sigma(t,X_t)\|^p \leqslant H\Big(t, \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r|^p\Big)\Big),$$

(1c) *for any constant $K>0$, the differential equation*

$$\frac{\mathrm{d}u}{\mathrm{d}t} = KH(t,u)$$

has a global solution for any initial value u_0;

(2) *there exists a function $F(t,u): \mathbb{R}_+ \times \mathbb{R}_+ \mapsto \mathbb{R}_+$ such that*

(2a) *for fixed t, $F(t,u)$ is continuous nondecreasing in u and $F(t,0)=0$,*

(2b) *for $T>0$, any $0<t\leqslant T$ and $X_t, Y_t \in L_G^p(\mathcal{F}_t)$,*

$$\begin{aligned}&\mathbb{E}|b(t,X_t)-b(t,Y_t)|^p + \mathbb{E}\|h(t,X_t)-h(t,Y_t)\|^p + \mathbb{E}\|\sigma(t,X_t)-\sigma(t,Y_t)\|^p\\ \leqslant\, & F\Big(t, \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r - Y_r|^p\Big)\Big),\end{aligned}$$

(2c) *for any constant $K>0$, if a non-negative function φ_t satisfies*

$$\varphi_t \leqslant K\int_0^t F(s,\varphi_s)\mathrm{d}s$$

for all $t\in\mathbb{R}_+$, then $\varphi_t = 0$.

Then (1.1) *has a unique solution X which is continuous q.s. and $X_t \in L_G^p(\mathcal{F}_t)$ for $t>0$.*

Remark 3.1 Fix $T>0$ and assume that b, h and σ satisfy, for all $x, x_1, x_2 \in \mathbb{R}^n$,

(H1) $|b(t,x)|^2 + \|h(t,x)\|^2 + \|\sigma(t,x)\|^2 \leqslant \beta_1^2(t) + \beta_2^2(t)|x|^2$;

(H2) $|b(t,x_1)-b(t,x_2)|^2 + \|h(t,x_1)-h(t,x_2)\|^2 + \|\sigma(t,x_1)-\sigma(t,x_2)\|^2 \leqslant \beta^2(t)\rho(|x_1-x_2|^2)$,

where $\beta_1 \in M_G^2(0,T)$, $\beta, \beta_2 : [0,T] \mapsto \mathbb{R}_+$ are square integrable and $\rho: (0,+\infty) \mapsto (0,+\infty)$ is a continuous, increasing and concave function so that

$$\rho(0+) = 0, \quad \int_0^1 \frac{\mathrm{d}r}{\rho(r)} = +\infty.$$

Under these conditions (H1) and (H2), Bai and Lin in [2, Theorem 3.1] showed the existence and uniqueness of the solution to (1.1). If we choose $H(t,u) = \overline{\mathbb{E}}\beta_1^2(t) + \beta_2^2(t)u$ and $F(t,u) = \beta^2(t)\rho(u)$, it can be easily justified that (1a)–(1c) and (2a)–(2c) hold for $p=2$. Therefore, our result is more general than that in [2].

We are now in a position to give the proof of Theorem 3.1.

Proof of Theorem 3.1 Let $X_t^0 := x$ and for $n \in \mathbb{N}$,

$$X_t^n := x + \int_0^t b(s, X_s^{n-1})\mathrm{d}s + \int_0^t h(s, X_s^{n-1})\mathrm{d}\langle B\rangle_s + \int_0^t \sigma(s, X_s^{n-1})\mathrm{d}B_s \tag{3.1}$$

$\forall T > 0$. First of all, we show that for $t < T$ and $n \in \mathbb{N}$,

$$X_t^n \in L_G^p(\mathcal{F}_t) \quad \text{and} \quad \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^n|^p\Big) \leqslant u_t \leqslant u_T, \tag{3.2}$$

where u_t satisfies

$$u_t = C_1(p,T)|x|^p + C_1(p,T)\int_0^t H(s,u_s)\mathrm{d}s,$$

and

$$C_1(p,T) := 4^{p-1}(1 + C(p)T^{p-1}).$$

Suppose

$$X_t^{n-1} \in L_G^p(\mathcal{F}_t)$$

and

$$\overline{\mathbb{E}}\big(\sup_{r\leqslant t}|X_r^{n-1}|^p\big) \leqslant u_t,$$

which together with the definition of the G-stochastic integral and (1b) yield $X_t^n \in L_G^p(\mathcal{F}_t)$. Secondly, by Theorems 2.1–2.2, the Hölder inequality and (1a)–(1b), we get

$$\begin{aligned}
&\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^n|^p\Big)\\
&\leqslant 4^{p-1}\Big(|x|^p + \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r b(s,X_s^{n-1})\mathrm{d}s\Big|^p\Big) + \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r h(s,X_s^{n-1})\mathrm{d}\langle B\rangle_s\Big|^p\Big)\\
&\quad + \overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r \sigma(s,X_s^{n-1})\mathrm{d}B_s\Big|^p\Big)\Big)\\
&\leqslant C_1(p,T)\Big(|x|^p + \int_0^t \overline{\mathbb{E}}(|b(s,X_s^{n-1})|^p)\mathrm{d}s + \int_0^t \overline{\mathbb{E}}(\|h(s,X_s^{n-1})\|^p)\mathrm{d}s\\
&\quad + \overline{\mathbb{E}}\Big[\Big(\int_0^t \|\sigma(s,X_s^{n-1})\|^2\mathrm{d}s\Big)^{\frac{p}{2}}\Big]\Big)\\
&\leqslant C_1(p,T)\Big(|x|^p + \int_0^t \overline{\mathbb{E}}(|b(s,X_s^{n-1})|^p)\mathrm{d}s + \int_0^t \overline{\mathbb{E}}(\|h(s,X_s^{n-1})\|^p)\mathrm{d}s\\
&\quad + T^{\frac{p}{2}-1}\int_0^t \overline{\mathbb{E}}(\|\sigma(s,X_s^{n-1})\|^p)\mathrm{d}s\Big)\\
&\leqslant C_1(p,T)\Big(|x|^p + \int_0^t H\Big(s,\overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r^{n-1}|^p\Big)\Big)\mathrm{d}s\Big)\\
&\leqslant C_1(p,T)\Big(|x|^p + \int_0^t H(s,u_s)\mathrm{d}s\Big)\\
&\leqslant u_t
\end{aligned}$$

for all $t \leqslant T$. By the induction method, (3.2) is proved.

Next, we have, by the same deduction as above,

$$
\begin{aligned}
\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^n-X_r^m|^p\Big) \leqslant\ & 3^{p-1}\Big(\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r(b(s,X_s^{n-1})-b(s,X_s^{m-1}))\mathrm{d}s\Big|^p\Big)\\
&+\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r(h(s,X_s^{n-1})-h(s,X_s^{m-1}))\mathrm{d}\langle B\rangle_s\Big|^p\Big)\\
&+\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r(\sigma(s,X_s^{n-1})-\sigma(s,X_s^{m-1}))\mathrm{d}B_s\Big|^p\Big)\Big)\\
\leqslant\ & C_2(p,T)\Big(\int_0^t\overline{\mathbb{E}}|b(s,X_s^{n-1})-b(s,X_s^{m-1})|^p\mathrm{d}s\\
&+\int_0^t\overline{\mathbb{E}}\|h(s,X_s^{n-1})-h(s,X_s^{m-1})\|^p\mathrm{d}s\\
&+T^{\frac{p}{2}-1}\int_0^t\overline{\mathbb{E}}\|\sigma(s,X_s^{n-1})-\sigma(s,X_s^{m-1})\|^p\mathrm{d}s\Big)\\
\leqslant\ & C_2(p,T)\int_0^t F\Big(s,\overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r^{n-1}-X_r^{m-1}|^p\Big)\Big)\mathrm{d}s,
\end{aligned}
$$

where $C_2(p,T):=3^{p-1}(1+C(p)T^{p-1})$. Let

$$
Y_t=\limsup_{n,m\to\infty}\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^{n-1}-X_r^{m-1}|^p\Big).
$$

It follows from the Fatou lemma and (2a) that

$$
Y_t\leqslant C_2(p,T)\int_0^t F(s,Y_s)\mathrm{d}s.
$$

By (2c), we obtain that $Y_t\equiv 0$, i.e.,

$$
\lim_{n,m\to\infty}\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^{n-1}-X_r^{m-1}|^p\Big)=0.
$$

Then there exists a subsequence $\{X_t^{n_k}\}$ such that for any $k\geqslant 1$,

$$
\Big\|\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big\|_p:=\Big(\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big)^p\Big)^{\frac{1}{p}}\leqslant\frac{1}{2^k}.
$$

Thus

$$
\begin{aligned}
\Big\|\sum_{k=1}^{\infty}\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big\|_p &=\sup_{\theta\in\mathcal{A}_{0,\infty}^{\Gamma}}\Big(E_{P_\theta}\Big(\sum_{k=1}^{\infty}\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big)^p\Big)^{\frac{1}{p}}\\
&\leqslant\sup_{\theta\in\mathcal{A}_{0,\infty}^{\Gamma}}\sum_{k=1}^{\infty}\Big(E_{P_\theta}\Big(\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big)^p\Big)^{\frac{1}{p}}\\
&\leqslant\sum_{k=1}^{\infty}\Big\|\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|\Big\|_p\leqslant 1,
\end{aligned}
$$

which implies

$$
\sum_{k=1}^{\infty}\sup_{r\leqslant t}|X_r^{n_{k+1}}-X_r^{n_k}|<\infty\quad\text{q.s.}
$$

Set $X_t = X_t^{n_1} + \sum\limits_{k=1}^{\infty}(X_t^{n_{k+1}} - X_t^{n_k})$. Then X is q.s. defined on Ω for all $t \in [0,T]$ and continuous q.s. Moreover, $\big\|\sup\limits_{0\leqslant t\leqslant T}|X_t|\big\|_p < \infty$, and

$$\Big(\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^{n_k} - X_r|^p\Big)\Big)^{\frac{1}{p}} \leqslant \sup_{\theta\in\mathcal{A}_{0,\infty}^{\Gamma}}\Big(E_{P_\theta}\Big(\sum_{l=k}^{\infty}\sup_{r\leqslant t}|X_r^{n_{l+1}} - X_r^{n_l}|\Big)^p\Big)^{\frac{1}{p}}$$
$$\leqslant \sum_{l=k}^{\infty}\Big\|\sup_{r\leqslant t}|X_r^{n_{l+1}} - X_r^{n_l}|\Big\|_p.$$

Letting $k \to \infty$ and taking limits on both sides of the above inequality, we get

$$\lim_{k\to\infty}\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r^{n_k} - X_r|^p\Big) = 0.$$

Now by the Hölder inequality, (2b) and Theorems 2.1–2.2, it holds that

$$\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r b(s, X_s^{n_k})\mathrm{d}s - \int_0^r b(s, X_s)\mathrm{d}s\Big|^p\Big)$$
$$\leqslant C_2(p,T)\int_0^t F\Big(s, \overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r^{n_k} - X_r|^p\Big)\Big)\mathrm{d}s,$$
$$\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r h(s, X_s^{n_k})\mathrm{d}\langle B\rangle_s - \int_0^r h(s, X_s)\mathrm{d}\langle B\rangle_s\Big|^p\Big)$$
$$\leqslant C_2(p,T)\int_0^t F\Big(s, \overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r^{n_k} - X_r|^p\Big)\Big)\mathrm{d}s$$

and

$$\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}\Big|\int_0^r \sigma(s, X_s^{n_k})\mathrm{d}B_s - \int_0^r \sigma(s, X_s)\mathrm{d}B_s\Big|^p\Big)$$
$$\leqslant C_2(p,T)\int_0^t F\Big(s, \overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r^{n_k} - X_r|^p\Big)\Big)\mathrm{d}s.$$

Taking limits on both sides of (3.1) in $L_G^p(\mathcal{F}_t)$, we attain that X satisfies (1.1).

Next, let X and X' be two solutions to (1.1), and then by the same way as above, we obtain that

$$\overline{\mathbb{E}}\Big(\sup_{r\leqslant t}|X_r - X_r'|^p\Big) \leqslant C_2(p,T)\int_0^t F\Big(s, \overline{\mathbb{E}}\Big(\sup_{r\leqslant s}|X_r - X_r'|^p\Big)\Big)\mathrm{d}s$$

for all $t \leqslant T$. We can apply (2c) and deduce that $\overline{\mathbb{E}}\big(\sup\limits_{r\leqslant t}|X_r - X_r'|^p\big) = 0$, which implies that $X_t = X_t',\ t \in [0,T]$ q.s. Thus the proof is completed.

Next assume that b, h and σ are independent of t, ω, and we study the cocycle property of (1.1) under these conditions of Theorem 3.1. The method comes from [1].

For $0 \leqslant s < t < \infty$, consider the following equation:

$$\Phi_{s,t}(x,\omega) = x + \int_s^t b(\Phi_{s,u}(x,\omega))\mathrm{d}u + \int_s^t h(\Phi_{s,u}(x,\omega))\mathrm{d}\langle B\rangle_u$$
$$+ \int_s^t \sigma(\Phi_{s,u}(x,\omega))\mathrm{d}B_u. \tag{3.3}$$

By Theorem 3.1 we know that (3.3) has a unique solution and denote it by $\Phi_{s,t}(x,\omega)$.

Lemma 3.1 *For $0 \leqslant r < s < t < \infty$,*

$$\Phi_{r,t}(x,\omega) = \Phi_{s,t}(\Phi_{r,s}(x,\omega),\omega) \quad q.s. \tag{3.4}$$

Proof Because $\Phi_{r,t}(x,\omega)$ solves (3.3), it follows from the additive property of G-stochastic integrals (see [7])

$$\begin{aligned}
\Phi_{r,t}(x,\omega) &= x + \int_r^t b(\Phi_{r,u}(x,\omega))\mathrm{d}u + \int_r^t h(\Phi_{r,u}(x,\omega))\mathrm{d}\langle B\rangle_u \\
&\quad + \int_r^t \sigma(\Phi_{r,u}(x,\omega))\mathrm{d}B_u \\
&= x + \int_r^s b(\Phi_{r,u}(x,\omega))\mathrm{d}u + \int_r^s h(\Phi_{r,u}(x,\omega))\mathrm{d}\langle B\rangle_u \\
&\quad + \int_r^s \sigma(\Phi_{r,u}(x,\omega))\mathrm{d}B_u + \int_s^t b(\Phi_{r,u}(x,\omega))\mathrm{d}u \\
&\quad + \int_s^t h(\Phi_{r,u}(x,\omega))\mathrm{d}\langle B\rangle_u + \int_s^t \sigma(\Phi_{r,u}(x,\omega))\mathrm{d}B_u \\
&= \Phi_{r,s}(x,\omega) + \int_s^t b(\Phi_{r,u}(x,\omega))\mathrm{d}u + \int_s^t h(\Phi_{r,u}(x,\omega))\mathrm{d}\langle B\rangle_u \\
&\quad + \int_s^t \sigma(\Phi_{r,u}(x,\omega))\mathrm{d}B_u.
\end{aligned}$$

However,

$$\begin{aligned}
\Phi_{s,t}(\Phi_{r,s}(x,\omega),\omega) &= \Phi_{r,s}(x,\omega) + \int_s^t b(\Phi_{s,u}(\Phi_{r,s}(x,\omega),\omega))\mathrm{d}u \\
&\quad + \int_s^t h(\Phi_{s,u}(\Phi_{r,s}(x,\omega),\omega))\mathrm{d}\langle B\rangle_u \\
&\quad + \int_s^t \sigma(\Phi_{s,u}(\Phi_{r,s}(x,\omega),\omega))\mathrm{d}B_u.
\end{aligned}$$

By the uniqueness of the solution to (3.3), (3.4) is proved.

Before stating another lemma, we introduce a notation $\Phi_{0,t}(x,\widehat{\omega})$, which solves the following equation:

$$\begin{aligned}
\Phi_{0,t}(x,\widehat{\omega}) &= x + \int_0^t b(\Phi_{0,u}(x,\widehat{\omega}))\mathrm{d}u + \int_0^t h(\Phi_{0,u}(x,\widehat{\omega}))\mathrm{d}\langle B^s\rangle_u \\
&\quad + \int_0^t \sigma(\Phi_{0,u}(x,\widehat{\omega}))\mathrm{d}B_u^s,
\end{aligned}$$

based on $\langle B^s\rangle_u = \langle B\rangle_{s+u} - \langle B\rangle_s \sim \langle B\rangle_u$, $B_u^s = B_{s+u} - B_s \sim B_u$ and Theorem 3.1.

Lemma 3.2 *For $0 \leqslant s, t < \infty$,*

$$\Phi_{s,s+t}(x,\omega) = \Phi_{0,t}(x,\widehat{\omega}) \quad \text{q.s.} \tag{3.5}$$

Proof Define $\Phi^0_{s,s+t}(x,\omega):=x,\ \Phi^0_{0,t}(x,\widehat{\omega}):=x$ and for $n\in\mathbb{N}$,

$$\begin{aligned}\Phi^n_{s,s+t}(x,\omega):=&\ x+\int_s^{s+t}b(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}u+\int_s^{s+t}h(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}\langle B\rangle_u\\&+\int_s^{s+t}\sigma(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}B_u,\\ \Phi^n_{0,t}(x,\widehat{\omega}):=&\ x+\int_0^t b(\Phi^{n-1}_{0,u}(x,\widehat{\omega}))\mathrm{d}u+\int_0^t h(\Phi^{n-1}_{0,t}(x,\widehat{\omega}))\mathrm{d}\langle B^s\rangle_u\\&+\int_0^t\sigma(\Phi^{n-1}_{0,t}(x,\widehat{\omega}))\mathrm{d}B^s_u.\end{aligned}$$

Then $\Phi^n_{s,s+t}(x,\omega),\Phi^n_{0,t}(x,\widehat{\omega})$ are well defined by the proof of Theorem 3.1. First of all, we prove

$$\Phi^n_{s,s+t}(x,\omega)=\Phi^n_{0,t}(x,\widehat{\omega})\quad \text{q.s.}\tag{3.6}$$

Assume that (3.6) holds for $n-1$. Taking a sequence of partitions of $[0,t]$: $\pi^N_t=\{t^N_0,t^N_1,\cdots,t^N_N\}$, $N=1,2,\cdots$, one gets by Proposition 5.3.5 in [9]

$$\sum_{j=1}^N b(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))((s+t^N_j)-(s+t^N_{j-1}))\to\int_s^{s+t}b(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}u\ \ \text{in } L^1_G(\mathcal{F}),$$

and by the definitions of G-stochastic integrals,

$$\begin{aligned}&\sum_{j=1}^N h(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))(\langle B\rangle_{s+t^N_j}-\langle B\rangle_{s+t^N_{j-1}})\to\int_s^{s+t}h(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}\langle B\rangle_u\ \ \text{in } L^1_G(\mathcal{F}),\\&\sum_{j=1}^N \sigma(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))(B_{s+t^N_j}-B_{s+t^N_{j-1}})\to\int_s^{s+t}\sigma(\Phi^{n-1}_{s,u}(x,\omega))\mathrm{d}B_u\ \ \text{in } L^2_G(\mathcal{F}),\end{aligned}$$

as $\mu(\pi^N_t)\to 0$. Thus, by (3.6), Proposition 5.3.5 in [9] and the definitions of G-stochastic integrals,

$$\begin{aligned}\Phi^n_{s,s+t}(x,\omega)=&\lim_{\mu(\pi^N_t)\to0}\Big[x+\sum_{j=1}^N b(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))((s+t^N_j)-(s+t^N_{j-1}))\\&+\sum_{j=1}^N h(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))(\langle B\rangle_{s+t^N_j}-\langle B\rangle_{s+t^N_{j-1}})\\&+\sum_{j=1}^N \sigma(\Phi^{n-1}_{s,s+t^N_{j-1}}(x,\omega))(B_{s+t^N_j}-B_{s+t^N_{j-1}})\Big]\\=&\lim_{\mu(\pi^N_t)\to0}\Big[x+\sum_{j=1}^N b(\Phi^{n-1}_{0,t^N_{j-1}}(x,\widehat{\omega}))(t^N_j-t^N_{j-1})\\&+\sum_{j=1}^N h(\Phi^{n-1}_{0,t^N_{j-1}}(x,\widehat{\omega}))(\langle B^s\rangle_{t^N_j}-\langle B^s\rangle_{t^N_{j-1}})\\&+\sum_{j=1}^N \sigma(\Phi^{n-1}_{0,t^N_{j-1}}(x,\widehat{\omega}))(B^s_{t^N_j}-B^s_{t^N_{j-1}})\Big]\end{aligned}$$

$$
\begin{aligned}
&= x + \int_0^t b(\Phi_{0,u}^{n-1}(x,\widehat{\omega}))\mathrm{d}u + \int_0^t h(\Phi_{0,u}^{n-1}(x,\widehat{\omega}))\mathrm{d}\langle B^s\rangle_u \\
&\quad + \int_0^t \sigma(\Phi_{0,u}^{n-1}(x,\widehat{\omega}))\mathrm{d}B_u^s \\
&= \Phi_{0,t}^n(x,\widehat{\omega}),
\end{aligned}
$$

where these limits hold in $L_G^1(\mathcal{F})$.

Finally, by the proof of Theorem 3.1 we know that $\Phi_{s,s+t}^n(x,\omega)$, $\Phi_{0,t}^n(x,\widehat{\omega})$ converge to $\Phi_{s,s+t}(x,\omega)$, $\Phi_{0,t}(x,\widehat{\omega})$ respectively in $L_G^2(\mathcal{F})$. So, (3.5) is proved by taking the limit to (3.6).

Theorem 3.2 (Cocycle Property) *For* $0 \leqslant s,t < \infty$,

$$
X_{t+s}(x,\omega) = X_t(X_s(x),\widehat{\omega}) \quad \text{q.s.}
$$

Proof By (3.4)–(3.5), we have

$$
\begin{aligned}
X_{t+s}(x,\omega) &= \Phi_{0,s+t}(x,\omega) = \Phi_{s,s+t}(\Phi_{0,s}(x,\omega),\omega) \\
&= \Phi_{0,t}(\Phi_{0,s}(x,\omega),\widehat{\omega}) = X_t(X_s(x),\widehat{\omega}) \quad \text{q.s.}
\end{aligned}
$$

4 A Special Case: $F(t,u) = Cq(t)\rho_\eta(u)$

For $0 < \eta < \frac{1}{\mathrm{e}}$, we define a concave function as

$$
\rho_\eta(u) := \begin{cases} u\log u^{-1}, & u \leqslant \eta, \\ \eta\log\eta^{-1} + (\log\eta^{-1} - 1)(u-\eta), & u > \eta. \end{cases}
$$

Then $Cq(t)\rho_\eta(u)$ satisfies (2a) and (2c), where $q(t)$ is a strictly positive and integrable function on $\mathbb{R}_+$. In this section, we consider the special case: $F(t,u) = Cq(t)\rho_\eta(u)$.

Lemma 4.1 *Suppose that* b, h *and* σ *satisfy those conditions in Theorem* 3.1 *for* $F(t,u) = Cq(t)\rho_\eta(u)$. *Then for any* $T > 0$, *there are three positive constants* $C_3 = C_3(p,T)$, $C_4 = C_4(p,T)$ *and* $C_5 = C_5(p,T)$ *such that for any* $x,y \in \mathbb{R}^n$ *and any* $s,t \in [0,T]$,

$$
\begin{aligned}
\overline{\mathbb{E}}(|X_s(x) - X_t(x)|^p) &\leqslant C_3|s-t|^{\frac{p}{2}}, \\
\overline{\mathbb{E}}\Big(\sup_{t\leqslant T}|X_t(x) - X_t(y)|^p\Big) &\leqslant C_5|x-y|^{p\cdot\exp\{-C_4\int_0^T q(s)\mathrm{d}s\}}.
\end{aligned}
$$

Its proof is similar to that of Theorem 3.1 and we omit it.

Proposition 4.1 *Suppose that* b, h *and* σ *satisfy those conditions in Theorem* 3.1 *for* $F(t,u) = Cq(t)\rho_\eta(u)$. *Moreover, they are independent of* t, ω. *Then the solution* $X_t(x)$ *to* (1.1) *is bi-continuous with respect to* t, x.

Proof In Lemma 4.1 we first choose p sufficiently large and secondly $T =: T_0$ sufficiently small such that

$$
p\cdot\exp\Big\{-C_4\int_0^{T_0} q(s)\mathrm{d}s\Big\} > d.
$$

Then by Theorem 31 in [5]

$$[0,T_0]\times\mathbb{R}^d \ni (t,x)\mapsto X_t(x,\omega)\in\mathbb{R}^d$$

has a bi-continuous modification which is still denoted by $X_t(x)$.

For $t\in[T_0,2T_0]$, by Theorem 3.2 we have that for all $s\in[0,T_0]$,

$$X_{s+T_0}(x,\omega)=X_s(X_{T_0}(x,\omega),\widehat{\widehat{\omega}}),\quad \text{q.s.,} \tag{4.1}$$

where $X_s(x,\widehat{\widehat{\omega}})$ solves the following equation:

$$\begin{aligned}X_s(x,\widehat{\widehat{\omega}}) &= x+\int_0^s b(X_u(x,\widehat{\widehat{\omega}}))\mathrm{d}u+\int_0^s h(X_u(x,\widehat{\widehat{\omega}}))\mathrm{d}\langle B^{T_0}\rangle_u\\ &\quad+\int_0^s \sigma(X_u(x,\widehat{\widehat{\omega}}))\mathrm{d}B_u^{T_0}.\end{aligned}$$

Set

$$\widetilde{X}_{s+T_0}(x,\omega):=X_s(X_{T_0}(x,\omega),\widehat{\widehat{\omega}}).$$

Since $(s,x)\mapsto X_s(x,\widehat{\widehat{\omega}})$ and $x\mapsto X_{T_0}(x,\omega)$ are continuous q.s., we obtain the continuity of $(s,x)\mapsto \widetilde{X}_{s+T_0}(x,\omega)$ q.s. We still denote the bi-continuous modification of $\widetilde{X}_{s+T_0}(x,\omega)$ by $\widetilde{X}_{s+T_0}(x,\omega)$. By (4.1), we have

$$\widetilde{X}_{s+T_0}(x,\omega)=X_{s+T_0}(x,\omega),\quad \text{q.s.}$$

Then there is a set $\widetilde{\widetilde{\Omega}}\subset\Omega$ with $\overline{C}(\widetilde{\widetilde{\Omega}}^c)=0$ such that for all $\omega\in\widetilde{\widetilde{\Omega}}$,

$$X_{s+T_0}(x,\omega)=X_s(X_{T_0}(x,\omega),\widehat{\widehat{\omega}}),\quad \forall s\in[0,T_0].$$

In the same way as $t\in[T_0,2T_0]$, we get that $X_t(x)$ has a bi-continuous modification for $t\in[2T_0,3T_0],[3T_0,4T_0],\cdots$.

Acknowledgements The author is very grateful to Professor Jinqiao Duan and Xicheng Zhang for their valuable discussions, and also wishes to thank the referees for their suggestions and improvements.

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