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Embedding Periodic Maps on Surfaces into Those on S^3*

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Abstract Call a periodic map h on the closed orientable surface Σ_g extendable if h extends to a periodic map over the pair (S^3, Σ_g) for possible embeddings $e : \Sigma_g \to S^3$. The authors determine the extendabilities for all periodical maps on Σ_2. The results involve various orientation preserving/reversing behalves of the periodical maps on the pair (S^3, Σ_g). To do this the authors first list all periodic maps on Σ_2, and indeed the authors exhibit each of them as a composition of primary and explicit symmetries, like rotations, reflections and antipodal maps, which itself should be interesting. A by-product is that for each even g, the maximum order periodic map on Σ_g is extendable, which contrasts sharply with the situation in the orientation preserving category.

Keywords Symmetry of surface, Symmetry of 3-sphere, Extendable action
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1 Introduction

Closed orientable surfaces are the most ordinary geometric and physical subjects to us, since they stay in our 3-dimensional space everywhere in various manners (often as the boundaries of 3-dimensional solids).

The study of symmetries on closed orientable surfaces is also a classical topic in mathematics. An interesting fact is that some of those symmetries are easy to see, and others are not, or to be more precise, some symmetries are more visible than others.

Let's have a look at the examples. Let Σ_g be the orientable closed surface with genus g. We always assume that $g > 1$ in this paper. It is easy to see that there exists a symmetry ρ of order 2 on Σ_2 indicated in the left side of Figure 1, and it is not easy to see that there exists a symmetry τ of order 2 on Σ_2 whose fixed-point set consists of two non-separating circles, indicated in the right side of Figure 1. A primary reason for this fact is that we can embed Σ_2 and ρ into the 3-space and its symmetry space simultaneously, that is to say, ρ is induced from a symmetry of our 3-space, or ρ extends to a symmetry over the 3-space; and on the other hand, as we will see, τ can never be induced by a symmetry of the 3-space for any embedding of Σ_2.

Now we make a precise definition: If a finite group action G on Σ_g can also act on the pair (S^3, Σ_g) for possible embeddings $e : \Sigma_g \to S^3$, that is to say, $\forall h \in G$, we have $h \circ e = e \circ h$, and we call such a group action on Σ_g extendable over S^3 (with respect to e).

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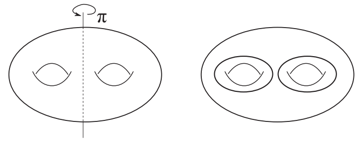


Figure 1

Such extendable finite group actions have been addressed in [13–14]. But those two papers focus on the problem of maximum orders in the orientation preserving category. In this paper we start to discuss the extendable problem for general finite group actions and we will delete the orientation preserving restriction. We will first focus on the simplest case: The cyclic group actions on the surface Σ_2.

We examine the extendabilities for all periodical maps on Σ_2. To do this we first need not only to exhibit all $\mathbb{Z}_n$-actions on Σ_2, but also to exhibit them in a very geometric and visible way.

Theorem 1.1 (1) *There exist twenty-one conjugacy classes of finite cyclic actions on* Σ_2 *which are generated by the following periodical maps:* $\rho_{2,1}$, $\rho_{2,2}$, $\tau_{2,1}$, $\tau_{2,2}$, $\tau_{2,3}$, $\tau_{2,4}$, $\tau_{2,5}$, ρ_3, ρ_4, $\tau_{4,1}$, $\tau_{4,2}$, ρ_5, $\rho_{6,1}$, $\rho_{6,2}$, $\tau_{6,1}$, $\tau_{6,2}$, $\tau_{6,3}$, ρ_8, τ_8, ρ_{10}, τ_{12}, *where each map presented by* ρ/τ *is orientation preserving/reversing, and the first subscript indicates the order.*

(2) *The extendability of twenty-one periodic maps in* (1) *are:* $\rho_{2,1}\{+\}$, $\rho_{2,2}\{+,-\}$, $\tau_{2,1}\{+,-\}$, $\tau_{2,2}\{+,-\}$, $\tau_{2,3}\{+\}$, $\tau_{2,4}\{\varnothing\}$, $\tau_{2,5}\{-\}$, $\rho_3\{+\}$, $\rho_4\{-\}$, $\tau_{4,1}\{+,-\}$, $\tau_{4,2}\{\varnothing\}$, $\rho_5\{\varnothing\}$, $\rho_{6,1}\{+\}$, $\rho_{6,2}\{-\}$, $\tau_{6,1}\{\varnothing\}$, $\tau_{6,2}\{-\}$, $\tau_{6,3}\{\varnothing\}$, $\rho_8\{\varnothing\}$, $\tau_8\{\varnothing\}$, $\rho_{10}\{\varnothing\}$, $\tau_{12}\{+\}$, *where the symbol* $\{+\}/\{-\}/\{+,-\}/\{\varnothing\}$ *indicates that the map has orientation preserving/orientation reversing/both orientation preserving and reversing/no extension.*

The geometric descriptions of (1) *and* (2) *are given in Figure* 2 *and Figure* 3*, respectively, where we exhibit each of them as a composition of rotations, reflections, and* (*semi-*)*antipodal maps*:

(i) *Each rotation in* 2*-space* (3*-space*) *is indicated by a circular arc with arrow around a point* (*an axis*)*.*

(ii) *Each reflection about a* 2*-sphere* = 2*-plane* $\cup\,\infty$ *or a circle* = *line* $\cup\,\infty$ *is indicated by an arc with two arrows.*

(iii) *Each* (*semi-*)*antipodal map is indicated by a point* (*another fixed point is* ∞)*.*

More concrete descriptions of those maps will be given in the proof of Theorem 1.1 *and Example* 2.1*.*

Let C_g and CE_g be the maximum orders of periodical maps and extendable periodical maps on Σ_g respectively; C_g^o and CE_g^o be the corresponding notions restricted to the orientation-preserving category. Then it is known that (1) C_g^o is $4g+2$. C_g is $4g+4$ when g is even and $4g+2$ when g is odd (see [10]). (2) $CE_g^o = 2g+2$ if g is even and $2g-2$ if g is odd (see [13]).

By the construction and argument provided for the maps τ_{12} and $\tau_{2,4}$ (see Example 2.1 and Case (2_-)), we can easily get the following facts which do not appear in the orientation preserving category.

Corollary 1.1 (1) *For each even* g*, the maximum order periodic map on* Σ_g *is extendable, that is to say,* $CE_g = 4g+4$*.*

(2) *For each* g*, there exists a non-extendable symmetry of order* 2 *on* Σ_g*.*

The following notions are convenient for our later discussion.

Let $G = \mathbb{Z}_n = \langle h \rangle$, where h is a periodic map of order n on (S^3, Σ_g). According to whether h preserves/reverses the orientation of S^3/Σ_g, we have four types of extendable group actions.

(1) Type $(+,+)$: h preserves the orientations of both S^3 and Σ_g.

(2) Type $(+,-)$: h preserves the orientation of Σ_g and reverses that of S^3.

(3) Type $(-,+)$: h reverses the orientation of Σ_g and preserves that of S^3.

(4) Type $(-,-)$: h reverses the orientations of both Σ_g and S^3.

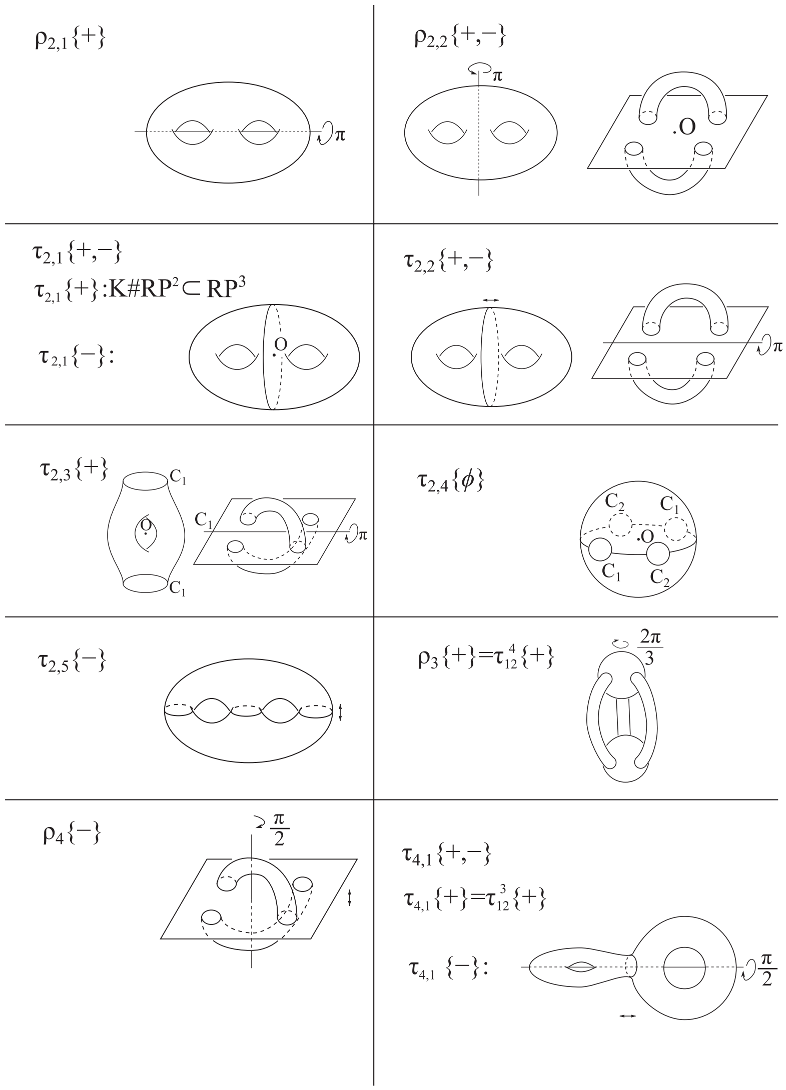


Figure 2

One can easily check that the type of the action $G = \langle h \rangle$ is independent of the choice of the periodical map h. Those notions remind us that Type $(+,+)$ and Type $(+,-)$ are extending orientation preserving maps on Σ_g to those of S^3 in the orientation preserving and reversing ways respectively, and Type $(-,+)$ and Type $(-,-)$ are extending orientation reversing maps on Σ_g to those of S^3 in the orientation preserving and reversing ways respectively. Periodical maps of Type $(+,+)$ and Type $(-,-)$ do not change the two sides of Σ_g, and periodical maps of the remaining two types do change the two sides of Σ_g.

Notice that if G is extendable and $h \in G$ is an element which equals the identity on Σ_g, then it is easy to see that h is the identity on the whole S^3. Hence we always assume that the group action is faithful on both Σ_g and S^3.

Suppose that S (resp. P) is a properly embedded $(n-1)$-manifold (resp. n-manifold) in an n-manifold M. We use $M \setminus S$ (resp. $M \setminus P$) to denote the resulting manifold obtained by splitting M along S (resp. removing int$P$, the interior of P).

The fixed point set Fix(G) for a finite group action on X is defined as $\{x \in X \mid g(x) = x, \forall g \in G\}$.

In Section 2, we recall some fundamental results, construct some examples, and prove some lemmas, which will be used in Section 3 to prove Theorem 1.1.

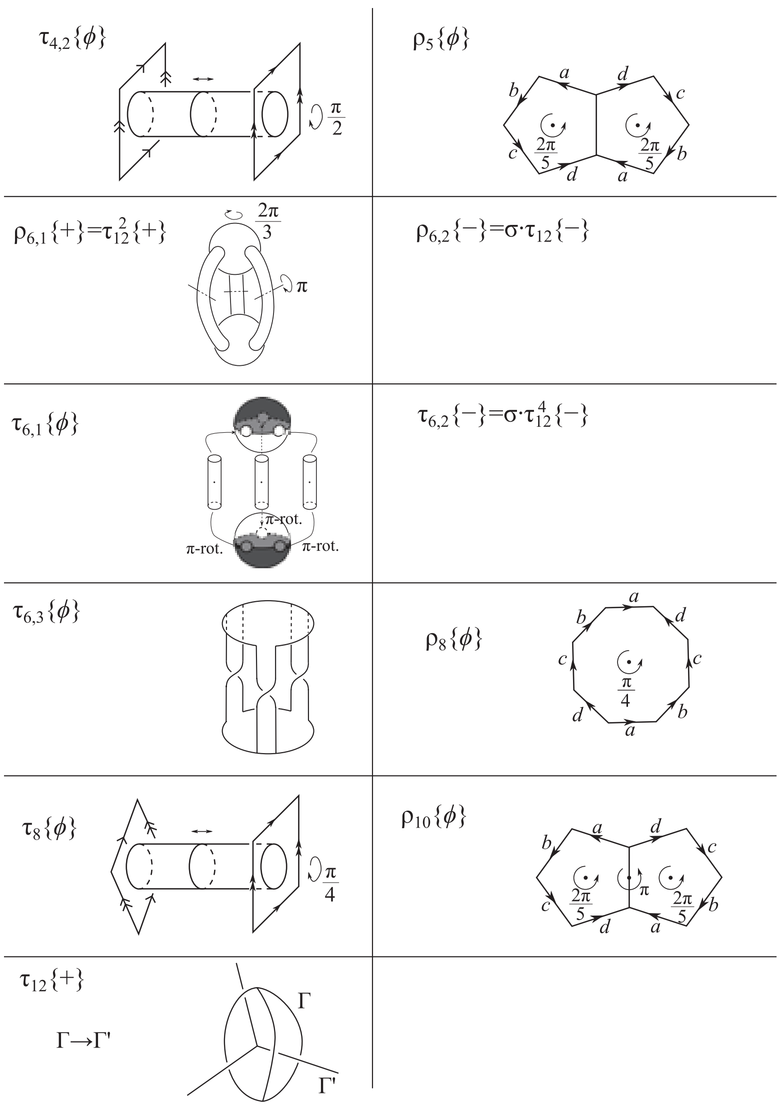


Figure 3

2 Some Facts and Examples

We first recall some fundamental results in this section.

Consider 2-disc D^2 (resp. 3-disc D^3) as the unit disc in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$); and 2-sphere S^2 (resp. 3-sphere S^3) as the unit sphere in $\mathbb{R}^3$ (resp. $\mathbb{R}^4$). Then each periodic map on D^2, S^2 and B^3, S^3 can be conjugated into $O(2)$, $O(3)$, $O(4)$ respectively. Those results in dimension 2 can be found in [3], and the results in dimension 3 can be found in [4] for actions of isolated fixed points, in [2, 11] for actions with fixed point set of dimension at least 1, and in [6–8] for fixed-point free actions.

There exists exactly one standard orientation-reversing $\mathbb{Z}_2$-action on D^2: A reflection about a diameter. There exist two standard orientation-reversing $\mathbb{Z}_2$-actions on D^3 (resp. S^2): One is a reflection about an equator 2-disc (circle), and the other is the antipodal map. There also exist two standard orientation-reversing $\mathbb{Z}_2$-actions on S^3: One is a reflection about an equator 2-sphere, and the other is the semi-antipodal map which has two fixed points $\{0, \infty\}$, which on every sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = R^2\}$ is an antipodal map.

The facts in the following statement, which can be found in [2–4, 6–9, 11], will be used repeatedly later.

Theorem 2.1 (1) *Any orientation reversing periodic map on the* 2*-disc* D^2 *is conjugate to a reflection about a diameter.*

(2) *An orientation reversing* $\mathbb{Z}_2$*-action on* $S^3(D^3)$ *must conjugate to either a reflection about a* 2*-sphere* (2*-disc*) *or a semi-antipodal* (*antipodal*) *map.*

(3) *The fixed-point set of an orientation preserving* $\mathbb{Z}_n$*-action on a closed orientable* 3*-manifold* M *is a disjoint union of circles* (*may be empty*).

(4) *In* (3) *if* $M = S^3$ *then the fixed-point set of an orientation preserving* $\mathbb{Z}_n$*-action must be either the empty set or an unknotted circle.*

We will also give a brief recall of orbifold theory for later use (see [1, 4–5, 16]).

Each orbifold we considered has the form M/G, where M is an n-manifold and G is a finite group acting faithfully on M. For a point $x \in M$, denote its stable subgroup by $St(x)$, and its image in M/G by x'. If $|St(x)| > 1$, x' is called a singular point and the singular index is $|St(x)|$, otherwise it is called a regular point. If we forget the singularity we get a topological space $|M/G|$ which is called an underlying space.

We can also define the covering space and the fundamental group for orbifold. There exists a one to one correspondence between orbifold covering spaces and conjugate classes of subgroups of orbifold fundamental groups, and regular covering spaces correspond to normal subgroups.

In the following, if we say covering spaces or fundamental groups, they always refer to the orbifold corresponding objects.

A simple picture we should keep in mind is the following: Suppose that G acts on (S^3, Σ_g). Let $\Gamma = \{x \in S^3 | \exists g \in G,\ g \neq \mathrm{id}, \text{s.t. } gx = x\}$. Then Γ/G is the singular set of the 3-orbifold S^3/G, and Σ_g/G is a 2-orbifold with a singular set $\Sigma_g/G \cap \Gamma/G$.

Suppose that a finite cyclic group $G = \mathbb{Z}_n$ acts on Σ_g. Then Σ_g/G is a 2-orbifold whose singular set contains isolated points $a_1, a_2, \cdots, a_k$, with indices $q_1 \leq q_2 \leq \cdots \leq q_k$. Suppose that the genus of $|\Sigma_g/G|$ is $\widehat{g}$. We have the Riemann-Hurwitz formula

$$2-2g=n\Big(2-2\widehat{g}-\sum_{i=1}^{k}\Big(1-\frac{1}{q_i}\Big)\Big),\quad q_i \text{ dividing } n. \tag{RH}$$

The following Hurwitz-type result is about the existence and classification of the actions of finite group G on Σ_g.

Theorem 2.2 (1) *A finite cyclic group G acts on the surface Σ_g to get an orbifold $X = \Sigma_g/G$ if and only if there exists an exact sequence*

$$1 \to \pi_1(\Sigma_g) \to \pi_1(X) \to G \to 1.$$

(2) *If two finite group actions G and G' are conjugate, then their exact sequences have the following diagram:*

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \pi_1(\Sigma_g) & \longrightarrow & \pi_1(X) & \longrightarrow & G & \longrightarrow & 1 \\ & & \cong\downarrow & & \cong\downarrow & & \cong\downarrow & & \\ 1 & \longrightarrow & \pi_1(\Sigma'_g) & \longrightarrow & \pi_1(X') & \longrightarrow & G' & \longrightarrow & 1 \end{array}$$

(3) *If two finite group actions G and G' have the above diagram, and both actions have no reflection fixed curves, then these actions are conjugate.*

Proof (1) It is parallel to the classical covering space theory, and is the fundamental property in orbifold theory.

(2) Suppose that two actions are conjugate induced by an homeomorphism between surfaces $\widetilde{f} : \Sigma_g \to \Sigma'_g$, and then there is a diagram about covering maps:

$$\begin{array}{ccc} \Sigma_g & \longrightarrow & X \\ \widetilde{f}\downarrow & & f\downarrow \\ \Sigma'_g & \longrightarrow & X' \end{array}$$

Then $\widetilde{f}_*$ and f_* give the first two vertical homomorphisms between the fundamental groups. The third one is also well defined as a quotient of f_*.

(3) Suppose that the two actions have the group-level diagram, and that both actions have no reflection-fixed curves. Then we have $f_* : \pi_1(X) \to \pi_1(X')$. By [15, Theorem 5.8.3], this homomorphism can be induced by some orbifold homeomorphism f. Now because the left square commutes, f can be lift to a homeomorphism $\widetilde{f}$ between covering surfaces such that it induces the first vertical homomorphism. Such $\widetilde{f}$ gives the desired conjugation between actions.

To prove Theorem 1.1, besides quoting the above results, we need more results and constructions.

Suppose that h is an orientation-reversing periodic map of order $2q$ on a compact p-manifold U, $p = 2, 3$. Then the index-two subgroup of $G = \langle h\rangle$ is the unique one $G^o = \langle h^2\rangle$ which acts on U orientation preservingly. Let $X = U/G^o$ be the corresponding p-orbifold, $\pi : U \to X$ be the cyclic branched covering of degree q, and $\langle\overline{h}\rangle$ be the induced order-2 orientation reversing action on X.

Lemma 2.1 *Under the setting above: Suppose that $x \subset X$ is of index n ($n = 1$ if x is a regular point), and x is a fixed point of $\overline{h}$. Then*

(1) $\frac{q}{n}$ must be odd.

(2) If $p = 2$, x must be a regular point of $\overline{h}$.

Proof (1) Let $N(x) = B^p \subset |X|$ be an $\overline{h}$-invariant p-ball. Then $\pi^{-1}(x) = \{\widetilde{x}_1, \cdots, \widetilde{x}_l\}$, where $l = \frac{q}{n}$, and $\pi^{-1}(B^p) = \{B_1, \cdots, B_l\}$ is a disjoint union p-ball which is an h-invariant set. Moreover, the action of h on $\{B_1, \cdots, B_l\}$ is transitive and orientation reserving (under the induced orientation). The stable subgroup of $\widetilde{x}_1$, $St(\widetilde{x}_1) \subset \mathbb{Z}_{2q} = \langle h \rangle$ is a cyclic group $\mathbb{Z}_{2n} = \langle h^l \rangle$. Since $St(\widetilde{x}_1)$ contains an orientation reversing element, l must be odd.

(2) In case of dimension 2, any orientation reversing periodic map must be conjugated to a reflection about a diameter L of B^2 by Theorem 2.1(1). Therefore any finite cyclic group containing an orientation reversing element must be $\mathbb{Z}_2$, that is to say, $n = 1$ and therefore x is a regular point.

Lemma 2.2 *Suppose that h is an orientation preserving periodic map on Σ_g and the number of singular points of $X = \Sigma_g/\langle h \rangle$ is odd. Then h can not extend to S^3 in the type $(+, +)$.*

Proof Otherwise let $\widetilde{h} : (S^3, \Sigma_g) \to (S^3, \Sigma_g)$ be such an extension. As an orientation preserving periodic map on S^3, its fixed-point set must be a disjoint union of circles by Theorem 2.1(3), and the singular-point set $\Gamma_{\langle \widetilde{h} \rangle}$ of the orbifold $S^3/\langle \widetilde{h} \rangle$ must be a disjoint union of circles. Then the singular-point set of the 2-orbifold $\Sigma_g/\langle h \rangle$, as the intersection of those circles and $|\Sigma_g/\langle h \rangle|$, must have an even number of points, which is a contradiction.

Lemma 2.3 *The Klein bottle K can not embed into $\mathbb{R}P^3$.*

Proof Otherwise there exists an embedding $K \subset \mathbb{R}P^3$. We can assume that K is transversal to some $\mathbb{R}P^2 \subset \mathbb{R}P^3$. Cutting $\mathbb{R}P^3$ along this $\mathbb{R}P^2$ we get the D^3, and K becomes an embedded proper surface $S \subset D^3$ with $\chi(S) = \chi(K) = 0$. Note that every embedded proper surface in D^3 must be orientable, so S must be an annulus. But the two boundaries of S in ∂D^3 must be identified by the antipodal map on ∂D^3 before the cutting, that is to say, we can only get a torus from S, not K, which is a contradiction.

The constructions below provide various extendable periodic maps on Σ_g.

Example 2.1 For every $g > 1$, we will construct some finite cyclic actions on a Heegaard splitting $S^3 = V_g \underset{\Sigma_g}{\cup} V'_g$.

Consider S^3 as the unit sphere in $\mathbb{C}^2$, then

$$S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \mid |z_1|^2 + |z_2|^2 = 1\}.$$

Let

$$\begin{aligned} a_m &= (\mathrm{e}^{\frac{m\pi \mathrm{i}}{2}}, 0), \quad m = 0, 1, 2, 3, \\ b_n &= (0, \mathrm{e}^{\frac{n\pi \mathrm{i}}{g+1}}), \quad n = 0, 1, \cdots, 2g+1. \end{aligned}$$

Connect each a_{2l} to each b_{2k} with a geodesic in S^3 and each a_{2l+1} to each b_{2k+1} with a geodesic in S^3, where $l = 0, 1$ and $k = 0, 1, \cdots, g$. Then we get two two-parted graphs $\Gamma, \Gamma' \in S^3$ each which has 2 vertices and $g+1$ edges, and is in the dual position (see Figure 4). The left one is a sketch map, and the right one presents exactly how the graphs look like. All the graphs are projected to $\mathbb{R}^3$ from S^3.

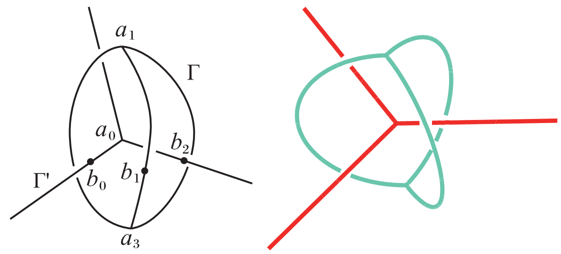


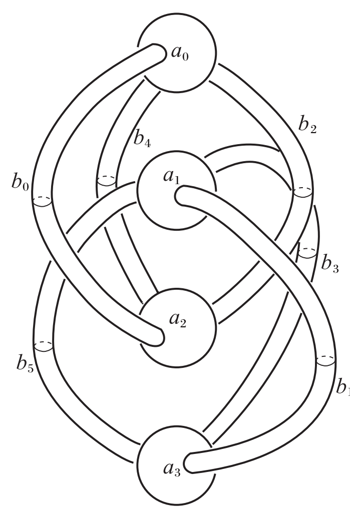
Figure 4

We have the following three isometries on S^3 which preserve the graph $\Gamma \cup \Gamma'$:

$$\tau(g) : (z_1, z_2) \mapsto (\mathrm{i}z_1, \mathrm{e}^{\frac{\pi \mathrm{i}}{g+1}} z_2),$$
$$\rho : (z_1, z_2) \mapsto (-z_1, z_2),$$
$$\sigma : (z_1, z_2) \mapsto (\overline{z}_1, z_2).$$

Here τ and ρ preserve the orientation of S^3 and σ reverses the orientation of S^3. If g is even, $\langle \tau(g) \rangle$ has order $4(g+1)$.

The points in S^3 have an equal distance to Γ and Γ' forms a closed subsurface having genus g. This is our Σ_g. It cuts S^3 into two handlebodies V_g and V_g' which are neighborhoods of Γ and Γ'. All the isometries $\tau(g), \rho, \sigma$ preserve $\Gamma \cup \Gamma'$, so they must also preserve Σ_g. One can check easily that τ, ρ and σ give examples of extendable periodical maps on (S^3, Σ_g) of types $(-,+)$ $(+,+)$ and $(-,-)$, respectively. In particular, $\tau(g)$ gives a periodic map on (S^3, Σ_g) of type $(-,+)$ and order $4(g+1)$ when g is even. Notice that this is also the maximum order of cyclic group action on Σ_g when g is even. A more concrete and intuitive picture to indicate how $\tau(2)$ acts on Σ_2 is given in Figure 5 (for another such description, see [12]).



$a_i \mapsto a_{i+1}, b_i \mapsto b_{i+1}$

Figure 5

Compositions of $\tau(g), \rho, \sigma$ provide extendable periodical maps on Σ_g of required orders and types, for example,

(1) $\sigma\tau(2)$ is a map on (S^3, Σ_2) of type $(+,-)$ and order 6.

(2) $\sigma\tau^4(2)$ on (S^3, Σ_2) is of type $(-,-)$ and order 6.

3 Extendabilities of Periodical Maps on Σ_2

Suppose that $\mathbb{Z}_n = \langle h \rangle$ acts on Σ_2, and we know $n \leq 12$. We will discuss all the periods $n \in \{2, 3, \cdots, 12\}$. For each period n, we will divide the discussion into two cases: The orientation preserving maps on Σ_2, denoted as (n_+); and the orientation reversing maps on Σ_2, denoted as (n_-). For each case (n_ϵ), $\epsilon = \pm$, we first discuss part (1) of Theorem 1.1, the classification of periodic maps; and then part (2) of Theorem 1.1, the extendabilities of those maps.

We remark that for each odd n, the situation is simpler, since all the possible actions must be orientation-preserving. If $n = 2k$, then h induces an involution $\overline{h}$ on $X = \Sigma_2/\langle h^2 \rangle$, the orbifold corresponding to the unique index-two sub-group $\mathbb{Z}_k$ of $\mathbb{Z}_{2k}$, and if h is orientation-reversing, n must be $2k$, and $\overline{h}$ is orientation-reversing on X.

(2_+) classification: Now $X = \Sigma_2/\mathbb{Z}_2$ is a closed orientable 2-orbifold with $\chi(X) = \frac{\chi(\Sigma_2)}{2} = -1$ by (RH), and $|X|$ must be a sphere or a torus. Every branched point of X must be of index-2. So X is either a sphere with six index-2 branched points, denoted by $X_1 = S^2(2,2,2,2,2,2)$, or a torus with two index-2 branched points, denoted by $X_2 = T(2,2)$.

By Theorem 2.2(1), we have an exact sequence $1 \to \pi_1(\Sigma_2) \to \pi_1(X_i) \to \mathbb{Z}_2 \to 1$, and for each branched point x of X, $St(x)$ must be mapped isomorphically onto $\mathbb{Z}_2$.

Note $\pi_1(X_1) = \langle a, b, c, d, e, f \mid abcdef = 1,\ a^2 = b^2 = c^2 = d^2 = e^2 = f^2 = 1 \rangle$, and the only surjection $\pi_1(X_1) \to \mathbb{Z}_2 = \langle t \mid t^2 = 1 \rangle$ is $(a, b, c, d, e, f) \mapsto (t, t, t, t, t, t)$, so this $\mathbb{Z}_2$ action is unique up to conjugacy, denoted by $\rho_{2,1}$, whose action on Σ_2 is indicated in the left of Figure 6.

Note $\pi_1(X_2) = \langle a, b, u, v \mid aba^{-1}b^{-1} = uv,\ u^2 = v^2 = 1 \rangle$, and the possible surjections from $\pi_1(X_2)$ to $\mathbb{Z}_2 = \langle t \mid t^2 = 1 \rangle$ are

$$(a, b, u, v) \mapsto \begin{cases} (1, 1, t, t), \\ (1, t, t, t), \\ (t, 1, t, t), \\ (t, t, t, t). \end{cases}$$

Consider the automorphisms of

$$\pi_1(X_2) : (a, b, u, v) \mapsto (a, ba, u, v), (a, b, u, v) \mapsto (a, uab, ava^{-1}, u),$$

and all these representations are equivalent. So this $\mathbb{Z}_2$ action is unique up to conjugacy, and we denote it by $\rho_{2,2}$, whose action on Σ_2 is indicated in the middle and right of Figure 6.

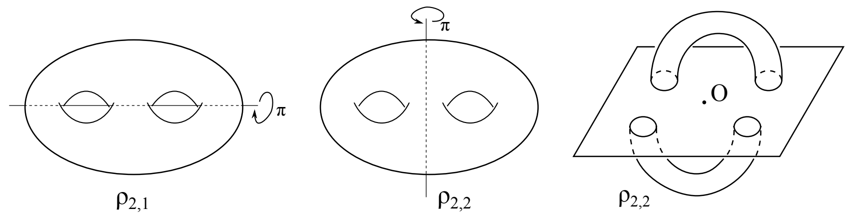


Figure 6

Extendibility: As indicated in Figure 6, $\rho_{2,1}$ can extend orientation-preservingly.

The fixed-point set of any orientation-reversing $\mathbb{Z}_2$ action on S^3 is either the set containing two points or the 2-sphere by Theorem 2.1(2), which can not intersect Σ_2 with 6 isolated points, and therefore $\rho_{2,1}$ can not extend orientation-reservingly. So we have $\rho_{2,1}\{+\}$.

As indicated in the middle of Figure 6, $\rho_{2,2}$ can extend orientation-preservingly. If we choose the embedding $\Sigma_2 \to S^3$ as the right-side of Figure 6, one can see that $\rho_{2,2}$ can also extend to S^3 orientation-reversingly, as a semi-antipodal map about the origin point O with two fixed points O and ∞. So we have $\rho_{2,2}\{+,-\}$.

(2_-) classification: Suppose τ is an order-2 orientation-reversing homeomorphism of Σ_2. Consider the fixed-point set fix(τ).

If fix$(\tau) = \varnothing$, then the map $\Sigma_2 \to \Sigma_2/\mathbb{Z}_2$ is a covering map, and $\Sigma_2/\mathbb{Z}_2$ is a non-orientable closed surface with $\chi = -1$, which is the connected sum of a torus and a projective plane. Because the covering map is unique, the action is unique up to conjugacy, and we denote it by $\tau_{2,1}$, whose action on Σ_2 is indicated in Figure 7. Here it is a semi-antipodal map with fixed points O and ∞.

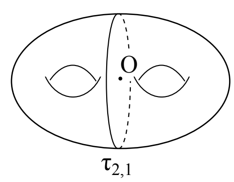


Figure 7

Now suppose fix$(\tau) \neq \varnothing$. Because τ is orientation-reversing, fix(τ) must be the union of disjoint circles on Σ_2. Suppose that fix(τ) contains at least one separating circle C_0, and then τ changes the two components of $\Sigma_2 \backslash C_0$. So fix$(\tau) = \{C_0\}$. In this case the action is also unique, and denote it by $\tau_{2,2}$, whose action on Σ_2 is indicated in Figure 8. On the left, the symbol $\leftrightarrow$ means a reflection about the middle plane, and on the right is a π-rotation.

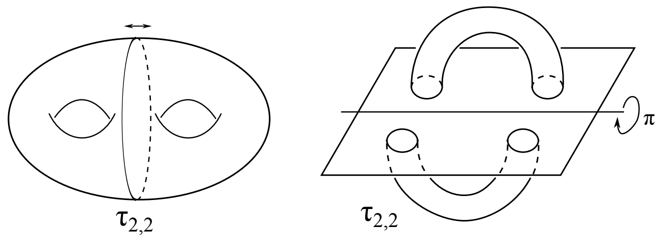


Figure 8

Now suppose that each component of $\text{fix}(\tau)$ is non-separating.

If $|\text{fix}(\tau)| = 1$, for example, $\text{fix}(\tau) = \{C_1\}$, then $\Sigma_2 \backslash C_1$ is a torus with two holes. Then the $\mathbb{Z}_2$ action on $\Sigma_2 \backslash C_1$ is fixed-point free and changes the two boundary components, so it induces a double cover onto a non-orientable surface. This is also unique, denoted by $\tau_{2,3}$, whose action on Σ_2 is indicated in Figure 9.

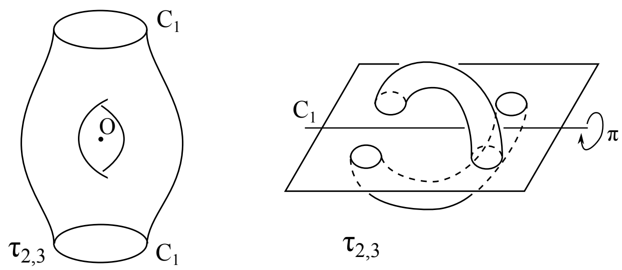


Figure 9

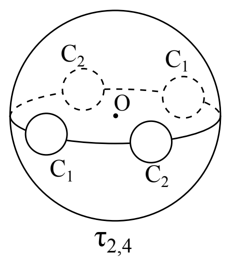


Figure 10

If $|\text{fix}(\tau)| = 2$, for example, $\text{fix}(\tau) = \{C_1, C_2\}$, then $\Sigma_2 \backslash \{C_1, C_2\}$ must be a sphere with four holes (see Figure 10). Then the $\mathbb{Z}_2$ action on $\Sigma_2 \backslash \{C_1, C_2\}$ is fixed-point free and changes the four boundary components into two pairs. The action is also unique, denoted by $\tau_{2,4}$.

If $|\text{fix}(\tau)| = 3$, for example, $\text{fix}(\tau) = \{C_1, C_2, C_3\}$, then $\Sigma_2 \backslash \{C_1, C_2, C_3\}$ are two 3-punctured spheres. The action is also unique, denoted by $\tau_{2,5}$.

If $|\text{fix}(\tau)| \geq 4$, then $\Sigma_2 \backslash \{C_i\}$ has more than two components, which is impossible.

Extendibility: $\tau_{2,1}$ can extend to S^3 as a semi-antipodal map under the embedding $\Sigma_2 \to S^3$ indicated in Figure 7. Choose an embedding of $\mathbb{R}P^2$ in $\mathbb{R}P^3$, and using a local connected sum with a torus T, we get an embedding of $RP^2 \# T$ into $\mathbb{R}P^3$. The double cover of $(\mathbb{R}P^3, RP^2 \# T)$ is (S^3, Σ_2). This shows that $\tau_{2,1}$ can also extend orientation-preservingly. So we have $\tau_{2,1}\{+, -\}$.

$\tau_{2,2}$ can obviously extend orientation-reversingly (the left side of Figure 8). It can also extend orientation-preservingly, as indicated in right side of Figure 8. So we have $\tau_{2,2}\{+, -\}$.

The fixed-point set of any orientation-reversing $\mathbb{Z}_2$ action on S^3 is either the set containing two points or a 2-sphere by Theorem 2.1(2), and the 2-sphere is separating, which can not intersect Σ_2 with a union of non-separating circles. Since the fixed-point sets of both $\tau_{2,3}$ and $\tau_{2,4}$ are unions of non-separation circles, neither $\tau_{2,3}$ nor $\tau_{2,4}$ can extend orientation-reversingly.

The fixed-point set of any orientation-preserving $\mathbb{Z}_2$ action on S^3 is either the empty set or a circle by Theorem 2.1(4), which can not intersect Σ_2 with more than one circle. Since the fixed-point sets of both $\tau_{2,4}$ and $\tau_{2,5}$ contain more than one circles, neither $\tau_{2,4}$ nor $\tau_{2,5}$ can extend orientation-preservingly.

Note that $\tau_{2,3}$ can extend orientation-preservingly as in Figure 9. and that $\tau_{2,5}$ can extend orientation-reversingly, as shown in Figure 11.

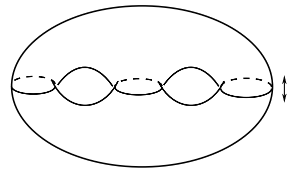


Figure 11

So we have $\tau_{2,3}\{+\}$, $\tau_{2,4}\{\varnothing\}$, and $\tau_{2,5}\{-\}$

(3_+) classification: Here $X = \Sigma_2/\mathbb{Z}_3$ is a closed orientable 2-orbifold with $\chi(X) = \frac{\chi(\Sigma_2)}{3} = -\frac{2}{3}$, and every branched point of X must has index 3. So X is either a sphere with four index-3 branched points, denoted by $X_1 = S^2(3,3,3,3)$, or a torus with one index-3 branched point, denoted by $X_2 = T(3)$.

As above, we have the exact sequence $1 \to \pi_1(\Sigma_2) \to \pi_1(X_i) \to \mathbb{Z}_3 \to 1$, and for each branched point x of X, $St(x)$ must be mapped isomorphically onto $\mathbb{Z}_3$.

For X_1, $\pi_1(X_1) = \langle a,b,c,d \mid abcd = 1,\ a^3 = b^3 = c^3 = d^3 = 1\rangle$, up to some permutation of the bases, the only possible surjections $\pi_1(X_1) \to \mathbb{Z}_3 = \langle t \mid t^3 = 1\rangle$ is $(a,b,c,d) \mapsto (t,t,t^2,t^2)$, and hence this $\mathbb{Z}_3$ action is unique up to conjugacy, denoted by ρ_3.

For X_2, $\pi_1(X_2) = \langle a,b,u \mid aba^{-1}b^{-1} = u,\ u^3 = 1\rangle$, from its abelianzation, we know that there is no finite-injective surjection to $\mathbb{Z}_3$, so there is no corresponding $\mathbb{Z}_3$ action.

Extendibility: In the embedding $\Sigma_2 \in S^3$ of Example 2.5, one can check directly that $\Sigma_2/\langle\tau_{12}^4\rangle = S^2(3,3,3,3)$, so ρ_3 can be the restriction of τ_{12}^4, where $\tau_{12} = \tau(2)$ in Example 2.5. So ρ_3 has the extension τ_{12}^4 over S^3, which is of type $(+,+)$, and we have $\rho_3\{+\}$.

(4_+) classification: Here $X = \Sigma_2/\mathbb{Z}_4$ is a closed orientable 2-orbifold with $\chi(X) = -\frac{1}{2}$. Every branched point of X has index either 2 or 4. So X is either $X_1 = S^2(2,2,2,2,2)$, or $X_2 = S^2(2,2,4,4)$, or $X_3 = T(2)$. As above we have an exact sequence $1 \to \pi_1(\Sigma_2) \to \pi_1(X_i) \to \mathbb{Z}_4 \to 1$, and for each branched point x of X, $St(x)$ must inject into $\mathbb{Z}_4$.

For X_1, $\pi_1(X_1) = \langle a,b,c,d,e \mid abcde = 1,\ a^2 = b^2 = c^2 = d^2 = e^2 = 1\rangle$, each generator corresponds to some branched point, and must be mapped to t^2 in $\mathbb{Z}_4 = \langle t \mid t^4 = 1\rangle$, which is impossible, so there is no corresponding $\mathbb{Z}_4$ action.

For X_3, $\pi_1(X_3) = \langle a,b,u \mid aba^{-1}b^{-1} = u, u^2 = 1\rangle$, u must be mapped to t^2 in $\mathbb{Z}_4 = \langle t \mid t^4 = 1\rangle$, which is impossible, so there is no corresponding $\mathbb{Z}_4$ action.

For X_2, $\pi_1(X_2) = \langle a,b,x,y \mid abxy = 1,\ a^2 = b^2 = x^4 = y^4 = 1\rangle$, up to some permutation of the bases, the only possible representation from $\pi_1(X_2)$ to $\mathbb{Z}_4 = \langle t \mid t^4 = 1\rangle$ is $(a,b,x,y) \mapsto (t^2,t^2,t,t^3)$, and hence this $\mathbb{Z}_4$ action is unique up to equivalence, denoted by ρ_4.

Extendibility: From Figure 12, ρ_4 can extend orientation-reversingly as a $\frac{\pi}{4}$-rotation together with a reflection. Note that ρ_4 has a fixed point. Suppose that ρ_4 can extend orientation-preservingly, and then by Lemma 2.1(4), the singular set of S^3/ρ_4 must be a circle of index 4. Therefore the index of singular points of X_2 must also be 4, which contradicts that $X_2 = S^2(2,2,4,4)$. So we have $\rho_4\{-\}$.

(4_-) classification: Consider the orbifolds $X = \Sigma_2/\langle h^2\rangle$. From the discussion in (2_+), either $X_1 = S^2(2,2,2,2,2,2)$, or $X_2 = T^2(2,2)$. By Lemma 2.1 (1), the orientation-reversing involution $\overline{h}_i$ on X_i has no regular fixed point, so the orbifold $X_i/\mathbb{Z}_2 = \Sigma_2/\mathbb{Z}_4$ is either a projective plane with three index-2 branched points, denoted by $Y_1 = \mathbb{R}P^2(2,2,2)$, or a Klein bottle K with one index-2 branched point, denoted by $Y_2 = K(2)$.

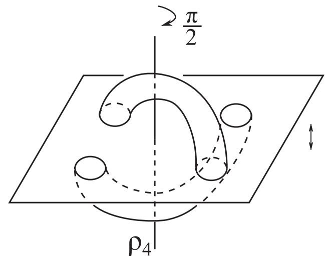


Figure 12

For Y_1, $\pi_1(Y_1) = \langle x, a, b, c \mid abc = x^2,\ a^2 = b^2 = c^2 = 1\rangle$, up to some permutation of the bases, the possible representations from $\pi_1(Y_1)$ to $\mathbb{Z}_4 = \langle t \mid t^4 = 1\rangle$ are

$$(x, a, b, c) \mapsto \begin{cases} (t, t^2, t^2, t^2), \\ (t^3, t^2, t^2, t^2), \end{cases}$$

and an automorphism of $\mathbb{Z}_4$: $t \mapsto t^3$ changes the two representations. Hence this $\mathbb{Z}_4$ action is unique up to equivalent, and we denote it by $\tau_{4,1}$.

For Y_2, $\pi_1(Y_2) = \langle x, a, b \mid aba^{-1}b = x,\ x^2 = 1\rangle$, the possible representations from $\pi_1(Y_2)$ to $\mathbb{Z}_4 = \langle t \mid t^4 = 1\rangle$ are

$$(x, a, b) \mapsto \begin{cases} (t^2, *, t), \\ (t^2, *, t^3), \end{cases}$$

where $*$ means that a may be mapped to any element in $\mathbb{Z}_4$. Consider the automorphism of $\pi_1(Y_2) : (x, a, b) \mapsto (x, ab, b)$ and some automorphism of $\mathbb{Z}_4$, and all these representations are equivalent. Hence this $\mathbb{Z}_4$ action is unique up to equivalent, and we denote it by $\tau_{4,2}$, whose action on Σ_2 is indicated in Figure 13.

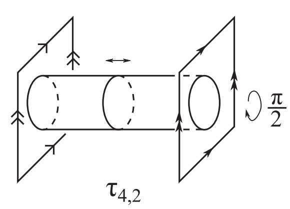


Figure 13

Extendibility: In the embedding $\Sigma_2 \in S^3$ of Example 2.1, one can check directly that $\Sigma_2/\langle \rho\tau_{12}^3\rangle = RP^2(2,2,2)$, so $\tau_{4,1}$ can be the restriction of τ_{12}^3. Therefore it has the extension τ_{12}^3 over S^3, which is of type $(-,+)$. It can also extend orientation-reversingly (see Figure 14). So we have $\tau_{4,1}\{+,-\}$.

$\tau_{4,2}$ can not extend orientation-preservingly, otherwise, there will be an embedding of Klein bottle K into $|S^3/\mathbb{Z}_4|$. Here $S^3/\mathbb{Z}_4$ has branched points, and it can not be a $\mathbb{Z}_4$-Len space, so $|S^3/\mathbb{Z}_4|$ must be S^3 or $\mathbb{R}P^3$. But K can not embed into either S^3 or $\mathbb{R}P^3$ by Lemma 2.3.

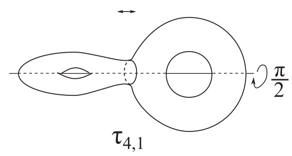


Figure 14

$\tau_{4,2}$ can not extend orientation-reversingly. By applying Dehn's lemma, the orbifold X_2 must bound a 3-orbifold Θ in $S^3/\mathbb{Z}_2$, with $|\Theta|$ a solid torus. There is a proper branched arc L of index 2 in Θ. It is easy to see $\chi(|\Theta \setminus N(L)|) = -1$, so $\overline{h}_2|$ on $|\Theta \setminus N(L)|$ must have fixed points. This means that $\overline{h}_2$ on Θ must have regular fixed points, which contradicts Lemma 2.1 (1). So we have $\tau_{4,2}\{\varnothing\}$.

(5_+) classfication: Now $X = \Sigma_2/\mathbb{Z}_5$ is a closed orientable 2-orbifold, $\chi(X) = \frac{\chi(\Sigma_2)}{5} = -\frac{2}{5}$, and every branched point of X must has index 5. So $X = S^2(5,5,5)$. As above, we have an exact sequence $1 \to \pi_1(\Sigma_2) \to \pi_1(X) \to \mathbb{Z}_5 \to 1$, and for each branched point x of X, $St(x)$ must be mapped isomorphically onto $\mathbb{Z}_5$.

Now $\pi_1(X) = \langle a, b, c \mid abc = 1,\ a^5 = b^5 = c^5 = 1\rangle$, and up to some permutation of the bases, the possible surjections from $\pi_1(x)$ to $\mathbb{Z}_5 = \langle t \mid t^5 = 1\rangle$ are

$$(a,b,c) \mapsto \begin{cases} (t,t,t^3), \\ (t,t^2,t^2), \\ (t^3,t^3,t^4), \\ (t^2,t^4,t^4). \end{cases}$$

Consider the automorphisms of $\mathbb{Z}_5 : t \mapsto t^2$ and $t \mapsto t^4$, and all these representations are conjugate. Hence this $\mathbb{Z}_5$ action is unique up to conjugacy, and we denote it by ρ_5, whose action on Σ_2 is indicated in Figure 21.

Extendibility: Suppose that ρ_5 is extendable, and then the extension on (S^3, Σ_2) must be of type $(+,+)$, which contradicts Lemma 2.2, since the orbifold $\Sigma_2/\langle\rho_5\rangle = S^2(5,5,5)$ contains three singular points. So we have $\rho_5\{\varnothing\}$.

(6_+) classification: The orbifolds that correspond to the index-2 subgroup must be $X = (S^2; 3,3,3,3)$, as we see in (3_+). Then orientation-preserving $\mathbb{Z}_2$ actions on X give the orbifold $X/\mathbb{Z}_2 = \Sigma_2/\mathbb{Z}_6$, which is either $Y_1 = S^2(2,2,3,3)$ or $Y_2 = S^2(3,6,6)$.

For Y_1, $\pi_1(Y_1) = \langle a, b, x, y \mid abxy = 1,\ a^2 = b^2 = x^3 = y^3 = 1\rangle$, up to some permutation of the bases, the only possible representation from $\pi_1(Y_1)$ to $\mathbb{Z}_6 = \langle t \mid t^6 = 1\rangle$ is $(a,b,x,y) \mapsto (t^3, t^3, t^2, t^4)$, so this $\mathbb{Z}_6$ action is unique up to equivalent, and we denote it by $\rho_{6,1}$, whose action on Σ_2 is indicated in Figure 15.

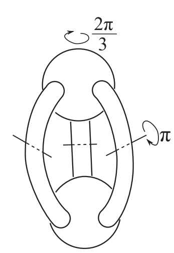


Figure 15

For Y_2, $\pi_1(Y_2) = \langle a, b, x \mid abx = 1,\ a^6 = b^6 = x^3 = 1\rangle$, the possible representations from $\pi_1(Y_2)$ to $\mathbb{Z}_6 = \langle t \mid t^6 = 1\rangle$ are

$$(a, b, x) \mapsto \begin{cases} (t, t, t^4), \\ (t^5, t^5, t^2). \end{cases}$$

Consider the automorphism of $\mathbb{Z}_6$: $t \mapsto t^5$, and these representations are equivalent. Hence this $\mathbb{Z}_6$ action is unique up to conjugacy, and we denote it by $\rho_{6,2}$.

Extendibility: In the embedding $\Sigma_2 \subset S^3$ in Example 2.1, one can check directly that $\Sigma_2/\langle\tau_{12}^2\rangle = S^2(2, 2, 3, 3)$, so $\rho_{6,1}$ can be the restriction of τ_{12}^2, and therefore has the extension τ_{12}^2 over S^3, which is of type $(+, +)$.

$\rho_{6,1}$ can not extend orientation-reversingly: Otherwise, the extension must be of type $(+, -)$, which must interchange two components of $S^3 \setminus \Sigma_2$. Denote by Θ_1 and Θ_2 the two 3-orbifolds bounded by $X = S^2(3, 3, 3, 3)$, and $\{A, B, C, D\}$ the four branched points on X. By applying the Smith theory, we may suppose that two branched arcs in Θ_1 are AB and CD, and that two branched arcs in Θ_2 are BC and DA (see Figure 16). Note that the induced involution $\overline{\rho}_{6,1}$ on X is a π-rotation about two ordinary points, and interchanges Θ_1 and Θ_2. So $\overline{\rho}_{6,1}(A) \neq A$. If $\overline{\rho}_{6,1}$ interchanges A and B, $\overline{\rho}_{6,1}$ will keep the singular arc AB invariant; if $\overline{\rho}_{6,1}$ interchanges the pair (A, B) and the pair (C, D), then $\overline{\rho}_{6,1}$ interchanges the singular arcs AB and CD. In either case we would have $\overline{\rho}_{6,1}(\Theta_1) = \Theta_1$, which is a contradiction. So we have $\rho_{6,1}\{+\}$.

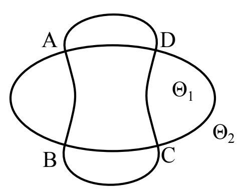


Figure 16

Since $\Sigma_2/\langle\rho_{6,2}\rangle = S^2(3, 6, 6)$ has three singular points, $\rho_{6,2}$ can not extend to S^3 in type $(+, +)$ by Lemma 2.2. In the embedding $\Sigma_2 \in S^3$ in Example 2.1, one can check that $\Sigma_2/\langle\sigma\tau_{12}\rangle = S^2(3, 6, 6)$, so $\rho_{6,2}$ can be the restriction of $\sigma\tau_{12}$, and therefore has the extension $\sigma\tau_{12}$ over S^3 which is of type $(+, -)$. So we have $\rho_{6,2}\{-\}$.

(6_-) classification: Consider the orbifolds $X = \Sigma_2/\langle h^2\rangle = (S^2; 3, 3, 3, 3)$. The action of $\overline{h}$ on X is either the antipodal map, corresponding to the orbifold $Y_1 = \mathbb{R}P^2(3, 3)$, or a reflection on a circle which contains no branched points, corresponding to the orbifold $Y_2 = \overline{D}^2(3, 3)$, a disk with a reflection boundary and branched points $(3, 3)$.

For Y_1, $\pi_1(Y_1) = \langle a, b, x \mid ab = x^2, a^3 = b^3 = 1\rangle$, the possible representations from $\pi_1(Y_1)$ to $\mathbb{Z}_6 = \langle t \mid t^6 = 1\rangle$ are

$$(a, b, x) \mapsto \begin{cases} (t^2, t^2, t^2), \\ (t^4, t^4, t^4), \\ (t^2, t^4, t^3). \end{cases}$$

But the first two respresentations are not surjective, so only the third one is possible. Hence this $\mathbb{Z}_6$ action is unique up to equivalent, and we denote it by $\tau_{6,1}$. An illustration of the action $\tau_{6,1}$ on Σ_2 is based on Figure 17: Denote the union of three tubes by its top and bottom boundary components by A, ∂A_+ and ∂A_-, and two 3-punctured 2-spheres by S_+ and S_-. Then

$$\Sigma_2 = S_+ \cup_{\partial S_+ = \partial A_+} A \cup_{\partial S_- = \partial A_-} \cup S_-,$$

where the identification $\partial S_+ = \partial A_+$ is given in the most obviously way, and the identifications $\partial S_- = \partial A_-$ and $\partial S_+ = \partial A_+$ differ by π. Then $\tau_{6,1}$ restricted on each tube is an antipodal map, and $\tau_{6,1}$ restricted to $S_+ \cup S_-$ is a reflection about the plane between them.

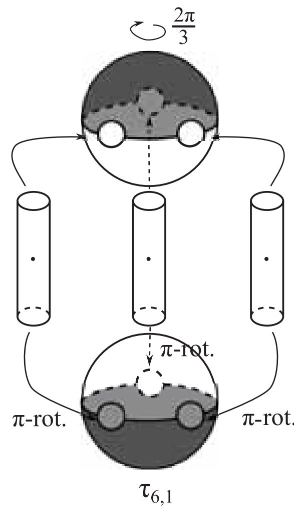


Figure 17

For Y_2, $\pi_1(Y_2) = \langle a, b, x, r \mid ab = x,\ r^2 = 1,\ xr = rx,\ a^3 = b^3 = 1\rangle$, the possible representations from $\pi_1(Y_1)$ to $\mathbb{Z}_6 = \langle t \mid t^6 = 1\rangle$ are

$$(a, b, x, r) \mapsto \begin{cases} (t^2, t^4, 1, t^3), \\ (t^2, t^2, t^4, t^3), \\ (t^4, t^4, t^2, t^3). \end{cases}$$

Denote the first one by $\tau_{6,2}$. The second and third ones are equivalent by an automorphism of $\mathbb{Z}_6$: $t \mapsto t^5$, and we denote it by $\tau_{6,3}$, whose action on Σ_2 is indicated in Figure 18: Joining two disks with 3 bands, each with a half twist, then we get a surface F, which can be viewed as the Seifert surface of the trefoil knot. Its neighborhood in S^3 is $V_2 \cong F \times [-1, 1]$. Then the action of $\tau_{6,3}$ on $\Sigma_2 - \partial V_2$ is the composition of a $\frac{2\pi}{3}$ rotation and a reflection about $F \times \{0\}$.

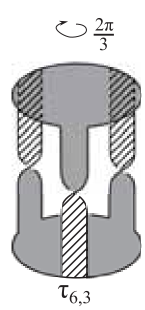


Figure 18

Extendibility: Just applying a similar proof for the non-existence of $\rho_{6,1}\{-\}$, we can show that $\tau_{6,1}$, $\tau_{6,2}$ and $\tau_{6,3}$ can not extend orientation-preservingly.

$\tau_{6,1}$ can not extend orientation-reversingly, otherwise $\tau_{6,1}$ acts on (S^3, Σ_2) in the type $(-,-)$. Since $Y_1 = \mathbb{R}P^2(3,3)$, the $\mathbb{Z}_2$-action $\tau_{6,1}^3$ has no fixed point on Σ_2, so $\mathrm{fix}(\tau_{6,1}^3)$ on S^3 contains two points $\{x, y\}$ by Theorem 2.1. Since $\tau_{6,1}^2(x)$ must be a fixed point of $\tau_{6,1}^3$ and $\tau_{6,1}^2$ is of order 3, we must have $\tau_{6,1}^2(x) = x$, and then $\tau_{6,1}(x) = x$. Hence the branched set $S^3/\langle\tau_{6,1}^2\rangle$ is a circle C of index 3 and $|S^3/\langle\tau_{6,1}^2\rangle| = S^3$. Denote by Θ a 3-orbifold bounded by $X = S^2(3,3,3,3)$ containing $\overline{x}$, the image of x. Then $|\Theta| = D^3$ and $\Theta \cap C$ is in two branched arcs of index 3. But the orientation revering involution $\overline{\tau}_{6,1}$ on (Θ, X) is the antipodal map, and hence $\overline{x}$ is the only fixed point of $\overline{\tau}_{6,1}$ on Θ. Clearly $\overline{x} \in \Theta \cap \overline{C}$, so $\overline{\tau}_{6,1}$ keeps each arc of $\Theta \cap \overline{C}$ invariant. Therefore $\overline{\tau}_{6,1}$ on Θ has at least two fixed points, which is a contradiction. So we have $\tau_{6,1}\{\varnothing\}$.

In the embedding $\Sigma_2 \in S^3$ in Example 2.1, one can check that $\Sigma_2/\langle\sigma\tau_{12}^4\rangle = \overline{D}^2(3,3)$, so $\tau_{6,2}$ can be the restriction of $\sigma\tau_{12}^4$. Therefore it has the extension $\sigma\tau_{12}^4$ over S^3 which is of type $(-,-)$. So we have $\tau_{6,2}\{-\}$.

$\tau_{6,3}$ can not extend orientation-reversingly. Otherwise $\tau_{6,3}$ acts on (S^3, Σ_2) in the type $(-,-)$. Now $\mathrm{fix}(\tau_{6,3}^3) \cap \Sigma_2$ is a separating circle C. $\mathrm{fix}(\tau_{6,3}^3) = S^2 \subset S^3$. Let $\Sigma_2 \backslash C = \Sigma_+ \cup \Sigma_-$ and $S^2 \backslash C = D_+ \cup D_-$. So the $\mathbb{Z}_3$-action $\tau_{6,3}^2$ on $\Sigma_+ \cup D_+ \cong T$ is an extendable action. But the orbifold $T/\mathbb{Z}_3 = (S^2; 3,3,3)$, which can not embed in $S^3/\langle\tau_{6,3}^2\rangle$ by Lemma 2.2. So we have $\tau_{6,3}\{\varnothing\}$.

(7_+) classification: Now $X = \Sigma_2/\mathbb{Z}_7$ is a closed orientable 2-orbifold with $\chi(X) = \frac{\chi(\Sigma_2)}{7} = -\frac{2}{7}$, and every branched point of X must be of index 7. There is no such orbifold.

(8_+) classification: Consider the orbifolds $X = \Sigma_2/\langle h^2\rangle = (S^2; 2,2,4,4)$. Let $Y = \Sigma_2/\mathbb{Z}_8 = X/\mathbb{Z}_2$, and then Y is either $Y_1 = (S^2; 2,2,2,4)$, or $Y_2 = (S^2; 4,4,4)$, or $Y_3 = (S^2; 2,8,8)$.

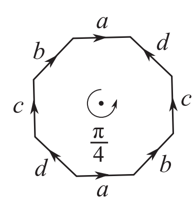
For Y_1, $\pi_1(Y_1) = \langle a, b, c, x \mid abcx = 1,\ a^2 = b^2 = c^2 = x^4 = 1\rangle$, there exists no surjection $\pi_1(Y_1) \to \mathbb{Z}_8$, so there is no corresponding $\mathbb{Z}_8$ action.

For Y_2, $\pi_1(Y_2) = \langle a, b, c \mid abc = 1,\ a^4 = b^4 = c^4 = 1\rangle$, there exists no surjection $\pi_1(Y_2) \to \mathbb{Z}_8$, so there is no corresponding $\mathbb{Z}_8$ action.

For Y_3, $\pi_1(Y_3) = \langle a, b, x \mid abx = 1,\ a^8 = b^8 = x^2 = 1\rangle$, the possible surjections from $\pi_1(Y_3)$ to $\mathbb{Z}_8 = \langle t \mid t^8 = 1\rangle$ are

$$(a,b,x) \mapsto \begin{cases} (t, t^3, t^4), \\ (t^5, t^7, t^4). \end{cases}$$

Consider the automorphism of $\mathbb{Z}_8$: $t \mapsto t^5$, and these representations are equivalent. Hence this $\mathbb{Z}_8$ action is unique up to equivalent, and we denote it by ρ_8, whose action on Σ_2 is indicated in Figure 19.



ρ_8

Figure 19

Extendibility: Note $\rho_8^2 = \rho_4$. If either $\rho_8(+)$ or $\rho_8(-)$ exists, we have $\rho_4(+)$, which contradicts $\rho_4\{\varnothing\}$. So we have $\rho_8\{\varnothing\}$.

(8_-) classification: Consider the orbifolds $X = \Sigma_2/\langle h^2\rangle = (S^2; 2, 2, 4, 4)$. By Lemma 2.3 (1), $\overline{h}$ on X has no regular fixed points on X, so it must be an antipodal map. Therefore $Y = X/\mathbb{Z}_2$ must be $(\mathbb{R}P^2; 2, 4)$. $\pi_1(Y) = \langle a, b, x \mid ab = x^2,\ a^2 = b^4 = 1\rangle$, and the possible representations from $\pi_1(Y)$ to $\mathbb{Z}_8 = \langle t \mid t^8 = 1\rangle$ are

$$(a, b, x) \mapsto \begin{cases} (t^4, t^2, t^3), \\ (t^4, t^2, t^7), \\ (t^4, t^6, t^5), \\ (t^4, t^6, t). \end{cases}$$

Consider the automorphisms of $\mathbb{Z}_8 : t \mapsto t^5$ and $t \mapsto t^7$, and all these representations are equivalent. Hence this $\mathbb{Z}_8$ action is unique up to equivalent, and we denote it by τ_8, whose action on Σ_2 is indicated in Figure 20.

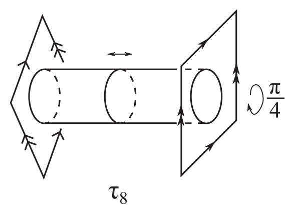


Figure 20

Extendibility: Still we have $\tau_8^2 = \rho_4$. The same reason used in (8_+) shows $\tau_8\{\varnothing\}$.

(9_+) classification: First consider the $\mathbb{Z}_3$ subgroup orbifold X. As we see in 3_+, $X = S^2(3, 3, 3, 3)$. The only possible orbifold $Y = X/\mathbb{Z}_3$ is $S^2(3, 3, 9)$. But its fundamental group can not surjectively map onto $\mathbb{Z}_9$. So there is no such action.

(10_+) classification: As we see in (5_+), the orbifolds $X = \Sigma_2/\langle h^2\rangle = S^2(5, 5, 5)$. Let $Y = \Sigma_2/\mathbb{Z}_{10} = X/\mathbb{Z}_2$, and then $Y = S^2(2, 5, 10)$. $\pi_1(Y) = \langle a, b, c \mid abc = 1,\ a^2 = b^5 = c^{10} = 1\rangle$, and the possible representations from $\pi_1(Y)$ to $\mathbb{Z}_{10} = \langle t \mid t^{10} = 1\rangle$ are

$$(a, b, c) \mapsto \begin{cases} (t^5, t^2, t^3), \\ (t^5, t^4, t), \\ (t^5, t^6, t^9), \\ (t^5, t^8, t^7). \end{cases}$$

Consider the automorphism of $\mathbb{Z}_{10} : t \mapsto t^3$ and $t \mapsto t^7$, and all these representations are equivalent. Hence this $\mathbb{Z}_{10}$ action is unique up to conjugacy, and we denote it by ρ_{10}, whose action on Σ_2 is indicated in Figure 21 (see [12] for detailed descriptions).

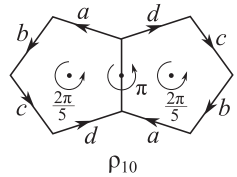


Figure 21

Extendibility: Since $\rho_{10}^2 = \rho_5$, the same reason used in (8_+) shows $\rho_{10}\{\varnothing\}$.

(10_-) classification: Consider the orbifolds $X = \Sigma_2/\langle h^2\rangle = S^2(5,5,5)$. By symmetric consideration, $\overline{h}$ can not be an antipodal map, since there are exactly three singular points of index 5, and $\overline{h}$ also can not be a reflection about a circle, since such a circle must pass a branched point of index 5, which contradicts Lemma 2.1(2).

(11_+) classification: Now $X = \Sigma_2/\mathbb{Z}_{11}$ is a closed orientable 2-orbifold $\chi(X) = \frac{\chi(\Sigma_2)}{11} = -\frac{2}{11}$, and every branched point of X must be of index 11. There is no such orbifold.

(12_+) classification: First consider the $\mathbb{Z}_6$ subgroup orbifold X. X is either $X_1 = S^2(2,2,3,3)$ or $X_2 = S^2(3,6,6)$ as we see in (6_+). So $Y = X/\mathbb{Z}_2$ is either $Y_1 = (S^2;3,4,4)$, or $Y_2 = (S^2;2,6,6)$, or $Y_3 = (S^2;2,2,2,3)$, and none of these fundamental groups has surjection onto $\mathbb{Z}_{12}$, so there is no orientation-preserving actions of Z_{12}.

(12_-) classification: Still the $\mathbb{Z}_6$ subgroup orbifold will be either $X_1 = (S^2;2,2,3,3)$ or $X_2 = (S^2;3,6,6)$. The orientation-reversing $\mathbb{Z}_2$-action on X can not have regular fixed points, so it must be an antipodal map. So only X_1 is possible and $Y = X_1/\mathbb{Z}_2 = (\mathbb{R}P^2;2,3)$. $\pi_1(Y) = \langle a,b,x \mid ab = x^2,\ a^2 = b^3 = 1\rangle$, and the possible representations from $\pi_1(Y)$ to $\mathbb{Z}_{12} = \langle t \mid t^{12} = 1\rangle$ are

$$(a,b,x) \mapsto \begin{cases} (t^6,t^4,t^5), \\ (t^6,t^4,t^{11}), \\ (t^6,t^8,t^7), \\ (t^6,t^8,t). \end{cases}$$

Consider the automorphisms of $\mathbb{Z}_{12} : t \mapsto t^5$ and $t \mapsto t^7$, and all these representations are equivalent. Hence this $\mathbb{Z}_{12}$ action is unique up to equivalent, which is the τ_{12} in Example 2.1.

Extendibility: τ_{12} can not extend orientation reversely. Otherwise τ_{12} acts on each component of $S^3 \setminus \Sigma_2$. Denote by Θ_1 and Θ_2 the two 3-orbifolds bounded by $X = (S^2;2,2,3,3)$, and then each Θ_i has two singular arcs of index 2 and index 3 respectively. We may assume $|\Theta_1| = B^3$ and the induced orientation reversing involution $\overline{\tau}_{12}$ acts on each Θ_i. Hence $\overline{\tau}_{12}$ on $|\Theta_1|$ must be a reflection about an equator disc, which has a regular fixed point, and this contradicts Lemma 2.1(1). From Example 2.5, we have $\tau_{12}\{+\}$.

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