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On s-Reflexive Spaces and Continuous Selections*

Rongxin SHEN[1] Heikki J K JUNNILA[2]

Abstract This paper deals with the s-reflexive spaces introduced by Yang and Zhao. The authors prove that every s-reflexive Hausdorff space is zero-dimensional, and indicate a close relationship between the theory of s-reflexive spaces and that of continuous selections. Several examples relating to s-reflexivity are given.

Keywords s-Reflexive, Continuous selection, Zero-Dimensional, Retractifiable
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1 Introduction

The notion of a reflexive family of sets originated in research on the invariant subspace problem in functional analysis (see [8, 19–21]). In [19], Yang and Zhao introduced the s-reflexive spaces in which reflexive families can be characterized in a simple way. Let X be a topological space. Denote by 2^X and by $\mathcal{F}(X)$, respectively, the family of all subsets and the family of all nonempty closed subsets of X. Let $C(X, X)$ be the set of all continuous mappings $X \to X$.

For all $\mathcal{A} \subset 2^X$ and $\mathcal{B} \subset C(X, X)$, let

$$\begin{aligned}\mathrm{Alg}_X(\mathcal{A}) &= \{f \in C(X, X) : f(A) \subset A \text{ for every } A \in \mathcal{A}\},\\ \mathrm{Lat}_X(\mathcal{B}) &= \{A \in 2^X : A \text{ is closed and } f(A) \subset A \text{ for every } f \in \mathcal{B}\}.\end{aligned}$$

When there is no possibility of confusion, we write $\mathrm{Alg}(\mathcal{A})$ and $\mathrm{Lat}(\mathcal{B})$ instead of $\mathrm{Alg}_X(\mathcal{A})$ and $\mathrm{Lat}_X(\mathcal{B})$.

Note that $\mathcal{A} \subset \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$ for every closed family $\mathcal{A}$. A closed family $\mathcal{A}$ is called reflexive if $\mathcal{A} = \mathrm{Lat}(\mathrm{Alg}(\mathcal{A}))$ (see [19]). As pointed out in [21], a closed family $\mathcal{A}$ is reflexive if and only if $\mathcal{A} = \mathrm{Lat}(\mathcal{B})$ for some $\mathcal{B} \subset C(X, X)$.

Lemma 1.1 (see [19]) *Let $\mathcal{A}$ be a reflexive closed family. Then*
(a) $X, \emptyset \in \mathcal{A}$.
(b) $\mathcal{B} \subset \mathcal{A}$ *implies* $\cap\mathcal{B} \in \mathcal{A}$.
(c) $\mathcal{B} \subset \mathcal{A}$ *implies* $\overline{\cup\mathcal{B}} \in \mathcal{A}$.

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[1]Department of Mathematics, Taizhou University, Taizhou 225300, Jiangsu, China.
E-mail: srx20212021@163.com
[2]Department of Mathematics and Statistics, University of Helsinki, FI-00014, Finland.
E-mail: heikki.junnila@helsinki.fi
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Definition 1.1 (see [19–20]) *A family $\mathcal{A}$ of closed subsets of a topological space X is a closed set lattice (briefly, a cs-lattice) if it satisfies the conditions (a)–(c) in Lemma* 1.1*. The space X is s-reflexive if every cs-lattice in X is reflexive.*

In [19], Yang and Zhao showed that every s-reflexive Hausdorff space is hereditarily disconnected, and they proved that all strongly zero-dimensional complete metric spaces and all countable metric spaces are s-reflexive.

In Section 2 below, we introduce the concept of a csl-carrier on a topological space, and we characterize s-reflexive spaces in terms of properties of csl-carriers. Some basic properties of s-reflexive spaces are obtained in this section. In particular, we prove that every s-reflexive Hausdorff space is zero-dimensional; this strengthens the result of Yang and Zhao mentioned above. In [20, Proposition 1], Yang and Zhao characterized reflexivity of cs-lattices by the existence of certain continuous selections. In Section 3, we study connections between s-reflexivity and the existence of continuous selections. We call a space X "self-selective" if every lower semi-continuous carrier $\Phi : X \to \mathcal{F}(X)$ has a continuous selection. We show that $X \oplus X$ is s-reflexive whenever X is self-selective and T_1. For an ultraparacompact space X, the converse obtains: If $X \oplus X$ is s-reflexive, then X is self-selective. In Section 4, we consider s-reflexivity of metrizable spaces. We show that some well-known selection theorems by E. Michael can be used to derive sufficient conditions for s-reflexivity of (locally) metrizable spaces. We raise the question whether every metrizable s-reflexive space is either completely metrizable or σ-discrete. As a partial answer to this question, we show that an s-reflexive absolutely Borel separable metrizable space is either completely metrizable or countable. In Section 5, we give examples on s-reflexive spaces, for instance, we describe a normal s-reflexive space which is not strongly zero-dimensional. We show that a dyadic space is s-reflexive if and only if the space is zero-dimensional and metrizable. We also indicate some open problems; some of them are essentially problems on continuous selections.

We denote the set of all rational numbers by $\mathbb{Q}$ and the set of all positive integers by $\mathbb{N}$.

2 On s-Reflexive Spaces

A carrier between spaces X and Y is a mapping $\Phi : X \to 2^Y$ such that $\Phi(x) \neq \emptyset$ for every $x \in X$. For a carrier $\Phi : X \to 2^Y$, set $\Phi[A] = \cup\Phi(A)$ for every $A \subset X$. A carrier $\Phi : X \to 2^Y$ is closed-valued if $\Phi(x)$ is closed in Y for every $x \in X$, and Φ is lower semi-continuous (briefly, lsc) if the set $\{x \in X : R\{x\} \cap G \neq \emptyset\}$ is open for every open $G \subset Y$. It is well known that a carrier $\Phi : X \to 2^Y$ is lsc if and only if $\Phi\big[\,\overline{A}\,\big] \subset \overline{\Phi[A]}$ for every $A \subset X$. A (continuous) selection of a carrier $\Phi : X \to 2^Y$ is a (continuous) mapping $f : X \to Y$ such that $f(x) \in \Phi(x)$ for each $x \in X$. We denote by $\mathrm{Sel}(\Phi)$ the set of all continuous selections of a carrier Φ.

In this paper, we deal with carriers $X \to 2^Y$ mainly in the situation where $X = Y$. A carrier $\Phi : X \to 2^X$ is called a carrier on X. For many purposes, it would be more convenient to represent carriers on X as (binary) relations on X, but we shall not do this because in the theory of continuous selections, carriers are almost always considered as set-valued mappings. Instead, we adopt some terminologies from the theory of relations for carriers on a space. We say that a carrier Φ on a space X is reflexive if $x \in \Phi(x)$ for every $x \in X$, and Φ is transitive if $y \in \Phi(x)$ implies $\Phi(y) \subset \Phi(x)$ for all $x, y \in X$.

Let $\mathcal{L}$ be a family of subsets of a space X. We define a carrier $\Delta\mathcal{L}$ on X by the formula

$\Delta\mathcal{L}(x) = \cap(\mathcal{L})_x$, where $(\mathcal{L})_x = \{A \in \mathcal{L} : x \in A\}$. Note that a carrier Φ on X has the form $\Delta\mathcal{L}$ for some $\mathcal{L} \subset 2^X$ if and only if Φ is reflexive and transitive (we agree that $\cap\mathcal{N} = X$ when $\mathcal{N} = \emptyset$).

Apart from notation, the following is essentially (see [20, Lemma 7]).

Lemma 2.1 (see [20]) *For every $\mathcal{L} \subset 2^X$, we have* $\mathrm{Sel}(\Delta\mathcal{L}) = \mathrm{Alg}(\mathcal{L})$.

We say that a carrier Φ on a space X is a csl-carrier provided that there exists a cs-lattice $\mathcal{A}$ in X such that $\Phi = \Delta\mathcal{A}$.

Lemma 2.2 *A carrier Φ on a space X is a csl-carrier if and only if Φ is reflexive, transitive, closed-valued and lsc.*

Proof Necessity. Suppose that Φ is a csl-carrier on X. Then Φ is obviously reflexive, transitive and closed-valued. By [19, Lemma 4], Φ is lsc.

Sufficiency. Suppose that a carrier Φ on X is reflexive, transitive, closed-valued and lsc. Let $\mathcal{A} = \{A \in 2^X : A \text{ is closed and } \Phi[A] \subset A\}$. Clearly $\emptyset, X \in \mathcal{A}$. Let $\mathcal{B} \subset \mathcal{A}$. Then we have $\Phi[\cap\mathcal{B}] \subset \cap\{\Phi[B] : B \in \mathcal{B}\} \subset \cap\mathcal{B}$, and thus $\cap\mathcal{B} \in \mathcal{A}$. Moreover, $\overline{\cup\mathcal{B}} \in \mathcal{A}$, because $\Phi\left[\overline{\cup\mathcal{B}}\right] \subset \overline{\Phi[\cup\mathcal{B}]} = \overline{\cup\{\Phi[B] : B \in \mathcal{B}\}} \subset \overline{\cup\mathcal{B}}$. We have shown that $\mathcal{A}$ is a cs-lattice. Let $x \in X$. Note that $\Phi(x) \in \mathcal{A}$ because Φ is transitive. Thus $\cap(\mathcal{A})_x \subset \Phi(x)$. Moreover, $\Phi(x) \subset \Phi[A] \subset A$ for every $A \in (\mathcal{A})_x$. Therefore $\cap(\mathcal{A})_x = \Phi(x)$. By the foregoing, $\Phi = \Delta\mathcal{A}$.

Note that if $\mathcal{F}$ is a closure-preserving family of closed subsets of X, then $\Delta\mathcal{F}$ is a csl-carrier.

By the proof of sufficiency for the above lemma, we have the following result.

Lemma 2.3 *Let Φ be a csl-carrier on a space X. Then the family $\mathcal{A} = \{A \subset X : A \text{ is closed and } \Phi[A] \subset A\}$ is a cs-lattice and $\Phi = \Delta\mathcal{A}$.*

By [20, Proposition 1], the following result obtains.

Proposition 2.1 (*see* [20]) *A space X is s-reflexive if and only if for every csl-carrier Φ on X, we have $\overline{\{f(x) : f \in \mathrm{Sel}(\Phi)\}} = \Phi(x)$ for each $x \in X$.*

For T_1-spaces, we can characterize s-reflexivity by a simpler condition.

Proposition 2.2 *A T_1-space X is s-reflexive if and only if for every csl-carrier Φ on X, we have $\{f(x) : f \in \mathrm{Sel}(\Phi)\} = \Phi(x)$ for each $x \in X$.*

Proof By Proposition 2.1, we only need to show the necessity. Assume that X is an s-reflexive T_1-space and Φ is a csl-carrier on X. Let $x \in X$ and $y \in \Phi(x)$. We show that there exists $f \in \mathrm{Sel}(\Phi)$ such that $f(x) = y$. If $y = x$, then $\mathrm{id}_X \in \mathrm{Sel}(\Phi)$ and $\mathrm{id}_X(x) = y$. Assume that $y \neq x$. Define a carrier $\Psi \subset X \times X$ by setting $\Psi(x) = \{x, y\}$, $\Psi(y) = \{y\}$ and $\Psi(z) = \Phi(z)$ for each $z \notin \{x, y\}$. It is easy to see that Ψ is a csl-carrier on X. The set $\{y\} = \Psi(x)\setminus\{x\}$ is open in $\Psi(x)$ and it follows, by Proposition 2.1, that there exists $f \in \mathrm{Sel}(\Psi)$ such that $f(x) = y$. Since $\Psi(z) \subset \Phi(z)$ for every z, we have $f \in \mathrm{Sel}(\Phi)$.

In [14, Proposition 2.2], it is observed that a carrier $\Phi : X \to 2^Y$ is lsc provided that $\{f(x) : f \in \mathrm{Sel}(\Phi)\} = \Phi(x)$ for every $x \in X$ (see also [17, Theorem 0.44]).

For zero-dimensional spaces, we can weaken the condition characterizing s-reflexivity in Proposition 2.2.

Proposition 2.3 *A zero-dimensional space X is s-reflexive provided that for each csl-carrier Φ on X and for all non-isolated points $x, y \in X$ with $y \in \Phi(x)$, there exists $f \in \mathrm{Sel}(\Phi)$ such that $f(x) = y$.*

Proof Assume that the stated condition holds. Let Φ be a csl-carrier on X, and let $x, y \in X$ be such that $y \in \Phi(x)$. We show that there exists $f \in \mathrm{Sel}(\Phi)$ with $f(x) = y$. If x is isolated, we can define f by setting $f(x) = y$ and $f(z) = z$ for $z \neq x$. If y is isolated, then the set $\{z \in X : y \in \Phi(z)\}$ is open. Since X is zero-dimensional, there is a clopen neighborhood U of x such that $U \subset \{z \in X : y \in \Phi(z)\}$. In this case we can define f by setting $f(z) = y$ for $z \in U$ and $f(z) = z$ for $z \notin U$. We have shown that the condition in Proposition 2.2 is satisfied. By Proposition 2.1, the space X is s-reflexive.

Since every space with at most one non-isolated point is zero-dimensional, we have the following consequence of Proposition 2.3.

Corollary 2.1 *A T_1-space with at most one non-isolated point is s-reflexive.*

Remarks 2.1 (1) Example 5.5 below shows that a regular space with only two non-isolated points can fail to be s-reflexive.

(2) Example 5.2 shows that the T_1-axiom in Corollary 2.1 can not be omitted.

We shall later give examples to show that s-reflexivity is not a hereditary property. Our next result shows that s-reflexivity is closed-hereditary.

Proposition 2.4 *A closed subspace of an s-reflexive space is s-reflexive.*

Proof Let S be a closed subspace of an s-reflexive space X, and let Φ be a csl-carrier on S. Define a carrier Ψ on X by setting $\Psi(x) = \Phi(x)$ for $x \in S$ and $\Psi(x) = X$ for $x \in X \backslash S$. Then Ψ is a csl-carrier on X. Let $x \in S$. By Proposition 2.1, we have $\overline{\{f(x) : f \in \mathrm{Sel}(\Psi)\}} = \Psi(x)$. Note that for every $f \in \mathrm{Sel}(\Psi)$, we have $f|S \in \mathrm{Sel}(\Phi)$. As a consequence, we have $\overline{\{g(x) : g \in \mathrm{Sel}(\Phi)\}} = \Phi(x)$. By Proposition 2.1, the space S is s-reflexive.

Proposition 2.5 *Let X be a space such that, for all $a, b \in X$, there exists a clopen s-reflexive $G \subset X$ with $a, b \in G$. Then X is s-reflexive.*

Proof Let Φ be a csl-carrier on X and let $a \in X$. By Proposition 2.1, we only need to show that $\{f(a) : f \in \mathrm{Sel}(\Phi)\}$ is dense in $\Phi(a)$. Let $b \in \Phi(a)$ and let U be a neighborhood of b. There exists a clopen s-reflexive $G \subset X$ such that $a, b \in G$. Define a carrier Ψ on G by setting $\Psi(x) = \Phi(x) \cap G$ for $x \in G$. Then Ψ is a csl-carrier. By Proposition 2.1, there exists $g \in \mathrm{Sel}(\Phi)$ such that $g(a) \in U$. Define a mapping $f : X \to X$ by setting $f(x) = g(x)$ for $x \in G$ and $f(x) = x$ for $x \in X \backslash G$. Then $f \in \mathrm{Sel}(\Phi)$ and $f(a) = g(a) \in U$.

In [19, Example 1], Yang and Zhao showed that a space is s-reflexive provided that the topology of the space is either indiscrete or cofinite. It follows that an s-reflexive space is not necessarily T_0 and an s-reflexive T_1-space is not necessarily Hausdorff. Example 5.1 below shows that an s-reflexive T_0-space may fail to be T_1, Example 5.3 shows that a regular s-reflexive space may fail to be normal, and Example 5.10 shows that a normal s-reflexive space may fail to be paracompact.

Next we shall prove that every s-reflexive Hausdorff space is zero-dimensional. As a consequence, s-reflexive Hausdorff spaces are regular.

Lemma 2.4 *Let F be a nonempty closed subset of an s-reflexive T_1-space X and let $p \in X \backslash F$. Then there exists a retraction f of X such that $p \notin f(X)$ and $F \subset f(X)$. If p has a non-dense neighborhood, then we can choose f so that the set $f(X)$ is clopen.*

Proof Let $a \in F$. Define a carrier Φ on X by setting $\Phi(x) = \{x\}$ for $x \in F$ and $\Phi(x) = \{x, a\}$ for $x \in X \backslash F$. Note that Φ is a csl-carrier and $\Phi(p) = \{p, a\}$. By Proposition 2.2, there exists $f \in \mathrm{Sel}(\Phi)$ such that $f(p) = a$. The mapping f is a retraction, because for each $x \in X$, either $f(x) = x$ or $f(x) = a$. Moreover, $F \subset f(X)$ and $p \notin f(X)$.

Assume that p has a non-dense open neighborhood G. Then $F' = F \cup (X \backslash G)$ is a closed set with non-empty interior and $p \notin F'$. Let $a \in \mathrm{Int}\, F'$. As above, there exists a retraction f on X such that $F' \subset f(X)$, $p \notin f(X)$ and for each $x \in X$, either $f(x) = x$ or $f(x) = a$. We show that $f(X)$ is clopen. Let $U = f^{-1}(\mathrm{Int}\, F')$ and note that U is open and $U = (X \backslash f(X)) \cup (\mathrm{Int}\, F')$. We have that $X \backslash f(X) = U \backslash F'$ and hence $X \backslash f(X)$ is open and $f(X)$ is closed. On the other hand, $f(X) = (\mathrm{Int}\, F') \cup f^{-1}(X \backslash \{a\})$ and hence $f(X)$ is open. As a consequence, $f(X)$ is clopen.

Corollary 2.2 *Let F be a nonempty closed subset of an s-reflexive Hausdorff space X and let $p \in X \backslash F$. Then there exists a retraction f of X such that $p \notin f(X)$, $F \subset f(X)$ and the set $f(X)$ is clopen.*

It follows from the above result that s-reflexive Hausdorff spaces have "many" retracts.

Corollary 2.3 *In an s-reflexive Hausdorff space, every closed subset is an intersection of clopen retracts.*

Corollary 2.4 *Every s-reflexive Hausdorff space is zero-dimensional.*

An infinite space with cofinite topology is s-reflexive and T_1, but not zero-dimensional. Example 5.4 below shows that a normal s-reflexive space is not necessarily strongly zero-dimensional.

3 Strongly s-Reflexive Spaces and Self-selective Spaces

In this section, we consider the relationship between the theory of s-reflexive spaces and that of continuous selections.

We call a space X self-selective if every lsc carrier $X \to \mathcal{F}(X)$ has a continuous selection. In the next section, we shall indicate some classes of self-selective spaces.

A space X is retractifiable (see [3]), if every nonempty closed subset of X is a retract. Every retractifiable space is strongly zero-dimensional and hereditarily collectionwise normal (see [3]). It is easy to see that every self-selective T_1-space X is retractifiable (see [14, Corollary 1.5]). Example 5.10 below gives a simple example of a retractifiable space which is not self-selective.

We call a space X strongly s-reflexive if the topological sum $X \oplus X$ is s-reflexive. Every strongly s-reflexive space is s-reflexive, but the converse does not hold, as we shall see in Section 4. It follows from Proposition 2.5 that if a space X is strongly s-reflexive, then $X \times D$ is s-reflexive for every discrete space D.

The following result indicates a connection between self-selective spaces and strongly s-reflexive spaces.

Proposition 3.1 *Every self-selective T_1-space is strongly s-reflexive.*

Proof Let X be a self-selective T_1-space. The topological sum $X \oplus X$ is homeomorphic with the space $Y = X \times \{0,1\}$, where $\{0,1\}$ is discrete. Set $X_0 = X \times \{0\}$ and $X_1 = X \times \{1\}$. Let Φ be a csl-carrier on Y, and let $a \in Y$. To show that $\{f(a) : f \in \mathrm{Sel}(\Phi)\} = \Phi(a)$, let $b \in \Phi(a)$. We have to show that there exists $f \in \mathrm{Sel}(\Phi)$ such that $f(a) = b$.

Without loss of generality, we can assume that $a \in X_0$. Define $j \in \{0,1\}$ by the condition $b \in X_j$. Since Φ is lsc and X is zero-dimensional, there exists a clopen subset U of X_0 such that $a \in U \subset \{p \in X_0 : \Phi(p) \cap X_j \neq \emptyset\}$. Define a carrier $\Psi : X_0 \to \mathcal{F}(X_j)$ by setting

$$\Psi(p) = \begin{cases} \{b\}, & p = a, \\ \Phi(p) \cap X_j, & p \in U \backslash \{a\}, \\ \{p\}, & p \in X_0 \backslash U. \end{cases}$$

It is easy to check that Ψ is lsc. Since X_0 and X_j are both homeomorphic to the self-selective space X, there exists $g \in \mathrm{Sel}(\Psi)$. Define $f : Y \to Y$ by setting $f(p) = g(p)$ for $p \in U$ and $f(p) = p$ for $p \in Y \backslash U$. Then $f \in \mathrm{Sel}(\Phi)$ and $f(a) = b$. It follows from the foregoing by Proposition 2.2 that Y is s-reflexive.

Example 5.4 below shows that not all strongly s-reflexive spaces are retractifiable, and Example 5.7 below shows that not all retractifiable spaces are s-reflexive. The following diagram summarizes the relationships between the previous properties in the class of Hausdorff spaces.

$$\begin{array}{ccccc} \text{self-selective} & \Rightarrow & \text{strongly s-reflexive} & \Rightarrow & \text{s-reflexive} \\ \Downarrow & & & & \Downarrow \\ \text{retractifable} & \Rightarrow & \text{strongly zero-dimensional} & \Rightarrow & \text{zero-dimensional} \end{array}$$

Next we shall indicate a situation in which strong s-reflexivity is equivalent to self-selectivity.

Recall that a topological space is ultraparacompact if every open cover of the space has a disjoint clopen refinement. A Hausdorff space is ultraparacompact if and only if the space is paracompact and strongly zero-dimensional.

Proposition 3.2 *Let X and Y be T_1-spaces. If X is ultraparacompact and $X \oplus Y$ is s-reflexive, then every lsc carrier $X \to \mathcal{F}(Y)$ has a continuous selection.*

Proof We assume that X and Y have no common points. Let $\Phi : X \to \mathcal{F}(Y)$ be an lsc carrier. Define a carrier Ψ on the space $Z{=}X \oplus Y$ by setting $\Psi(z) = \{z\} \cup \Phi(z)$ for each $z \in X$ and $\Psi(z) = \{z\}$ for each $z \in Y$. To see that Ψ is a csl-carrier, it suffices to show that $\Psi\big[\,\overline{A}\,\big] \subset \overline{\Psi[A]}$ for every $A \subset X \oplus Y$. Let $A \subset X \oplus Y$, and set $A_1 = A \cap X$ and $A_2 = A \cap Y$. Then

$$\begin{aligned} \Psi\big[\,\overline{A}\,\big] &= \Psi\big[\,\overline{A_1} \cup \overline{A_2}\,\big] = \Psi\big[\,\overline{A_1}\,\big] \cup \Psi\big[\,\overline{A_2}\,\big] = \overline{A_1} \cup \Phi\big[\,\overline{A_1}\,\big] \cup \overline{A_2} \\ &\subset \overline{A_1} \cup \overline{\Phi[A_1]} \cup \overline{A_2} = \overline{A_1 \cup \Phi[A_1]} \cup \overline{A_2} = \overline{\Psi[A_1]} \cup \overline{\Psi[A_2]} \\ &= \overline{\Psi[A_1] \cup \Psi[A_2]} = \overline{\Psi[A]}. \end{aligned}$$

Let $x \in X$. Pick $y_x \in \Phi(x)$, and note that $y_x \in \Psi(x)$. By Proposition 2.2, there exists $f_x \in \mathrm{Sel}(\Psi)$ such that $f_x(x) = y_x$. Let $U_x = f_x^{-1}(Y) \cap X$. Note that U_x is an open neighborhood of x in X and $f_x|_{U_x}$ is a continuous mapping $U_x \to Y$ with $f_x(z) \in \Phi(z)$ for each $z \in U_x$. Let $\mathcal{V}$ be a disjoint clopen refinement of the open cover $\{U_x : x \in X\}$ of X. For each $V \in \mathcal{V}$, let $p_V \in X$ such that $V \subset U_{p_V}$. Define $f : X \to Y$ by the condition that $f(x) = f_{p_V}(x)$ when $x \in V \in \mathcal{V}$. It is easy to check that $f \in \mathrm{Sel}(\Phi)$.

Corollary 3.1 *Every ultraparacompact strongly s-reflexive T_1-space is self-selective.*

Problem 3.1 Is every self-selective space paracompact?

V. Gutev obtained a partial solution to the above problem. An argument from Gutev's proof establishes the following result.

Lemma 3.1 *Let X be a self-selective space, D a closed discrete subset of X and $\{U_d : d \in D\}$ an open cover of X such that $d \in U_d$ for every $d \in D$. Then there exists an open partition $\{G_d : d \in D\}$ of X such that $d \in G_d \subset U_d$ for each $d \in D$.*

Proof For every $d \in D$, let $V_d = U_d \backslash (D \backslash \{d\})$. Note that $\{V_d : d \in D\}$ is an open cover of X. Define a carrier $\Phi : X \to \mathcal{F}(X)$ by setting $\Phi(x) = \{d \in D : x \in V_d\}$. The carrier Φ is lsc, because $\{x \in X : \Phi(x) \cap L \neq \emptyset\} = \cup\{V_d : d \in D \cap L\}$ for every $L \subset X$. Since X is self-selective, there exists $f \in \mathrm{Sel}(\Phi)$. Note that f is a continuous mapping from X to the discrete subspace D. It follows that the family $\mathcal{G} = \left\{f^{-1}\{d\} : d \in D\right\}$ is an open partition of X. Moreover, we have $d \in f^{-1}\{d\} \subset V_d \subset U_d$ for every $d \in D$.

Proposition 3.3 (see [6]) *Every self-selective T_1-space is countably paracompact.*

Proof Let $\{U_n : n \in \mathbb{N}\}$ be an open cover of a self-selective T_1-space X. Set $A = \{n \in \mathbb{N} : U_n \not\subset \bigcup\limits_{i<n} U_i\}$, and note that the family $\{U_n : n \in A\}$ covers X. For every $n \in A$, let $d_n \in U_n \backslash \bigcup\limits_{i<n} U_i$. Note that the set $D = \{d_n : n \in A\}$ is closed and discrete in X. By Lemma 3.1, there exists an open partition $\{G_n : n \in A\}$ of X such that $G_n \subset U_n$ for every $n \in A$.

We shall give another application of Lemma 3.1.

A cover $\mathcal{L}$ of a set L is a minimal cover of L if no proper subfamily of $\mathcal{L}$ covers L. A topological space X is irreducible if every open cover of X has a minimal open refinement. For background on irreducible spaces, see [2]. We only mention here that irreducibility is a rather weak covering property, and it is implied by such better known properties as submetacompactness and van Douwen's D-space property.

Proposition 3.4 *Every irreducible self-selective T_1-space is paracompact.*

Proof Let $\mathcal{U}$ be an open cover of an irreducible self-selective T_1-space X. Let $\mathcal{V}$ be a minimal open refinement of $\mathcal{U}$. For every $V \in \mathcal{V}$, the family $\mathcal{V} \backslash \{V\}$ fails to cover X, and hence there exists a point $d_V \in V \backslash \big(\mathcal{V} \backslash \{V\}\big)$. Note that the set $D = \{d_V : V \in \mathcal{V}\}$ is closed and discrete in X. By Lemma 3.1, there exists an open partition $\{G_V : V \in \mathcal{V}\}$ of X such that $G_V \subset V$ for every $V \in \mathcal{V}$.

Remark 3.1 The above result remains valid without the T_1-assumption.

We give one more partial solution to Problem 3.1.

A space X is called monotonically normal (see [10]) if for each pair of disjoint closed subsets (A, B), there is an open set $G(A, B)$ with the properties $A \subset G(A, B) \subset \overline{G(A, B)} \subset (X \backslash B)$ and $G(A, B) \subset G(A', B')$, whenever $A \subset A'$ and $B' \subset B$.

Proposition 3.5 *Every monotonically normal self-selective T_1-space is paracompact.*

Proof Let X be a monotonically normal self-selective T_1-space. By Proposition 3.1, X is s-reflexive. It follows from Proposition 2.4 and Example 5.9 that X contains no closed subspace homeomorphic with a stationary subset of a regular uncountable ordinal. Paracompactness of X now follows from the famous Balogh-Rudin theorem (see [1, Theorem I]) on paracompactness of monotonically normal spaces.

4 On s-Reflexivity of Metrizable Spaces

We can use results from the theory of selections to study s-reflexivity of metrizable spaces. The following two classical selection theorems of Michael are especially useful for this purpose.

Zero-Dimensional Selection Theorem (see [15]) *Let X be a strongly zero-dimensional paracompact space and Y be a completely metrizable space. Then every lsc carrier $X \to \mathcal{F}(Y)$ has a continuous selection.*

Recall that a space X is σ-discrete (F_σ-discrete) if X is the union of countably many (closed and) discrete subsets. In a metrizable space, these two properties are mutually equivalent.

F_σ-Discrete Selection Theorem (see [16]) *Let X be an F_σ-discrete paracompact Hausdorff space and Y be a first-countable space. Then every lsc carrier $X \to \mathcal{F}(Y)$ has a continuous selection.*

Note that it follows from the two selection theorems above that all strongly zero-dimensional completely metrizable spaces and all σ-discrete metrizable spaces are self-selective.

In light of Proposition 2.5, Proposition 3.1 and the fact that every locally zero-dimensional T_3-space is zero-dimensional, the following is a direct consequence of the two selection theorems mentioned above.

Theorem 4.1 *A T_3-space X is strongly s-reflexive provided that one of the following conditions holds:*

(A) *Every point of X has a neighborhood which is strongly zero-dimensional and completely metrizable.*

(B) *Every point of X has a neighborhood which is σ-discrete and metrizable.*

Corollary 4.1 (see [19]) (1) *Every countable metrizable space is s-reflexive.*

(2) *Every strongly zero-dimensional completely metrizable space is s-reflexive.*

We do not know if there are any other metrizable s-reflexive spaces except those provided by Michael's theorems.

Problem 4.1 Is every s-reflexive metrizable space either completely metrizable or σ-discrete?

By Corollary 2.4, zero-dimensionality is a necessary condition for a metrizable space to be s-reflexive. We shall now show that some of the simplest zero-dimensional non-complete and non-σ-discrete metrizable spaces fail to be s-reflexive.

Example 4.1 The spaces $2^\omega \oplus \mathbb{Q}$, $2^\omega \times \mathbb{Q}$ and $\mathbb{Q}^\omega$ are not s-reflexive.

Proof By [7, Theorem 1], there exists an lsc carrier $\Phi : 2^\omega \to \mathcal{F}(\mathbb{Q})$ without a continuous selection. It follows from Theorem 3.2 that $2^\omega \oplus \mathbb{Q}$ is not s-reflexive. Note that both $2^\omega \times \mathbb{Q}$ and $\mathbb{Q}^\omega$ contain a closed copy of $2^\omega \oplus \mathbb{Q}$. Thus neither of these spaces is s-reflexive.

Remarks 4.1 (1) The non-s-reflexive space $2^\omega \oplus \mathbb{Q}$ is σ-compact, zero-dimensional, metrizable and locally s-reflexive. The space $2^\omega \oplus \mathbb{Q}$ can be embedded in the space $2^\omega \oplus 2^\omega$ and this space is s-reflexive by Corollary 4.1. This shows that s-reflexivity is not a hereditary property.

(2) It is claimed in [17, Theorem 5.47] that if X is a strongly zero-dimensional GO-space and Y is a GO-space, then every lsc carrier $\Phi : X \to \mathcal{F}(Y)$ has a continuous selection. However, with $X = Y = 2^\omega \oplus \mathbb{Q}$ we have a counterexample, since $2^\omega \oplus \mathbb{Q}$ is linearly orderable.

We have a partial solution to Problem 4.1.

Proposition 4.1 *Let X be an absolutely Borel separable metrizable space. If X is s-reflexive, then X is either completely metrizable or countable.*

Proof Let X be an absolutely Borel separable metrizable space. Suppose that X is not completely metrizable. It follows from the classical Hurewicz theorem (see [11]) that X contains a closed subspace A homeomorphic to $\mathbb{Q}$. Suppose also that X is uncountable. It follows from a result of Souslin (see [12, Theorem 94]) that X contains a subspace B homeomorphic to the Cantor set 2^ω. Since A is closed, we have that $A \backslash B \neq \emptyset$. It follows that there exists a closed set $A' \subset A \backslash B$ such that A' is homeomorphic to $\mathbb{Q}$. As a consequence, X contains a closed copy of $2^\omega \oplus \mathbb{Q}$. By Proposition 2.4 and Example 4.1, X is not s-reflexive.

Results in [13] show that it is consistent with ZFC that an analytic metrizable space contains a closed copy of $\mathbb{Q}$ provided that the space is not completely metrizable. Similarly as in the proof of Proposition 4.1, we can then obtain the following consistency result.

Proposition 4.2 *It is consistent with ZFC that every s-reflexive analytic metrizable space is either completely metrizable or countable.*

Problems 4.2 (1) *Is every F_σ-discrete, first-countable regular space s-reflexive?*

(2) *Is every zero-dimensional completely metrizable space s-reflexive?*

5 Examples

Example 5.1 Every two-point space is s-reflexive.

Proof Let $X = \{a, b\}$ and let Φ be a csl-carrier on X with $b \in \Phi(a)$. Then the constant mapping $f : X \to \{b\}$ is a continuous selection for Φ with $f(a) = b$. By Proposition 2.1, we see that X is s-reflexive.

Example 5.2 There exists a T_0-space with three points, which is not s-reflexive.

Proof Let $X = \{a, b, c\}$ with the topology $\{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Define a carrier Φ on X by setting $\Phi(a) = X$, $\Phi(b) = \{b\}$ and $\Phi(c) = \{c\}$. Then Φ is a csl-carrier. Note that Φ has no continuous selection f with $f(a) = b$. By Proposition 2.1, X is not s-reflexive.

Example 5.3 There exists a strongly s-reflexive separable regular space which is not normal.

Proof Let $L = \{(x, y) : x, y \in \mathbb{R} \text{ and } y \geq 0\}$ be the Niemytzki plane (see [5, Example 1.2.4]) and let X be the subspace $\{(x, y) \in L : y = 0 \text{ or } x, y \in \mathbb{Q}\}$. The space X is separable and regular. Moreover, X is first countable and locally countable, and it follows from Theorem

4.1 that X is strongly s-reflexive. Similarly as in [5, Example 1.5.10], we see that X is not normal.

Our next example describes a normal s-reflexive space which is not strongly zero-dimensional. We shall modify an example of a normal zero-dimensional, but not strongly zero-dimensional space constructed by Dowker. We shall make the space locally completely metrizable and hence s-reflexive. For this purpose, we need the following result.

Lemma 5.1 *There exists an ascending transfinite sequence $\langle S_\alpha \rangle_{\alpha<\omega_1}$ of zero-dimensional G_δ-subsets of the interval $\mathbb{I} = [0,1]$ such that $\mathbb{I} = \cup\{S_\alpha : \alpha < \omega_1\}$.*

Proof Without "zero-dimensional", this result is due to Hausdorff. The proof below is a slight modification of Hausdorff's (see [9, 18]). Let $A, B \in [\mathbb{N}]^\omega$. Write $A \subset_* B$ if $B \backslash A$ is finite. If $A \subset_* B$ but $B \not\subset_* A$, then write $A \prec B$. In [9], Hausdorff constructed two transfinite sequences $\{A_\alpha : \alpha < \omega_1\}$ and $\{B_\alpha : \alpha < \omega_1\}$ in $[\mathbb{N}]^\omega$ such that

(i) $A_\alpha \prec A_\beta \prec B_\beta \prec B_\alpha$ for all $\alpha < \beta < \omega_1$, and

(ii) there is no $E \in [\mathbb{N}]^\omega$ such that $A_\alpha \subset_* E \subset_* B_\alpha$ for every $\alpha < \omega_1$.

Define $f : [\mathbb{N}]^\omega \to \mathbb{I}$ by the formula

$$f(E) = \sum_{n=1}^{\infty} \chi_E(n) 2^{-n},$$

where χ_E is the characteristic function of E. Similarly as in [18, p. 173], we can show that, for each $\alpha < \omega_1$, the set $Q_\alpha = \{f(E) : E \in [\mathbb{N}]^\omega \text{ and } A_\alpha \subset_* E \subset_* B_\alpha\}$ is an F_σ-set, and hence the set $S_\alpha = \mathbb{I} \backslash Q_\alpha$ is a G_δ-set.

Let $\alpha < \omega_1$. To verify that S_α is zero-dimensional, we need to show that Q_α is dense in $\mathbb{I}$. Let $i_0 \in \mathbb{N}$ such that $A_\alpha \backslash \{n \in \mathbb{N} : n < i_0\} \subset B_\alpha$. Let $G \subset \mathbb{I}$ be open and nonempty. There exist $i, k \in \mathbb{N}$ such that $i \geq i_0$ and $[k2^{-i}, (k+1)2^{-i}] \subset G$. We can write k as $\sum_{j=1}^{i} k_j 2^{i-j}$, where $k_j \in \{0, 1\}$ for each $j \leq i$. Let $E = (A_\alpha \backslash \{n \in \mathbb{N} : n < i\}) \cup \{j \leq i : k_j = 1\}$ and note that $A_\alpha \subset_* E \subset_* B_\alpha$. Now $f(E) \in Q_\alpha$ and $f(E) \in [k2^{-i}, (k+1)2^{-i}] \subset G$. Hence $Q_\alpha \cap G \neq \emptyset$. We have shown that Q_α is dense in $\mathbb{I}$.

Clearly, $S_\alpha \subset S_\beta$ whenever $\alpha < \beta$. By (ii), we have $\cap\{Q_\alpha : \alpha < \omega_1\} = \emptyset$. As a consequence, $\mathbb{I} = \cup\{S_\alpha : \alpha < \omega_1\}$.

Example 5.4 There exists a normal, $\aleph_1$-compact, locally completely metrizable strongly s-reflexive space which is not strongly zero-dimensional.

Proof We modify an example due to Dowker (see [5, Example 6.2.20]). Let $\langle S_\alpha \rangle_{\alpha<\omega_1}$ be the sequence constructed in Lemma 5.1. For each $\alpha < \omega_1$, let $Y_\alpha = \cup\{\{\gamma\} \times S_\gamma : \gamma \leq \alpha\}$. Consider $Y = \cup\{\{\gamma\} \times S_\gamma : \gamma < \omega_1\}$ as a subspace of $(\omega_1 + 1) \times \mathbb{I}$. Similarly as in [5, Example 6.2.20], we see that Y is normal and zero-dimensional, but not strongly zero-dimensional, and that Y_α is clopen in Y for each $\alpha < \omega_1$.

Let $\alpha < \omega_1$. Then $\mathbb{I} \backslash S_\alpha$ is an F_σ-subset of $\mathbb{I}$. It follows that $((\alpha+1) \times \mathbb{I}) \backslash Y_\alpha = \cup\{\{\gamma\} \times (\mathbb{I} \backslash S_\gamma) : \gamma \leq \alpha\}$ is an F_σ-subset of $(\alpha + 1) \times \mathbb{I}$. Since $(\alpha + 1) \times \mathbb{I}$ is compact and metrizable, the space Y_α is separable and completely metrizable. By Theorem 4.1, Y is strongly s-reflexive.

It remains to show that Y is $\aleph_1$-compact. Assume to the contrary that Y contains a closed discrete subspace $D = \{(\gamma_\alpha, r_\alpha) : \alpha < \omega_1\}$, where $(\gamma_\alpha, r_\alpha) \neq (\gamma_\beta, r_\beta)$ for $\alpha \neq \beta$. There exists $r \in \mathbb{I}$ such that every neighborhood of r contains r_α for uncountable many α's. Let $\beta = \min\{\delta < \omega_1 : (\delta, r) \in Y\}$. For every $\delta \geq \beta$, there exist $n_\delta \in \mathbb{N}$ and $\gamma_\delta < \delta$ such that the neighborhood $(\gamma_\delta, \delta] \times (r - \frac{1}{n_\delta}, r + \frac{1}{n_\delta})$ of (δ, r) contains at most one element of D. There exists $m \in \mathbb{N}$ such that the set $E_m = \{\delta \geq \beta : n_\delta = m\}$ is stationary. By the pressing down lemma, there exists $\rho < \omega_1$ such that the set $\{\delta \in E_m : \gamma_\delta = \rho\}$ is uncountable. Note that the set $[\rho, \omega_1) \times (r - \frac{1}{m}, r + \frac{1}{m})$ contains at most one element of D. Since D is closed discrete, the set $D \cap Y_\rho$ is countable. It follows that the set $D \cap (\omega_1 \times (r - \frac{1}{m}, r + \frac{1}{m}))$ is countable. As a consequence, the neighborhood $(r - \frac{1}{m}, r + \frac{1}{m})$ of r contains r_α for only countably many α's — a contradiction.

Remark 5.1 Note that if the continuum hypothesis is assumed, then the space Y can be defined in a simpler way and Lemma 5.1 is not needed.

Problem 5.1 Is every metrizable s-reflexive space strongly zero-dimensional?

Next we give three examples of regular spaces with only one non-isolated point which are not strongly s-reflexive. Since the topological sum $X \oplus X$ is homeomorphic with $X \times \{0, 1\}$, these examples also show that the product of two s-reflexive spaces is not necessarily s-reflexive.

Example 5.5 There exists a countable s-reflexive regular space which is not strongly s-reflexive.

Proof The idea of the following construction and proof comes from [19, Example 2]. Let X be the set $\{x\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{nm} : n, m \in \mathbb{N}\}$ equipped with the topology in which x is the only non-isolated point, and a neighborhood base at x is formed by the sets $\{x\} \cup \{x_n : n \in \mathbb{N}\backslash K\} \cup \{x_{nm} : n \in \mathbb{N}\backslash K \text{ and } m > \varphi(n)\}$, where K is finite and $\varphi : \mathbb{N} \to \mathbb{N}$. It follows from Corollary 2.1 that X is s-reflexive. Note that the sets $X_1 = \{x\} \cup \{x_{nm} : n, m \in \mathbb{N}\}$ and $X_2 = \{x\} \cup \{x_n : n \in \mathbb{N}\}$ are closed in X, and therefore $X_1 \oplus X_2$ is closed in $X \oplus X$. Rewrite X_2 as $Y_2 = \{y\} \cup \{y_n : n \in \mathbb{N}\}$, and let $Z = X_1 \oplus Y_2$. To show that $X \oplus X$ is not s-reflexive, it suffices to verify that Z does not have this property. In fact, Z is homeomorphic with the space X of [19, Example 2], and Yang and Zhou showed that this X is not s-reflexive. For the sake of convenience, we give a brief proof here. Assume to the contrary that Z is s-reflexive. For $z \in Z$, we put

$$\Phi(z) = \begin{cases} \{z\}, & z \in X_1, \\ \{x, y\}, & z = y, \\ \{y_n\} \cup \{x_{nm} : m \in \mathbb{N}\}, & z = y_n. \end{cases}$$

Note that Φ is a csl-carrier on Z. By s-reflexivity, there exists $f \in \mathrm{Sel}(\Phi)$ with $f(y) = x$. Since $y_n \to y$, we have $f(y_n) \to f(y) = x$. This is a contradiction, since $\{f(y_n) : n \in \mathbb{N}\} \subset Z\backslash\{x\}$ and no sequence from $Z\backslash\{x\}$ converges to x.

Denote by $A(\omega_1)$ (by $L(\omega_1)$) the one-point compactification (the one-point Lindelöfication) of the discrete space ω_1, that is, the space $\{\infty\} \cup \omega_1$ with base $\big\{\{\alpha\} : \alpha < \omega_1\big\} \cup \{B : \omega_1 \backslash B$ is finite$\}$ (with base $\big\{\{\alpha\} : \alpha < \omega_1\big\} \cup \{B : \omega_1 \backslash B$ is countable$\}$).

Example 5.6 The spaces $A(\omega_1)$ and $L(\omega_1)$ are retractifiable and s-reflexive, but not strongly s-reflexive.

Proof It is easy to check that every space with at most one non-isolated point is retractifiable. It follows from Corollary 2.1 that both $A(\omega_1)$ and $L(\omega_1)$ are s-reflexive. Note that both $A(\omega_1)$ and $L(\omega_1)$ are ultraparacompact. To show that $A(\omega_1)$ and $L(\omega_1)$ are not strongly s-reflexive, it suffices, by Corollary 3.1, to show that neither of them is self-selective.

First we show that $A(\omega_1)$ is not self-selective. Instead of $A(\omega_1)$, we consider the homeomorphic space $A(\omega_1 \times \omega_1)$, the one-point compactification $\{\infty\} \cup (\omega_1 \times \omega_1)$ of the discrete space $\omega_1 \times \omega_1$. Define $\Phi : A(\omega_1 \times \omega_1) \to \mathcal{F}(A(\omega_1 \times \omega_1))$ by setting

$$\Phi(x) = \begin{cases} \{\infty\}, & x = \infty, \\ \{(\alpha, 0), (0, \beta)\}, & x = (\alpha, \beta) \in \omega_1 \times \omega_1. \end{cases}$$

It is easy to see that Φ is lsc. We show that Φ has no continuous selection. Suppose to the contrary that Φ has a continuous selection f. Note that $f^{-1}\{\infty\} = \{\infty\}$. It follows, since the point ∞ is in the closure of every infinite subset of $A(\omega \times \omega_1)$, that the mapping f is finite-to-one. As a consequence, the set $E = \{\alpha < \omega_1 : f((\alpha, n)) = (0, n) \text{ for some } n < \omega\}$ is countable. Let $\gamma \in \omega_1 \backslash E$. Then $f(\gamma, n) = (\gamma, 0)$ for every $n < \omega$. This is a contradiction.

To show that $L(\omega_1)$ is not self-selective, define a carrier $\Phi : L(\omega_1) \to \mathcal{F}(L(\omega_1))$ by setting

$$\Phi(\alpha) = \begin{cases} \{\alpha\}, & \alpha \in \{0, \infty\}, \\ [0, \alpha)\,, & 0 < \alpha < \omega_1. \end{cases}$$

Then Φ is lsc. To show that $\mathrm{Sel}(\Phi) = \emptyset$, assume on the contrary that there exists $f \in \mathrm{Sel}(\Phi)$. Then $f(\alpha) < \alpha$ for every $0 < \alpha < \omega_1$. By the pressing down lemma, there exists an uncountable set $A \subset \omega_1$ and $\beta < \omega_1$ such that $f(\alpha) = \beta$ for every $\alpha \in A$. Since $f(\infty) = \infty$, we have a contradiction with continuity of f.

Example 5.7 The Arens' space S_2 is retractifiable, but not s-reflexive.

Proof The ground-set of S_2 is $\{y\} \cup \{y_n : n \in \mathbb{N}\} \cup \{y_{nm} : n, m \in \mathbb{N}\}$, and the topology is defined as follows: (i) each y_{nm} is isolated; (ii) a neighborhood base at y_n is formed by the sets $\{y_n\} \cup \{y_{nm} : m \in \mathbb{N}\backslash K\}$, where K is finite; (iii) a neighborhood base at y is formed by the sets $\{y\} \cup \{y_n : n \in \mathbb{N}\backslash K\} \cup \{y_{nm} : n \in \mathbb{N}\backslash K \text{ and } m > \varphi(n)\}$, where K is finite and $\varphi : \mathbb{N} \to \mathbb{N}$.

To show that S_2 is retractifiable, let F be a nonempty closed subset of S_2. We define a mapping $f : S_2 \to F$ as follows. For every $x \in F$, we set $f(x) = x$. If $y \notin F$, then we set $f(y) = q$, where $q \in F$. For every $y_n \notin F$, we set $f(y_n) = f(y)$, and for every $y_{nm} \notin F$, we set $f(y_{nm}) = f(y_n)$. It is easy to check that if a sequence $\langle p_n \rangle_{n\in\mathbb{N}}$ in S_2 converges to p, then the sequence $\langle f(p_n) \rangle_{n\in\mathbb{N}}$ converges to $f(p)$ in F. Since S_2 is a sequential space, the mapping f is continuous, and hence f is a retraction $S_2 \to F$.

To prove that S_2 is not s-reflexive, let $X = \{x\} \cup \{x_m : m \in \mathbb{N}\}$ be the space formed by a convergent sequence together with its limit. Set $Z = X \oplus S_2$. Note that Z is homeomorphic to S_2. It suffices to show that Z is not s-reflexive. Define a carrier Φ on Z by setting

$$\Phi(z) = \begin{cases} \{x, y\}, & z = x, \\ \{x_m\} \cup \{y_{nm} : n \in \mathbb{N}\}, & z = x_m, \\ \{z\}, & z \in S_2. \end{cases}$$

Then Φ is a csl-carrier. Similarly as in Example 5.5, we see that $f(x) \neq y$ for every $f \in \mathrm{Sel}(\Phi)$. It follows from Proposition 2.2 that Z is not s-reflexive.

Examples 5.5 and 5.7 show that not all countable regular spaces are s-reflexive.

Example 5.8 The Michael line is not s-reflexive.

Proof According to [3, Corollary 4.4], the Michael line M is not retractifiable. Hence M is not self-selective. Note that M is ultraparacompact and the topological sum $M \oplus M$ can be embedded in M as a closed subspace. It follows from Proposition 2.4 and Corollary 3.1 that M is not s-reflexive.

Problem 5.2 Is the Sorgenfrey line s-reflexive?

Note that, like the Michael line, the Sorgenfrey line S is ultraparacompact and $S \oplus S$ embeds in S as a closed subspace. Hence Proposition 2.4 and Corollary 3.1 show that if S is s-reflexive, then S is self-selective.

Example 5.9 Let A be a stationary subset of an uncountable regular cardinal κ. The subspace A of the ordinal space κ is retractifiable but not self-selective.

Proof By [3, Theorem III.8], every subspace of an ordinal space is retractifable. To show that A is not self-selective, define a carrier $\Phi : A \to \mathcal{F}(A)$ by the formula $\Phi(\alpha) = \{\beta \in A : \beta > \alpha\}$. It is easy to see that Φ is lsc. However, no selection for Φ is continuous. This is a consequence of the result that if $g : \kappa \to \kappa$ is continuous and $g(\alpha) \geq \alpha$ for every α, then the set of fixed points of g is a cub set (see [12, Exercise 7.9]). Hence A is not self-selective.

The following is a consequence of the preceding example and Theorem 4.1.

Example 5.10 The ordinal space ω_1 is retractifiable and strongly s-reflexive, but not self-selective.

Example 5.11 The ordinal space γ is s-reflexive when $\gamma \leq \omega_1 + \omega_1$.

Proof Let Φ be a csl-carrier on γ, and let α and β be distinct limit ordinals less than γ with $\beta \in \Phi(\alpha)$. We shall prove that there exists $f \in \mathrm{Sel}(\Phi)$ with $f(\alpha) = \beta$. It then follows from Proposition 2.3 that γ is s-reflexive.

Case 1 $\beta = \omega_1$.

If $\alpha < \omega_1$, then we set $A = \{\zeta \in [0, \alpha) : \beta \notin \Phi(\zeta)\}$ and if $\alpha > \omega_1$, then we set $A = \{\zeta \in [\omega_1 + 1, \alpha) : \beta \notin \Phi(\zeta)\}$. The set A is countable and it follows that $\beta \notin \overline{\Phi[A]}$. Since $\beta \in \Phi(\alpha)$, we have $\alpha \notin \overline{A}$. As a consequence, there exists a clopen neighborhood $(\alpha', \alpha]$ of α such that $\beta \in \Phi(\zeta)$ for each $\zeta \in (\alpha', \alpha]$. Define $f : \gamma \to \gamma$ by setting $f(\zeta) = \beta$ for each $\zeta \in (\alpha', \alpha]$ and $f(\zeta) = \zeta$ for each $\zeta \in \gamma \backslash (\alpha', \alpha]$. Then $f \in \mathrm{Sel}(\Phi)$ and $f(\alpha) = \beta$.

Case 2 $\beta \neq \omega_1$.

Let U be a countable clopen neighborhood of β. Since $\Phi(\alpha) \cap U \neq \emptyset$, there exists a clopen neighborhood V of α such that $\Phi(\zeta) \cap U \neq \emptyset$ for each $\zeta \in V$. The subspace V of γ is compact and zero-dimensional, and the subspace U is compact and metrizable. Define a carrier $\Psi : V \to \mathcal{F}(U)$ by setting $\Psi(\alpha) = \{\beta\}$ and $\Psi(\zeta) = \Phi(\zeta) \cap U$ for $\zeta \in V \backslash \{\alpha\}$. The carrier Ψ is lsc, because for every $G \subset U$, we have either $\{\delta \in V : \Psi(\delta) \cap G \neq \emptyset\} = \{\delta \in V : \Phi(\delta) \cap G \neq \emptyset\}$ or $\{\delta \in V : \Psi(\delta) \cap G \neq \emptyset\} = \{\delta \in V : \Phi(\delta) \cap G \neq \emptyset\} \backslash \{\alpha\}$. It follows from the zero-dimensional selection theorem that Ψ has a continuous selection f. Note that $f(\alpha) = \beta$. Define $g : \gamma \to \gamma$ by setting $g(\zeta) = f(\zeta)$ for $\zeta \in V$ and $g(\zeta) = \zeta$ for $\zeta \notin V$. Then $g \in \mathrm{Sel}(\Phi)$ and $g(\alpha) = \beta$.

Problem 5.3 Is every ordinal space s-reflexive? In particular, is $\omega_1 + \omega_1 + 1$ s-reflexive?

Note that $\omega_1 + \omega_1 + 1$ is homeomorphic with $(\omega_1 + 1) \oplus (\omega_1 + 1)$. It follows, by Proposition 3.1 and Corollary 3.1, that $\omega_1 + \omega_1 + 1$ is s-reflexive, if and only if $\omega_1 + 1$ is self-selective.

We close this paper with a result which provides many examples of non-s-reflexive zero-dimensional compact spaces.

Proposition 5.1 *An s-reflexive dyadic space is metrizable.*

Proof Let X be an s-reflexive dyadic space. Suppose that X is not metrizable. Then by [4, Theorem 15 and Corollary 1 to Theorem 14], X contains a copy of $A(\omega_1) \oplus A(\omega_1)$. This, however, contradicts Proposition 2.4 and Example 5.6.

Example 5.12 2^{ω_1} is not s-reflexive.

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