

Hypercontinuous Posets*

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Abstract The concepts of hypercontinuous posets and generalized completely continuous posets are introduced. It is proved that for a poset P the following three conditions are equivalent: (1) P is hypercontinuous; (2) the dual of P is generalized completely continuous; (3) the normal completion of P is a hypercontinuous lattice. In addition, the relational representation and the intrinsic characterization of hypercontinuous posets are obtained.

Keywords Hypercontinuous posets, Generalized completely continuous posets,
Normal completion

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1 Introduction

To generalize Dedekind's pioneer construction of the real line by cuts of rational numbers (see [3]), MacNeille [12] introduced the famous normal completion for arbitrary posets. It is well-known that modularity and distributivity are not preserved under the formation of normal completions (see [2]). However, other notions of distributivity, such as Boolean algebras and Heyting algebras, are closed under normal completions (see [5–7, 11, 13, 18–19]). In [4], Ern  observed that continuity was not completion-invariant (see [5]), that is, the normal completion of a continuous domain is not always a continuous lattice. To obtain the completion-invariant property, Ern  introduced a new concept of precontinuous posets by taking Frink ideals (see [8]) instead of directed lower sets, and proved that a poset is precontinuous if and only if its normal completion is a continuous lattice. Generally speaking, continuity defined by the usual way is not completion-invariant.

As a common generalization of completely distributive lattices (see [14]) and generalized continuous lattices (see [10]) which were called quasicontinuous lattices in [9], Venugopalan introduced the concept of generalized completely distributive lattices which have many properties similar to those of completely distributive lattices (see [15]). In [10], Gierz and Lawson introduced the concept of a hypercontinuous lattice, which is also among the most successful generalizations of continuous lattices, to characterize a continuous lattice with the Hausdorff interval topology. In [17], Yang and Xu proved that a complete lattice is hypercontinuous if and only if its order dual is generalized completely distributive.

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In this paper, we generalize the concepts of hypercontinuous lattices and generalized completely distributive lattices to the setting of posets, and introduce the concepts of hypercontinuous posets and generalized completely continuous posets. It is proved that for a poset P the following three conditions are equivalent: (1) P is hypercontinuous; (2) the dual of P is generalized completely continuous; (3) the normal completion of P is a hypercontinuous lattice. So the hypercontinuity and generalized complete continuity are completion-invariant. Also in this paper, the relational representation and the intrinsic characterization of hypercontinuous posets are obtained.

For a poset P , let $P^{(<\omega)} = \{F \subseteq P : F \text{ is finite}\}$. The order dual of P is written as P^{op} . For all $x \in P$, $A \subseteq P$, let $\uparrow x = \{y \in P : x \leq y\}$ and $\uparrow A = \bigcup_{a \in A} \uparrow a$; $\downarrow x$ and $\downarrow A$ are defined dually. A^\uparrow and A^\downarrow denote the sets of all upper and lower bounds of A , respectively. Let $A^\delta = (A^\uparrow)^\downarrow$ and $\delta(P) = \{A^\delta : A \subseteq P\}$. $(\delta(P), \subseteq)$ is called the normal completion, or the Dedekind-MacNeille completion of P . By a completion-invariant property, we mean a property that holds for a poset P if and only if it holds for the normal completion of P . For all $U \in \delta(P)$ and $\mathcal{F} \subseteq \delta(P)$, let $\uparrow_{\delta(P)} \{U\} = \{V \in \delta(P) : U \subseteq V\}$ and $\uparrow_{\delta(P)} \mathcal{F} = \{V \in \delta(P) : \text{there exists } U \in \mathcal{F} \text{ with } U \subseteq V\}$; $\downarrow_{\delta(P)} \{U\}$ and $\downarrow_{\delta(P)} \mathcal{F}$ are defined dually.

Let P be a poset. The topology generated by the collection of sets $P \setminus \downarrow x$ (as a subbase) is called the upper topology and denoted by $v(P)$; the lower topology $\omega(P)$ on P is defined dually. The topology $\theta(P) = v(P) \vee \omega(P)$ is called the interval topology on P . For $x, y \in P$, define a relation \prec on P by $x \prec y \Leftrightarrow y \in \text{int}_{v(P)} \uparrow x$.

The following lemma is well-known (see [5]).

Lemma 1.1 *Let P be a poset.*

(1) *The maps $(-)^{\uparrow} : (2^P)^{\text{op}} \rightarrow 2^P$, $A \mapsto A^\uparrow$ and $(-)^{\downarrow} : 2^P \rightarrow (2^P)^{\text{op}}$, $A \mapsto A^\downarrow$ are order preserving.*

(2) *$((-)^{\uparrow}, (-)^{\downarrow})$ is a Galois connection between $(2^P)^{\text{op}}$ and 2^P , that is, for all $A, B \subseteq P$, $B^\uparrow \supseteq A \Leftrightarrow B \subseteq A^\downarrow$. Thus both $\delta : 2^P \rightarrow 2^P$, $A \mapsto A^\delta = (A^\uparrow)^\downarrow$ and $\delta^* : 2^P \rightarrow 2^P$, $A \mapsto (A^\downarrow)^\uparrow$ are closure operators.*

(3) *For all $\{C_j : j \in J\} \subseteq 2^P$, $(\bigcup_{j \in J} C_j)^\uparrow = \bigcap_{j \in J} C_j^\uparrow$, $(\bigcup_{j \in J} C_j)^\downarrow = \bigcap_{j \in J} C_j^\downarrow$.*

(4) *Let $L = \delta(P)$. For all $\{A_i^\delta : i \in I\} \subseteq L$, $\bigwedge_L \{A_i^\delta : i \in I\} = \bigcap \{A_i^\delta : i \in I\}$, $\bigvee_L \{A_i^\delta : i \in I\} = (\bigcup \{A_i^\delta : i \in I\})^\delta = (\bigcup_{i \in I} A_i)^\delta$.*

Corollary 1.1 *Let P be a poset. Then the map $e_P : P \rightarrow \delta(P)$, $x \mapsto \downarrow x$ is an order embedding of P in the normal completion $\delta(P)$ and*

(1) *e_P preserves all existing joins and meets;*

(2) *for all $A^\delta \in \delta(P)$, $A^\delta = \bigvee_{a \in A} e_P(a) = \bigvee_{a \in A^\delta} e_P(a)$.*

Definition 1.1 (see [16]) *A binary relation ρ on X is called finitely regular if for all $(x, y) \in \rho$, there exist $u \in X$ and $\{v_1, v_2, \dots, v_k\} \in X^{(<\omega)}$ such that*

(1) *$(u, y) \in \rho$, and $(x, v_i) \in \rho$ for each $i \in \{1, 2, \dots, k\}$, and*

(2) *for all $\{s_1, s_2, \dots, s_k\} \in X^{(<\omega)}$ and $t \in X$, if $(u, t) \in \rho$, $(s_i, v_i) \in \rho$ for each $i \in \{1, 2, \dots, k\}$, then there exists $j \in \{1, 2, \dots, k\}$ such that $(s_j, t) \in \rho$.*

Definition 1.2 (see [9–10]) *A complete lattice L is called hypercontinuous if and only if $x = \bigvee \{y \in L : y \prec x\}$ for all $x \in L$.*

Definition 1.3 (see [15]) For a complete lattice L , define a relation \blacktriangleleft on 2^L by $A \blacktriangleleft B$ if and only if for all $S \subseteq L$, $\bigvee S \in \uparrow B$ implies $S \cap \uparrow A \neq \emptyset$. L is called a generalized completely distributive lattice if and only if $\uparrow x = \bigcap \{\uparrow F : F \in P^{(<\omega)} \text{ and } F \blacktriangleleft x\}$.

2 Hypercontinuous Posets

In this section, the concept of hypercontinuous posets is introduced, and we give intrinsic characterizations of hypercontinuous posets and show that a poset is hypercontinuous if and only if its normal completion is a hypercontinuous lattice.

Definition 2.1 A poset P is called hypercontinuous if $x \in \{u \in P : u \prec x\}^\delta$ for all $x \in P$. Let $i(x) = \{u \in P : u \prec x\}$.

Remark 2.1 If $X \subseteq \downarrow x$, then $x = \sup X$ if and only if $x \in X^\delta$. Thus for complete lattices, the preceding definition of hypercontinuity is equivalent to Definition 1.2.

Theorem 2.1 Let P be a poset. Then the following conditions are equivalent:

- (1) P is hypercontinuous;
- (2) If $x, y \in P$ with $x \not\leq y$, then there exist $F \in P^{(<\omega)}$ and $u \in P$ such that (i) $x \notin \downarrow F$, $y \notin \uparrow u$, and (ii) $\downarrow F \cup \uparrow u = P$;
- (3) The relation $\not\leq$ on P is finitely regular.

Proof (1) \Rightarrow (2). Let $x, y \in P$ with $x \not\leq y$. Then by (1), there exists $u \in P$ with $u \prec x$ such that $u \not\leq y$. Thus $x \in \text{int}_{v(P)} \uparrow u$. Choose $F \in P^{(<\omega)}$ such that $x \in P \setminus \downarrow F \subseteq \text{int}_{v(P)} \uparrow u \subseteq \uparrow u$. Then F and u satisfy the conditions of (i) and (ii) in (2).

(2) \Rightarrow (1). Suppose that there exists $x \in P$ with $x \notin i(x)^\delta$, and then there exists $y \in P$ with $i(x) \subseteq \downarrow y$ such that $x \not\leq y$. By (2), there exist $F \in P^{(<\omega)}$ and $u \in P$ that satisfy the conditions (i)–(ii) in (2). Then $u \prec x$ and $u \not\leq y$, a contradiction to $i(x) \subseteq \downarrow y$. Therefore P is hypercontinuous.

(2) \Rightarrow (3). Let $x, y \in P$ with $x \not\leq y$. By (2), there exist $F = \{v_1, v_2, \dots, v_k\} \in P^{(<\omega)}$ and $u \in P$ that satisfy the conditions (i)–(ii) in (2). For all $\{s_1, s_2, \dots, s_k\} \in P^{(<\omega)}$ and $t \in P$, if $u \not\leq t$ and $s_i \not\leq v_i$ for each $i \in \{1, 2, \dots, k\}$, then there exists $j \in \{1, 2, \dots, k\}$ such that $t \leq v_j$. Thus $s_j \not\leq t$ since $s_j \not\leq v_j$.

(3) \Rightarrow (2). Let $x, y \in P$ with $x \not\leq y$. By (3), there exist $F = \{v_1, v_2, \dots, v_k\} \in P^{(<\omega)}$ and $u \in P$ such that $1^\circ u \not\leq y$, $x \notin \downarrow F$, and 2° for all $\{s_1, s_2, \dots, s_k\} \in P^{(<\omega)}$ and $t \in P$, if $u \not\leq t$, $s_i \not\leq v_i$ for each $i \in \{1, 2, \dots, k\}$, then there exists $j \in \{1, 2, \dots, k\}$ such that $s_j \not\leq t$.

For all $z \in P$, let $t = z$, and $s_i = z$ for all $i \in \{1, 2, \dots, k\}$. By 2° , we have $u \leq t = z$ or there exists $j \in \{1, 2, \dots, k\}$ such that $z = s_j \leq v_j$, i.e., $\uparrow u \cup \downarrow F = P$.

Corollary 2.1 Let P be a hypercontinuous poset. Then $(P, \theta(P))$ is T_2 .

Theorem 2.2 For a poset P , the following two conditions are equivalent:

- (1) P is hypercontinuous;
- (2) $(\delta(P), \subseteq)$ is a hypercontinuous lattice.

Proof (1) \Rightarrow (2). If $A^\delta, B^\delta \in \delta(P)$ with $A^\delta \not\subseteq B^\delta$, then $A \not\subseteq B^\delta$. Thus there exists $x \in A$ with $x \notin B^\delta$. Hence there exists $y \in P$ with $B \subseteq \downarrow y$ such that $x \not\leq y$. By Theorem 2.1, there exist $F \in P^{(<\omega)}$ and $u \in P$ such that (i) $x \notin \downarrow F$, $y \notin \uparrow u$, and (ii) $\downarrow F \cup \uparrow u = P$.

Let $\mathcal{F} = \{\downarrow a : a \in F\}$. Then $\mathcal{F} \in \delta(P)^{(<\omega)}$. We show that \mathcal{F} and $\downarrow u$ satisfy the conditions (i)–(ii) in (2) of Theorem 2.1. Firstly, if $A^\delta \in \downarrow_{\delta(P)} \mathcal{F}$, then there exists $a \in F$ with $A^\delta \subseteq \downarrow a$. Thus $x \in A \subseteq A^\delta \subseteq \downarrow a$, a contradiction to $x \notin \downarrow F$; if $B^\delta \in \uparrow_{\delta(P)} \{\downarrow u\}$, then $u \in \downarrow u \subseteq B^\delta \subseteq$

$\downarrow y$, a contradiction to $y \not\leq \uparrow u$. Then we show that $\downarrow_{\delta(P)} \mathcal{F} \cup \uparrow_{\delta(P)} \{\downarrow u\} = \delta(P)$. For all $C^\delta \in \delta(P)$, if $\downarrow u \not\subseteq C^\delta$, i.e., $u \notin C^\delta$, then there exists $m \in P$ with $C \subseteq \downarrow m$ such that $u \not\leq m$. Thus $m \in \downarrow F$. Then there exists $a \in F$ such that $m \leq a$. Therefore, $C^\delta \subseteq \downarrow m \subseteq \downarrow a$, which implies $C^\delta \in \downarrow_{\delta(P)} \mathcal{F}$.

(2) \Rightarrow (1). If $x, y \in P$ with $x \not\leq y$, then $\downarrow x \not\subseteq \downarrow y$. By Theorem 2.1, there exist $\mathcal{F} = \{A_1^\delta, A_2^\delta, \dots, A_k^\delta\} \in \delta(P)^{(<\omega)}$ and $B^\delta \in \delta(P)$ such that (i) $\downarrow x \not\subseteq \downarrow_{\delta(P)} \mathcal{F}$, $\downarrow y \notin \uparrow_{\delta(P)} \{B^\delta\}$, and (ii) $\downarrow_{\delta(P)} \mathcal{F} \cup \uparrow_{\delta(P)} \{B^\delta\} = \delta(P)$.

Since $\downarrow x \not\subseteq \downarrow_{\delta(P)} \mathcal{F}$, $x \notin A_i^\delta$ for all $i \in \{1, 2, \dots, k\}$. Thus there exists $m_i \in P$ with $A_i \subseteq \downarrow m_i$ such that $x \not\leq m_i$. Let $F = \{m_1, m_2, \dots, m_k\}$. Since $\downarrow y \notin \uparrow_{\delta(P)} \{B^\delta\}$, there exists $u \in B^\delta$ with $u \not\leq y$. Then we show that F and u satisfy the conditions of (i)–(ii) in (2) of Theorem 2.1. Obviously, $x \notin \downarrow F$ and $y \notin \uparrow u$. We show that $\downarrow F \cup \uparrow u = P$. For all $z \in P$, if $u \not\leq z$, then $B^\delta \not\subseteq \downarrow z$ since $u \in B^\delta$. Thus there exists $i \in \{1, 2, \dots, k\}$ with $z \in \downarrow z \subseteq A_i^\delta \subseteq \downarrow m_i$. So $z \in \downarrow F$. Therefore, $\downarrow F \cup \uparrow u = P$.

Definition 2.2 (see [5]) *A map f between posets P and Q is said to be cut-stable if $f[A^\uparrow]^\downarrow = f[A]^\uparrow^\downarrow$ and $f[A^\downarrow]^\uparrow = f[A]^\downarrow^\uparrow$ for all $A \subseteq P$.*

Proposition 2.1 (see [5]) *A map f from a poset P into a complete lattice L is cut-stable if and only if there exists a (unique) complete homomorphism g from $\delta(P)$ into L such that $f = g \circ e_P$.*

A subcategory \mathbf{A} of a category \mathbf{C} is called a reflective subcategory (see [1]) of \mathbf{C} , if for each \mathbf{C} -object C , there exists an \mathbf{A} -object A_0 and a \mathbf{C} -morphism $r : C \rightarrow A_0$ such that for each \mathbf{A} -object A and \mathbf{C} -morphism $f : C \rightarrow A$ there exists a unique \mathbf{A} -morphism $g : A_0 \rightarrow A$ such that $f = g \circ r$.

By Proposition 2.1 and Theorem 2.2, we immediately have the following theorem.

Theorem 2.3 *The category of hypercontinuous lattices with complete homomorphisms is a full reflective subcategory of the category of hypercontinuous posets with cut-stable maps.*

3 Generalized Completely Continuous Posets

In this section, the concept of generalized completely continuous posets is introduced, and we give intrinsic characterizations of generalized completely continuous posets and show that a poset is generalized completely continuous if and only if its normal completion is a generalized completely distributive lattice if and only if its order dual is a hypercontinuous poset.

Definition 3.1 *Let P be a poset, $x \in P$, $A, B \subseteq P$. We say that:*

(1) *A is completely way below B , in symbols $A \triangleleft B$ if for all $S \subseteq P$ $\uparrow B \cap S^\delta \neq \emptyset$ implies $\uparrow A \cap S \neq \emptyset$. We write $F \triangleleft \{x\}$ for $F \triangleleft x$. Let $\nabla(x) = \{F \in P^{(<\omega)} : F \triangleleft x\}$.*

(2) *P is generalized completely continuous if for all $x \in P$, $\uparrow x = \cap \{\uparrow F : F \in \nabla(x)\}$.*

Remark 3.1 For complete lattices, the preceding definition of generalized completely continuity is equivalent to Definition 1.3.

Proposition 3.1 *For a poset P , $x \in P$, $A \subseteq P$, the following conditions are equivalent:*

- (1) $A \triangleleft x$;
- (2) $x \notin (P \setminus \uparrow A)^\delta$;
- (3) $x \in P \setminus (P \setminus \uparrow A)^\delta \subseteq \uparrow A$.

Proof (1) \Rightarrow (2). If $x \in (P \setminus \uparrow A)^\delta$, then by the definition of \triangleleft , $\uparrow A \cap (P \setminus \uparrow A) \neq \emptyset$, which is impossible.

(2) \Rightarrow (1). If there exists $S \subseteq P$ with $x \in S^\delta$ such that $\uparrow A \cap S = \emptyset$, then $x \in S^\delta \subseteq (P \setminus \uparrow A)^\delta$, a contradiction to (2).

(2) \Leftrightarrow (3). Obviously.

Proposition 3.2 For a poset P , $A \subseteq P$, the following two conditions are equivalent:

- (1) $A \triangleleft A$;
- (2) $\uparrow A = P \setminus (P \setminus \uparrow A)^\delta$.

Proof (1) \Rightarrow (2). Obviously, $P \setminus (P \setminus \uparrow A)^\delta \subseteq \uparrow A$ by Lemma 1.1. If there exists $y \in \uparrow A$ but $y \notin P \setminus (P \setminus \uparrow A)^\delta$, then by (1), $\uparrow A \cap (P \setminus \uparrow A) \neq \emptyset$, which is impossible.

(2) \Rightarrow (1). If there exists $S \subseteq P$ with $\uparrow A \cap S^\delta \neq \emptyset$ such that $\uparrow A \cap S = \emptyset$, then $S^\delta \subseteq (P \setminus \uparrow A)^\delta$. By (2), we have $\uparrow A \subseteq P \setminus S^\delta$, a contradiction to $\uparrow A \cap S^\delta \neq \emptyset$.

Theorem 3.1 For a poset P , the following two conditions are equivalent:

- (1) P is generalized completely continuous;
- (2) $(\delta(P), \subseteq)$ is a generalized completely distributive lattice.

Proof (1) \Rightarrow (2). We show that $\uparrow_{\delta(P)} \{A^\delta\} = \cap \{\uparrow_{\delta(P)} \mathcal{F} : \mathcal{F} \in \delta(P)^{(<\omega)}, \mathcal{F} \triangleleft A^\delta\}$ for all $A^\delta \in \delta(P)$. Obviously, $\uparrow_{\delta(P)} \{A^\delta\} \subseteq \cap \{\uparrow_{\delta(P)} \mathcal{F} : \mathcal{F} \in \delta(P)^{(<\omega)}, \mathcal{F} \triangleleft A^\delta\}$. We show that $\cap \{\uparrow_{\delta(P)} \mathcal{F} : \mathcal{F} \in \delta(P)^{(<\omega)}, \mathcal{F} \triangleleft A^\delta\} \subseteq \uparrow_{\delta(P)} \{A^\delta\}$. If $A^\delta \not\subseteq B^\delta$, then $A \not\subseteq B^\delta$. Thus there exists $x \in A$ with $x \notin B^\delta$. Hence there exists $y \in P$ with $B \subseteq \downarrow y$ such that $x \not\leq y$. By (1), there exists $F \in P^{(<\omega)}$ with $F \triangleleft x$ such that $y \notin \uparrow F$. Let $\mathcal{F} = \{\downarrow u : u \in F\}$. Then $\mathcal{F} \in \delta(P)^{(<\omega)}$. We show that $\mathcal{F} \triangleleft A^\delta$ and $B^\delta \notin \uparrow_{\delta(P)} \mathcal{F}$. If $B^\delta \in \uparrow_{\delta(P)} \mathcal{F}$, then there exists $u \in F$ with $\downarrow u \subseteq B^\delta$. Thus $u \in B^\delta \subseteq \downarrow y$, a contradiction to $y \notin \uparrow F$. Then we show that $\mathcal{F} \triangleleft A^\delta$. For all $\{A_i^\delta : i \in I\} \subseteq \delta(P)$ with $A^\delta \subseteq \bigvee_{i \in I} A_i^\delta$, we have $A^\delta \subseteq (\bigcup_{i \in I} A_i^\delta)^\delta$ by Lemma 1.1. Since $x \in A \subseteq A^\delta$ and $F \triangleleft x$, we have $F \cap \bigcup_{i \in I} A_i^\delta \neq \emptyset$. Thus there exist $u \in F$ and $i \in I$ with $u \in A_i^\delta$. Since A_i^δ is a lower set, $\downarrow u \subseteq A_i^\delta$. Hence $\uparrow_{\delta(P)} \mathcal{F} \cap \{A_i^\delta : i \in I\} \neq \emptyset$. Therefore, $\delta(P)$ is a generalized completely distributive lattice.

(2) \Rightarrow (1). Obviously, $\uparrow x \subseteq \cap \{\uparrow F : F \in \nabla(x)\}$ for all $x \in P$. If $x \not\leq y$, then $\downarrow x \not\subseteq \downarrow y$. By (2), there exists $\mathcal{F} = \{A_1, A_2, \dots, A_k\} \in \delta(P)^{(<\omega)}$ with $\mathcal{F} \triangleleft \downarrow x$ such that $\downarrow y \notin \uparrow_{\delta(P)} \mathcal{F}$, i.e., $A_i \not\subseteq \downarrow y$ for all $i \in \{1, 2, \dots, k\}$. Thus there exists $y_i \in A_i$ with $y_i \not\leq y$ for all $i \in \{1, 2, \dots, k\}$. Let $F = \{y_1, y_2, \dots, y_k\}$. Obviously, $y \notin \uparrow F$. We show that $F \triangleleft x$ for all $S \subseteq P$ with $x \in S^\delta$. Since S^δ is a lower set, $\downarrow x \subseteq S^\delta = (\bigcup_{s \in S} \downarrow s)^\delta = \bigvee_{s \in S} \downarrow s$. Since $\mathcal{F} \triangleleft \downarrow x$, $\uparrow_{\delta(P)} \mathcal{F} \cap \{\downarrow s : s \in S\} \neq \emptyset$. Thus there exist $i \in \{1, 2, \dots, k\}$ and $s \in S$ such that $y_i \in A_i \subseteq \downarrow s$. So $\uparrow F \cap S \neq \emptyset$. Hence $F \triangleleft x$. Since $y \notin \uparrow F$, $y \notin \cap \{\uparrow F : F \in \nabla(x)\}$. Therefore, $\uparrow x = \cap \{\uparrow F : F \in \nabla(x)\}$.

Theorem 3.2 (see [17]) A complete lattice L is a hypercontinuous lattice if and only if L^{op} is a generalized completely distributive lattice.

By Theorems 2.2, 3.1–3.2, we obtain the following result.

Corollary 3.1 A poset P is hypercontinuous if and only if P^{op} is generalized completely continuous.

By Theorem 2.1 and Corollary 3.1, we obtain the following result.

Corollary 3.2 For a poset P , the following conditions are equivalent:

- (1) P is generalized completely continuous;
 (2) for all $x, y \in P$ with $x \not\leq y$, there exist $u \in P$ and finite set $F \in P^{(<\omega)}$ such that (i) $x \notin \downarrow u$, $y \notin \uparrow F$, and (ii) $\downarrow u \cup \uparrow F = P$;
 (3) $\not\leq$ on P is finitely regular.

Corollary 3.3 *Let P be a generalized completely continuous poset. Then $(P, \theta(P))$ is T_2 .*

Corollary 3.4 *The category of generalized completely distributive lattices with complete homomorphisms is a full reflective subcategory of the category of generalized completely continuous posets with cut-stable maps.*

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