Chin. Ann. Math.
36B(2), 2015, 213–224
DOI: 10.1007/s11401-015-0888-6

Chinese Annals of Mathematics, Series B

Some Remarks on Hom-Modules and Hom-Path Algebras*

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Abstract This paper deals with injective and projective right Hom-H-modules for a Hom-algebra H. In particular, Baer Criterion of injective Hom-module is obtained, and it is shown that HomModH is an Abelian category. Next, the authors define Hom-path algebras and construct Hom-path algebras of some quivers.

Keywords Hom-Module, Hom-Algebra, Quiver
2000 MR Subject Classification 16G20, 16D99, 17A30

1 Introduction

A Hom-algebra structure is a multiplication on a vector space where the structure is twisted by a homomorphism. Hom-Lie algebras and general quasi-Hom-Lie and quasi-Lie algebras were introduced by Hartwig, Larsson and Silvestrov as algebras embracing Lie algebras, super and color Lie algebras and their quasi-deformations by twisted derivations. Makhlouf and Silvestrov introduced and studied Hom-associative, Hom-Leibniz and Hom-Lie admissible algebraic structures generalizing associative, Leibniz and Lie admissible algebras in [1]. At the same time, they developed the theory of Hom-coalgebras and related structures in [2]. In [3–5], Yau constructed enveloping algebras of Hom-Lie and Hom-Leibniz algebras, researched G-Hom-associative algebras as deformations of G-associative algebras along algebra endomorphisms, and studied Hom-bialgebras and objects admitting coactions by Hom-bialgebras.

In this paper, we extend Hom-modules and Hom-algebras to the category of modules and the representation of quivers respectively, by using the ideas of [6–8]. This paper is organized as follows. In Section 2, we summarize the definitions of Hom-algebra, Hom-module and path algebra of quivers. In Section 3, we define right Hom-H-module for a Hom-algebra H and prove that HomMod_H is an Abelian category. For injective and projective right Hom-H-modules, we research some of their essential properties and give the Baer Criterion of injective Hom-module. In Section 4, we define the concept of Hom-path algebra and give the types of quivers whose path algebras can be made into (nontrivial) Hom-path algebras.

2 Preliminaries

Throughout this paper, let K denote a field of characteristic 0. Firstly, we introduce the definitions of Hom-algebra and Hom-module as follows.

Manuscript received October 22, 2013. Revised February 24, 2014.
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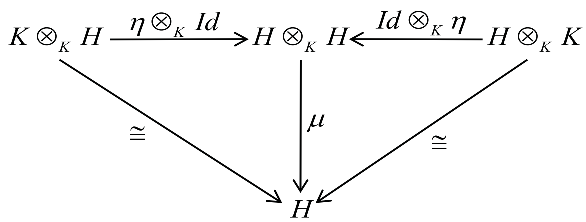
*This work was supported by the National Natural Science Foundation of China (Nos. 11271119, 11201314) and the Natural Science Foundation of Beijing (No. 1122002).

Definition 2.1 (see [3]) *A Hom-algebra is a triple (H, μ, α, η) in which:*
(1) *H is a K-module;*
(2) *$\mu : H \otimes_K H \longrightarrow H$ is a bilinear map;*
(3) *$\alpha : H \longrightarrow H$ is a linear endomorphism;*
(4) *$\eta : K \longrightarrow H$, the unit, is a linear map such that the following diagram commutes.*



When there is no danger of confusion, we will denote a Hom-algebra (H, μ, α, η) simply by H.

Definition 2.2 (see [3]) *By a Hom-module, we mean a pair (V, α) consisting of*
(1) *a K-module V, and*
(2) *a linear endomorphism $\alpha : V \longrightarrow V$.*

A morphism $f : (M, \alpha_M) \longrightarrow (N, \alpha_N)$ of Hom-modules is a linear map $f : M \longrightarrow N$ such that $f \circ \alpha_M = \alpha_N \circ f$.

Next, we recall some points about quivers and path (co)algebras. By a quiver Q, we mean a quadruple (Q_0, Q_1, h, s), where Q_0 is the set of vertices (points), Q_1 is the set of arrows and for each arrow $a \in Q_1$, the vertices $h(a)$ and $s(a)$ are the source (or start point) and the sink (or end point) of a, respectively. If i and j are vertices in Q, an (oriented) path in Q of length m from i to j is a formal composition of arrows

$$p = a_m \cdots a_2 a_1,$$

where $h(a_1) = i,\ s(a_m) = j$ and $s(a_{k-1}) = h(a_k)$, for $k = 2, \cdots, m$. To any vertex $i \in Q_0$, we attach a trivial path of length 0, say e_i, starting and ending at i such that $ae_i = a$ (resp. $e_i b = b$) for any arrow a (resp. b) with $h(a) = i$ (resp. $s(b) = i$). We identify the set of vertices and the set of trivial paths. An (oriented) cycle is a path in Q which starts and ends at the same vertex. Q is said to be acyclic if there is no oriented cycle in Q.

Let KQ be the K-vector space generated by the set of all paths in Q. Then KQ can be endowed with the structure of a (unnecessarily unitary) K-algebra with multiplication induced by concatenation of paths, that is,

$$(a_m \cdots a_2 a_1)(b_n \cdots b_2 b_1) = \begin{cases} a_m \cdots a_2 a_1 b_n \cdots b_2 b_1, & \text{if } s(b_n) = h(a_1), \\ 0, & \text{otherwise.} \end{cases}$$

KQ is the path algebra of the quiver Q. The algebra KQ can be graded by

$$KQ = KQ_0 \oplus KQ_1 \oplus \cdots \oplus KQ_m \oplus \cdots,$$

where Q_m is the set of all paths of length m.

Following [9], the path algebra KQ can be viewed as a K-coalgebra with comultiplication induced by the decomposition of path, that is, if $p = a_m \cdots a_1$ is a path from the vertex i to the vertex j, then $\Delta(p) = \sum_{\eta\tau = p} \eta \otimes \tau$ and for a stationary path e_i, we have $\Delta(e_i) = e_i \otimes e_i$. The counit of KQ is defined by the formula

$$\epsilon(a) = \begin{cases} 1, & \text{if } a \in Q_0, \\ 0, & \text{if } a \text{ is a path of length} \geq 1. \end{cases}$$

The coalgebra (KQ, Δ, ϵ) (shortly KQ) is called the path coalgebra of the quiver Q.

Definition 2.3 (see [10]) *A relation subcoalgebra of a path coalgebra KQ is any subcoalgebra S of KQ satisfying the following two conditions:*

(a) *The subcoalgebra $KQ_{\leq 1} = KQ_0 \oplus KQ_1$ of KQ is a subcoalgebra of S;*

(b) $S = \bigoplus_{i,j \in Q_0} S(i,j)$, *where* $S(i,j) = S \cap KQ(i,j)$.

3 Injective and Projective Hom-Modules

The main purpose of this section is to study injective and projective Hom-modules and some of their fundamental properties which are similar to those in the homological algebra. First we need some preliminary concepts.

Definition 3.1 *Let H be a Hom-algebra. By a right Hom-H-module, we mean a Hom-module (M, α_M) equipped with a right H-action, $\rho_M : M \otimes_K H \longrightarrow M (m \otimes h \longmapsto mh)$, such that $\alpha_M(mh) = \alpha_M(m)\alpha_H(h)$ for $m \in M,\ h \in H$.*

A morphism $f : (M, \alpha_M) \longrightarrow (N, \alpha_N)$ of right Hom-H-modules is a morphism of Hom-modules such that $f(mh) = f(m)h$ for $m \in M,\ h \in H$.

Remark 3.1 The morphism f is well defined, that is, $f \circ \alpha_M(mh) = f(\alpha_M(m)\alpha_H(h)) = f(\alpha_M(m))\alpha_H(h) = f \circ \alpha_M(m)\alpha_H(h) = \alpha_N \circ f(m)\alpha_H(h) = \alpha_N(f(m))\alpha_H(h) = \alpha_N(f(m)h) = \alpha_N \circ f(mh)$. We denote the set of morphisms of right Hom-H-modules from (M, α_M) to (N, α_N) by $\mathrm{Hom}_H((M, \alpha_M), (N, \alpha_N))$.

All right Hom-H-modules and their morphisms form a category which is denoted by HomMod_H.

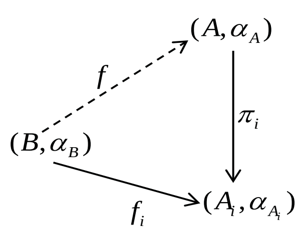
Definition 3.2 *For a right Hom-H-module (M, α_M), we define that (U, α_U) is a Hom-submodule of (M, α_M) if*

(1) *$U \subseteq M$ is a K-submodule;*

(2) *$\alpha_U = \alpha_M|_U$, and $\alpha_U(U) \subseteq U$;*

(3) *$\rho_U = \rho_M|_U$ and $\rho_U(U \otimes_K H) \subseteq U$.*

Definition 3.3 *A (direct) product of a family of right Hom-H-modules (A_i, α_{A_i}) is (A, α_A), if there exist morphisms $\pi_i : (A, \alpha_A) \longrightarrow (A_i, \alpha_{A_i})$ such that for any (B, α_B) and $f_i : (B, \alpha_B) \longrightarrow (A_i, \alpha_{A_i})$, there is a unique morphism $f : B \longrightarrow A$ such that the following diagram commutes for all $i \in I$, where I is an index set.*

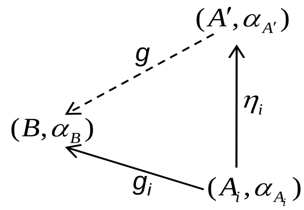


Remark 3.2 By the category of modules, we know $A = \prod_{i \in I} A_i$. Define $\prod_{i \in I} \alpha_{A_i}(a) = \prod_{i \in I} \alpha_{A_i}(a_i)$ for $a \in A,\ a_i \in A_i$. It is easy to see that $\alpha_A = \prod_{i \in I} \alpha_{A_i}$. Then (A, α_A) is the (direct) product $\big(\prod_{i \in I} A_i, \prod_{i \in I} \alpha_{A_i}\big)$.

Similarly, we can define the concept of coproduct.

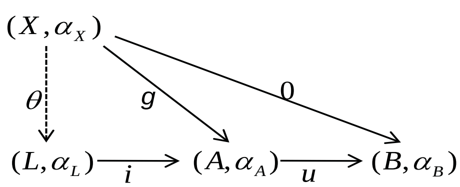
Definition 3.4 *A coproduct of a family of right Hom-H-modules (A_i, α_{A_i}) is $(A', \alpha_{A'})$, if there exist morphisms $\eta_i : (A_i, \alpha_{A_i}) \longrightarrow (A', \alpha_{A'})$ such that for any (B, α_B) and g_i :*

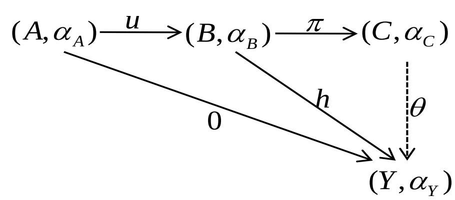
$(A_i, \alpha_{A_i}) \longrightarrow (B, \alpha_B)$, there is a unique morphism $g : A' \longrightarrow B$ such that the following diagram commutes for all $i \in I$.



Remark 3.3 In the category of modules, we know $A' = \coprod_{i\in I} A_i$ and $\coprod_{i\in I} A_i \subseteq \prod_{i\in I} A_i$. Define $\alpha_{A'} = \alpha_A|_{A'}$ and $\coprod_{i\in I} \alpha_{A_i}(a') = \sum_{i\in I} \alpha_{A_i}(a_i)$ for $a' \in A',\ a_i \in A_i$. It is easy to see that $\alpha_{A'} = \coprod_{i\in I} \alpha_{A_i}$. Then $(A', \alpha_{A'})$ is the coproduct $\big(\coprod_{i\in I} A_i, \coprod_{i\in I} \alpha_{A_i}\big)$. Normally, we also denote $(A', \alpha_{A'})$ by the direct sum $\big(\bigoplus_{i\in I} A_i, \bigoplus_{i\in I} \alpha_{A_i}\big)$.

Definition 3.5 *If $u : (A, \alpha_A) \longrightarrow (B, \alpha_B)$ is a morphism of right Hom-H-modules, then its kernel $Keru$ is a morphism $i : (L, \alpha_L) \longrightarrow (A, \alpha_A)$ that satisfies the following universal mapping property: $ui = 0$ and for every $g : (X, \alpha_X) \longrightarrow (A, \alpha_A)$ with $ug = 0$, there exists a unique $\theta : (X, \alpha_X) \longrightarrow (L, \alpha_L)$ with $i\theta = g$. There is a dual definition for cokernel (the morphism π in the diagram).*





Theorem 3.1 HomMod_H *is an Abelian category.*

Proof Firstly, let us show that HomMod_H is an additive category. For any $f_1, f_2 \in \mathrm{Hom}_H((A, \alpha_A), (B, \alpha_B))$, we define $(f_1 + f_2)(a) = f_1(a) + f_2(a)$ for $a \in A$, then

$$\begin{aligned}(f_1 + f_2)(a_1 + a_2) &= f_1(a_1 + a_2) + f_2(a_1 + a_2)\\ &= f_1(a_1) + f_1(a_2) + f_2(a_1) + f_2(a_2) = (f_1 + f_2)(a_1) + (f_1 + f_2)(a_2),\\ (f_1 + f_2)(ah) &= f_1(ah) + f_2(ah) = f_1(a)h + f_2(a)h = (f_1 + f_2)(a)h\end{aligned}$$

for $a_1, a_2 \in A,\ h \in H$.

$$(f_1 + f_2) \circ \alpha_A = f_1 \circ \alpha_A + f_2 \circ \alpha_A = \alpha_B \circ f_1 + \alpha_B \circ f_2 = \alpha_B \circ (f_1 + f_2).$$

Thus, $f_1 + f_2 \in \mathrm{Hom}_H((A, \alpha_A), (B, \alpha_B))$ and $\mathrm{Hom}_H((A, \alpha_A), (B, \alpha_B))$ is an additive Abelian group; zero morphism is the zero element, and $-f$ is the negative element of f. It is easy to see that the distribution laws are established and any finite right Hom-H-module has a coproduct (see Remark 3.3).

Next, we will show that the additive category HomMod_H is an Abelian category. Suppose $f \in \mathrm{Hom}_H((A, \alpha_A), (B, \alpha_B))$, and let

$$N = \{a \in A \mid f(a) = 0\}, \quad \alpha_N = \alpha_A|_N,$$

where $f(\alpha_N(a)) = f \circ \alpha_A|_N(a) = f \circ \alpha_A(a) = \alpha_B \circ f(a) = \alpha_B(f(a)) = 0$ for $a \in N$, that is, $\alpha_N(N) \subseteq N$;

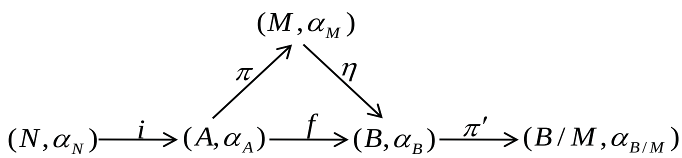
$$M = \{b \in B \mid \text{there is an } a \text{ in } A, \text{ such that } f(a) = b\}, \quad \alpha_M = \alpha_B|_M,$$

where $\alpha_M(b) = \alpha_B|_M(b) = \alpha_B(f(a)) = \alpha_B \circ f(a) = f \circ \alpha_A(a) = f(\alpha_A(a))$ for $b \in M$, that is, $\alpha_M(M) \subseteq M$. Then (N, α_N) is a Hom-submodule of (A, α_A) and (M, α_M) is a Hom-submodule of (B, α_B) because $f(ah) = f(a)h = 0$ and $bh = f(a)h = f(ah)$. Note that $(N, \alpha_N) = \mathrm{Ker} f$ is the kernel of f. We denote $(M, \alpha_M) = \mathrm{Im} f$ and the cokernel of f is $\mathrm{Cok} f = (B/M, \alpha_{B/M})$, where $\alpha_{B/M}(b + M) = \alpha_B(b) + M$, which is well defined. In fact, if $b + M = b' + M$, then $b - b' \in M$ and $\alpha_B(b - b') \in M$, so $\alpha_B(b) + M = \alpha_B(b') + M$, as desired.

If $N = 0$, f is a monomorphism; if $M = B$, f is an epimorphism; and if $M = 0$, f is a zero morphism. We define

$$i : (N, \alpha_N) \longrightarrow (A, \alpha_A), \text{ such that } i(a) = a \in N;$$
$$\pi : (A, \alpha_A) \longrightarrow (M, \alpha_M), \text{ such that } \pi(a) = f(a) \in M;$$
$$\eta : (M, \alpha_M) \longrightarrow (B, \alpha_B), \text{ such that } \eta(b) = b \in B;$$
$$\pi' : (B, \alpha_B) \longrightarrow (B/M, \alpha_{B/M}) \text{ is the natural epimorphism.}$$

Then i and η are monomorphisms; π and π' are epimorphisms. Since $f = \eta\pi$, the following diagram commutes.



Thus, $i = \mathrm{Ker}\, f = \mathrm{Ker}\, \pi$, and $\pi' = \mathrm{Cok}\, f = \mathrm{Cok}\, \eta$.

Definition 3.6 *A covariant functor T is an exact functor if for every exact sequence*

$$0 \longrightarrow (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \alpha_C) \longrightarrow 0$$

in HomMod_H*, the sequence*

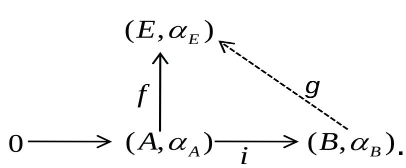
$$0 \longrightarrow (T(A), \alpha_{T(A)}) \xrightarrow{T(i)} (T(B), \alpha_{T(B)}) \xrightarrow{T(\pi)} (T(C), \alpha_{T(C)}) \longrightarrow 0$$

is also exact.

A contravariant functor F is an exact functor if there is always exactness of

$$0 \longrightarrow (F(C), \alpha_{F(C)}) \xrightarrow{F(\pi)} (F(B), \alpha_{F(B)}) \xrightarrow{F(i)} (F(A), \alpha_{F(A)}) \longrightarrow 0.$$

Definition 3.7 *A right Hom-H-module (E, α_E) is injective if, whenever i is an injection, a dashed arrow exists such that the following diagram commutes.*



Remark 3.4 If (E, α_E) is an injective right Hom-H-module, then E is an injective right H-module.

Proposition 3.1 *A right Hom-H-module (E, α_E) is injective if and only if*

$$\mathrm{Hom}_H(-, (E, \alpha_E))$$

is an exact functor.

Proof If (E, α_E) is injective, for an exact sequence of right Hom-H-modules

$$0 \longrightarrow (A, \alpha_A) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \alpha_C) \longrightarrow 0,$$

we can get that

$$0 \longrightarrow \mathrm{Hom}_H(C, E) \xrightarrow{\pi^*} \mathrm{Hom}_H(B, E) \xrightarrow{i^*} \mathrm{Hom}_H(A, E) \longrightarrow 0$$

is an exact sequence. We must prove the exactness of

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_H((C, \alpha_C), (E, \alpha_E)) \xrightarrow{\pi^*} \mathrm{Hom}_H((B, \alpha_B), (E, \alpha_E)) \\ &\xrightarrow{i^*} \mathrm{Hom}_H((A, \alpha_A), (E, \alpha_E)) \longrightarrow 0. \end{aligned}$$

For $f : (C, \alpha_C) \longrightarrow (E, \alpha_E)$, i.e., $f : C \longrightarrow E$, $f \circ \alpha_C = \alpha_E \circ f$, and $f(ch') = f(c)h'$ for $c \in C$, $h' \in H$, let $\pi^*(f) = f \circ \pi$, and then $f \circ \pi \circ \alpha_B = f \circ \alpha_C \circ \pi = \alpha_E \circ f \circ \pi$ and $f \circ \pi(bh') = f(\pi(b)h') = f(\pi(b))h' = f \circ \pi(b)h'$.

For $g : (A, \alpha_A) \longrightarrow (E, \alpha_E)$, i.e., $g : A \longrightarrow E$, $g \circ \alpha_A = \alpha_E \circ g$, and $g(ah') = g(a)h'$ for $a \in A$, $h' \in H$, there exists a map $h : B \longrightarrow E$, such that $i^*(h) = g = h \circ i$. Since (E, α_E) is injective and i is an injection, by the definition, we obtain $h \circ \alpha_B = \alpha_E \circ h$ and $h(bh') = h(b)h'$ for $b \in B$, $h' \in H$. Therefore, $\mathrm{Hom}_H(-, (E, \alpha_E))$ is an exact functor.

For the converse, assume that $\mathrm{Hom}_H(-, (E, \alpha_E))$ is an exact functor. For any $g \in \mathrm{Hom}_H((A, \alpha_A), (E, \alpha_E))$, there exists a morphism $h \in \mathrm{Hom}_H((B, \alpha_B), (E, \alpha_E))$ such that $g = h \circ i$, that is, (E, α_E) is an injective right Hom-H-module.

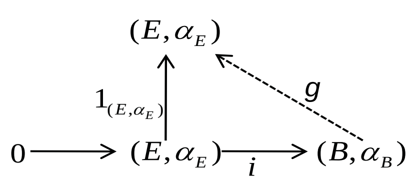
Corollary 3.1 *For any right Hom-H-module (M, α_M), $\mathrm{Hom}_H(-, (M, \alpha_M))$ is a left exact contravariant functor.*

Proposition 3.2 *If a right Hom-H-module (E, α_E) is injective, then every short exact sequence*

$$0 \longrightarrow (E, \alpha_E) \xrightarrow{i} (B, \alpha_B) \xrightarrow{\pi} (C, \alpha_C) \longrightarrow 0$$

splits.

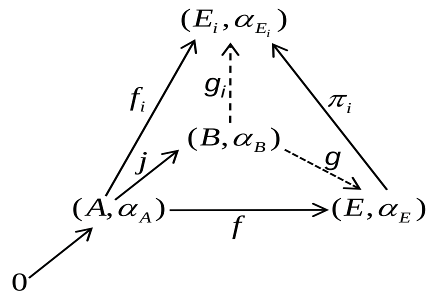
Proof Since (E, α_E) is injective, there exists a morphism $g : (B, \alpha_B) \longrightarrow (E, \alpha_E)$ such that the following diagram commutes,



that is, $g \circ i = 1_{(E, \alpha_E)}$.

Theorem 3.2 $(E, \alpha_E) = \big(\prod_{i \in I} E_i, \prod_{i \in I} \alpha_{E_i}\big)$ *is injective if and only if every right Hom-H-module (E_i, α_{E_i}) is injective.*

Proof Consider the diagram



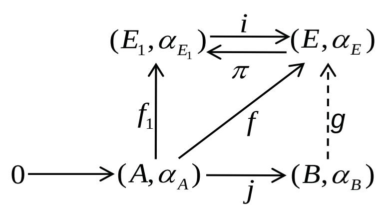
If (E_i, α_{E_i}) is injective, then there exists a morphism g_i such that $f_i = g_i \circ j$. Since (E, α_E) is the product of $(E_i, \alpha_{E_i}),\ i \in I$, there is a unique morphism f such that $\pi_i \circ f = f_i$ and a unique morphism g such that $\pi_i \circ g = g_i$. So $g \circ j = f$ and (E, α_E) is injective.

Conversely, if (E, α_E) is injective, then there exists a morphism g such that $g \circ j = f$. For $f_i : (A, \alpha_A) \longrightarrow (E_i, \alpha_{E_i})$, we must prove that there is a morphism g_i such that $g_i \circ j = f_i$. Since (E, α_E) is the product of $(E_i, \alpha_{E_i}),\ i \in I$, there is a unique morphism f such that $\pi_i \circ f = f_i$. We set $g_i = \pi_i \circ g$. Then $g_i \circ \alpha_B = \pi_i \circ g \circ \alpha_B = \pi_i \circ \alpha_E \circ g = \alpha_{E_i} \circ \pi_i \circ g = \alpha_{E_i} \circ g_i$ and $g_i(bh) = \pi_i \circ g(bh) = \pi_i(g(b)h) = \pi_i(g(b))h = \pi_i \circ g(b)h = g_i(b)h$ for $b \in B,\ h \in H$. Thus $g_i \circ j = f_i$ and (E_i, α_{E_i}) is injective.

Corollary 3.2 (1) *Every direct summand of an injective right Hom-H-module (E, α_E) is injective.*

(2) *A finite direct sum of an injective right Hom-H-module is injective.*

Proof (1) Assume that $(E, \alpha_E) = (E_1, \alpha_{E_1}) \oplus (E_2, \alpha_{E_2})$, $i : (E_1, \alpha_{E_1}) \longrightarrow (E, \alpha_E)$ is the inclusion and $\pi : (E, \alpha_E) \longrightarrow (E_1, \alpha_{E_1})$ is the projection. From the following diagram



we can conclude that (E_1, α_{E_1}) is injective. Similarly, (E_2, α_{E_2}) is also injective.

(2) Let I be a finite set, and then $\bigoplus\limits_{i \in I} E_i = \prod\limits_{i \in I} E_i$. So the conclusion holds.

Definition 3.8 *Let (M, α_M) be a right Hom-H-module. A right Hom-H-module (E, α_E) containing (M, α_M), that is, $M \subseteq E$ and $\alpha_E|_M = \alpha_M$, is an injective envelope of (M, α_M), if (E, α_E) is injective and there is no proper injective Hom-submodule $(E', \alpha_{E'})$ such that $(M, \alpha_M) \subseteq (E', \alpha_{E'}) \subsetneq (E, \alpha_E)$.*

Definition 3.9 (see [11]) *Let $H = (H, \mu, \alpha, \eta)$ be a Hom-algebra. A Hom-subalgebra S of H is a triple $(S, \mu|_S, \alpha|_S, \eta|_S)$ in which:*

(1) *$S \subseteq H$ is a K-submodule;*

(2) *$\mu|_S : S \otimes_K S \longrightarrow S$ is a bilinear map;*

(3) *$\alpha|_S : S \longrightarrow S$ is a linear endomorphism;*

(4) *$\eta|_S : K \longrightarrow S$ is the unit.*

Definition 3.10 (see [11]) *Let $H = (H, \mu, \alpha, \eta)$ be a Hom-algebra. A right Hom-ideal D of H is a triple $(D, \mu_D, \alpha_D, \eta_D)$ in which:*

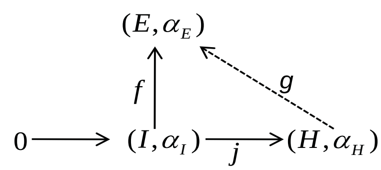
(1) *$D \subseteq H$ is a K-submodule;*

(2) *$\mu_D = \mu|_D$ and $\mu_D(D \otimes_K H) \subseteq D$;*

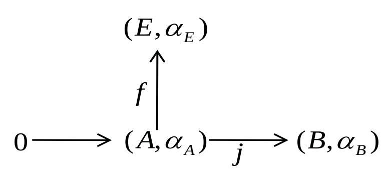
(3) $\alpha|_D = \alpha_D$.

Theorem 3.3 (Baer Criterion of Injective Hom-Module) *A right Hom-H-module (E, α_E) is injective if and only if every right Hom-H-module morphism $f : (D, \alpha_D) \longrightarrow (E, \alpha_E)$, where D is a right Hom-ideal of H, which can be extended to (H, α_H).*

Proof Assume that (E, α_E) is injective, and there exists a morphism $g : (H, \alpha_H) \longrightarrow (E, \alpha_E)$ such that $g \circ j = f$.



Conversely, consider the diagram



where (A, α_A) is a Hom-submodule of (B, α_B) such that $\alpha_A = \alpha_B|_A$. Let X be the set of all ordered pairs $((A_i, \alpha_{A_i}), g_i)$, where $A \subseteq A_i \subseteq B$, and $g_i : (A_i, \alpha_{A_i}) \longrightarrow (E, \alpha_E)$ extends f, that is, $g_i|_{(A,\alpha_A)} = f$, $g_i \circ \alpha_{A_i} = \alpha_E \circ g_i$ and $g_i(a_i h) = g_i(a_i)h$ for $a_i \in A_i$, $h \in H$. Note that $X \neq \emptyset$ because $((A, \alpha_A), f) \in X$. The partial order on X is defined by

$$((A_i, \alpha_{A_i}), g_i) \leq ((A_l, \alpha_{A_l}), g_l)$$

in which $A_i \subseteq A_l$, g_l extends g_i and $\alpha_{A_l}|_{A_i} = \alpha_{A_i}$. By Zorn's lemma, there exists a maximal element $((A_n, \alpha_{A_n}), g_n)$ in X.

If $A_n = B$, we are done. Otherwise, we may assume that there is some $b \in B$ with $b \notin A_n$. Define

$$D = \{x \in H : bx \in A_n\}, \quad \alpha_D = \alpha_H|_D.$$

It is easy to see that (D, α_D) is a right Hom-ideal of (H, α_H). In fact, for $x \in D$, $h \in H$, $bxh \in A_n$, because (A_n, α_{A_n}) is a Hom-submodule of (B, α_B), we have $xh \in D$. Define $q : (D, \alpha_D) \longrightarrow (E, \alpha_E)$ by $q(x) = g_n(bx)$ and $b\alpha_D(x) = \alpha_{A_n}(bx)$. By the hypothesis, there is a map $q^* : (H, \alpha_H) \longrightarrow (E, \alpha_E)$ extending q. We set $A' = A_n + \langle b \rangle$ and $g' : A' \longrightarrow E$ is given by $g'(a_n + bx) = g_n(a_n) + q^*(1)x$. It is easy to see that g' is well defined by [6]. Clearly, $g'(a_n) = g_n(a_n)$ for all $a_n \in A_n$.

We set $\alpha_{A'}(a_n + bx) = \alpha_{A_n}(a_n) + b\alpha_D(x)$. Let us show that $\alpha_{A'}$ is well defined. If $a_n + bx = a'_n + bx'$, then $b(x - x') = a'_n - a_n \in A_n$ and $x - x' \in D$. We have

$$\alpha_{A_n}(a'_n - a_n) = \alpha_{A_n}(b(x - x')) = b\alpha_D(x - x').$$

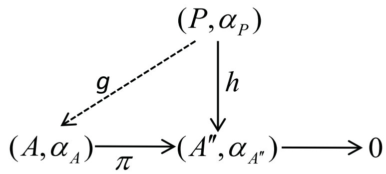
Thus, $\alpha_{A_n}(a'_n) - \alpha_{A_n}(a_n) = b\alpha_D(x) - b\alpha_D(x')$ and $\alpha_{A_n}(a'_n) + b\alpha_D(x') = \alpha_{A_n}(a_n) + b\alpha_D(x)$, as desired.

$$g' \circ \alpha_{A'}(a_n + bx) = g'(\alpha_{A_n}(a_n) + b\alpha_D(x)) = g_n(\alpha_{A_n}(a_n)) + q^*(1)\alpha_D(x),$$
$$\alpha_E \circ g'(a_n + bx) = \alpha_E(g_n(a_n) + q^*(1)x) = \alpha_E(g_n(a_n)) + q^*(1)\alpha_D(x).$$

So $g'\circ\alpha_{A'}=\alpha_E\circ g'$ and $g'((a_n+bx)h)=g_n(a_n)h+b\alpha_D(x)h=g'(a_n+bx)h$. We conclude that $((A_n,\alpha_{A_n}),g_n)\le((A',\alpha_{A'}),g')$, contradicting the maximality of $((A_n,\alpha_{A_n}),g_n)$. Therefore $A_n=B$, g_n extends f and (E,α_E) is injective.

Next we consider the projective right Hom-H-module which is dual to the injective right Hom-H-module.

Definition 3.11 *A right Hom-H-module (P,α_P) is projective if, whenever π is surjective and h is any map, there exists a map g such that the following diagram commutes.*



Remark 3.5 If (P,α_P) is a projective right Hom-H-module, then P is a projective right H-module.

Proposition 3.3 *A right Hom-H-module (P,α_P) is projective if and only if*

$$\mathrm{Hom}_H((P,\alpha_P),-)$$

is an exact functor.

Proof Assume that there exists an exact sequence in HomMod_H

$$0\longrightarrow(A',\alpha_{A'})\xrightarrow{i}(A,\alpha_A)\xrightarrow{\pi}(A'',\alpha_{A''})\longrightarrow 0.$$

Since P is projective, we have an exact sequence

$$0\longrightarrow\mathrm{Hom}_H(P,A')\xrightarrow{i_*}\mathrm{Hom}_H(P,A)\xrightarrow{\pi_*}\mathrm{Hom}_H(P,A'')\longrightarrow 0.$$

We must prove the exactness of

$$\begin{aligned}0\longrightarrow\mathrm{Hom}_H((P,\alpha_P),(A',\alpha_{A'}))&\xrightarrow{i_*}\mathrm{Hom}_H((P,\alpha_P),(A,\alpha_A))\\ &\xrightarrow{\pi_*}\mathrm{Hom}_H((P,\alpha_P),(A'',\alpha_{A''}))\longrightarrow 0.\end{aligned}$$

For a morphism $f\in\mathrm{Hom}_H((P,\alpha_P),(A',\alpha_{A'}))$, i.e., $f:P\longrightarrow A'$ such that $f\circ\alpha_P=\alpha_{A'}\circ f$, $f(ph')=f(p)h'$ for $p\in P$, $h'\in H$. Let $i_*(f)=i\circ f$. Then $i\circ f\circ\alpha_P=i\circ\alpha_{A'}\circ f=\alpha_A\circ i\circ f$ and $i\circ f(ph')=i(f(p)h')=i(f(p))h'=i\circ f(p)h'$.

For a morphism $g\in\mathrm{Hom}_H((P,\alpha_P),(A'',\alpha_{A''}))$, i.e., $g:P\longrightarrow A''$ such that $g\circ\alpha_P=\alpha_{A''}\circ g$ and $g(ph')=g(p)h'$, there is a morphism $h:P\longrightarrow A$ such that $\pi\circ h=g$. Since (P,α_P) is projective, we have $h\circ\alpha_P=\alpha_A\circ h$ and $h(ph')=h(p)h'$. So $\mathrm{Hom}_H((P,\alpha_P),-)$ is an exact functor.

Conversely, assume that $\mathrm{Hom}_H((P,\alpha_P),-)$ is an exact functor. So π_* is surjective: If $g\in\mathrm{Hom}_H((P,\alpha_P),(A'',\alpha_{A''}))$ and there exists a morphism $h\in\mathrm{Hom}_H((P,\alpha_P),(A,\alpha_A))$ with $g=\pi_*(h)=\pi\circ h$, $h\circ\alpha_P=\alpha_A\circ h$ and $h(ph')=h(p)h'$ for $p\in P$, $h'\in H$, then (P,α_P) is a projective right Hom-H-module.

Corollary 3.3 *For any right Hom-H-module (N,α_N), $\mathrm{Hom}_H((N,\alpha_N),-)$ is a covariant left exact functor.*

Corollary 3.4 *A right Hom-H-module (P,α_P) is projective, and then every short exact sequence $0\longrightarrow(A,\alpha_A)\xrightarrow{i}(B,\alpha_B)\xrightarrow{\pi}(P,\alpha_P)\longrightarrow 0$ splits.*

Dual to Theorem 3.2, we get the following theorem.

Theorem 3.4 $(P, \alpha_P) = \left(\bigoplus_{i\in I} P_i, \bigoplus_{i\in I} \alpha_{P_i}\right)$ *is projective if and only if every right Hom-H-module* (P_i, α_{P_i}) *is projective.*

4 Hom-Path Algebras

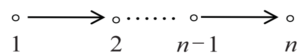
The purpose of this section is to define the concept of Hom-path algebra and give the types of quivers whose path algebras can be made into (nontrivial) Hom-path algebras.

Definition 4.1 *A Hom-path algebra is a pair* (KQ, α) *in which:*
(1) KQ *is a path algebra of a quiver* Q*, and*
(2) $\alpha : KQ \longrightarrow KQ$ *is a linear endomorphism.*

In the following, we consider the Hom-path algebras of some Dynkin diagrams and other quiver diagrams.

1. A_n-type

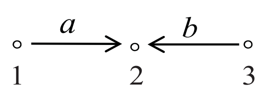
(1) Let Q be the quiver



and let KQ be the path algebra. $\alpha : KQ \longrightarrow KQ$ is an endomorphism. It is easy to see that (KQ, α) is a Hom-path algebra if and only if $\alpha = \mathrm{Id}_{KQ}$.

Remark 4.1 If the arrows are all reversed in the quiver Q above, the conclusion also holds.

(2) Let Q be the quiver

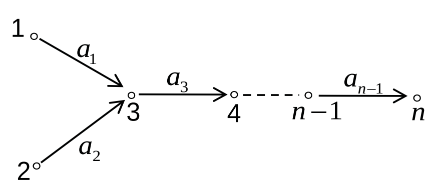


and let KQ be the path algebra. We construct $\alpha : KQ \longrightarrow KQ$, the endomorphism of KQ, by $\alpha(e_1) = e_3,\ \alpha(e_2) = e_2,\ \alpha(e_3) = e_1,\ \alpha(a) = b$ and $\alpha(b) = a$. Then (KQ, α) is the nontrivial Hom-path algebra and this case can be extended to all situations of centrosymmetry.

The indecomposable injective Hom-KQ-modules are $(E_1, \alpha|_{E_1}),\ (E_2, \alpha|_{E_2})$ and $(E_3, \alpha|_{E_3})$, where E_1 has a K-basis $\{e_1\}$, E_2 has a K-basis $\{e_2, a, b\}$ and E_3 has a K-basis $\{e_3\}$. The indecomposable projective Hom-KQ-modules are $(P_1, \alpha|_{P_1}),\ (P_2, \alpha|_{P_2})$ and $(P_3, \alpha|_{P_3})$, where P_1 has a K-basis $\{e_1, a\}$, P_2 has a K-basis $\{e_2\}$ and P_3 has a K-basis $\{e_3, b\}$.

2. D_n-type

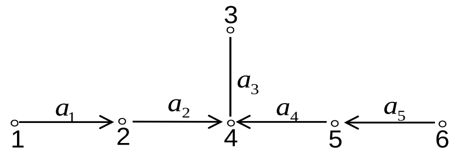
Let Q be the quiver



and let KQ be the path algebra. We set $\alpha : KQ \longrightarrow KQ$ by $\alpha(e_1) = e_2,\ \alpha(e_2) = e_1,\ \alpha(e_3) = e_3,\ \cdots,\ \alpha(e_n) = e_n,\ \alpha(a_1) = a_2,\ \alpha(a_2) = a_1,\ \alpha(a_3) = a_3,\ \cdots$, and $\alpha(a_{n-1}) = a_{n-1}$. Thus (KQ, α) is the nontrivial Hom-path algebra.

3. E_6-type

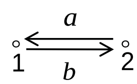
Let Q be the quiver



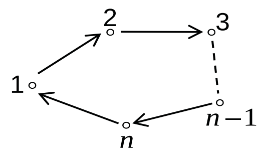
and let KQ be the path algebra. We set $\alpha : KQ \longrightarrow KQ$ by $\alpha(e_1) = e_6,\ \alpha(e_2) = e_5,\ \alpha(e_3) = e_3,\ \alpha(e_4) = e_4,\ \alpha(e_5) = e_2,\ \alpha(e_6) = e_1,\ \alpha(a_1) = a_5,\ \alpha(a_2) = a_4,\ \alpha(a_3) = a_3,\ \alpha(a_4) = a_2,$ and $\alpha(a_5) = a_1$. Thus (KQ, α) is the nontrivial Hom-path algebra.

4. Type of $\widetilde{A}_n$ with cyclic paths

Let Q be the quiver



and let KQ be the path algebra. We set $\alpha : KQ \longrightarrow KQ$ by $\alpha(e_1) = e_2,\ \alpha(e_2) = e_1,\ \alpha(a) = b,$ and $\alpha(b) = a$. Thus (KQ, α) is the nontrivial Hom-path algebra. This case can be extended to the following situation.



Definition 4.2 (see [2]) *A Hom-coalgebra is a triple* $(C, \Delta, \beta, \epsilon)$ *in which:*

(1) C *is a* K*-comodule;*

(2) $\Delta : C \longrightarrow C \otimes_K C$ *is a bilinear map;*

(3) $\beta : C \longrightarrow C$ *is a linear endomorphism;*

(4) $\epsilon : C \longrightarrow K$*, the counit, is a linear map.*

Definition 4.3 *By a Hom-comodule, we mean a pair* (W, β) *consisting of*

(1) *a* K*-comodule* W*, and*

(2) *a linear endomorphism* $\beta : W \longrightarrow W$.

A morphism $f : (M, \beta_M) \longrightarrow (N, \beta_N)$ *of Hom-comodules is a linear map* $f : M \longrightarrow N$ *such that* $f \circ \beta_M = \beta_N \circ f$.

Definition 4.4 *A Hom-path coalgebra is a pair* (KQ, β) *in which:*

(1) KQ *is a path coalgebra of* Q*, and*

(2) $\beta : KQ \longrightarrow KQ$ *is a linear endomorphism.*

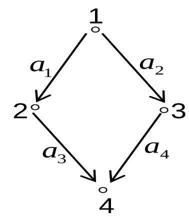
Remark 4.2 The Hom-path algebra can be viewed as a Hom-path coalgebra.

In fact, let Q be a quiver and let (KQ, α) be a Hom-path algebra, that is, KQ is a path algebra of Q and $\alpha : KQ \longrightarrow KQ$ is a linear endomorphism. First, we know that KQ can be a path coalgebra of Q, which is described in the preliminaries. Then, it is only to construct a linear endomorphism $\beta : KQ \longrightarrow KQ$ of the path coalgebra KQ. We can see the case of (2) in A_n-type on the 10th page where the endomorphism α of path algebra is also an endomorphism of path coalgebra by

$$\Delta(\alpha(e_i)) = \alpha(e_i) \times \alpha(e_i),$$
$$\Delta(\alpha(a)) = \alpha(a) \otimes \alpha(e_1) + \alpha(e_2) \otimes \alpha(a),$$
$$\Delta(\alpha(b)) = \alpha(b) \otimes \alpha(e_3) + \alpha(e_2) \otimes \alpha(b).$$

Definition 4.5 *A Hom-path coalgebra* (S, β_S) *is said to be a relation Hom-subcoalgebra of a Hom-path Coalgebra* (KQ, β_{KQ}) *if* S *is a relation subcoalgebra of a path coalgebra* KQ *and* $\beta_{KQ}|_S = \beta_S$.

Example 4.1 Let Q be the quiver



and let KQ be the path coalgebra. We define $\beta : KQ \longrightarrow KQ$ by $\beta(e_1) = e_1$, $\beta(e_2) = e_3$, $\beta(e_3) = e_2$, $\beta(e_4) = e_4$, $\beta(a_1) = a_2$, $\beta(a_2) = a_1$, $\beta(a_3) = a_4$, and $\beta(a_4) = a_3$. Let S be generated by $\{e_1, e_2, e_3, e_4, a_1, a_2, a_3, a_4\}$ as a K-basis. It is easy to see that S is a relation subcoalgebra of KQ. Then $(S, \beta|_S)$ is a relation Hom-subcoalgebra.

Acknowledgement The authors would like to thank the referees for the careful reading and valuable suggestions.

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