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Sharp Inequalities for BMO Functions*

Adam OSĘKOWSKI[1]

Abstract The purpose of the paper is to study sharp weak-type bounds for functions of bounded mean oscillation. Let $0 < p < \infty$ be a fixed number and let I be an interval contained in $\mathbb{R}$. The author shows that for any $\varphi : I \to \mathbb{R}$ and any subset $E \subset I$ of positive measure,

$$\frac{|I|^{-\frac{1}{p}}}{|E|^{1-\frac{1}{p}}}\int_E \Big|\varphi - \frac{1}{|I|}\int_I \varphi \mathrm{d}y\Big| \mathrm{d}x \le \|\varphi\|_{\mathrm{BMO}(I)}, \quad 0 < p \le 2,$$

$$\frac{|I|^{-\frac{1}{p}}}{|E|^{1-\frac{1}{p}}}\int_E \Big|\varphi - \frac{1}{|I|}\int_I \varphi \mathrm{d}y\Big| \mathrm{d}x \le \frac{p}{2^{\frac{2}{p}}}\mathrm{e}^{\frac{2}{p}-1}\|\varphi\|_{\mathrm{BMO}(I)}, \quad p \ge 2.$$

For each p, the constant on the right-hand side is the best possible. The proof rests on the explicit evaluation of the associated Bellman function. The result is a complement of the earlier works of Slavin, Vasyunin and Volberg concerning weak-type, L^p and exponential bounds for the BMO class.

Keywords BMO, Bellman function, Weak-type inequality, Best constants
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1 Introduction

A real-valued locally integrable function φ defined on $\mathbb{R}^n$ is said to be in BMO, the space of functions of bounded mean oscillation, if

$$\sup_Q \langle|\varphi - \langle\varphi\rangle_Q|\rangle_Q < \infty, \tag{1.1}$$

where the supremum is over all cubes Q in $\mathbb{R}^n$ with edges parallel to the coordinate axes, and

$$\langle\varphi\rangle_Q = \frac{1}{|Q|}\int_Q \varphi(x)\mathrm{d}x$$

denotes the average of φ over Q. We will consider a slightly less restrictive setting in which only the cubes Q within a given Q^0 are considered; to stress the dependence on Q^0, we will use the notation $\mathrm{BMO}(Q^0)$.

The BMO class, introduced by John and Nirenberg in [8], plays an important role in analysis and probability, since many classical operators (maximal, singular integral, etc.) map L^∞ into BMO. Another remarkable result, due to Fefferman [4], asserts that BMO is a dual to the Hardy

Manuscript received July 20, 2013. Revised February 18, 2014.
[1]Faculty of Mathematics, Informatics and Mechanics, University of Warsaw, Banacha 2, 02-097 Warsaw, Poland. E-mail: ados@mimuw.edu.pl
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space H^1. It is well-known that the functions of bounded mean oscillation have very strong integrability properties (see [8]). In particular, the p-oscillation

$$\|\varphi\|_{\mathrm{BMO}^p} := \sup_Q \langle |\varphi - \langle\varphi\rangle_Q|^p\rangle_Q^{\frac{1}{p}}, \quad 1 < p < \infty$$

is finite for any $\varphi \in \mathrm{BMO}$. It turns out that $\|\cdot\|_{\mathrm{BMO}^p}$ forms an equivalent norm on $\mathrm{BMO}(\mathbb{R}^n)$. In what follows, we will work with $\|\cdot\|_{\mathrm{BMO}^2}$ and denote it simply by $\|\cdot\|_{\mathrm{BMO}}$. One of the reasons we choose this particular norm is that we have the identity

$$\|\varphi\|_{\mathrm{BMO}^2} = \sup_Q \{\langle\varphi^2\rangle_Q - \langle\varphi\rangle_Q^2\}^{\frac{1}{2}}, \tag{1.2}$$

which makes the norm very convenient to handle; see below. Furthermore, from now on, we restrict ourselves to the case $n = 1$. Then the cubes become intervals, and to stress that we work in the one-dimensional setting, we will replace the letter Q with I.

In the recent years, there has been a considerable interest in obtaining various sharp estimates for the BMO class. Probably the first result in this direction is that of Slavin [14] and Slavin and Vasyunin [15], which identifies the optimal constants in the so-called integral form of John-Nirenberg inequality. Namely, it was shown there that if $\varphi : I \to \mathbb{R}$ satisfies $\|\varphi\|_{\mathrm{BMO}(I)} < 1$, then

$$\langle \mathrm{e}^\varphi\rangle_I \le \frac{\exp(-\|\varphi\|_{\mathrm{BMO}(I)})}{1 - \|\varphi\|_{\mathrm{BMO}(I)}}\, \mathrm{e}^{\langle\varphi\rangle_I}.$$

Furthermore, this bound is sharp in the sense that for each $\varepsilon < 1$ there is a function φ satisfying $\|\varphi\|_{BMO(I)} = \varepsilon$ and $\langle \mathrm{e}^\varphi\rangle_I = \frac{\mathrm{e}^{-\varepsilon}\mathrm{e}^{\langle\varphi\rangle_I}}{1-\varepsilon}$. In particular, this shows that there exists no exponential estimate of the above type when $\|\varphi\|_{\mathrm{BMO}(I)} \ge 1$.

The following sharp version of the related classical weak form of John-Nirenberg inequality is due to Vasyunin [18] and Vasyunin and Volberg [20]. Namely, if $\varepsilon := \|\varphi\|_{\mathrm{BMO}(I)} < \infty$, then

$$\frac{1}{|I|}|\{s \in I : |\varphi(s) - \langle\varphi\rangle_I| \ge \lambda\}| \le \begin{cases} 1, & \text{if } 0 \le \lambda \le \varepsilon, \\ \dfrac{\varepsilon^2}{\lambda^2}, & \text{if } \varepsilon \le \lambda \le 2\varepsilon, \\ \dfrac{\mathrm{e}^2}{4}\mathrm{e}^{-\frac{\lambda}{\varepsilon}}, & \text{if } \lambda \ge 2\varepsilon, \end{cases}$$

and for each value of ε and λ, the equality can be attained. Optimizing over λ, we get the sharp weak-type inequality

$$\|\varphi - \langle\varphi\rangle_I\|_{L^{p,\infty}(I)} \le C_p \|\varphi\|_{\mathrm{BMO}(I)}. \tag{1.3}$$

Here

$$C_p = \begin{cases} 1, & \text{if } 0 < p < 2, \\ \dfrac{p\mathrm{e}^{\frac{2}{p}-1}}{2^{\frac{2}{p}}}, & \text{if } p \ge 2, \end{cases} \tag{1.4}$$

and

$$\|\varphi\|_{L^{p,\infty}(I)} = \sup_{\lambda > 0} \lambda \Big[\frac{1}{|I|}\big|\{s \in I : |\varphi(s)| \ge \lambda\}\big|\Big]^{\frac{1}{p}}$$

is the usual weak p-th quasinorm. We should also mention here a result of Korenovskii [9], who studied the weak-type constant for BMO space equipped with the norm $\|\cdot\|_{\mathrm{BMO}^1}$. He showed that the optimal (i.e., the largest) value of the constant c_2 in the inequality

$$\frac{1}{|I|}|\{s\in I:|\varphi(s)-\langle\varphi\rangle_I|\geq\lambda\}|\leq c_1\exp\Big(-\frac{c_2\lambda}{\|\varphi\|_{\mathrm{BMO}^1(I)}}\Big),\quad \lambda>0$$

equals $\frac{2}{\mathrm{e}}$. The reasoning rests on the careful analysis of the nonincreasing rearrangement of the function φ.

There is a related work of Slavin and Vasyunin [16], which provides the sharp comparison of the norms $\|\cdot\|_{\mathrm{BMO}^p}$ and $\|\cdot\|_{\mathrm{BMO}^2}$. Among other things, that paper contains the proof of the following statement:

$$\begin{aligned}
&2^{1-\frac{2}{p}}\|\varphi\|_{\mathrm{BMO}(I)}\leq\|\varphi\|_{\mathrm{BMO}^p(I)}\leq\|\varphi\|_{\mathrm{BMO}(I)} && \text{for } 0<p\leq 1,\\
&\Big(\frac{p}{2}\Gamma(p)\Big)^{\frac{1}{p}}\|\varphi\|_{\mathrm{BMO}(I)}\leq\|\varphi\|_{\mathrm{BMO}^p(I)}\leq\|\varphi\|_{\mathrm{BMO}(I)} && \text{for } 1<p\leq 2,\\
&\|\varphi\|_{\mathrm{BMO}(I)}\leq\|\varphi\|_{\mathrm{BMO}^p(I)}\leq\Big(\frac{p}{2}\Gamma(p)\Big)^{\frac{1}{p}}\|\varphi\|_{\mathrm{BMO}(I)} && \text{for } 2<p<\infty.
\end{aligned}$$

Furthermore, the right-hand side inequalities for $p<2$ and both left- and right-hand side inequalities for $p>2$ are sharp and attainable.

It should be pointed out here that except for Korenovskii's result, all the estimates formulated above were established by use of a powerful technique, the so-called Bellman function method. Roughly speaking, the approach turns the problem of proving a given estimate for a BMO class into the search of a certain special function, enjoying appropriate majorization and concavity conditions. The method originates from certain extremal constructions in the dynamic programming (see the recent edition of the classical monograph of Bellman [1]). As observed by Burkholder [2–3] in the 1980s, the framework of optimal stochastic control can be modified appropriately and used in the study of sharp inequalities for martingale transforms. In the 1990s, in the works [11–12], Nazarov, Treil and Volberg noticed some deep connections between the Bellman approach and various aspects of harmonic analysis, and formulated the general modern framework of the technique. Since then, the method has been applied in numerous papers, in both harmonic analysis and probability. The literature on the subject is very large and it is impossible to review it here. We only refer the reader to the works [10, 13, 19], the papers mentioned above and the references therein.

Finally, we would like to mention here the recent work of Ivanishvili et al. [7], which treats the above BMO estimates from a much wider perspective. More specifically, it provides the detailed description of the machinery which can be used to prove a general estimate in the BMO setting (under some regularity conditions on the boundary value function). Consult also [6] for the short discussion on the subject.

We turn our attention to the main results of this paper. As in [18, 20], we will be interested in the weak-type estimates for BMO functions, but we will work under a different norming of the weak spaces. Namely, for $0<p<\infty$ and $\varphi\in L^{p,\infty}(I)$, define

$$|||\varphi|_{L^{p,\infty}(I)}=\sup\Big\{\frac{|I|^{-\frac{1}{p}}}{|E|^{1-\frac{1}{p}}}\int_E|\varphi(x)|\mathrm{d}x\Big\},$$

where the supremum is taken over all measurable subsets E of I satisfying $|E| > 0$. It is well-known (see [5]) that $|||\cdot|||_{L^{p,\infty}(I)}$ is an equivalent norm in the space $L^{p,\infty}$ (for $1 < p < \infty$). Furthermore, one easily verifies that $\|\varphi\|_{L^{p,\infty}(I)} \leq |||\varphi$
$|_{L^{p,\infty}(I)}$ for any function φ and any $0 < p < \infty$.

Our main result can be stated as follows.

Theorem 1.1 *Suppose that φ belongs to the class* BMO(I)*. Then for any $0 < p < \infty$ we have*

$$\big|\big|\big|\varphi - \langle\varphi\rangle_I\big|\big|\big|_{L^{p,\infty}(I)} \leq C_p\|\varphi\|_{\mathrm{BMO}(I)}, \tag{1.5}$$

where C_p is given by (1.4)*. For each p the inequality is sharp.*

Thus, in comparison to (1.3), we see that the optimal constant under the new stronger norming of $L^{p,\infty}$ remains the same. As in the aforementioned papers, our approach will rest on the Bellman function method. We would like to point out here that the desired estimate does not fall into the scope of the (general) bounds covered by [6–7], since the corresponding boundary value function is not sufficiently regular. However, the arguments developed in these papers have turned out to be quite helpful during the search.

We have organized this paper as follows. The next section is devoted to the proof of (1.5). The validity of this estimate is deduced from the existence of a certain special function. In Section 3, we exhibit some examples which show that for each p equality can hold in (1.5); thus the constant C_p appearing in this estimate can not be replaced by a smaller number.

2 A Locally Concave Function and the Proof of (1.5)

We start from the observation that it suffices to prove the inequality (1.5) for $p \geq 2$. Indeed, having successfully done this, we pick $0<p<2$ and apply the weak-type BMO $\to L^{2,\infty}$ estimate to get

$$|||\varphi - \langle\varphi\rangle_I|||_{L^{p,\infty}(I)} \leq |||\varphi - \langle\varphi\rangle_I|||_{L^{2,\infty}(I)} \leq \|\varphi\|_{\mathrm{BMO}(I)} = C_p\|\varphi\|_{\mathrm{BMO}(I)}.$$

Thus, in the remainder of this section, we assume that $p \geq 2$. Suppose that $\lambda \geq 1$ is a fixed parameter and consider the parabolic strip

$$\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 \leq y \leq x^2 + 1\}.$$

A key step in the analysis of the inequality (1.5) is to introduce the following Bellman function $\mathbb{B}_{p,\lambda} : \Omega \to [0, \infty)$:

$$\mathbb{B}_{p,\lambda}(x, y) = \sup\Big\{\frac{1}{|I|}\int_I(\varphi(s) - \lambda)_+\mathrm{d}s : \ \|\varphi\|_{\mathrm{BMO}(I)} \leq 1,\ \langle\varphi\rangle_I = x,\ \langle\varphi^2\rangle_I = y\Big\}.$$

It is easy to see that $\mathbb{B}_{p,\lambda}$ is well defined (i.e., for each $(x, y) \in \Omega$ there exists at least one function φ satisfying the required properties). Indeed, define

$$\varphi = (x - \sqrt{y - x^2})\chi_{I^-} + (x + \sqrt{y - x^2})\chi_{I^+},$$

where I^- and I^+ are the left and right half of I, respectively. Then $\langle\varphi\rangle_I = x$, $\langle\varphi^2\rangle_I = y$, and to show the upper bound for the BMO norm of φ, pick an arbitrary subinterval J of I. Denoting

$J^{\pm} = J \cap I^{\pm}$, we compute that

$$\begin{aligned}\langle \varphi^2 \rangle_J &= \frac{|J^-|}{|J|}(x - \sqrt{y-x^2})^2 + \frac{|J^+|}{|J|}(x + \sqrt{y-x^2})^2 \\ &= y + 2x\sqrt{y-x^2} \cdot \frac{|J^+| - |J^-|}{|J|} \\ &= 2x\langle \varphi \rangle_J - 2x^2 + y \le 2x\langle \varphi \rangle_J - x^2 + 1 \le \langle \varphi \rangle_J^2 + 1,\end{aligned}$$

and hence $\|\varphi\|_{\mathrm{BMO}} \le 1$, as J is arbitrary.

Actually, to show (1.5), we do not need the explicit formula for $\mathbb{B}_{p,\lambda}$, but it suffices to have an accurate upper bound for this function. To provide such an estimate, let us first split Ω into the union of the following sets (see Figure 2 below):

$$\begin{aligned} D_1 &= \{(x,y) \in \Omega : y > 2|x|\}, \\ D_2 &= \{(x,y) \in \Omega : 2|x| \ge y > 2\lambda|x| - \lambda^2 + 1\}, \\ D_3 &= \{(x,y) \in \Omega : 2\lambda|x| - \lambda^2 + 1 > y\}, \\ D_4 &= \{(x,y) \in \Omega : |x| \ge \lambda,\ y > 2\lambda|x| - \lambda^2 + 1\}. \end{aligned}$$

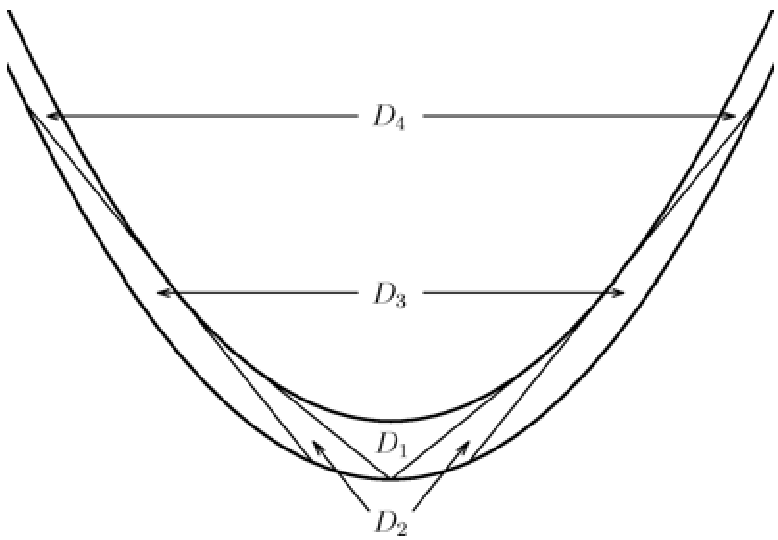

Figure 1 The regions D_1–D_4

Now, consider the function $B_{p,\lambda} : \Omega \to [0,\infty)$ given by

$$B_{p,\lambda}(x,y) = \begin{cases} \frac{1}{4} y \mathrm{e}^{1-\lambda} & \text{on } D_1, \\ \frac{1}{2}(1 - \sqrt{x^2+1-y}) \exp(|x| - \lambda + \sqrt{x^2+1-y}) & \text{on } D_2, \\ \frac{1}{2}(\sqrt{y - 2\lambda|x| + \lambda^2} - \lambda + |x|) & \text{on } D_3, \\ |x| - \lambda + \frac{1 - \sqrt{x^2+1-y}}{2} \exp(\lambda - |x| + \sqrt{x^2+1-y}) & \text{on } D_4. \end{cases}$$

This function enjoys the following properties. First, observe that

$$B_{p,\lambda}(x,x^2) = (|x| - \lambda)_+ \quad \text{for all } x \in \mathbb{R}. \tag{2.1}$$

Second, note that we obviously have

$$B_{p,\lambda}(0,y)=\frac{1}{4}y\mathrm{e}^{1-\lambda}\le\frac{1}{4}\mathrm{e}^{1-\lambda}. \tag{2.2}$$

The final property of $B_{p,\lambda}$ is studied in a separate lemma below.

Lemma 2.1 *The function $B_{p,\lambda}$ is locally concave, i.e., it is concave along any line segment contained in Ω.*

Proof Clearly, $B_{p,\lambda}$ is of class C^∞ on each D_i^o (where A^o denotes the interior of a set A). Moreover, it is straightforward to check that the function $B_{p,\lambda}$ is continuous and of class C^1 in the interior of Ω: One needs to verify that the partial derivatives of $B_{p,\lambda}$ match appropriately at the common boundaries of the sets D_i. We leave the necessary calculations to the reader. Thus, to establish the assertion, it suffices to show that the Hessian matrix of $B_{p,\lambda}$ is nonpositive-definite in the interior of each D_i. Since $B_{p,\lambda}(x,y)=B_{p,\lambda}(-x,y)$, it is enough to handle the matrix $D^2B_{p,\lambda}(x,y)$ for $x\ge 0$ only. We start from observing that if (x,y) lies in the interior of one of D_i's, then there exists a (short) line segment passing through this point, along which $B_{p,\lambda}$ is linear. This is clear when $(x,y)\in D_1^o$ (any line segment contained in D_1 does the job) or $(x,y)\in D_3^o$ (take a line segment of slope 2λ: Then the square root $\sqrt{y-2\lambda x+\lambda^2}$ is constant). When $(x,y)\in D_2^o$, we consider the line segment of slope $a=2x+2\sqrt{x^2+1-y}$. Then, for t sufficiently close to 0, we see that

$$B_{p,\lambda}(x+t,y+at)=\frac{1}{2}(1+\sqrt{x^2+1-y}-t)\exp(x-\lambda+\sqrt{x^2+1-y})$$

is a linear function of t. A similar calculation shows that for $(x,y)\in D_4^o$, the function $B_{p,\lambda}$ is linear along a (short) line segment of slope $2x-2\sqrt{x^2+1-y}$. This "local linearity" implies that the Hessian matrix of $B_{p,\lambda}$ has determinant 0. Therefore we will be done if we prove that the second-order partial derivative $\partial_y^2B_{p,\lambda}$ is nonpositive on $D_1^o\cup D_2^o\cup D_3^o\cup D_4^o$. But this is simple: A little calculation shows that $\partial_y^2B_{p,\lambda}(x,y)$ equals

$$\begin{cases}0, & \text{if } (x,y)\in D_1^o,\\ -\dfrac{1}{8}(x^2+1-y)^{-\frac{1}{2}}\exp(x-\lambda+\sqrt{x^2+1-y}), & \text{if } (x,y)\in D_2^o,\\ -\dfrac{1}{8}(y-2x\lambda+\lambda^2)^{-\frac{3}{2}}, & \text{if } (x,y)\in D_3^o,\\ -\dfrac{1}{8}(x^2+1-y)^{-\frac{1}{2}}\exp(-x+\lambda+\sqrt{x^2+1-y}), & \text{if } (x,y)\in D_4^o,\end{cases}$$

and all the expressions are nonpositive.

Now we will prove that $\mathbb{B}_{p,\lambda}\le B_{p,\lambda}$. To accomplish this, we will require the following lemma, which can be found in [15] (consult Lemma 4c there).

Lemma 2.2 *Fix $\varepsilon<1$. Then for every interval I and every $\varphi:I\to\mathbb{R}$ with $\|\varphi\|_{\mathrm{BMO}(I)}\le\varepsilon$, there exists a splitting $I=I_-\cup I_+$ such that the whole straight-line segment with the endpoints $(\langle\varphi\rangle_\pm,\langle\varphi^2\rangle_\pm)$ is contained within Ω. Moreover, the splitting parameter $\alpha=\frac{|I_+|}{|I|}$ can be chosen uniformly (with respect to φ and I) separated from 0 and 1.*

Now we will show the key majorization.

Theorem 2.1 *For any $p \geq 2$ and $\lambda \geq 1$, we have $\mathbb{B}_{p,\lambda} \leq B_{p,\lambda}$.*

Proof Pick an arbitrary $(x, y) \in \Omega$ and let $\varphi : I \to \mathbb{R}$ be an arbitrary function as in the definition of $\mathbb{B}_{p,\lambda}(x, y)$. Next, let $\varepsilon \in (0, 1)$ be a fixed parameter and put $\widetilde{\varphi} = \varepsilon\varphi$; then, clearly, $\|\widetilde{\varphi}\|_{\mathrm{BMO}(I)} \leq \varepsilon$. Consider the following family $\{\mathcal{I}^n\}_{n\geq 0}$ of partitions of I, generated by the inductive use of Lemma 2.2. We start with $\mathcal{I}^0 = \{I\}$; then, given $\mathcal{I}^n = \{I^{n,1}, I^{n,2}, \cdots, I^{n,2^n}\}$, we split each $I^{n,k}$ according to Lemma 2.2, applied to the function $\widetilde{\varphi}$, and put

$$\mathcal{I}^{n+1} = \{I_-^{n,1}, I_+^{n,1}, I_-^{n,2}, I_+^{n,2}, \cdots, I_-^{n,2^n}, I_+^{n,2^n}\}.$$

Next, we define functional sequences $(\varphi_n)_{n\geq 0}$ and $(\psi_n)_{n\geq 0}$ by the formulas

$$\varphi_n(x) = \langle \widetilde{\varphi} \rangle_{I^n(x)} \quad \text{and} \quad \psi_n(x) = \langle \widetilde{\varphi}^2 \rangle_{I^n(x)},$$

where $I^n(x) \in \mathcal{I}^n$ is an interval containing x (if there exist two such intervals, we pick the one which has x as its right endpoint). An important observation, which is the consequence of the fact that we work with $\|\cdot\|_{\mathrm{BMO}^2}$-norm, is that for each n the pair (φ_n, ψ_n) takes values in Ω. Indeed, for any $J \in \mathcal{I}^n$ we have

$$0 \leq \langle \widetilde{\varphi}^2 \rangle_J - \langle \widetilde{\varphi} \rangle_J^2 \leq 1,$$

where the left bound is due to Schwarz inequality, and the right follows from (1.2) and the assumption $\|\widetilde{\varphi}\|_{\mathrm{BMO}(I)} = \varepsilon\|\varphi\|_{\mathrm{BMO}(I)} \leq 1$.

Now, we will show that for any $n \geq 0$ and any $1 \leq k \leq 2^n$, we have

$$\int_{I^{n,k}} B_{p,\lambda}(\varphi_n(s), \psi_n(s))\mathrm{d}s \geq \int_{I^{n,k}} B_{p,\lambda}(\varphi_{n+1}(s), \psi_{n+1}(s))\mathrm{d}s. \tag{2.3}$$

To do this, note that φ_n and ψ_n are constant on $I^{n,k}$, while φ_{n+1} and ψ_{n+1} are constant on the intervals $I_\pm^{n,k}$ into which $I^{n,k}$ splits. Therefore, dividing both sides by $|I^{n,k}|$, we see that the above estimate is equivalent to

$$\begin{aligned} B_{p,\lambda}(\langle \widetilde{\varphi} \rangle_{I^{n,k}}, \langle \widetilde{\varphi}^2 \rangle_{I^{n,k}}) \geq{}& \frac{|I_-^{n,k}|}{|I^{n,k}|} B_{p,\lambda}(\langle \widetilde{\varphi} \rangle_{I_-^{n,k}}, \langle \widetilde{\varphi}^2 \rangle|_{I_-^{n,k}}) \\ &+ \frac{|I_+^{n,k}|}{|I^{n,k}|} B_{p,\lambda}(\langle \widetilde{\varphi} \rangle_{I_+^{n,k}}, \langle \widetilde{\varphi}^2 \rangle_{I_+^{n,k}}). \end{aligned}$$

This bound follows from the local concavity of $B_{p,\lambda}$ and the fact that the whole line segment with endpoints $\big(\langle \widetilde{\varphi} \rangle_{I_\pm^{n,k}}, \langle \widetilde{\varphi}^2 \rangle_{I_\pm^{n,k}}\big)$ is contained in Ω (which is guaranteed by Lemma 2.2). Summing (2.3) over all $k = 1, 2, \cdots, 2^n$, we get

$$\int_I B_{p,\lambda}(\varphi_n(s), \psi_n(s))\mathrm{d}s \geq \int_I B_{p,\lambda}(\varphi_{n+1}(s), \psi_{n+1}(s))\mathrm{d}s.$$

Hence, by induction,

$$\int_I B_{p,\lambda}(\varphi_0(s), \psi_0(s))\mathrm{d}s \geq \int_I B_{p,\lambda}(\varphi_n(s), \psi_n(s))\mathrm{d}s \tag{2.4}$$

for any $n = 0, 1, 2, \cdots$. To handle the left-hand side, observe that

$$\frac{1}{|I|}\int_I B_{p,\lambda}(\varphi_0(s), \psi_0(s))\mathrm{d}s = B_{p,\lambda}(\langle \widetilde{\varphi} \rangle_I, \langle \widetilde{\varphi}^2 \rangle_I) = B_{p,\lambda}(\varepsilon x, \varepsilon^2 y).$$

To deal with the right-hand side of (2.4), let n go to infinity. Since the splitting ratio of Lemma 2.2 is bounded away from 0 and 1, we see that the diameter of the partition $\mathcal{I}^n$ (i.e., $\sup_{1\leq k\leq 2^n} |I^{n,k}|$) tends to 0. Consequently, by Lebesgue's differentiation theorem, we have $\varphi_n \to \widetilde{\varphi}$ and $\psi_n \to \widetilde{\varphi}^2$ almost everywhere on I. Combining the above facts with Fatou's lemma, we see that (2.4) leads to

$$\frac{1}{|I|}\int_I B_{p,\lambda}(\widetilde{\varphi}(s),\widetilde{\varphi}^2(s))\mathrm{d}s \leq B_{p,\lambda}(\varepsilon x,\varepsilon^2 y),$$

so, by (2.1),

$$\frac{1}{|I|}\int_I (\varepsilon|\varphi(s)|-\lambda)_+\mathrm{d}s \leq B_{p,\lambda}(\varepsilon x,\varepsilon^2 y).$$

It remains to let $\varepsilon \to 1$ and use continuity of $B_{p,\lambda}$ and Fatou's lemma again. As the result, we obtain

$$\frac{1}{|I|}\int_I (|\varphi(s)|-\lambda)_+\mathrm{d}s \leq B_{p,\lambda}(x,y),$$

and since φ was arbitrary, the bound $\mathbb{B}_{p,\lambda}(x,y) \leq B_{p,\lambda}(x,y)$ follows.

Remark 2.1 Using examples similar to those which will appear in Section 3 below, one can show that we actually have an equality: $\mathbb{B}_{p,\lambda} = B_{p,\lambda}$. However, this is quite elaborate and we will not need this; hence, we have decided not to include the details here.

We turn to the inequality of Theorem 1.1.

Proof of (1.5) By homogeneity, it is enough to show that if $\|\varphi\|_{\mathrm{BMO}(I)} \leq 1$, then

$$\frac{|I|^{\frac{1}{p}}}{|E|^{1-\frac{1}{p}}}\int_E |\varphi-\langle\varphi\rangle_I|\mathrm{d}x \leq \frac{p}{2}\mathrm{e}^{1-\frac{2}{p}} \tag{2.5}$$

for any $E \subset I$ of positive measure. Without loss of generality we may assume that φ has integral 0 and that I has length one. Pick E as above and decompose it into the union of

$$E^+ = \{s\in I : |\varphi(s)| \geq \lambda\}\cap E \quad \text{ and } \quad E^- = \{s\in I : |\varphi(s)| < \lambda\}\cap E.$$

Using the estimates $\mathbb{B}_{p,\lambda} \leq B_{p,\lambda}$ and (2.2), we get

$$\int_{E^+}(|\varphi(s)|-\lambda)\mathrm{d}s \leq \int_I \big(|\varphi(s)|-\lambda\big)_+\mathrm{d}s \leq B_{p,\lambda}(0,y) \leq \frac{1}{4}\mathrm{e}^{1-\lambda}.$$

Furthermore, we obviously have

$$\int_{E^-}(|\varphi(s)|-\lambda)\mathrm{d}s \leq 0.$$

Adding the two estimates above, we obtain an inequality which can be equivalently transformed into

$$\int_E |\varphi(s)|\mathrm{d}s \leq \lambda|E| + \frac{1}{4}\mathrm{e}^{1-\lambda}. \tag{2.6}$$

Now, suppose that $|E| \leq \frac{1}{4}$. By a straightforward analysis of a derivative, we check that the right-hand side above, considered as a function of $\lambda \in [1,\infty)$, attains its minimum for

$\lambda = 1 - \ln(4|E|)$. Plugging this value into (2.6) and dividing throughout by $|E|^{1-\frac{1}{p}}$, we obtain the inequality

$$\frac{1}{|E|^{1-\frac{1}{p}}}\int_E |\varphi(s)|\mathrm{d}s \le |E|^{\frac{1}{p}}(2-\ln(4|E|)).$$

However, the function $t \mapsto t^{\frac{1}{p}}(2-\ln(4t))$, $t \in \left(0, \frac{1}{4}\right]$, attains its maximum $\frac{p\mathrm{e}^{\frac{2}{p}-1}}{2^{\frac{2}{p}}}$ at $t = \frac{\mathrm{e}^{2-p}}{4}$. This shows (2.5) for small $|E|$. Next, suppose that the converse bound $|E| > \frac{1}{4}$ holds true. Since $\langle\varphi^2\rangle_I \le \|\varphi\|^2_{\mathrm{BMO}(I)} \le 1$, Schwarz inequality yields

$$\int_E |\varphi|\mathrm{d}s = \int_I |\varphi|\chi_E \mathrm{d}s \le \langle\varphi^2\rangle_I^{\frac{1}{2}}|E|^{\frac{1}{2}} \le |E|^{\frac{1}{2}}.$$

On the other hand, we have

$$\frac{p}{2^{\frac{2}{p}}}\mathrm{e}^{\frac{2}{p}-1}|E|^{1-\frac{1}{p}} = |E|^{\frac{1}{2}} \cdot \frac{p}{2}\mathrm{e}^{\frac{2}{p}-1} \cdot (4|E|)^{\frac{1}{2}-\frac{1}{p}} \ge |E|^{\frac{1}{2}}.$$

This completes the proof of (2.5) and thus (1.5) is established.

3 Sharpness

For each $0 < p < \infty$, we will give an example of $E \subset [0,1]$ and $\varphi : [0,1] \to \mathbb{R}$ with $\|\varphi\|_{\mathrm{BMO}([0,1])} \le 1$ for which both sides of (2.5) are equal. This will clearly give the optimality of the constant C_p in (1.5).

We start with the easy case $0 < p \le 2$. Pick $\varphi = \chi_{[0,\frac{1}{2}]} - \chi_{(\frac{1}{2},1]}$ and $E = [0,1]$. Then $\langle\varphi\rangle_I = 0$ and the left-hand side of (2.5) is equal to 1. Furthermore, φ^2 is identically 1, so

$$\|\varphi\|_{\mathrm{BMO}([0,1])} \le \sup_{J \subseteq I}\langle\varphi^2\rangle_J = 1.$$

Thus we must have the equality in (2.5) and the sharpness follows.

Next, we turn to the more difficult case $p > 2$. Consider the function $\varphi : [0,1] \to \mathbb{R}$, given by the condition $\varphi(s) = -\varphi(1-s)$ and

$$\varphi(s) = \begin{cases} p, & \text{if } 0 \le s \le \dfrac{\mathrm{e}^{2-p}}{8}, \\[2mm] p-2, & \text{if } \dfrac{\mathrm{e}^{2-p}}{8} < s \le \dfrac{\mathrm{e}^{2-p}}{4}, \\[2mm] -\ln(4s), & \text{if } \dfrac{\mathrm{e}^{2-p}}{4} < s \le \dfrac{1}{4}, \\[2mm] 0, & \text{if } \dfrac{1}{4} < s \le \dfrac{1}{2}. \end{cases}$$

If we take $E = \left[0, \frac{\mathrm{e}^{2-p}}{8}\right] \cup \left[1 - \frac{\mathrm{e}^{2-p}}{8}, 1\right]$, then $|E| = \frac{\mathrm{e}^{2-p}}{4}$ and $|\varphi| \equiv p$ on E, so

$$\frac{|[0,1]|^{-\frac{1}{p}}}{|E|^{1-\frac{1}{p}}}\int_E |\varphi|\mathrm{d}s = \frac{p}{2^{\frac{2}{p}}}\mathrm{e}^{\frac{2}{p}-1}.$$

Therefore, all we need is the bound $\|\varphi\|_{\mathrm{BMO}([0,1])} \le 1$. In other words, we must show that for each $0 \le a < b \le 1$ we have

$$(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]}) \in \Omega. \tag{3.1}$$

Actually, by the symmetry of φ, we may restrict ourselves to $a \in \big[0, \frac{1}{2}\big)$. We split the reasoning into three separate parts.

Step 1 Let us start with the case $a = 0$, $b \in \big(0, \frac{1}{2}\big]$. If $b \leq \frac{e^{2-p}}{8}$, then φ is constant on $[a, b]$ and $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]}) = (p, p^2) \in \Omega$. If $\frac{e^{2-p}}{8} < b \leq \frac{e^{2-p}}{4}$, then the point $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ belongs to the line segment $\mathcal{S}$ joining (p, p^2) and $((p-2), (p-2)^2)$. This is a consequence of

$$\begin{aligned} \langle\varphi\rangle_{[a,b]} &= \frac{|[0, \frac{e^{2-p}}{8}]|}{|[0,b]|}\langle\varphi\rangle_{[0,\frac{e^{2-p}}{8}]} + \frac{|[\frac{e^{2-p}}{8}, b]|}{|[0,b]|}\langle\varphi\rangle_{[\frac{e^{2-p}}{8},b]} \\ &= \frac{|[0, \frac{e^{2-p}}{8}]|}{|[0,b]|}p + \frac{|[\frac{e^{2-p}}{8}, b]|}{|[0,b]|}(p-2) \end{aligned} \tag{3.2}$$

and a similar identity with the same weights, which holds for the average of φ^2. Since the segment $\mathcal{S}$ is contained within Ω (actually, it is tangent to the upper boundary of Ω), we see that (3.1) holds for $b \in \big(\frac{e^{2-p}}{8}, \frac{e^{2-p}}{4}\big]$ as well. Next, suppose now that $b \in \big(\frac{e^{2-p}}{4}, \frac{1}{4}\big]$. Integrating by parts, we obtain

$$\langle\varphi\rangle_{[0,b]} = 1 - \ln(4b) \quad \text{ and } \quad \langle\varphi^2\rangle_{[0,b]} = (1 - \ln(4b))^2 + 1,$$

so (3.1) is satisfied. Finally, if $b \in \big(\frac{1}{4}, \frac{1}{2}\big]$, then arguing as in (3.2), we see that the point $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$ belongs to the line segment with the endpoints

$$(\langle\varphi\rangle_{[0,\frac{1}{4}]}, \langle\varphi^2\rangle_{[0,\frac{1}{4}]}) = (1, 2) \quad \text{ and } \quad (\langle\varphi\rangle_{[\frac{1}{4},b]}, \langle\varphi^2\rangle_{[\frac{1}{4},b]}) = (0, 0).$$

Again, this line segment is contained in Ω, so (3.1) is valid.

Step 2 Next, we turn our attention to the case $0 < a < b \leq \frac{1}{2}$. Analogous to (3.2), we may write

$$(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]}) = \frac{a}{b}(\langle\varphi\rangle_{[0,a]}, \langle\varphi^2\rangle_{[0,a]}) + \frac{b-a}{b}(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]}).$$

Therefore, we see that the points $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$, $(\langle\varphi\rangle_{[0,a]}, \langle\varphi^2\rangle_{[0,a]})$ and $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ are colinear. In addition, $(\langle\varphi\rangle_{[0,a]}, \langle\varphi^2\rangle_{[0,a]})$ and $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ lie at the opposite sides of the point $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$. Furthermore, by Schwarz inequality, $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ lies on or above the lower boundary of Ω. Combining these observations with the analysis of the position of the point $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$ carried out in Step 1 gives (3.1). Let us be a little bit more specific about this and take a look at the line segment $\mathcal{T}$ joining $(\langle\varphi\rangle_{[0,a]}, \langle\varphi^2\rangle_{[0,a]})$ and $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$. If $b \leq \frac{e^{2-p}}{4}$, then it is tangent to the upper boundary of Ω, so (3.1) holds. If $b \in \big(\frac{e^{2-p}}{4}, \frac{1}{4}\big)$, then there exists a point p lying on the intersection of the upper boundary Ω and the interior of $\mathcal{T}$. This implies that the part of $\mathcal{T}$, which lies between p and $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$, lies above the upper boundary of Ω. Hence, by the convexity of the function $s \mapsto s^2 + 1$, the whole line segment with endpoints $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$ and $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ must be contained in Ω. Finally, if $b \in \big(\frac{1}{4}, \frac{1}{2}\big]$, then the point $(\langle\varphi\rangle_{[0,b]}, \langle\varphi^2\rangle_{[0,b]})$ belongs to the line segment joining $(0, 0)$ and $(1, 2)$ (tangent to the upper boundary of Ω), while $(\langle\varphi\rangle_{[0,a]}, \langle\varphi^2\rangle_{[0,a]})$ lies on or above the line passing through $(0, 0)$ and $(1, 2)$. Therefore $(\langle\varphi\rangle_{[a,b]}, \langle\varphi^2\rangle_{[a,b]})$ must lie on or below this line, and hence also lie on or below the upper boundary of Ω.

Step 3 Finally, we consider the case $a < \frac{1}{2} < b$. It follows from the analysis in Step 1 that $\langle\varphi^2\rangle_{[0,a]} > 1$ $\big($since this average is a decreasing function of $a \in \big(0, \frac{1}{2}\big]$ and $\langle\varphi^2\rangle_{[0,\frac{1}{2}]} = 1\big)$.

Furthermore, observe that

$$\begin{aligned}\Big(\frac{1}{2},1\Big) &= \frac{1}{2}(1,2)+\frac{1}{2}(0,0)\\ &= \frac{1}{2}(\langle\varphi\rangle_{[0,\frac14]},\langle\varphi^2\rangle_{[0,\frac14]})+\frac{1}{2}(\langle\varphi\rangle_{[\frac14,\frac12]},\langle\varphi^2\rangle_{[\frac14,\frac12]})\\ &= (\langle\varphi\rangle_{[0,\frac12]},\langle\varphi^2\rangle_{[0,\frac12]})\\ &= 2a(\langle\varphi\rangle_{[0,a]},\langle\varphi^2\rangle_{[0,a]})+(1-2a)(\langle\varphi\rangle_{[a,\frac12]},\langle\varphi^2\rangle_{[a,\frac12]}).\end{aligned}$$

Combining the two facts above, we conclude that $\langle\varphi^2\rangle_{[a,\frac12]}\le 1$. By symmetry of φ, we also have $\langle\varphi^2\rangle_{[\frac12,b]}\le 1$. Since

$$(\langle\varphi\rangle_{[a,b]},\langle\varphi^2\rangle_{[a,b]})=\frac{\frac12-a}{b-a}(\langle\varphi\rangle_{[a,\frac12]},\langle\varphi^2\rangle_{[a,\frac12]})+\frac{b-\frac12}{b-a}(\langle\varphi\rangle_{[\frac12,b]},\langle\varphi^2\rangle_{[\frac12,b]}),$$

the point $(\langle\varphi\rangle_{[a,b]},\langle\varphi^2\rangle_{[a,b]})$ lies on or below the line $\mathbb{R}\times\{1\}$, which is tangent to the upper boundary of Ω. Thus, (3.1) follows.

This completes the proof of the bound $\|\varphi\|_{\mathrm{BMO}([0,1])}\le 1$ and establishes Theorem 1.1.

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