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Moments of L-Functions Attached to the Twist of Modular Form by Dirichlet Characters*

Guanghua JI[1] Haiwei SUN[2]

Abstract Let $f(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL_2(\mathbb{Z})$. Let $L(s,f)$ be the automorphic L-function associated with $f(z)$ and χ be a Dirichlet character modulo q. In this paper, the authors prove that unconditionally for $k=\frac{1}{n}$ with $n\in\mathbb{N}$,

$$M_k(q,f)=\sum_{\substack{\chi(\bmod q)\\ \chi\neq\chi_0}}\Big|L\Big(\frac{1}{2},f\otimes\chi\Big)\Big|^{2k}\ll_k \phi(q)(\log q)^{k^2},$$

and the result also holds for any real number $0<k<1$ under the GRH for $L(s,f\otimes\chi)$. The authors also prove that under the GRH for $L(s,f\otimes\chi)$,

$$M_k(q,f)\gg_k \phi(q)(\log q)^{k^2}$$

for any real number $k>0$ and any large prime q.

Keywords Moments, Automorphic L-functions, Convexity theorem
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1 Introduction

An important problem in number theory is to study the moments of central values of L-functions. Many authors considered this problem for several families of L-functions (see [1, 3, 6, 8, 10, 16–17] etc). Among them, the family of twisting L-functions has received much attention in recent years. The aim of this paper is to consider the moments of L-functions attached to the twist of the modular form by Dirichlet characters.

Let $f(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL_2(\mathbb{Z})$. Moreover, we assume that $f(z)$ is a normalized eigenfunction for all Hecke operators. In this case $f(z)$ has the following Fourier series expansion:

$$f(z)=\sum_{n=1}^{\infty}\lambda_f(n)n^{\frac{\kappa-1}{2}}\mathrm{e}(nz)$$

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[1]School of Mathematics, Shandong University, Jinan 250100, China. E-mail: ghji@sdu.edu.cn
[2]Corresponding author. School of Mathematics and Statistics, Shandong University, Weihai 264209, Shandong, China. E-mail: hwsun@sdu.edu.cn
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with $\lambda_f(1) = 1$.

Such an f is called a holomorphic Hecke eigenform. Associated with each Hecke eigenform f, there exists an L-function $L(s, f)$, which is defined as

$$L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} \tag{1.1}$$

for $\mathrm{Re}s > 1$.

Let q be a positive integer and χ be a Dirichlet character modulo q. For $\mathrm{Re}s > 1$, the automorphic L-functions $L(s, f \otimes \chi)$ are defined by

$$L(s, f \otimes \chi) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)\chi(n)}{n^s}.$$

For any positive real number k, we define

$$M_k(q, f) = \sum_{\substack{\chi (\bmod q) \\ \chi \neq \chi_0}} \left| L\Big(\frac{1}{2}, f \otimes \chi\Big) \right|^{2k}.$$

In this paper, we will estimate the upper and lower bounds of $M_k(q, f)$.

Recently, Heath-Brown [6] proved that

$$M_k(q) = \sum_{\substack{\chi (\bmod q) \\ \chi \neq \chi_0}} \left| L\Big(\frac{1}{2}, \chi\Big) \right|^{2k} \ll \phi(q)(\log q)^{k^2}$$

for $k = \frac{1}{n}$ with $n \in \mathbb{N}$, and under the GRH the estimate also holds for all positive real numbers $0 < k < 2$. In the proof, the author transformed the sum into the corresponding integral, and estimated the upper bounds of the integral using the method of [5], which is based on a convexity theorem for mean-value integrals.

Using the method of [6], we hope to get the same upper bounds for $M_k(q, f)$ with $k = \frac{1}{n}$ unconditionally and $0 < k < 1$ under GRH. For $M_k(q, f)$, we need to consider the effect of f. Firstly, since L-functions $L(s, f \otimes \chi)$ have degree 2 and not 1, we need to improve the estimates of some integrals used in the proof (see Section 2). We also need to estimate the average of the Dirichlet coefficients of $L(s, f \otimes \chi)^k$, which is

$$\sum_{n} \frac{A_f(n)^2}{n^{2\sigma}}, \tag{1.2}$$

where (see Section 2 for the details)

$$L(s, f \otimes \chi)^k = \sum_{n=1}^{\infty} \frac{A_f(n)\chi(n)}{n^s}$$

and $\sigma \leq 1$. In [13], Pi considered the last average for any real number k (see Lemma 2.1). Using this lemma and the upper bounds of corresponding integrals of $M_k(q, f)$, we will prove the following theorem.

Theorem 1.1 *For $k = \frac{1}{n}$ with $n \in \mathbb{N}$, we have*

$$M_k(q, f) \ll_k \phi(q)(\log q)^{k^2},$$

and the estimate also holds for any real number $0 < k < 1$ under the GRH for $L(s, f \otimes \chi)$.

To consider the lower bounds we will use the method of [15]. In [15], Rudnick and Soundararajan considered the lower bounds of $M_k(q)$ and proved that unconditionally

$$M_k(q) \gg \phi(q)(\log q)^{k^2}$$

for any rational number $k \geq 1$, at least when q is prime.

In Section 3, we will establish different constitutions of the two sums and use the average estimates of Lemma 2.1, to prove the lower bound of $M_k(q, f)$. But here we can not get unconditionally the lower bounds, since unconditionally the analytic continuation of $L(s, f \otimes \chi)^k$ and the estimates of error terms are not good enough (see Section 3). So we assume that the GRH is ture for $L(s, f \otimes \chi)$, and under this condition we can get the lower bounds for a more general k.

Theorem 1.2 *Let k be a fixed positive real number and q be any large prime, and under GRH for $L(s, f \otimes \chi)$, we have*

$$M_k(q, f) \gg_k \phi(q)(\log q)^{k^2}.$$

2 Proof of Theorem 1.1

2.1 Introduction

It is known that for a primitive χ, $L(s, f \otimes \chi)$ admits analytic continuation to $\mathbb{C}$ as an entire function and satisfies the functional equation (see Proposition 14.20 of [7])

$$\Lambda(s, f \otimes \chi) = \mathrm{i}^{\kappa} \frac{\tau(\chi)^2}{q} \overline{\eta}_f \Lambda(1 - s, \overline{f} \otimes \overline{\chi}), \tag{2.1}$$

where

$$\Lambda(s, f \otimes \chi) = \Big(\frac{q}{2\pi}\Big)^s \Gamma\Big(s + \frac{\kappa - 1}{2}\Big) L(s, f \otimes \chi)$$

is the complete L-function, $\tau(\chi)$ is the Gauss sum, and η_f is the eigenvalue of f for the operator $\overline{W}$ with $|\eta_f| = 1$. In addition, $L(s, f \otimes \chi)$ has an Euler product of degree 2, which is

$$L(s, f \otimes \chi) = \prod_p \Big(1 - \frac{\alpha_f(p)\chi(p)}{p^s}\Big)^{-1} \Big(1 - \frac{\beta_f(p)\chi(p)}{p^s}\Big)^{-1} \tag{2.2}$$

for $\mathrm{Re}s > 1$, where $\alpha_f(p), \beta_f(p) \in \mathbb{C}$. For $\mathrm{Re}s > 1$, we have

$$L(s, f \otimes \chi)^k = \prod_p \Big(1 - \frac{\alpha_f(p)\chi(p)}{p^s}\Big)^{-k} \Big(1 - \frac{\beta_f(p)\chi(p)}{p^s}\Big)^{-k}$$

$$
\begin{aligned}
&= \prod_p \Big(\sum_{i=0}^{\infty} \frac{d_k(p^i)\alpha_f(p)^i \chi(p^i)}{p^{is}} \Big) \Big(\sum_{j=0}^{\infty} \frac{d_k(p^j)\beta_f(p)^j \chi(p^j)}{p^{js}} \Big) \\
&= \prod_p \sum_{l=0}^{\infty} \frac{A_f(p^l)\chi(p^l)}{p^{ls}} \\
&= \sum_{n=1}^{\infty} \frac{A_f(n)\chi(n)}{n^s},
\end{aligned}
$$

where

$$
d_k(p^j) = \frac{k(k+1)\cdots(k+j-1)}{j!} \tag{2.3}
$$

and

$$
A_f(p^l) = \sum_{j=0}^{l} d_k(p^j)\alpha_f(p)^j d_k(p^{l-j})\beta_f(p)^{l-j}. \tag{2.4}
$$

For the definition of function $d_k(\cdot)$ we can refer to [5]. If we assume the generalized Riemann hypothesis hold, then there exist no zeros for $\sigma > \frac{1}{2}$, so that one can define a holomorphic extension of

$$
L(s, f \otimes \chi)^k = \sum_{n=1}^{\infty} \frac{A_f(n)\chi(n)}{n^s}
$$

in the half-plane $\sigma > \frac{1}{2}$.

For the proof of Theorem 1.1, we will use the following integral:

$$
J(\sigma, f \otimes \chi) = \int_{-\infty}^{+\infty} |L(\sigma + \mathrm{i}t, f \otimes \chi)|^{2k} |W(\sigma + \mathrm{i}t)|^6 \mathrm{d}t,
$$

where the weight function $W(s)$ is defined by

$$
W(s) := \frac{q^{\delta(s-\frac{1}{2})} - 1}{\left(s - \frac{1}{2}\right) \log q}
$$

with $\delta > 0$ to be specified later. In addition to $J(\sigma, f \otimes \chi)$ we will also consider its average over non-principal characters

$$
J(\sigma) := \sum_{\substack{\chi \neq \chi_0 \\ \chi (\bmod q)}} J(\sigma, f \otimes \chi).
$$

In this section, we are mainly to estimate the upper bound of $J(\sigma)$, i.e., Lemma 2.6. Using this lemma we can prove Theorem 1.1.

2.2 Necessary lemmas

In [13], the author gave the estimate of the average (1.2), which we state as the following lemma whose proof is included for completeness.

Lemma 2.1 *Let $\frac{1}{2} < \sigma \leq 1$. For any real number $k > 0$ we have*

$$\min\Big\{\Big(\sigma - \frac{1}{2}\Big)^{-k^2}, (\log x)^2\Big\} \ll \sum_{n \leq x} \frac{A_f(n)^2}{n^{2\sigma}} \ll \Big(\sigma - \frac{1}{2}\Big)^{-k^2}$$

and

$$(\log x)^{k^2} \ll \sum_{n \leq x} \frac{A_f(n)^2}{n} \ll (\log x)^{k^2}.$$

Proof Firstly, we use the result of [18] to prove the following asymptotic formula:

$$\sum_{n \leq x} A_f(n)^2 \sim \frac{\mathrm{e}^{-\gamma_0 k^2}}{\Gamma(k^2)} \frac{x}{\log x} \prod_{p \leq x} \Big(1 + \sum_{l=1}^{\infty} \frac{|A_f(p^l)|^2}{p^l}\Big). \tag{2.5}$$

From [18] we know that if a multiplicative and nonnegative function $\lambda(n)$ satisfies the following three conditions:

(i) For some constant $\tau > 0$,

$$\sum_{p \leq x} \frac{\log p}{p} \lambda(p) \sim \tau \log x;$$

(ii) For some constant $G > 0$ and any prime p,

$$\lambda(p) \leq G;$$

(iii)

$$\sum_{\substack{p,v \\ v \geq 2}} \frac{1}{p^v} \lambda(p^v) < \infty,$$

then

$$\sum_{n \leq x} \lambda(n) \sim \frac{\mathrm{e}^{-c\tau}}{\Gamma(\tau)} \frac{x}{\log x} \prod_{p \leq x} \Big(1 + \sum_{l=1}^{\infty} \frac{\lambda(p^l)}{p^l}\Big).$$

Hence we just need to show that $A_f(n)$ satisfies the last three conditions.

To check the condition (i), we apply (2.3)–(2.4) to see that

$$A_f(p)^2 = d_k(p)^2 (\alpha_f(p) + \beta_f(p))^2 = k^2 (\alpha_f(p) + \beta_f(p))^2 \leq 4k^2, \tag{2.6}$$

where we use Deligne's estimate (see [2]):

$$|\alpha_f(p)|, \ |\beta_f(p)| = 1.$$

From the following estimate (see [11] or [14])

$$\sum_{p \leq x} \frac{(\alpha_f(p) + \beta_f(p))^2}{p} = \log\log x + O(1),$$

and the partial summation formulae we have

$$\sum_{p\le x}\frac{\log p}{p}A_f(p)^2=k^2\int_1^x\log t\mathrm{d}\Big(\sum_{p\le t}\frac{(\alpha_f(p)+\beta_f(p))^2}{p}\Big)\sim k^2\log x, \tag{2.7}$$

from which the condition (ii) follows. From (2.4) and Deligne's estimate we have

$$A_f(p^l)\le\sum_{j=0}^{l}d_k(p^j)d_k(p^{k-j})=d_{2k}(p^l)\ll p^{\epsilon l},$$

and now we can easily check that the condition (iii) is also true:

$$\sum_{\substack{p,l\\ l\ge 2}}\frac{1}{p^l}A_f(p^l)^2\ll\sum_p\sum_{l\ge 2}\frac{1}{p^{l(1-\epsilon)}}\ll\sum_p\frac{1}{p^{2(1-\epsilon)}\log p}<\infty. \tag{2.8}$$

Then from (2.6)–(2.8) we get the asymptotic formula (2.5). So now we just need to estimate the asymptotic formula. From the above estimates we can see that

$$\begin{aligned}\prod_{p\le x}\Big(1+\sum_{l=1}^{\infty}\frac{A_f(p^l)^2}{p^l}\Big)&=\prod_{p\le x}\Big(1+\frac{A_f(p)^2}{p}+O\Big(\frac{1}{p^{2(1-\epsilon)}\log p}\Big)\Big)\\&=\exp\Big\{\sum_{p\le x}\log\Big(1+\frac{A_f(p)^2}{p}+O\Big(\frac{1}{p^{2(1-\epsilon)}\log p}\Big)\Big)\Big\}\\&=\exp\Big(k^2\sum_{p\le x}\frac{(\alpha(p)+\beta(p))^2}{p}+O(1)\Big)\\&=\exp(k^2\log\log x+O(1)).\end{aligned}$$

Now from (2.5) we get that

$$c_1x(\log x)^{k^2-1}\le\sum_{n\le x}A_f(n)^2\le c_2x(\log x)^{k^2-1}$$

with some constants $0<c_1<c_2$. We can easily get the first part of the theorem from the last estimate and the partial summation. Note that when $\sigma=\frac{1}{2}+\frac{c}{\log x}$, $m^{2\sigma}\ll m\ll m^{2\sigma}$, the second result of the theorem can be easily proved from the first part.

The following lemma is a convexity estimate (see [4] or [6]), which plays an important role in the proof.

Lemma 2.2 *Let f and g be complex-valued functions which are regular in the strip $\{s\in\mathbb{C}:\alpha<\sigma<\beta\}$, and continuous in the closed strip $\{s=\sigma+\mathrm{i}t:\alpha\le\sigma\le\beta\}$. Let b and c be positive real numbers. Suppose that $|f(s)|^b|g(s)|^c$ and $|g(s)|$ tend to zero as $t\to\infty$, uniformly in $\{s=\sigma+\mathrm{i}t:\alpha\le\sigma\le\beta\}$. Set*

$$I(\eta):=\int_{-\infty}^{+\infty}|f(\eta+\mathrm{i}t)|^b|g(\eta+\mathrm{i}t)|^c\mathrm{d}t.$$

Then, for $\alpha\le\gamma\le\beta$,

$$I(\gamma)\le I(\alpha)^{\frac{\beta-\gamma}{\beta-\alpha}}I(\beta)^{\frac{\gamma-\alpha}{\beta-\alpha}}.$$

The following lemma gives the upper bounds of $J(\sigma)$. In the proof, we use some properties and a mean value estimate of $L(s, f\otimes\chi)$, which are similar to the case of $L(s,\chi)$. We just give the outline of the proof. For the details one can refer to Lemma 4 in [6].

Lemma 2.3 *Let $\frac{1}{2}\le\sigma\le 1$ and $1-\sigma\le\gamma\le\sigma$. We have*

$$J(\gamma)\ll q^{2k(\sigma-\gamma)}\Big(J(\sigma)+\frac{q}{(\log q)^4}\Big). \tag{2.9}$$

Proof By Lemma 2.2, we get that if $\frac{1}{2}\le\sigma\le\frac{3}{4}$ and $1-\sigma\le\gamma\le\sigma$,

$$J(\gamma, f\otimes\chi)\le J(\sigma, f\otimes\chi)^{\frac{\gamma-1+\sigma}{2\sigma-1}}J(1-\sigma, f\otimes\chi)^{\frac{\sigma-\gamma}{2\sigma-1}}.$$

By Hölder's inequality, we get

$$J(\gamma)\le J(\sigma)^{\frac{\gamma-1+\sigma}{2\sigma-1}}J(1-\sigma)^{\frac{\sigma-\gamma}{2\sigma-1}}.$$

Firstly, we estimate $J(1-\sigma)$. Following the argument of Lemma 4 in [6], we can get the similar result

$$J(1-\sigma)\ll q^{2k(2\sigma-1)}((\log q)^{-6}J^*(\sigma)+J(\sigma))$$

with

$$J^*(\sigma):=\sum_{\chi\ne\chi_0\,(\mathrm{mod}\,q)}\int_{-\infty}^{+\infty}|L(\sigma+\mathrm{i}t, f\otimes\chi)|^{2k}\frac{\mathrm{d}t}{1+t^2}.$$

Note that the exponent of q is $2k(2\sigma-1)$ but not $k(2\sigma-1)$, because $L(s, f\otimes\chi)$ is of degree 2 and the Stirling formula gives a doubled exponent.

Then it is sufficient to prove the following inequality:

$$J^*(\sigma)\ll q(\log q)^2. \tag{2.10}$$

From Lemma 4 of [6], we know that a mean value estimate is required. The following estimate (see [9]) is the result corresponding to [12] for the fourth power moment of $L(s,\chi)$, which is

$$\sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}\int_{-T}^{+T}\Big|L\Big(\frac{1}{2}+\mathrm{i}t, f\otimes\chi\Big)\Big|^2\mathrm{d}t\ll\phi(q)T(\log qT)^4$$

for $T\ge 2$, where $\sideset{}{^*}\sum_{\chi(\mathrm{mod}\,q)}$ indicates that only primitive characters are to be considered. From this mean value estimate we can prove (2.10) following the argument in [6]. Therefore we prove that the result is true.

In the following lemma, we will consider two other integrals. For the proof of the unconditional result of Theorem 1.1 we use

$$H(\sigma, f\otimes\chi):=\int_{-\infty}^{+\infty}|L(\sigma+\mathrm{i}t, f\otimes\chi)-S(s, f\otimes\chi)^n|^{\frac{2}{n}}|W(\sigma+\mathrm{i}t)|^6\mathrm{d}t$$

for $\sigma > \frac{1}{2}$. Under the GRH we will employ

$$G(\sigma, f\otimes\chi) := \int_{-\infty}^{+\infty} |L(\sigma+\mathrm{i}t, f\otimes\chi)^k - S(s, f\otimes\chi)|^2 |W(\sigma+\mathrm{i}t)|^6 \mathrm{d}t$$

for $\sigma > \frac{1}{2}$. Their averages over non-principal characters are

$$H(\sigma) := \sum_{\substack{\chi\neq\chi_0\\ \chi(\bmod q)}} H(\sigma, f\otimes\chi), \quad G(\sigma) := \sum_{\substack{\chi\neq\chi_0\\ \chi(\bmod q)}} G(\sigma, f\otimes\chi).$$

Note that for $H(\sigma)$, here we get an upper bound which is different from the result of Lemma 5 in [6]. This improvement is the main point to get the unconditional result in our theorem. For $G(\sigma)$, the proof is exactly the same as Lemma 5 in [6]. So we only give the proof for $H(\sigma)$ here.

Lemma 2.4 *Let $\sigma \in [\frac{1}{2}, \frac{3}{2}]$. Under the GRH we have*

$$G(\sigma) \ll q^{-(2\sigma-1)(1-4\delta)}\Big(\frac{q}{\log q} + G\Big(\frac{1}{2}\Big)\Big),$$

and unconditionally

$$H(\sigma) \ll q^{-(2\sigma-1)(\frac{5}{4}k-4\delta)}\Big(\frac{q}{\log q} + H\Big(\frac{1}{2}\Big)\Big).$$

Proof We only consider $H(\sigma, f\otimes\chi)$. By Lemma 2.2 we have

$$H(\sigma, f\otimes\chi) \le H\Big(\frac{1}{2}, f\otimes\chi\Big)^{\frac{3}{2}-\sigma} H\Big(\frac{3}{2}, f\otimes\chi\Big)^{\sigma-\frac{1}{2}}, \quad \frac{1}{2}\le\sigma\le\frac{3}{2}.$$

Therefore

$$H(\sigma) \le H\Big(\frac{1}{2}\Big)^{\frac{3}{2}-\sigma} H\Big(\frac{3}{2}\Big)^{\sigma-\frac{1}{2}}. \tag{2.11}$$

For $H\big(\frac{1}{2}\big)$, we follow the argument in [6] and get that

$$H\Big(\frac{1}{2}\Big)^{\frac{3}{2}-\sigma} \ll q^{-(1-\delta)(\sigma-\frac{1}{2})}\Big(\frac{q}{\log q} + H\Big(\frac{1}{2}\Big)\Big).$$

Now we estimate $H\big(\frac{3}{2}\big)$. By Hölder's inequality we have

$$\begin{aligned} H\Big(\frac{3}{2}, f\otimes\chi\Big) \le &\Big\{\int_{-\infty}^{+\infty} |W(\sigma+\mathrm{i}t)|^6 \mathrm{d}t\Big\}^{1-\frac{k}{2}} \\ &\times \Big\{\int_{-\infty}^{+\infty} |L(\sigma+\mathrm{i}t, f\otimes\chi) - S(s, f\otimes\chi)^n|^4 |W(\sigma+\mathrm{i}t)|^6 \mathrm{d}t\Big\}^{\frac{k}{2}}. \end{aligned}$$

The first integral on the right is trivially $O(q^{6\delta})$. Moreover,

$$L(\sigma+\mathrm{i}t, f\otimes\chi) - S(s, f\otimes\chi)^n = \sum_{n>q} \frac{\lambda_f(n)\chi(n)}{n^{\frac{3}{2}+\mathrm{i}t}}$$

with certain coefficients $\lambda_f(n) \ll n^{\epsilon}$. The argument then proceeds as [6], noting that

$$\sum_{\substack{m_1,n_1,m_2,n_2>q\\ q|m_1n_1-m_2n_2}} \frac{\lambda_f(m_1)\lambda_f(n_1)\overline{\lambda_f(m_2)\lambda_f(n_2)}}{\max\{(m_1n_1)^{\frac{1}{2}}(m_2n_2)^{\frac{5}{2}}, (m_1n_1)^{\frac{5}{2}}(m_2n_2)^{\frac{1}{2}}\}} \ll q^{2\epsilon-5}.$$

It follows that

$$\sum_{\chi (\operatorname{mod} q)} \int_{-\infty}^{+\infty} |L(\sigma + \mathrm{i}t, f \otimes \chi) - S(s, f \otimes \chi)^v|^4 |W(\sigma + \mathrm{i}t)|^6 \mathrm{d}t \ll q^{6\delta + 2\epsilon - 4}.$$

Then from (2.11) we deduce that

$$H(\sigma) \ll q^{-(2\sigma-1)(\frac{5}{4}k - 4\delta)} \Big(\frac{q}{\log q} + H\Big(\frac{1}{2}\Big)\Big).$$

Now we consider the last integral. Let

$$K(\sigma, f \otimes \chi) := \int_{-\infty}^{+\infty} |S(\sigma + \mathrm{i}t, f \otimes \chi)|^2 |W(\sigma + \mathrm{i}t)|^6 \mathrm{d}t,$$

where

$$S(s, f \otimes \chi) = \sum_{n \le q} \frac{A_f(n)\chi(n)}{n^s}.$$

Let

$$K(\sigma) := \sum_{\substack{\chi \ne \chi_0 \\ \chi (\operatorname{mod} q)}} K(\sigma, f \otimes \chi)$$

be the average over non-principal characters. From Lemma 2.1 we can estimate the upper bound of $K(\sigma)$.

Lemma 2.5 *Let* $\frac{1}{2} < \sigma \le \frac{3}{2}$. *We have*

$$K(\sigma) \ll \frac{\phi(q) q^{6\delta(\sigma - \frac{1}{2})} \Big(\sigma - \dfrac{1}{2}\Big)^{-k^2}}{\log q}$$

and

$$K\Big(\frac{1}{2}\Big) \ll \phi(q) (\log q)^{k^2 - 1}.$$

Proof By the definition we get

$$K(\sigma) \le \sum_{\chi (\operatorname{mod} q)} K(\sigma, f \otimes \chi) = \sum_{m,n \le q} \frac{A_f(m)\overline{A_f(n)}}{(mn)^\sigma} S(m,n) I(m,n),$$

where

$$S(m,n) = \sum_{\chi (\operatorname{mod} q)} \chi(m) \overline{\chi(n)}$$

and

$$I(m,n) = \int_{-\infty}^{+\infty} \Big(\frac{n}{m}\Big)^{\mathrm{i}t} |W(\sigma + \mathrm{i}t)|^6 \mathrm{d}t.$$

Evaluating the sum $S(m,n)$ we find that

$$\begin{aligned}&\sum_{m,n\le q}\frac{A_f(m)\overline{A_f(n)}}{(mn)^\sigma}S(m,n)I(m,n)\\&=\phi(q)\sum_{\substack{m,n\le q\\ q|m-n,(mn,q)=1}}\frac{A_f(m)\overline{A_f(n)}}{(mn)^\sigma}I(m,n)\\&=\phi(q)\sum_{\substack{n\le q\\ (n,q)=1}}\frac{|A_f(m)|^2}{n^{2\sigma}}\int_{-\infty}^{+\infty}|W(\sigma+\mathrm{i}t)|^6\mathrm{d}t.\end{aligned}$$

By Lemma 2.1 and observing that

$$\int_{-\infty}^{+\infty}|W(\sigma+\mathrm{i}t)|^6\mathrm{d}t\ll q^{6\delta(\sigma-\frac{1}{2})}(\log q)^{-1},$$

we can easily complete the lemma.

From the above results, we can prove our main Lemma.

Lemma 2.6 *Let* $\sigma_0=\frac{1}{2}+\frac{c}{\log q}$ *with some constant* $c>0$. *For* $1-\sigma_0\le\gamma\le\sigma_0$ *we have*

$$J(\gamma)\ll\phi(q)(\log q)^{k^2-1}.$$

Proof By the definitions of $G(\sigma,f\otimes\chi)$ and $H(\sigma,f\otimes\chi)$, we have

$$J(\sigma)\ll K(\sigma)+G(\sigma)$$

under the GRH, and

$$J(\sigma)\ll K(\sigma)+H(\sigma)$$

unconditionally. In view of Lemma 2.4 we have

$$J(\sigma)\ll K(\sigma)+q^{-(2\sigma-1)(1-4\delta)}\Big(\frac{q}{\log q}+G\Big(\frac{1}{2}\Big)\Big)$$

and

$$J(\sigma)\ll K(\sigma)+q^{-(2\sigma-1)(\frac{5k}{4}-4\delta)}\Big(\frac{q}{\log q}+H\Big(\frac{1}{2}\Big)\Big),$$

respectively. However we also have

$$G\Big(\frac{1}{2}\Big)\ll K\Big(\frac{1}{2}\Big)+J\Big(\frac{1}{2}\Big)$$

and

$$H\Big(\frac{1}{2}\Big)\ll K\Big(\frac{1}{2}\Big)+J\Big(\frac{1}{2}\Big),$$

and therefore

$$J(\sigma)\ll K(\sigma)+q^{-(2\sigma-1)(1-4\delta)}\Big(\frac{q}{\log q}+K\Big(\frac{1}{2}\Big)+J\Big(\frac{1}{2}\Big)\Big)$$

and

$$J(\sigma) \ll K(\sigma) + q^{-(2\sigma-1)(k-4\delta)}\Big(\frac{q}{\log q} + +K\Big(\frac{1}{2}\Big) + J\Big(\frac{1}{2}\Big)\Big)$$

in the two cases, respectively.

Using Lemma 2.5, we find that

$$J(\sigma) \ll \frac{\phi(q) q^{6\delta(\sigma-\frac{1}{2})}\Big(\sigma - \frac{1}{2}\Big)^{-k^2}}{\log q} + q^{-(2\sigma-1)(1-4\delta)}\Big(\phi(q)(\log q)^{k^2-1} + J\Big(\frac{1}{2}\Big)\Big)$$

under the GRH, since

$$\frac{q}{\log q} \ll \phi(q)(\log q)^{k^2-1} \tag{2.12}$$

for $0 < k < 1$. Similarly we have

$$J(\sigma) \ll \frac{\phi(q) q^{6\delta(\sigma-\frac{1}{2})}\Big(\sigma - \frac{1}{2}\Big)^{-k^2}}{\log q} + q^{-(2\sigma-1)(\frac{5k}{4}-4\delta)}\Big(\phi(q)(\log q)^{k^2-1} + J\Big(\frac{1}{2}\Big)\Big)$$

unconditionally.

Finally we take $\sigma = \sigma_0 := \frac{1}{2} + \frac{c}{\log q}$, then apply Lemma 2.3 with $\gamma = \frac{1}{2}$ and use (2.12) again, to deduce that under GRH

$$J(\sigma) \ll_k \mathrm{e}^{8c\delta+2c(k-1)} J(\sigma) + \phi(q)(\log q)^{k^2-1}(\mathrm{e}^{6c\delta-k^2\log c} + \mathrm{e}^{2c(4\delta-1)} + \mathrm{e}^{2c(4\delta-1)+2kc}),$$

and unconditionally

$$J(\sigma) \ll_k \mathrm{e}^{c(8\delta-\frac{1}{2}k)} J(\sigma) + \phi(q)(\log q)^{k^2-1}(\mathrm{e}^{6c\delta-k^2\log c} + \mathrm{e}^{c(8\delta-\frac{5}{2}k)} + \mathrm{e}^{c(8\delta-\frac{1}{2}k)}).$$

We are now ready to choose the value of δ. We write $c_{k,1}$ and $c_{k,2}$ for the implied constants in the last two estimates respectively, and note that they depend only on k.

Under the GRH, we take

$$\delta = \frac{1-k}{8} \quad \text{and} \quad c = \max\Big\{\frac{\log 2c_{k,1}}{1-k}, 1\Big\},$$

which ensure that

$$c_{k,1}\mathrm{e}^{8c\delta+2c(k-1)} \leq \frac{1}{2},$$

and hence imply that

$$J(\sigma_0) \ll_k \phi(q)(\log q)^{k^2-1}.$$

Unconditionally, we take

$$\delta = \frac{k}{32} \quad \text{and} \quad c = \max\Big\{\frac{4}{k}\log 2c_{k,2}, 1\Big\},$$

which ensure that

$$c_{k,2}\mathrm{e}^{c(8\delta-\frac{1}{2}k)} \leq \frac{1}{2},$$

and hence also imply that

$$J(\sigma_0) \ll_k \phi(q)(\log q)^{k^2-1}.$$

At last using Lemma 2.3, we can easily prove our result.

2.3 Proof of the theorem

Now following the argument in [6], we can extract the sum $M_k(q,f)$ from the integral $J(\gamma)$ and prove Theorem 1.1. Since $|L(s,f\otimes\chi)|^{2k}$ is subharmonic we have

$$\Big|L\Big(\frac{1}{2},f\otimes\chi\Big)\Big|^{2k}\le\frac{1}{2\pi}\int_0^{2\pi}\Big|L\Big(\frac{1}{2}+r\mathrm{e}^{\mathrm{i}\theta},f\otimes\chi\Big)\Big|^{2k}\mathrm{d}\theta.$$

We now multiply by r and integrate for $0\le r\le R$ to show that

$$\Big|L\Big(\frac{1}{2},f\otimes\chi\Big)\Big|^{2k}\le\frac{1}{\mathrm{Meas}(D)}\int_D\Big|L\Big(\frac{1}{2}+z,f\otimes\chi\Big)\Big|^{2k}\mathrm{d}A,$$

where $D=D(0,R)$ is the disc of radius of R about the origin, and $\mathrm{d}A$ is the measure of area. We take

$$R=\frac{\min\{c,\delta^{-1}\}}{\log q},$$

so that if $z\in D$ then $1-\sigma_0\le \mathrm{R}\big(\frac{1}{2}+z\big)\le\sigma_0$ and $W\big(\frac{1}{2}+z\big)\gg 1$. It follows that

$$\int_D\Big|L\Big(\frac{1}{2}+z,f\otimes\chi\Big)\Big|^{2k}\mathrm{d}A\ll\int_{1-\sigma_0}^{\sigma_0}J(\gamma,f\otimes\chi)\mathrm{d}\gamma,$$

whence

$$M_k(q,f)\ll\frac{1}{\mathrm{Meas}(D)}\int_{1-\sigma_0}^{\sigma_0}J(\gamma)\mathrm{d}\gamma.$$

Since $\mathrm{Meas}(D)\gg(\log q)^{-2}$ we now deduce from Lemma 2.6 that

$$M_k(q,f)\ll_k\phi(q)(\log q)^{k^2},$$

as required.

3 Proof of Theorem 1.2

In this section, we will give the lower bound of $M_k(q,f)$. Let x be a small power of q, and set

$$A(\chi)=\sum_{n\le x}\frac{A_f(n)\chi(n)}{\sqrt{n}}.$$

We will evaluate

$$S_1:=\sum_{\substack{\chi(\bmod q)\\ \chi\ne\chi_0}}L\Big(\frac{1}{2},f\otimes\chi\Big)^k\overline{A(\chi)}\quad\text{and}\quad S_2:=\sum_{\substack{\chi(\bmod q)\\ \chi\ne\chi_0}}|A(\chi)|^2,$$

and show that $S_2\ll q(\log q)^{k^2}\ll S_1$. Then Theorem 1.2 follows from Hölder's inequality:

$$\sum_{\substack{\chi(\bmod q)\\ \chi\ne\chi_0}}\Big|L\Big(\frac{1}{2},f\otimes\chi\Big)\Big|^{2k}\ge\frac{|S_1|^2}{|S_2|}\gg q(\log q)^{k^2}.$$

We start with S_2. By Lemma 2.1 we get $|A(\chi_0)|^2 \ll x(\log q)^{k^2}$ and thus

$$\begin{aligned}\sum_{\substack{\chi (\mathrm{mod}\, q)\\ \chi\neq\chi_0}} |A(\chi)|^2 &= \sum_{\chi(\mathrm{mod}\, q)} |A(\chi)|^2 + O(x(\log q)^{k^2})\\ &= \sum_{m,n\le x} \frac{A_f(m)\overline{A_f(n)}}{\sqrt{mn}} \sum_{\chi(\mathrm{mod}\, q)} \chi(m)\overline{\chi(n)} + O(x(\log q)^{k^2}).\end{aligned}$$

Since $x < q$, the orthogonality relation for characters (mod q) gives that only the diagonal terms $m = n$ survive. Thus

$$S_2 = \phi(q) \sum_{n\le x} \frac{|A_f(n)|^2}{n} + O(x(\log q)^{k^2}).$$

Then using Lemma 2.1 we find that $S_2 \ll \phi(q)(\log q)^{k^2}$.

We now turn to S_1. If $\mathrm{Re}(s) > 1$, integration by parts gives

$$\begin{aligned}L(s, f\otimes\chi)^k &= \sum_{n\le X} \frac{A_f(n)\chi(n)}{n^s} + \int_X^\infty \frac{1}{y^s}\mathrm{d}\Big(\sum_{X<n\le y} A_f(n)\chi(n)\Big)\\ &= \sum_{n\le X} \frac{A_f(n)\chi(n)}{n^s} + s\int_X^\infty \frac{\sum_{X<n\le y} A_f(n)\chi(n)}{y^{s+1}}\mathrm{d}y,\end{aligned} \tag{3.1}$$

where $X \ge x$. To deal with the sum in the second integration we use Perron's formula and get

$$\sum_{n\le y} A_f(n)\chi(n) = \frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}T}^{c+\mathrm{i}T} L(s, f\otimes\chi)^k \frac{y^s}{s}\mathrm{d}s + O\Big(\frac{y^{1+\epsilon}}{T}\Big)$$

for $c = 1 + \frac{1}{\log y}$. Then we shift the path of integration to $\mathrm{Re}(s) = \sigma_1 > 0$. Since $L(s, f\otimes\chi)^k$ has no pole, we have

$$\sum_{n\le y} A_f(n)\chi(n) = R(y) + \frac{1}{2\pi\mathrm{i}}\Big\{\int_{\sigma_1+\mathrm{i}T}^{c+\mathrm{i}T} + \int_{c-\mathrm{i}T}^{\sigma_1-\mathrm{i}T}\Big\},$$

where

$$R(y) = \frac{1}{2\pi\mathrm{i}}\int_{\sigma_1-\mathrm{i}T}^{\sigma_1+\mathrm{i}T} L(s, f\otimes\chi)^k \frac{y^s}{s}\mathrm{d}s.$$

Note that under GRH we have

$$L(s, f\otimes\chi) \ll_\kappa (qt)^\epsilon,$$

where ϵ is a positive number. Hence we have

$$\int_{\sigma_1+\mathrm{i}T}^{c+\mathrm{i}T} + \int_{c-\mathrm{i}T}^{\sigma_1-\mathrm{i}T} \ll q^\epsilon \frac{y}{T^{1-\epsilon}\log y}$$

and

$$R(y) \ll q^\epsilon y^{\sigma_1} T^\epsilon.$$

If we take $T = y^{\sigma_1 - 2\epsilon}$, then we conclude that under GRH

$$\sum_{n \le y} A_f(n)\chi(n) = O(q^\epsilon y^{\sigma_1 - \epsilon} + q^\epsilon y^{1-\sigma_1-\epsilon}).$$

If $\sigma_1 = \frac{1}{2}$, then (3.1) furnishes an analytic continuation of $L(s, f \otimes \chi)$ to $\mathrm{Re}(s) \ge \frac{1}{2}$. Thus we have

$$L\Big(\frac{1}{2}, f \otimes \chi\Big)^k = \sum_{n \le X} \frac{A_f(n)\chi(n)}{\sqrt{n}} + O(q^\epsilon X^{-\epsilon}).$$

Therefore

$$S_1 = \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} \sum_{n \le X} \frac{A_f(n)\chi(n)}{\sqrt{n}} \overline{A(\chi)} + O\Big(q^\epsilon X^{-\epsilon} \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} |A(\chi)|\Big).$$

Obviously the main term is

$$\sum_{\chi (\mathrm{mod}\ q)} \sum_{n \le X^k} \frac{A_f(n)\chi(n)}{\sqrt{n}} \overline{A(\chi)} + O(X^{(\frac{1}{2}+\epsilon)} |A(\chi)|).$$

Then using the orthogonality relation for characters we conclude that

$$\begin{aligned} S_1 = \phi(q) \sum_{n \le X} \sum_{\substack{m \le x \\ n \equiv m (\mathrm{mod}\ q)}} \frac{A_f(n)\overline{A_f(m)}}{\sqrt{mn}} + O(X^{(\frac{1}{2}+\epsilon_1)} |A(\chi)|) \\ + O\Big(q^\epsilon X^{-\epsilon} \sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} |A(\chi)|\Big). \end{aligned}$$

By Hölder's inequality we have

$$\sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} |A(\chi)| \ll q^{\frac{1}{2}} \Big(\sum_{\substack{\chi (\mathrm{mod}\ q) \\ \chi \ne \chi_0}} |A(\chi)|^2 \Big)^{\frac{1}{2}} \ll q (\log q)^{\frac{k^2}{2}}$$

and

$$|A(\chi)| \ll x^{\frac{1}{2}} \Big(\sum_{n \le x} \frac{|A_f(n)|^2}{n} \Big)^{\frac{1}{2}} \ll x^{\frac{1}{2}} (\log q)^{\frac{k^2}{2}}.$$

So we get

$$\begin{aligned} S_1 = \phi(q) \sum_{m \le x} \sum_{\substack{n \le X \\ n \equiv m (\mathrm{mod}\ q)}} \frac{A_f(n)\overline{A_f(m)}}{\sqrt{mn}} + O(X^{(\frac{1}{2}+\epsilon_1)} x^{\frac{1}{2}} (\log q)^{\frac{k^2}{2}}) \\ + O(q^\epsilon X^{-\epsilon} q (\log q)^{\frac{k^2}{2}}). \end{aligned}$$

Now we first estimate the contribution of the off-diagonal terms. Here we may write $n = m + ql$, where $1 \le l \le \frac{X}{q}$. The contribution of these off-diagonal terms is

$$\ll q \sum_{m \le x} \frac{|A_f(m)|}{\sqrt{m}} \sum_{l \le \frac{X}{q}} \frac{|A_f(m+ql)|}{\sqrt{m+ql}} \ll X^{(\frac{1}{2}+\epsilon_2)} x^{\frac{1}{2}} (\log q)^{\frac{k^2}{2}}.$$

Therefore

$$S_1 = \phi(q) \sum_{m \le x} \frac{|A_f(m)|^2}{m} + O(X^{(\frac{1}{2}+\epsilon_3)} x^{\frac{1}{2}} (\log q)^{\frac{k^2}{2}})$$
$$+ O(q^{\epsilon} X^{-\epsilon} q (\log q)^{\frac{k^2}{2}}).$$

From Lemma 2.5, we know that

$$\sum_{m \le x} \frac{|A_f(m)|^2}{m} \gg (\log q)^{k^2}.$$

At last taking $X = q$ and $x = q^{1-2\epsilon_3}$, we deduce that

$$S_1 \gg \phi(q)(\log q)^{k^2}.$$

This proves the theorem.

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