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The Uniqueness of Inverse Problem for the Dirac Operators with Partial Information*

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Abstract The inverse spectral problem for the Dirac operators defined on the interval $[0,\pi]$ with self-adjoint separated boundary conditions is considered. Some uniqueness results are obtained, which imply that the pair of potentials $(p(x), r(x))$ and a boundary condition are uniquely determined even if only partial information is given on $(p(x), r(x))$ together with partial information on the spectral data, consisting of either one full spectrum and a subset of norming constants, or a subset of pairs of eigenvalues and the corresponding norming constants. Moreover, the authors are also concerned with the situation where both $p(x)$ and $r(x)$ are C^n-smoothness at some given point.

Keywords Eigenvalue, Norming constant, Boundary condition, Inverse spectral problem
2000 MR Subject Classification 34A55, 34L40

1 Introduction

In this paper, we are concerned with the Dirac operator $H := H(p(x), r(x); \alpha, \beta)$ which is formulated as

$$HY := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\mathrm{d}Y}{\mathrm{d}x} + \begin{pmatrix} p(x) & 0 \\ 0 & r(x) \end{pmatrix} Y(x) \tag{1.1}$$

for $x \in [0,\pi]$, subject to the self-adjoint separated boundary conditions

$$\cos\alpha y_1(0) + \sin\alpha y_2(0) = 0, \tag{1.2}$$

$$\cos\beta y_1(\pi) + \sin\beta y_2(\pi) = 0. \tag{1.3}$$

Here $Y(x) = (y_1(x), y_2(x))^{\mathrm{T}}$, $\alpha, \beta \in [0,\pi)$ and the potentials $p(x), r(x) \in L^2[0,\pi]$ are all real-valued. It is known (see [1]) that the operator H is self-adjoint in $L^2[0,\pi] \times L^2[0,\pi]$ and has a real simple discrete spectrum, denoted by $\sigma(H) := \{\lambda_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, accumulating at $-\infty$ and $+\infty$.

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It is well known (see [1–2]) that the potentials p and r of the Dirac operator H defined by (1.1)–(1.3) are uniquely determined in terms of one of the following three sets of spectral data:

$$\Gamma_1 := \{\lambda_n, \alpha_n, n \in \mathbb{Z}\backslash\{0\}\}, \tag{1.4}$$

$$\Gamma_2 := \{\lambda_n, \kappa_n, n \in \mathbb{Z}\backslash\{0\}\}, \tag{1.5}$$

$$\Gamma_3 := \{\lambda_n, \widetilde{\lambda}_n, n \in \mathbb{Z}\backslash\{0\}\}, \tag{1.6}$$

where $\widetilde{\lambda}_n$ is the eigenvalue of the same problem as (1.1)–(1.3) but with β in (1.3) replaced by $\widetilde{\beta}$ ($\widetilde{\beta} \neq \beta$), κ_n is called the terminal velocitie (or a norming constant) and α_n is called a normalized constant corresponding to eigenvalue λ_n, which are defined as

$$\kappa_n = \frac{u_1(0, \lambda_n)}{u_1(\pi, \lambda_n)} \tag{1.7}$$

and

$$\alpha_n^2 = \int_0^\pi u_1^2(x, \lambda_n) + u_2^2(x, \lambda_n)\mathrm{d}x, \tag{1.8}$$

respectively. Here $(u_1(x, z), u_2(x, z))^{\mathrm{T}} =: U(x, z)$ is the solution of the Dirac equation $H(Y) = zY$ with the initial conditions

$$u_1(0) = \sin\alpha, \qquad u_2(0) = -\cos\alpha. \tag{1.9}$$

The present paper will mainly investigate the uniqueness problem of the determination of the potentials p and r under the circumstances, where only partial information of (p, r), the eigenvalues $\{\lambda_n\}_{n\in\mathbb{Z}\backslash\{0\}}$ and the norming constants $\{\kappa_n\}_{n\in\mathbb{Z}\backslash\{0\}}$ are available.

In 1996, Amour [3] proved the half-inverse theorem for the Dirac operators, which says that the Dirichlet spectrum (that is $\alpha = 0 = \beta$ in (1.2)–(1.3)) determines the potentials (p, r) uniquely provided that the potentials (p, r) are given a priori on the half-left (or half-right) of the interval $[0, \pi]$. This result is a generation of Hochstadt and Lieberman's theorem (see [4]) for Sturm-Liouville operators. Furthermore, in 2001, Delrio and Grbert [5] considered the case where the potentials are a priori known on $[a, \pi]$ with $0 \le a < \pi$ and proved that this together with a part of two spectra determines the potentials (p, r) uniquely on $[0, \pi]$. This result can be viewed as a parallel one of Gesztesy and Simon's theorems (see [6, Theorem 1.3]) for the Sturm-Liouville problems.

For the question of uniqueness of the inverse Sturm-Liouville problems, Wei and Xu in [7] showed that norming constants play an equal role as eigenvalues. They obtained some uniqueness results for Sturm-Liouville problems, analogous to the theorems of Gesztesy-Simon (see [6]) and Hochstadt-Lieberman (see [4]), which imply that the potential q can be completely determined even if partial information is given on q together with partial information on the spectral data Γ_1 or Γ_2.

The main aim of the present paper is to generalize the results of [7] to the Dirac operators. More specifically, we will show that the pair of potentials $(p(x), r(x))$ and a boundary condition are uniquely determined even if only partial information is given on $(p(x), r(x))$ together with partial information on the spectral data, consisting of either one full spectrum and a subset of norming constants, or a subset of pairs of eigenvalues and the corresponding norming constants. Moreover, we also concern with the situation in which both $p(x)$ and $r(x)$ are C^n-smoothness at some given point.

Throughout this paper, for any $S \subset \sigma(H)$, the statement that the set S is almost symmetric means that $\lambda_n \in S$ for $n \in \mathbb{N}$ imply $\lambda_{-n} \in S$ with finitely many possible exceptions. For each $t \geq 0$, we define

$$n_S(t) = \begin{cases} \sum\limits_{\substack{0<n<t \\ \lambda_n \in S}} 1, & t > 0, \\ -\sum\limits_{\substack{t<n<0 \\ \lambda_n \in S}} 1, & t < 0. \end{cases}$$

We state the main results of this paper through two cases. We first treat the case of one full spectrum and a subset of norming constants to be known. The other case that the known eigenvalues and norming constants are pairs will be treated in Theorem 1.2 below.

Theorem 1.1 *Let $a \in (0, \frac{\pi}{2})$, and $p(x), r(x) \in C^n(a-\varepsilon, a+\varepsilon)$ for some $n \in \mathbb{N}$ and $\varepsilon > 0$. Let the subset $S \subset \sigma(H)$ be almost symmetric such that κ_j are known for $\lambda_j \in S$. Suppose that*

$$\lim_{t\to\infty} \frac{n_S(t)}{t} = \gamma \tag{1.10}$$

exists, and for $\mu_1 \in \mathbb{R}$, $t_0 > 0$,

$$n_S(t) \begin{cases} \geq \Big(1 - \dfrac{2a}{\pi}\Big)[t] + \mu_1 + \Big(1 - \dfrac{2a}{\pi}\Big) - (n+1), & t \geq t_0, \\ \leq -\Big(1 - \dfrac{2a}{\pi}\Big)[-t] + \mu_1, & t \leq -t_0 \end{cases} \tag{1.11}$$

for all $t \in \mathbb{R}$. Then the potentials $(p(x), r(x))$ on $[0, a]$ together with $(p^{(j)}(a), r^{(j)}(a))$ $(j = 1, 2, \cdots, n)$, κ_j corresponding to $\lambda_j \in S$, and $\sigma(H)$ uniquely determine β and the potentials $(p(x), r(x))$ on $[0, \pi]$.

Let us mention that if $p(x), r(x) \in C^n(a-\varepsilon, a+\varepsilon)$, then n values of norming constants can be replaced by the values of $(p^{(j)}(a), r^{(j)}(a))$ $(j = 1, 2, \cdots, n)$, that is, in the set S, n norming constants can be missed. It should be noted that if $a = 0$ in the above theorem, that is, the knowledge of the potentials $(p(x), r(x))$ is missing, then $(p(x), r(x))$ on $[0, \pi]$ are uniquely determined by $\Gamma_2 := \{\lambda_n, \kappa_n, n \in \mathbb{Z}\backslash\{0\}\}$.

The following theorem treats the case that the known eigenvalues and norming constants are pairs.

Theorem 1.2 *Let $a \in (0, \pi)$. Set the subset $S \subset \sigma(H)$ be almost symmetric such that κ_j is known for $\lambda_j \in S$. Suppose that*

$$\lim_{t\to\infty} \frac{n_S(t)}{t} = \gamma \tag{1.12}$$

exists, and for $\mu_2 \in \mathbb{R}$, $t_0 > 0$, $\varepsilon > 0$ being an arbitrary number,

$$n_S(t) \begin{cases} \geq \Big(1 - \dfrac{a}{\pi}\Big)[t] + \mu_2 + \Big(1 - \dfrac{a}{\pi}\Big) + \varepsilon, & t \geq t_0, \\ \leq -\Big(1 - \dfrac{a}{\pi}\Big)[-t] + \mu_2, & t \leq -t_0. \end{cases} \tag{1.13}$$

Then the potentials $(p(x), r(x))$ on $[0, a]$, $\lambda_j \in S$ and the corresponding norming constants κ_j determine uniquely β and the potentials $(p(x), r(x))$ on $[0, \pi]$.

It is worth pointing out the special case where $a = \frac{\pi}{2}$, we only need half (e.g. the even or the odd) of the full spectrum and the corresponding norming constants to uniquely determine $(p(x), r(x))$ on $[0, \pi]$ and β. In this situation, the problem reduces to Theorem 1 in [3], for which one can see that the other half of the spectrum is replaced by half of the norming constants.

The method we use to obtain our results is based on the uniqueness theorem of Weyl-Titchmarsh-m-function (see [8–9]). This approach has been employed skillfully by Del Rio, Gesztesy and Simon in a series of papers (see [10–12]) to deal with inverse problems. The key technique relies on the asymptotic expansion of an m-function.

The organization of this paper is as follows. In Section 2, we give some preliminaries to our problems. The proofs of our theorems are presented in Section 3.

2 Preliminaries

We begin by recalling some classical results, which will be needed later. Let $U(x, z) = (u_1(x, z), u_2(x, z))^{\mathrm{T}}$ and $V(x, z) = (v_1(x, z), v_2(x, z))^{\mathrm{T}}$ denote the solutions of the equation

$$HY = zY \tag{2.1}$$

for $x \in [0, \pi]$, with the initial conditions

$$u_1(0, z) = \sin\alpha, \quad u_2(0, z) = -\cos\alpha \tag{2.2}$$

and

$$v_1(\pi, z) = \sin\beta, \quad v_2(\pi, z) = -\cos\beta, \tag{2.3}$$

respectively. It is known (see [1]) that, as $|z| \to \infty$ in $\mathbb{C}$, the following asymptotic formulae are uniformly in $x \in [0, \pi]$:

$$\begin{aligned} u_1(x, z) &= \sin\Big\{zx - \frac{1}{2}\int_0^x [p(\tau) + r(\tau)]\mathrm{d}\tau + \alpha\Big\} + O\Big(\frac{\mathrm{e}^{|\mathrm{Im}\, z|x}}{|z|}\Big), \\ u_2(x, z) &= -\cos\Big\{zx - \frac{1}{2}\int_0^x [p(\tau) + r(\tau)]\mathrm{d}\tau + \alpha\Big\} + O\Big(\frac{\mathrm{e}^{|\mathrm{Im}\, z|x}}{|z|}\Big) \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} v_1(x, z) &= \sin\Big\{z(\pi - x) - \frac{1}{2}\int_x^\pi [p(\tau) + r(\tau)]\mathrm{d}\tau - \beta\Big\} + O\Big(\frac{\mathrm{e}^{|\mathrm{Im}\, z|(\pi - x)}}{|z|}\Big), \\ v_2(x, z) &= \cos\Big\{z(\pi - x) - \frac{1}{2}\int_x^\pi [p(\tau) + r(\tau)]\mathrm{d}\tau - \beta\Big\} + O\Big(\frac{\mathrm{e}^{|\mathrm{Im}\, z|(\pi - x)}}{|z|}\Big). \end{aligned} \tag{2.5}$$

As is well known, the eigenvalues $\{\lambda_j\}_{j\in\mathbb{Z}\setminus\{0\}}$ of operator H have the asymptotic formula

$$\lambda_j = j + \frac{\vartheta}{\pi} + O\Big(\frac{1}{j}\Big) \tag{2.6}$$

as $j \to \infty$, where $\vartheta = \beta - \alpha + \frac{1}{2}\int_0^\pi (p(\tau) + r(\tau))\mathrm{d}\tau$. Note that $U(x,\lambda_j)$ and $V(x,\lambda_j)$ are eigenfunctions corresponding to the eigenvalue λ_j. From (1.2) we see that if $\sin\alpha\sin\beta \neq 0$, then the norming constant κ_j associated with λ_j is

$$\frac{v_1(0,\lambda_j)}{v_1(\pi,\lambda_j)} = \kappa_j = \frac{u_1(0,\lambda_j)}{u_1(\pi,\lambda_j)}, \tag{2.7}$$

that is, $\kappa_j = \frac{v_1(0,\lambda_j)}{\sin\beta} = \frac{\sin\alpha}{u_1(\pi,\lambda_j)}$; otherwise, κ_j can be represented by

$$\frac{v_2(0,\lambda_j)}{v_2(\pi,\lambda_j)} = \kappa_j = \frac{u_2(0,\lambda_j)}{u_2(\pi,\lambda_j)}.$$

Let us introduce the Weyl-Titchmarsh-m function (see [13]) for the operator H, which reads as

$$m(x,z) = \frac{v_2(x,z)}{v_1(x,z)}. \tag{2.8}$$

Note that $m(x,z)$ is the Herglotz function, that is, $m : \mathbb{C}_+ \to \mathbb{C}_+$ is analytic, and it has the following asymptotic formula:

$$m(x,z) = \mathrm{i} + o(1), \tag{2.9}$$

which is uniformly in $x \in [0,\pi-\delta]$ for $\delta > 0$, as $|z| \to \infty$ in any sector $\varepsilon < \operatorname{Arg}(z) < \pi - \varepsilon$ for $\varepsilon > 0$. Furthermore, let ω be a number from the interval $\frac{N+2}{N+3} < \omega < 1$, and let the set $D \subset \mathbb{Z}$ be determined thus by

$$D = \{x + \mathrm{i}y \in D : |x| > 1,\ k|x|^\omega < y < k|x|,\ k > 0\}.$$

If the potentials $p(x), r(x)$ belong to the class $C^N[0,\delta)$ for some $\delta > 0$, then, for all $z \in D$, the high-energy asymptotic expansion of the Weyl-Titchmarsh-m function holds (see [14]):

$$m(x,z) = \mathrm{i} - \sum_{j=1}^{N} \frac{b_j(x)}{z^j} + O\Big(\frac{1}{|z|^{N+\widetilde{\theta}}}\Big) \tag{2.10}$$

as $z = x + \mathrm{i}y (\in D) \to \infty$, where $\widetilde{\theta} = 1 - (1-\omega)(N+3)$ and the functions $b_j(x)$ are given by the recursive equalities:

$$\begin{aligned} b_1(x) &= \frac{1}{2\mathrm{i}}[r(x) - \mathrm{i}p(x)], \\ b_{n+1}(x) &= \Big[\frac{1}{2}\mathrm{i}b_n'(x) - \mathrm{i}p(x)b_n(x)\Big] \\ &\quad - \frac{1}{2}\mathrm{i}\sum_{j=1}^{n} b_j(x)b_{n+1-j}(x) - \frac{1}{2}\mathrm{i}p(x)\sum_{j=1}^{n-1} b_j(x)b_{n-j}(x). \end{aligned}$$

With the above preliminaries out of way, in order to prove Theorem 1.1 and Theorem 1.2, let $(p(x), r(x))$ be given on $[0,a]$ and let $(p_1(x), r_1(x)), (p_2(x), r_2(x))$ be two candidates for

$(p(x), r(x))$ extended to the interval $[0, \pi]$. Denote by $m_j(x, z)$ the m-function corresponding to operators $H_j := H((p_j(x), r_j(x)); \alpha, \beta_j)$ for $j = 1, 2$.

A similar statement of the Dirac operator to the uniqueness theorem of Marchenko (see [8]) is the following lemma.

Lemma 2.1 (see [8]) *If* $m_1(a, z) = m_2(a, z)$, *then* $p_1(x) = p_2(x), r_1(x) = r_2(x)$ *on* $[a, \pi]$ *and* $\tan\beta_1 = \tan\beta_2$.

Finally, for functions $p(x), r(x) \in L^2[0, \pi]$, we introduce the Green's formula associating with (1.1) which reads as following:

$$\begin{aligned} &(HY(x), \bar{Z}(x)) - (Y(x), H\bar{Z}(x)) \\ &= \int_0^x (y_2' z_1 - y_1' z_2)\mathrm{d}t - \int_0^x (y_1 z_2' - y_2 z_1')\mathrm{d}t \\ &= [Y, Z]|_0^x, \end{aligned} \tag{2.11}$$

where $Y(x) = (y_1(x), y_2(x))^{\mathrm{T}}$, and $Z(x) = (z_1(x), z_2(x))^{\mathrm{T}}$. Particularly, if both $Y(x) = (y_1(x), y_2(x))^{\mathrm{T}}$ and $Z(x) = (z_1(x), z_2(x))^{\mathrm{T}}$ are solutions of (2.1), then $[Y, Z](x) = [Y, Z](0)$ is a constant for $x \in [0, \pi]$.

3 Proof of Theorems

In this section we give the proofs of our main results. Let $U_j = (u_{j,1}(x, z), u_{j,2}(x, z))^{\mathrm{T}}$ and $V_j = (v_{j,1}(x, z), v_{j,2}(x, z))^{\mathrm{T}}$ for $j = 1, 2$ be solutions of the equation

$$H_j Y(x) = zY(x) \ (x \in [0, \pi]) \tag{3.1}$$

subject to the initial conditions

$$u_{j,1}(0, z) = \sin\alpha, \quad u_{j,2}(0, z) = -\cos\alpha \tag{3.2}$$

and

$$v_{j,1}(\pi, z) = \sin\beta_j, \quad v_{j,2}(\pi, z) = -\cos\beta_j, \tag{3.3}$$

respectively. Let

$$W_j(z) = v_{j,1}(0, z)\cos\alpha + v_{j,2}(0, z)\sin\alpha. \tag{3.4}$$

Then the zeros of equations $W_j(z) = 0$ coincide with the eigenvalues $\{\lambda_n^{(j)}\}_{n\in\mathbb{Z}\setminus\{0\}}$ of the operators H_j, where $W_j(z)$ are called the characteristic functions of H_j.

In virtue of (2.5), we infer that if $z = \mathrm{i}y$ with $y \in \mathbb{R}$, we have

$$\begin{aligned} |v_{j,1}(a, \mathrm{i}y)| &= \mathrm{e}^{|y|(\pi - a)}\Big(1 + O\Big(\frac{1}{|y|}\Big)\Big), \\ |v_{j,2}(a, \mathrm{i}y)| &= \mathrm{e}^{|y|(\pi - a)}\Big(1 + O\Big(\frac{1}{|y|}\Big)\Big) \end{aligned} \tag{3.5}$$

as $y \to \infty$. This together with (3.4) yields

$$|W_j(\mathrm{i}y)| = \mathrm{e}^{|y|\pi}\Big(1 + O\Big(\frac{1}{|y|}\Big)\Big). \tag{3.6}$$

For our purpose of this paper, we need the following Lemmas 3.1–3.4.

Lemma 3.1 (see [15]) *Let $z_n,\ n \geq 1$ be complex numbers with*

$$\lim_{n\to\infty} \frac{n}{z_n} = b \in \mathbb{R}.$$

Suppose further that for some $c > 0$, $|z_n - z_m| \geq c|n - m|$. Let

$$F(z) = \prod_{n=1}^{\infty} \Big(1 - \frac{z^2}{z_n^2}\Big). \tag{3.7}$$

Then for any $\varepsilon > 0$, as $r = |z| \to \infty$,

$$F(r\mathrm{e}^{\mathrm{i}\varphi}) = O(\mathrm{e}^{\pi br|\sin\varphi| + \varepsilon r}) \tag{3.8}$$

and

$$\frac{1}{F(r\mathrm{e}^{\mathrm{i}\varphi})} = O(\mathrm{e}^{-\pi br|\sin\varphi| + \varepsilon r}), \quad if\ |r\mathrm{e}^{\mathrm{i}\varphi} - z_n| \geq \frac{1}{8}c. \tag{3.9}$$

Lemma 3.2 (see [16]) *Let $F(z)$ be an entire function of the zero exponential type, i.e.,*

$$\limsup_{r\to\infty} \frac{lnM(r)}{r} \leq 0, \quad M(r) = \max_{\varphi}|F(r\mathrm{e}^{\mathrm{i}\varphi})|.$$

If $F(z)$ is bounded along a line, then $F(z)$ is a constant. In particular, if $F(z) \to 0$ when $|z| \to \infty$ along a line, then $F(z) \equiv 0$.

Lemma 3.3 *Let $\sigma(H_j)$ be the sets of eigenvalues of operators H_j for $j = 1, 2$. If $\sigma(H_1) = \sigma(H_2)$, then the characteristic functions $W_1(z) = W_2(z)$.*

Proof Let $\{\lambda_n^{(j)}\}_{n\in\mathbb{Z}\setminus\{0\}}$ be the eigenvalues of H_j for $j = 1, 2$. It is known that the characteristic functions $W_j(z)$ of the operators H_j, defined by (3.4), are entire functions in z of order 1, and consequently by Hadamard's factorization theorem (see [17, p. 289]), which are uniquely determined up to a multiplicative constant by their zeros:

$$W_j(z) = C_j \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{\lambda_k^{(j)}}\Big) := C_j \lim_{N\to\infty} \prod_{\substack{k=-N \\ k\neq 0}}^{+N} \Big(1 - \frac{z}{\lambda_k^{(j)}}\Big), \tag{3.10}$$

where C_j is some constant. Furthermore, recalling Green's formula (2.11) and (3.4), $W_j(z) = \cos\beta_j u_{j,1}(\pi) + \sin\beta_j u_{j,2}(\pi)$. It follows from (2.4) that

$$W_j(z) = \sin(z\pi - \vartheta_j) + O\Big(\frac{\mathrm{e}^{|\mathrm{Im}\, z|\pi}}{|z|}\Big), \tag{3.11}$$

where $\vartheta_j = \beta_j - \alpha + \frac{1}{2}\int_0^\pi (p_j(\tau) + r_j(\tau))\mathrm{d}\tau$. Consider the functions

$$\Delta_j(z) := \sin(z\pi - \vartheta_j) = -\sin\vartheta_j \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \Big(1 - \frac{z}{k + \frac{\vartheta_j}{\pi}}\Big). \tag{3.12}$$

These together with (3.10) yield

$$\begin{aligned}\frac{W_j(z)}{\Delta_j(z)} &= -\frac{C_j}{\sin\vartheta_j} \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \frac{k + \frac{\vartheta_j}{\pi}}{\lambda_k^{(j)}} \frac{\lambda_k^{(j)} - z}{k + \frac{\vartheta_j}{\pi} - z}\\ &= -\frac{C_j}{\sin\vartheta_j} \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \frac{k + \frac{\vartheta_j}{\pi}}{\lambda_k^{(j)}} \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \Big[1 - \frac{(k + \frac{\vartheta_j}{\pi}) - \lambda_k^{(j)}}{k + \frac{\vartheta_j}{\pi} - z}\Big].\end{aligned}$$

Since $\lim\limits_{\substack{z\to\infty\\ z\notin\mathbb{R}}} \frac{W_j(z)}{\Delta_j(z)} = 1$, from (3.11) and (3.12), it follows that

$$C_j = -\sin\vartheta_j \cdot \text{p.v.} \prod_{k\in\mathbb{Z}\setminus\{0\}} \frac{\lambda_k^{(j)}}{k + \frac{\vartheta_j}{\pi}}. \tag{3.13}$$

Since $\lambda_k^{(1)} = \lambda_k^{(2)}$, taking the asymptotic formulae of $\lambda_k^{(1)}$ and $\lambda_k^{(2)}$ (see(2.6)) into account we calculate $\vartheta_1 = \vartheta_2$ and therefore $C_1 = C_2$. This implies $W_1(z) = W_2(z)$ and completes the proof.

Lemma 3.4 *For j=1, 2, let $\{\lambda_n^{(j)}\}_{n\in\mathbb{Z}\setminus\{0\}}$ and $\{\kappa_n^{(j)}\}_{n\in\mathbb{Z}\setminus\{0\}}$ be the eigenvalues and norming constants of operators H_j, respectively. If $\lambda_n^{(1)} = \lambda_n^{(2)}$ for $n \in \mathbb{Z}\setminus\{0\}$, and $\kappa_l^{(1)} = \kappa_l^{(2)}$ for l large enough, then* $\sin\beta_1 = \sin\beta_2$.

Proof Taking (2.5) into $\kappa_l^{(j)} = \frac{v_{j,1}(0,\lambda_l^{(j)})}{v_{j,1}(\pi,\lambda_l^{(j)})}$ (see (2.7)), we have

$$\kappa_l^{(j)} = \frac{1}{\sin\beta_j}\sin(\lambda_l^{(j)}\pi - \vartheta_j - \alpha) + O\Big(\frac{1}{|\lambda_l^{(j)}|}\Big).$$

From the proof of Lemma 3.3, we know $\vartheta_1 = \vartheta_2$. It follows from (2.6) that

$$\lim_{l\to\infty} \frac{\kappa_l^{(1)}}{\kappa_l^{(2)}} = \lim_{l\to\infty} \frac{\sin\beta_2}{\sin\beta_1}\Big(1 + O\Big(\frac{1}{l}\Big)\Big) = 1.$$

This shows $\sin\beta_1 = \sin\beta_2$ and completes the proof.

Basing on the above preliminaries, now we give the proofs of Theorems 1.1–1.2.

Proof of Theorem 1.1 Define

$$G_S(z) = \text{p.v.} \prod_{\lambda_n\in S} \Big(1 - \frac{z}{\lambda_n}\Big). \tag{3.14}$$

We prove first that $G_S(z)$ is an entire function. Set

$$F^{\pm}(z) = \prod_{\substack{n\in\mathbb{Z}^+\\ \lambda_{\pm n}\in S}} \Big(1 + \frac{z}{\lambda_{\pm n}}\Big)\Big(1 - \frac{z}{\lambda_{\pm n}}\Big).$$

From Lemma 3.1, for any $\varepsilon > 0$, we have

$$\frac{1}{F^{\pm}(r\mathrm{e}^{\mathrm{i}\theta})} = O(\mathrm{e}^{-\pi r|\sin\theta| + \varepsilon r})$$

as $r = |z| \to \infty$, $\theta = \mathrm{Arg} z$. Note that the set S is almost symmetrical, which implies

$$\begin{aligned}\Big|\frac{1}{G_S^2(r\mathrm{e}^{\mathrm{i}\theta})}\Big| &= \Big|\frac{1}{F^{+}(r\mathrm{e}^{\mathrm{i}\theta})F^{-}(r\mathrm{e}^{\mathrm{i}\theta})}\Big| \\ &= O(\mathrm{e}^{-2\pi r|\sin\theta| + 2\varepsilon r}),\end{aligned}$$

that is,

$$\Big|\frac{1}{G_S(r\mathrm{e}^{\mathrm{i}\theta})}\Big| = O(\mathrm{e}^{-\pi r|\sin\theta| + \varepsilon r}) \le M_0 \mathrm{e}^{-\pi r|\sin\theta| + r\varepsilon}.$$

This shows that $G_S(z)$ is of locally uniform convergence and hence it is an entire function with zeros $\{\lambda_n\}$.

Second, we give the estimation about the lower bound of $|G_S(\mathrm{i}y)|$. From (3.14), for $z = x + \mathrm{i}y$,

$$\begin{aligned}\ln|G_S(z)| &= \mathrm{p.v.}\sum_{\lambda_n \in S}\frac{1}{2}\ln\Big[\Big(1 - \frac{x}{\lambda_n}\Big)^2 + \Big(\frac{y^2}{\lambda_n^2}\Big)\Big] \\ &= \mathrm{p.v.}\sum_{\lambda_n \in S}\frac{1}{2}\ln\Big(1 - \frac{2x}{\lambda_n} + \frac{|z|^2}{\lambda_n^2}\Big) \\ &= \frac{1}{2}\int_{-\infty}^{+\infty}\ln\Big(1 - \frac{2x}{t} + \frac{|z|^2}{t^2}\Big)\mathrm{d}n_S(t). \qquad (3.15)\end{aligned}$$

Integrating (3.15) by parts, we obtain

$$\begin{aligned}\ln|G_S(z)| &= \int_{-\infty}^{+\infty} n_S(t)\frac{\frac{|z|^2}{t^3} - \frac{x}{t^2}}{1 - \frac{2x}{t} + \frac{|z|^2}{t^2}}\mathrm{d}t \\ &= \int_{-\infty}^{+\infty}\frac{n_S(t)}{t}\frac{y^2 - x(t-x)}{y^2 + (t-x)^2}\mathrm{d}t.\end{aligned}$$

This yields

$$\begin{aligned}\ln|G_S(\mathrm{i}y)| &= \int_{-\infty}^{+\infty}\frac{n_S(t)}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t \\ &= \int_{-\infty}^{-1} n_S(t)\frac{y^2}{t(y^2+t^2)}\mathrm{d}t + \int_{1}^{+\infty} n_S(t)\frac{y^2}{t(y^2+t^2)}\mathrm{d}t. \qquad (3.16)\end{aligned}$$

It is noted that

$$\begin{aligned}&\int_1^{+\infty}\frac{[t]}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t + \int_{-\infty}^{-1}\frac{-[-t]}{t}\frac{y^2}{y^2+t^2}\mathrm{d}t \\ &= \ln\Big|\frac{\sin(\mathrm{i}\pi y)}{\mathrm{i}\pi y}\Big| \\ &= \pi|y| - \ln|y| + O(1), \qquad (3.17)\end{aligned}$$

because of the known formula

$$\frac{\sin(\pi z)}{\pi z} = \prod_{n=1}^{\infty}\Big(1-\frac{z^2}{n^2}\Big).$$

On the other hand, we have

$$\int_1^{+\infty}\frac{y^2}{t(y^2+t^2)}\mathrm{d}t = \frac{1}{2}\ln(y^2+1) = \ln|y| + O(1) \tag{3.18}$$

from the relation

$$\frac{y^2}{t(y^2+t^2)} = -\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{2}\ln\Big(1+\frac{y^2}{t^2}\Big)\Big).$$

Analogously, we obtain

$$\int_{-\infty}^{-1}\frac{y^2}{t(y^2+t^2)}\mathrm{d}t = -\ln|y| + O(1). \tag{3.19}$$

Taking the inequality (1.11) into (3.16) and using (3.18)–(3.19) and (3.23), we deduce that

$$\begin{aligned}\ln|G_S(\mathrm{i}y)| &\geq \Big(1-\frac{2a}{\pi}\Big)(\pi|y| - \ln|y|) + \Big[\mu_1 - (n+1) + \Big(1-\frac{2a}{\pi}\Big)\Big]\ln|y| - \mu_1\ln|y| + O(1)\\ &= (\pi-2a)|y| - (n+1)\ln|y| + O(1).\end{aligned}$$

It follows that

$$G_S(\mathrm{i}y) \geq c_1\mathrm{e}^{(\pi-2a)|y|}|y|^{-(n+1)}. \tag{3.20}$$

Finally, we complete the proof. Let $\widetilde{V}(x,z) = (\widetilde{v}_1(x,z), \widetilde{v}_2(x,z))^{\mathrm{T}}$ be another solution of the equation $H_2Y = zY$ with the initial conditions

$$\widetilde{v}_1(\pi,z) = \sin\widetilde{\beta}, \quad \widetilde{v}_2(\pi,z) = -\cos\widetilde{\beta},$$

where $\widetilde{\beta} \in [0,\pi]$ and $\widetilde{\beta} \neq \beta_1, \beta_2$. It should be noted that, at the boundary $x=\pi$, $\widetilde{V}(x,z)$ satisfies

$$\widetilde{v}_1(\pi,z)\cos\widetilde{\beta} + \widetilde{v}_2(\pi,z)\sin\widetilde{\beta} = 0.$$

Consider the operator $H(p_2, r_2; \alpha, \widetilde{\beta})$. It is easy to see that its eigenvalues, denoted by $\{\mu_n\}_{n\in\mathbb{Z}\setminus\{0\}}$, are the zeros of the following equation:

$$\widetilde{W}(0,z) := \widetilde{v}_1(0,z)\cos\alpha + \widetilde{v}_2(0,z)\sin\alpha = 0, \tag{3.21}$$

and the eigenvalues are interlaced and disjoint to the eigenvalues of $H(p_2, r_2; \alpha, \beta)$ (see [18]). Define

$$F(z) = \Big(\frac{v_{1,1}(0,z)}{\sin\beta_1} - \frac{v_{2,1}(0,z)}{\sin\beta_2}\Big)\widetilde{W}(0,z),$$

which implies that $F(\mu_n) = 0$ for all $n \in \mathbb{Z}\setminus\{0\}$. Furthermore, we have $F(\lambda_l) = 0$ for all $\lambda_l \in S$ because of $\frac{v_{1,1}(0,\lambda_l)}{\sin\beta_1} = \kappa_l^{(1)} = \kappa_l^{(2)} = \frac{v_{2,1}(0,\lambda_l)}{\sin\beta_2}$ (see (2.7)). Noting that $W_1(z) = W_2(z)$ and $\sin\beta_1 = \sin\beta_2$ from Lemmas 3.3–3.4, by using the equations (3.4) and (3.21), we get

$$\begin{aligned}F(z) &= \Big[\frac{v_{1,1}(0,z)}{\sin\beta_1}(\widetilde{v}_1(0,z)\cos\alpha + \widetilde{v}_2(0,z)\sin\alpha) - \frac{\widetilde{v}_1(0,z)}{\sin\beta_1}W_1(z)\Big]\\ &\quad - \Big[\frac{v_{2,1}(0,z)}{\sin\beta_2}(\widetilde{v}_1(0,z)\cos\alpha + \widetilde{v}_2(0,z)\sin\alpha) - \frac{\widetilde{v}_1(0,z)}{\sin\beta_2}W_2(z)\Big]\\ &= \frac{\sin\alpha}{\sin\beta_1}\begin{vmatrix} v_{1,1} & \widetilde{v}_1\\ v_{1,2} & \widetilde{v}_2\end{vmatrix}_{(0,z)} - \frac{\sin\alpha}{\sin\beta_2}\begin{vmatrix} v_{2,1} & \widetilde{v}_1\\ v_{2,2} & \widetilde{v}_2\end{vmatrix}_{(0,z)}.\end{aligned}$$

Since $p_1(x) = p_2(x)$ and $r_1(x) = r_2(x)$ on $[0, a]$, using the Green's formulae (2.11) and (2.8), we have

$$\begin{aligned} F(z) &= \frac{\sin\alpha}{\sin\beta_1}\begin{vmatrix} v_{1,1} & \widetilde{v}_1 \\ v_{1,2} & \widetilde{v}_2 \end{vmatrix}_{(a,z)} - \frac{\sin\alpha}{\sin\beta_2}\begin{vmatrix} v_{2,1} & \widetilde{v}_1 \\ v_{2,2} & \widetilde{v}_2 \end{vmatrix}_{(a,z)} \\ &= \frac{\sin\alpha}{\sin\beta_1} v_{1,1}(a,z)\widetilde{v}_1(a,z)(\widetilde{m}(a,z) - m_1(a,z)) \\ &\quad - \frac{\sin\alpha}{\sin\beta_2} v_{2,1}(a,z)\widetilde{v}_1(a,z)(\widetilde{m}(a,z) - m_2(a,z)), \end{aligned}$$

where $\widetilde{m}(a,z) = \frac{\widetilde{v}_2(a,z)}{\widetilde{v}_1(a,z)}$ is the Weyl m-function associating with the solution $\widetilde{V}(x,z)$. Furthermore, since $p_1^{(j)}(a) = p_2^{(j)}(a)$, $r_1^{(j)}(a) = r_2^{(j)}(a)$ for $j = 1, 2, \cdots, n$, by using (2.10) and (3.5), we infer that

$$\begin{aligned} |F(z)| &\le \frac{\sin\alpha}{\sin\beta_1}|v_{1,1}(a,z)\widetilde{v}_1(a,z)||\widetilde{m}(a,z) - m_1(a,z)| \\ &\quad + \frac{\sin\alpha}{\sin\beta_2}|v_{2,1}(a,z)\widetilde{v}_1(a,z)||\widetilde{m}(a,z) - m_2(a,z)| \\ &\le \mathrm{e}^{2|y|(\pi-a)}\Big(1 + O\Big(\frac{1}{|y|}\Big)\Big)O\Big(\frac{1}{y^{n+1+\widetilde{\theta}}}\Big). \end{aligned} \tag{3.22}$$

Let us define $H(z)$ by

$$H(z) = \frac{F(z)}{G_S(z)\widetilde{W}(z)}.$$

The cross ratio $F(z)$ vanishes at each point where $G_S(z)\widetilde{W}(z)$ vanishes, and also $G_S(z)\widetilde{W}(z)$ necessarily has simple zeros since $H(p_2, r_2; \alpha, \beta_2)$ and $H(p_2, r_2; \alpha, \widetilde{\beta})$ have simple spectra, respectively, and their spectra are interlaced and disjointed. Thus $H(z)$ is an entire function, and from (3.6), (3.20) and (3.22), we have

$$\begin{aligned} |H(\mathrm{i}y)| &\le \frac{\mathrm{e}^{2|y|(\pi-a)}\big(1 + O\big(\frac{1}{|y|}\big)\big)O\big(\frac{1}{y^{n+1+\widetilde{\theta}}}\big)}{\mathrm{e}^{(\pi-2a)|y|}|y|^{-(n+1)}\mathrm{e}^{|y|\pi}\big(1 + O\big(\frac{1}{|y|}\big)\big)} \\ &= O\Big(\frac{1}{y^{\widetilde{\theta}}}\Big). \end{aligned} \tag{3.23}$$

It turns out that $|H(\mathrm{i}y)| \to 0$ as $y \to \infty$. By Lemma 3.2, we obtain $H(z) \equiv 0$ for all $z \in \mathbb{C}$. We can multiply $H(z)$ by $G_S(z)$, which has isolated zeros, so we conclude that $m_1(a,z) = m_2(a,z)$. From Lemma 2.1, we have $\beta_1 = \beta_2$ and $p_1(x) = p_2(x)$, $r_1(x) = r_2(x)$ on $[a, \pi]$. The proof is completed.

Proof of Theorem 1.2 Set

$$F_V(z) = (\cot\beta_2 - \cot\beta_1)\widetilde{v}_1(\pi,z) + \frac{1}{\sin\beta_1}\int_0^\pi V_1^{\mathrm{T}}(x,z)Q(x)\widetilde{V}(x,z)\mathrm{d}x, \tag{3.24}$$

where $V_1(x) = (v_{1,1}(x), v_{1,2}(x))^{\mathrm{T}}$, $\widetilde{V}(x) = (\widetilde{v}_1(x), \widetilde{v}_2(x))^{\mathrm{T}}$ and $Q(x) = \mathrm{diag}(p_1(x) - p_2(x), r_1(x) -$

$r_2(x)$). Then from the initial conditions (3.3), we have

$$\begin{aligned} F_V(z) &= \Big(\frac{v_{1,2}(\pi,z)}{\sin\beta_1} - \frac{v_{2,2}(\pi,z)}{\sin\beta_2}\Big)\widetilde{v}_1(\pi,z) + \frac{1}{\sin\beta_1}\int_0^\pi V_1^{\mathrm{T}}(x,z)Q(x)\widetilde{V}(x,z)\mathrm{d}x \\ &= \Big(\frac{v_{1,2}(\pi,z)}{\sin\beta_1} - \frac{v_{2,2}(\pi,z)}{\sin\beta_2}\Big)\widetilde{v}_1(\pi,z) + \frac{1}{\sin\beta_1}\int_0^\pi \mathrm{d}W\{V_1(x,z),\widetilde{V}(x,z)\} \\ &= \frac{1}{\sin\beta_1}v_{1,1}\widetilde{v}_2(\pi,z) - \frac{1}{\sin\beta_2}v_{2,2}\widetilde{v}_1(\pi,z) - \frac{1}{\sin\beta_1}\begin{vmatrix} v_{1,1} & \widetilde{v}_1 \\ v_{1,2} & \widetilde{v}_2 \end{vmatrix}_{(0,z)}, \end{aligned} \tag{3.25}$$

where $W\{V_1(x,z),\widetilde{V}(x,z)\}$ is the Wronskian of $V_1(x,z)$ and $\widetilde{V}(x,z)$. Since

$$v_{1,1}(\pi) = \sin\beta_1,\ \ v_{2,1}(\pi) = \sin\beta_2,$$

by using the Green's formula (2.11), we have

$$\begin{aligned} F_V(z) &= \frac{1}{\sin\beta_2}v_{2,1}\widetilde{v}_2(\pi,z) - \frac{1}{\sin\beta_2}v_{2,2}\widetilde{v}_1(\pi,z) - \frac{1}{\sin\beta_1}\begin{vmatrix} v_{1,1} & \widetilde{v}_1 \\ v_{1,2} & \widetilde{v}_2 \end{vmatrix}_{(0,z)} \\ &= \frac{1}{\sin\beta_2}\begin{vmatrix} v_{2,1} & \widetilde{v}_1 \\ v_{2,2} & \widetilde{v}_2 \end{vmatrix}_{(\pi,z)} - \frac{1}{\sin\beta_1}\begin{vmatrix} v_{1,1} & \widetilde{v}_1 \\ v_{1,2} & \widetilde{v}_2 \end{vmatrix}_{(0,z)} \\ &= \begin{vmatrix} \frac{1}{\sin\beta_2}v_{2,1} - \frac{1}{\sin\beta_1}v_{1,1} & \widetilde{v}_1 \\ \frac{1}{\sin\beta_2}v_{2,2} - \frac{1}{\sin\beta_1}v_{1,2} & \widetilde{v}_2 \end{vmatrix}_{(0,z)} \\ &= \frac{1}{\sin\alpha}\begin{vmatrix} \frac{1}{\sin\beta_2}v_{2,1} - \frac{1}{\sin\beta_1}v_{1,1} & \widetilde{v}_1 \\ \frac{1}{\sin\beta_2}W_2 - \frac{1}{\sin\beta_1}W_1 & \widetilde{W} \end{vmatrix}_{(0,z)}. \end{aligned}$$

Note that if $\lambda_j \in S$, then $\frac{v_{1,1}(0,\lambda_l)}{\sin\beta_1} = \kappa_l^{(1)} = \kappa_l^{(2)} = \frac{v_{2,1}(0,\lambda_l)}{\sin\beta_2}$ (see (2.7)). This shows that $F_V(\lambda_j) = 0$. Furthermore, since $p_1(x) = p_2(x)$, and $r_1(x) = r_2(x)$ on $[0,a]$, it follows from (3.25) that

$$\begin{aligned} F_V(z) &= \Big(\frac{v_{1,2}(\pi,z)}{\sin\beta_1} - \frac{v_{2,2}(\pi,z)}{\sin\beta_2}\Big)\widetilde{v}_1(\pi,z) + \frac{1}{\sin\beta_1}\int_a^\pi \mathrm{d}W\{V_1(x,z),\widetilde{V}(x,z)\} \\ &= \frac{1}{\sin\alpha}\begin{vmatrix} \frac{1}{\sin\beta_2}v_{2,1} - \frac{1}{\sin\beta_1}v_{1,1} & \widetilde{v}_1 \\ \frac{1}{\sin\beta_2}W_2 - \frac{1}{\sin\beta_1}W_1 & \widetilde{W} \end{vmatrix}_{(a,z)}. \end{aligned}$$

Let $V_D(x,z) := \widetilde{V}$ be the solution of the equation $H_2Y = zY$ satisfying the initial conditions $\widetilde{V}(a,z) = (0,1)^{\mathrm{T}}$. Then

$$F_{V_D}(z) = \frac{1}{\sin\beta_2}v_{2,1}(a,z) - \frac{1}{\sin\beta_1}v_{1,1}(a,z).$$

Defining

$$G_S(z) = \text{p.v.}\prod_{\lambda_j\in S}\Big(1 - \frac{z}{\lambda_j}\Big), \quad H_D(z) = \frac{F_{V_D}(z)}{G_S(z)},$$

by the hypothesis on S and the arguments of the proof of Theorem 1.1, we have

$$|G_S(\mathrm{i}y)| \geq c_2 \mathrm{e}^{|y|(\pi-a)}|y|^{\varepsilon}$$

and

$$|F_{V_D}(\mathrm{i}y)| \leq c_3 \mathrm{e}^{|y|(\pi-a)}\Big(1+O\Big(\frac{1}{|y|}\Big)\Big),$$

which yields

$$|H_D(\mathrm{i}y)| = O(|y|^{-\varepsilon}) \to 0$$

as $y \to \infty$. Hence $H_D(z) = 0$ and $\frac{v_{1,1}(a,z)}{\sin\beta_1} = \frac{v_{2,1}(a,z)}{\sin\beta_2}$ for all $z \in \mathbb{C}$.

Let $V_N(x,z) := \widetilde{V}$ be the solution of the equation $H_2 Y = zY$ satisfying the initial conditions $\widetilde{V}(a,z) = (1,0)^{\mathrm{T}}$. We have

$$F_{V_N}(z) = \frac{1}{\sin\beta_2} v_{2,2}(a,z) - \frac{1}{\sin\beta_1} v_{1,2}(a,z). \tag{3.26}$$

Define

$$H_N(z) = \frac{F_{V_N}(z)}{G_S(z)}.$$

Since $\frac{v_{1,1}(a,z)}{\sin\beta_1} = \frac{v_{2,1}(a,z)}{\sin\beta_2}$ for all $z \in \mathbb{C}$, from (2.9), (3.5), (3.20) and (3.26), we have

$$\begin{aligned} |H_N(\mathrm{i}y)| &= \frac{|m_2(a,\mathrm{i}y) - m_1(a,\mathrm{i}y)||v_{1,1}(a,\mathrm{i}y)|}{|G_S(\mathrm{i}y)||\sin\beta_1|} \\ &\leq \frac{O(1)\mathrm{e}^{|y|(\pi-a)}\Big(1+O\Big(\frac{1}{|y|}\Big)\Big)}{\mathrm{e}^{|y|(\pi-a)}|y|^{\varepsilon}} \\ &= O(|y|^{-\varepsilon}) \to 0 \end{aligned}$$

as $y \to \infty$. This shows $H_N(z) = 0$, and $\frac{v_{1,2}(a,z)}{\sin\beta_1} = \frac{v_{2,2}(a,z)}{\sin\beta_2}$ for all $z \in \mathbb{C}$. Thus we conclude that $m_1(a,z) = m_2(a,z)$. By Lemma 2.1, we have $p_1(x) = p_2(x)$, $r_1(x) = r_2(x)$ on $[0,\pi]$, and $\beta_1 = \beta_2$. The proof is completed.

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