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Perturbed Riemann Problem for a Scalar Chapman-Jouguet Combustion Model*

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Abstract The author considers the perturbed Riemann problem for a scalar Chapman-Jouguet combustion model which comes from Majda's model with a modified, bump-type ignition function proposed in the results of Lyng and Zumbrun in 2004. Under the entropy conditions, the unique solution in a neighborhood of the origin on the (x, t) plane $(t > 0)$ is obtained. It is found that, for some cases, the perturbed Riemann solutions are essentially different from the corresponding Riemann solutions. The perturbation may transform a strong detonation into a weak deflagration in the neighborhood of the origin. Especially, it can be observed that burning happens although the corresponding Riemann solution does not contain combustion wave, which exhibits the instability for the unburnt state.

Keywords Scalar Chapman-Jouguet combustion model, Perturbed Riemann problem, Detonation, Deflagration
2000 MR Subject Classification 35L65, 35L67, 76N15

1 Introduction

The Chapman-Jouguet (CJ for short) combustion theory describing the combustible gas with an infinite reaction rate plays an important role in gas dynamics (see [2, 5, 22]). Because of the difficulty of combustion problems in gas dynamics, there are few results for the CJ gas dynamic combustion except for the Riemann problems in [4] (a solution involving only detonation), [21] (a solution involving only deflagration), and [25] (a solution satisfying the so-called geometrical entropy conditions).

In [13], from the one-dimensional combustion equation written in Lagrangian coordinates for the simple reactant to the product mechanism, Majda proposed the simplified scalar combustion model as

$$\begin{cases} (u+qz)_t + f(u)_x = \beta u_{xx}, \\ z_t = -k\varphi(u)z. \end{cases} \tag{1.1}$$

It is hoped that this qualitative model retains most of the essential features of the Lagrangian equations, except the species diffusion (see [14]). Here u is a lumped gas-dynamical variable combining aspects of specific volume, particle velocity and temperature, z measures the mass fraction of unburnt gas, and the flux $f(u)$ is required to satisfy

$$f'(u) > 0, \quad f''(u) > 0. \tag{1.2}$$

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The viscosity coefficient β, the chemical binding energy q and the reaction rate k are all positive constants. The ignition function $\varphi(u)$ is the Heaviside function as follows:

$$\varphi(u) = 0 \ \text{ for } u \le u_i, \quad \varphi(u) = 1 \ \text{ for } u > u_i, \tag{1.3}$$

where u_i is the ignition temperature. This model was extensively studied as it is perhaps the most amenable one of all the models to rigorous mathematical analysis. Ying and Teng [23] studied the Riemann solution of (1.1) at $\beta = 0$ where the flux $f(u)$ was relaxed to satisfy $f''(u) > 0$. They obtained the limit of the solution as k tends to infinity and defined the limit function as the solution of the Riemann problem for the corresponding CJ model

$$\begin{cases} (u + qz)_t + f(u)_x = 0, \\ z(x,t) = \begin{cases} z(x,0), & \sup\limits_{0 \le \tau \le t} u(x,\tau) \le u_i, \\ 0, & \text{otherwise.} \end{cases} \end{cases} \tag{1.4}$$

Based on Ying and Teng's results, Liu and Zhang [9] summarized a set of entropy conditions, including pointwise and global entropy conditions, with which they obtained the uniqueness of the Riemann solution of the CJ model (1.4). The CJ model (1.4) with convex or nonconvex fluxes $f(u)$ was also considered in [1, 6–7, 16–17, 19–20, 24].

Although there is some good qualitative agreement between the behaviors of solutions of the model system (1.1) and the observed phenomena such as the appearance of "detonation spikes" in Majda model solutions, unfortunately deflagration waves do not occur in the Majda model and this is one limit of its utility. In [10], Lyng and Zumbrun developed a stability index for weak and strong detonation waves in the context of the Majda model with a modified, bump-type ignition function

$$\varphi(u) = 0 \ \text{ for } u \le u_i \text{ or } u \ge u^i, \quad \varphi(u) > 0 \ \text{ for } u_i < u < u^i, \tag{1.5}$$

instead of the step-type function in [13], where u_i and u^i denote lower and upper ignition thresholds respectively. Here u can be interpreted more precisely as a particle velocity, and then φ takes the form $\varphi = \psi(T(u))$, where $T(u)$ denotes temperature, ψ is a standard, step-type ignition function, $\psi = 0$ below some ignition temperature T_i and positive above, and $T(u)$ is quadratic and concave-down with $T(u_i) = T(u^i) = T_i$. This agrees qualitatively with the physical dependence of temperature on velocity along the one-dimensional flow of the traveling wave ODE for the Zeldovich-von Neumann-Döring (ZND for short) model (see [10–12] for more detailed physical background). Under the same requirement (1.2), the Majda model allows for the existence of weak deflagration profiles when choosing this bump-type ignition function, while the step-type ignition function does not [10].

In this paper, we consider the CJ model for which β is set zero, k is taken to be infinite and $\varphi(u)$ has a bump-type form as

$$\varphi(u) = 0 \ \text{ for } u \le u_i \text{ or } u \ge u^i, \quad \varphi(u) = 1 \ \text{ for } u_i < u < u^i \tag{1.6}$$

in (1.1). So it can be expressed as

$$\begin{cases} (u + qz)_t + f(u)_x = 0, \\ z(x,t) = \begin{cases} 0, & u_i < \sup\limits_{0 \le \tau \le t} u(x,\tau) < u^i, \\ z(x,0), & \text{otherwise,} \end{cases} \end{cases} \tag{1.7}$$

where $f(u)$ satisfies (1.2) throughout the present paper following [13]. The Riemann problem for (1.7) was studied in [18].

For reactive gas, it is very interesting to study the nonlinear stability and instability of flows with combustion waves. In this paper, our purpose is to investigate whether the combustion waves in the Riemann solutions are stable or not due to the perturbation of the initial data. Therefore we deal with the perturbed Riemann problem (1.7) and the following initial data:

$$(u,z)(x,0) = (u_0^{\pm}(x), z_0^{\pm}(x)), \quad \pm x > 0, \tag{1.8}$$

where $u_0^{\pm}(x)$ are arbitrary smooth functions with $u_0^{\pm}(0\pm) = u^{\pm}$, and $z_0^{\pm}(x)$ equal a constant 1 for unburnt gas and 0 for burnt gas. The initial data (1.8) is a perturbation of the corresponding Riemann data

$$(u,z)(x,0) = (u^{\pm}, z^{\pm}), \quad \pm x > 0 \tag{1.9}$$

in the neighborhood of the origin. There is another motivation to study the perturbed Riemann problem (1.7)–(1.8). As is well-known that an error is unavoidable in numerical simulation and the error forms a perturbation of the initial data. So there may exist some numerical results which are unreadable and puzzle numerical analysts. For instance, the error may transform a strong detonation to a weak deflagration in the numerical solutions in [1, 26], while the theoretical results in [16] give a reasonable explanation for this phenomenon. Thus the results of the present paper provide a preparation of theoretical analysis for the numerical simulation for the Majda model with a bump-type ignition function.

Under the entropy conditions proposed in [18], we obtain the unique perturbed Riemann solution of (1.7)–(1.8) by using the method of characteristic analysis (also called planar phase portrait analysis). We find that for most cases, the combustion waves in the corresponding Riemann solutions retain their forms after perturbation, while for some other cases, the perturbation brings essential changes to the combustion waves. That is, the perturbation may transfer a Chapman-Jouguet detonation into a strong detonation, a strong detonation into a weak deflagration following a shock wave under some conditions which does not happen for (1.4) with convex flux. The interesting phenomenon that burning happens although the corresponding Riemann solution does not contain combustion waves can also be observed, which exhibits the instability for the unburnt state. The perturbed Riemann problem for (1.4) was investigated in [16, 20], and the perturbation on initial binding energy for (1.7) was considered in [15].

This paper is organized as follows. In Section 2, we present some preliminaries including the pointwise and global entropy conditions, and the Riemann solutions of (1.7) and (1.9). In Section 3, the construction of the perturbed Riemann solution of (1.7)–(1.8) is exhibited. Finally, the conclusion and discussion are drawn in Section 4.

2 Preliminaries

In this section, we sketch some results on the Riemann solutions of (1.7) and (1.9) for completeness, and the detailed study can be found in [18].

We seek self-similar solutions $(u(\xi), z(\xi))$ of the Riemann problem for (1.7) and (1.9) where $\xi = \frac{x}{t}$. It is easy to show that $z(\xi)$ is piecewise constant, 0 or 1, and the smooth solutions $u(\xi)$ are constant states or rarefaction waves (abbr. R): $u(\xi) = (f')^{-1}(\xi)$.

A jump of the solution $(u,z)=(u,z)(x,t)$ at $x=x(t)$ should satisfy the Rankine-Hugoniot condition

$$s[u+qz]=[f], \tag{2.1}$$

where $s:=\frac{\mathrm{d}x}{\mathrm{d}t}$, $[f]=f(u_r)-f(u_l)$, $u_r=u(x(t)+0,t)$, $u_l=u(x(t)-0,t)$, etc., and the assumption (1.2) determines that $s\geq 0$ if $[z]=0$.

Besides (2.1), we need some criteria to determine which kinds of discontinuities are admissible. Then the following pointwise entropy conditions are suggested in [10, 18].

(1) Pointwise entropy conditions.

(A) For noncombustion discontinuities across which particles do not burn, the following two kinds are admissible:

(a) $[z]=0$, $[u]\neq 0\Rightarrow s=\frac{[f]}{[u]}$, it is a shock (abbr. S) supplemented by the Lax characteristic condition $f'(u_l)>s>f'(u_r)$;

(b) $[z]\neq 0$, $[u]=0\Rightarrow s=0$, it is a contact discontinuity (abbr. J);

(B) For combustion jumps across which particles burn: $[z]\neq 0$, $[u]\neq 0$, the following four kinds are admissible. Two states (u_l,z_l) and (u_r,z_r) connected by a combustion wave with $s>0$ must satisfy $z_l=0,\ u_i<u_l<u^i$ and $z_r=1,\ u_r\geq u^i$ or $u_r\leq u_i$.

(a) If $u_l>u_r$, it is compressive, called

(i) a strong detonation (abbr. SDT) for $f'(u_l)>s>f'(u_r)$;

(ii) a Chapman-Jouguet detonation (abbr. CJDT) for $f'(u_l)=s>f'(u_r)$.

(b) If $u_l<u_r$, it is expansive, called

(i) a weak deflagration (abbr. WDF) for $s<f'(u_l),\ f'(u_r)$ and $u_r=u^i$;

(ii) a Chapman-Jouguet deflagration (abbr. CJDF) for $f'(u_l)=s<f'(u_r)$ and $u_r=u^i$.

We call R, S, J, SDT, CJDT, WDF and CJDF elementary waves for (1.7).

In [18], we find that the aforementioned entropy conditions cannot guarantee the uniqueness of Riemann solutions of (1.7) and (1.9). Therefore the global entropy condition analogous to [7, 9, 25] is needed.

(2) Global entropy condition.

If the Riemann problem for (1.7) and (1.9) has several solutions satisfying the pointwise entropy conditions, we choose the one for which the propagation speed of the combustion wave is as slow as possible.

In all, a combustion solution (u,z) is admissible, if it satisfies both the pointwise entropy conditions and the global entropy condition. The Riemann problem of (1.7) and (1.9) subject to the above entropy conditions can be solved uniquely.

Now we briefly review the Riemann solutions of (1.7) and (1.9) for the most interesting situation $z^+=1$, $z^-=0$, $u^+\leq u_i<u^-$. Here and below, $w(u,v):=\frac{f(u)-f(v)}{u-v}$, "+" means "following", $\mathrm{R}(u_l,z_l;u_r,z_r)$ means the rarefaction wave varying from (u_l,z_l) to (u_r,z_r), and $\mathrm{CJDT}(u_l,z_l;u_r,z_r)$ means the combustion wave CJDT with states (u_l,z_l) and (u_r,z_r) on its left and right sides respectively, etc.

For the case $u^+\leq u_i<u^-<u^i$, the structure of the solution will strongly depend on the value of q, which can be described as follows.

When $u^+\leq u_i<u^-<u^i$ as well as $q\leq q^*$, the structure of the solution can be denoted by

(i) $\mathrm{R}(u^-,0;u_1,0)+\mathrm{CJDT}(u_1,0;u^+,1)$, if $u_i<u^-\leq u_1$;

(ii) $\mathrm{SDT}(u^-,0;u^+,1)$, if $u_1<u^-\leq u_2$;

(iii) $\mathrm{WDF}(u^-,0;u^i,1)+S(u^i,1;u^+,1)$, if $u_2<u^-<u^i$,

where the notations $q^* > 0$, $u^* \in (u^+, u^i)$, u_1 and u_2 satisfy

$$f'(u^*) = w(u^+, u^i) = \frac{f(u^+) - f(u^*)}{u^+ + q^* - u^*}, \tag{2.2}$$

$$f'(u_1) = \frac{f(u^+) - f(u_1)}{u^+ + q - u_1}, \quad w(u^+, u^i) = \frac{f(u^+) - f(u_2)}{u^+ + q - u_2}, \tag{2.3}$$

respectively, and $u^+ < u_1 \le u^* \le u_2 < u^i$ for $q \le q^*$ (see Figures 2.2–2.3 in [18]).

When $u^+ \le u_i < u^- < u^i$ as well as $q > q^*$, the structure of the solution can be denoted by

(i) $\mathrm{R}(u^-, 0; \widehat{u}, 0) + \mathrm{CJDF}(\widehat{u}, 0; u^i, 1) + \mathrm{S}(u^i, 1; u^+, 1)$, if $u_i < u^- \le \widehat{u}$;

(ii) $\mathrm{WDF}(u^-, 0; u^i, 1) + \mathrm{S}(u^i, 1; u^+, 1)$, if $\widehat{u} < u^- < u^i$,

where the notation $\widehat{u} < u^*$ is defined by $f'(\widehat{u}) = \frac{f(u^i) - f(\widehat{u})}{u^i + q - \widehat{u}}$ (see Figure 2.5 in [18]).

For the case $u^+ \le u_i$, $u^- \ge u^i$, the unique admissible solution of (1.7) and (1.9) is the noncombustion solution which can be denoted by $J(u^-, 0; u^-, 1) + S(u^-, 1; u^+, 1)$ (see Figure 1).

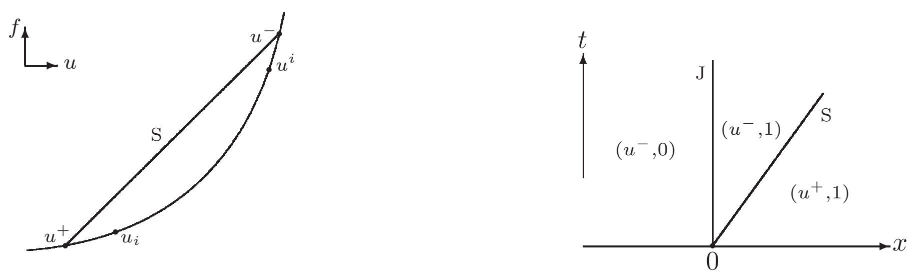


Figure 1 Solution for $u^+ \le u_i$, $u^- \ge u^i$.

3 Solution of the Perturbed Riemann Problem

We will investigate the solutions for the discontinuous initial value problem (1.7)–(1.8) in a neighborhood of the origin ($t > 0$) in the (x, t) plane. In fact, for the smooth solution ($z =$ constant), (1.7) reduces to the scalar conservation law

$$u_t + f(u)_x = 0, \tag{3.1}$$

for which the perturbed Riemann problem was studied in [3]. Hence by [8], the classical solution $(u_l, z_l)(x, t)$ (resp. $(u_r, z_r)(x, t)$) can be defined in a strip domain D_l (resp. D_r) locally in time. The strip domain D_l has the characteristic OA: $x = \lambda(u^-)t$ as the right boundary, and D_r has the characteristic OB : $x = \lambda(u^+)t$ as the left boundary, where $\lambda(u) := f'(u)$. For example, see Figure 2 for the case $0 < \lambda(u^-) < \lambda(u^+)$.

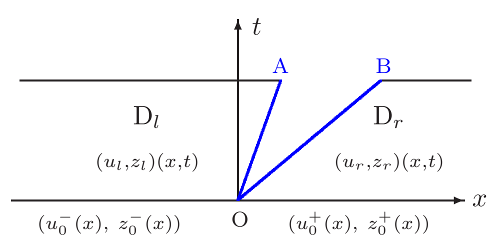


Figure 2 The strip domain $\mathrm{D}_{l,r}$ for $0 < \lambda(u^-) < \lambda(u^+)$.

By the one-side nature of model (1.7), we find that only three situations as follows are interesting (see [18]):

(i) $u_0^+(x) \leq u_i < u^-$, $z_0^-(x) = 0 < z_0^+(x) = 1$;

(ii) $u_0^+(x) \geq u^i > u^-$, $z_0^-(x) = 0 < z_0^+(x) = 1$;

(iii) $u_0^+(x) \geq u^i$, $u^- \leq u_i$, $z_0^-(x) = z_0^+(x) = 1$.

In the following, we focus our attention just on the typical case (i), for the reason that the discussions for the other two cases can be reduced to that of this typical one. In fact, for cases (ii) and (iii), a combustion wave CJDF or WDF appears in the corresponding Riemann solution (see [18]), so the discussions for these two cases are the same as those of the following Cases 3–4.

We will construct the perturbed Riemann solution of (1.7)–(1.8) case by case according to the different solutions of the corresponding Riemann problem (1.7) and (1.9).

Case 1 A combustion wave CJDT appears in the corresponding Riemann solution.

The occurrence of this case depends on the condition $u^+ \leq u_i < u^- \leq u_1$ and $q \leq q^*$ (see Figure 2.2 in [18]).

It can be proved that the perturbation of the initial data has influence on the CJDT which turns to be an SDT for $\dot{u}_0^+(0) < 0$, whereas the CJDT remains for $\dot{u}_0^+(0) > 0$. Hereafter, the dot $(\cdot)$ means derivative.

Lemma 3.1 *The perturbed solution of* (1.7)–(1.8) *is an* SDT *if and only if* $\dot{u}_0^+(0) < 0$.

Proof Suppose that there is a combustion jump $x = x(t)$ with $\dot{x}(0) = \lambda(u_1)$ and $\ddot{x}(0) \leq 0$ in the domain $\lambda(u^-)t < x < \lambda(u_1)t$, where $x = x(t)$ satisfies

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{f(u_r(x,t)) - f(u_l(x,t))}{u_r(x,t) + q - u_l(x,t)}, \\ x(0) = 0, \end{cases} \tag{3.2}$$

and $u_l(x,t)$ is a centered rarefaction wave defined by

$$\frac{x}{t} = \lambda(u_l(x,t)), \quad \lambda(u^-) \leq \frac{x}{t} \leq \lambda(u_1). \tag{3.3}$$

We now prove that $\dot{u}_0^+(0) \leq 0$. In fact, $\ddot{x}(0)$ can be calculated as follows. We may write the first equation in (3.2) in the form

$$(u_r(x,t) + q - u_l(x,t))\frac{\mathrm{d}x}{\mathrm{d}t} = f(u_r(x,t)) - f(u_l(x,t)). \tag{3.4}$$

Then differentiating (3.4) with respect to t, one obtains

$$(u_r(x,t) + q - u_l(x,t))\ddot{x}(t) = (\lambda(u_r) - \dot{x}(t))\,\frac{\mathrm{d}u_r}{\mathrm{d}t} - (\lambda(u_l) - \dot{x}(t))\,\frac{\mathrm{d}u_l}{\mathrm{d}t}. \tag{3.5}$$

Note that along $x = x(t)$,

$$\lim_{t\to 0} \lambda(u_l(x,t)) = \lim_{t\to 0} \frac{x}{t} = \dot{x}(0) = \lambda(u_1),$$

which means

$$\lim_{t\to 0} u_l(x(t),t) = u_1.$$

Thus if we consider (3.5) at $t = 0$, we find

$$(u^+ + q - u_1)\ddot{x}(0) = (\lambda(u^+) - \dot{x}(0)) \left.\frac{\mathrm{d}u_r}{\mathrm{d}t}\right|_{t=0} - (\lambda(u_1) - \dot{x}(0)) \left.\frac{\mathrm{d}u_l}{\mathrm{d}t}\right|_{t=0}, \tag{3.6}$$

where we have

$$\left.\frac{\mathrm{d}u_r}{\mathrm{d}t}\right|_{t=0} = \frac{\partial u_r}{\partial t} + \frac{\partial u_r}{\partial x}\dot{x}(0) = (\dot{x}(0) - \lambda(u^+))\frac{\partial u_r}{\partial x} = (\lambda(u_1) - \lambda(u^+))\dot{u}_0^+(0). \tag{3.7}$$

As for $\frac{\mathrm{d}u_l}{\mathrm{d}t}\big|_{t=0}$, by differentiating (3.3) with respect to t along $x = x(t)$ and setting $t = 0$, we get

$$\lambda'(u_1)\left.\frac{\mathrm{d}u_l}{\mathrm{d}t}\right|_{t=0} = \lim_{t\to 0}\frac{t\dot{x}(t) - x(t)}{t^2} = \frac{\ddot{x}(0)}{2}, \tag{3.8}$$

which implies

$$\left.\frac{\mathrm{d}u_l}{\mathrm{d}t}\right|_{t=0} = \frac{\ddot{x}(0)}{2\lambda'(u_1)} \neq \infty. \tag{3.9}$$

Substituting (3.7) and (3.9) into (3.6) yields

$$(u^+ + q - u_1)\ddot{x}(0) = -(\lambda(u_1) - \lambda(u^+))^2\dot{u}_0^+(0), \tag{3.10}$$

which shows that $\ddot{x}(0)$ has the same sign as $\dot{u}_0^+(0)$ by noticing that $u^+ + q < u_1$.

Conversely, if $\dot{u}_0^+(0) < 0$, it is possible by the analysis above to construct a combustion jump $x = x(t)$, which is a solution of (3.2) and lies in the domain $\lambda(u^-)t < x < \lambda(u_1)t$ near the origin. The combustion jump $x = x(t)$ is an SDT because it satisfies the entropy condition $\lambda(u_r) < \frac{\mathrm{d}x}{\mathrm{d}t} < \lambda(u_l)$ near the origin (see Figure 3(a)).

We turn our attention now to the perturbed solution of (1.7)–(1.8) for $\dot{u}_0^+(0) > 0$.

Lemma 3.2 *The perturbed solution of* (1.7)–(1.8) *is a* CJDT *if and only if* $\dot{u}_0^+(0) > 0$.

Proof If $\dot{u}_0^+(0) > 0$, by the entropy conditions, such a CJDT: $x = x(t)$ with $\dot{x}(0) = \lambda(u_1)$ as

$$\begin{cases} \dfrac{\mathrm{d}x}{\mathrm{d}t} = \dfrac{f(u_r(x,t)) - f(\overline{u}(x,t))}{(u_r(x,t)+q) - \overline{u}(x,t)} = \lambda(\overline{u}(x,t)), \quad \lambda(u_1)t \le x(t) < +\infty, \\ x(0) = 0, \end{cases} \tag{3.11}$$

should be constructed near the origin. Similarly the sign of $\ddot{x}(0)$ can be determined as follows. We consider the equation

$$(u_r(x,t) + q - \overline{u}(x,t))\frac{\mathrm{d}x}{\mathrm{d}t} = f(u_r(x,t)) - f(\overline{u}(x,t)),$$

differentiate with respect to t along $x = x(t)$, and set $t = 0$. This gives (3.10), where we have used the fact that along $x = x(t)$,

$$\lim_{t\to 0}\lambda(\overline{u}(x,t)) = \dot{x}(0) = \lambda(u_1),$$

namely,

$$\lim_{t\to 0}\overline{u}(x(t),t) = u_1.$$

Thus, from (3.10), we see that $\ddot{x}(0) > 0$. A solution of (1.7)–(1.8) can then be defined as $(u_r(x,t),1)$ on the right-hand side of $x = x(t)$ and $(u_l(x,t),0)$ on the left-hand side of $x = \lambda(u^-)t$. The state on the left-hand side of $x = x(t)$ between $x = \lambda(u^-)t$ and $x = x(t)$ is $(\overline{u}(x,t),0)$. The characteristic of the solution in the domain $\lambda(u_1)t \le x < x(t)$ is the tangent of $x = x(t)$ (see Figure 3(b)).

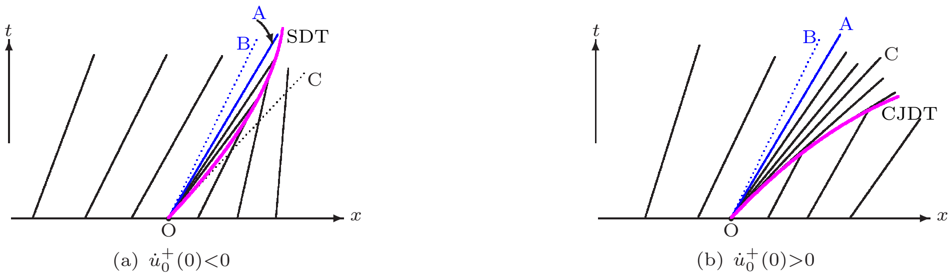


Figure 3 The perturbed solutions for Case 1, where OC: $x = \lambda(u_1)t$.

In the same way as in Lemma 3.1, it can be proved that the condition $\dot{u}_0^+(0) > 0$ is necessary for a solution constructed as above.

Case 2 A combustion wave SDT appears in the corresponding Riemann solution.

This case happens if and only if $u^+ \le u_i$, $u_1 < u^- \le u_2$ and $q \le q^*$ (see Figure 2.3 in [18]).

It is obvious that the SDT retains its form when $u_1 < u^- < u_2$. Namely, a strong detonation $x = x(t)$ satisfying (3.2) can be constructed in the interior of the domain AOB.

Now we consider the complicated case $u^- = u_2$. Motivated by the Riemann solution, it is possible to define a combustion jump $x = x(t)$ by solving the problem (3.2). Whether an $x = x(t)$ so defined is an SDT depends on whether the stability condition $\lambda(u_r) < \frac{\mathrm{d}x}{\mathrm{d}t} < \lambda(u_l)$ holds for it.

Let us do some analysis first. For any $u_r(x,t)$, we may define $\widetilde{u}(u_r) > u^*$, such that

$$\begin{cases} w(u_r, u^i) = \dfrac{f(u^i) - f(\widetilde{u})}{u^i + q - \widetilde{u}}, \\ \widetilde{u}(u^+) = u_2. \end{cases} \tag{3.12}$$

Then $x = x(t)$ is an SDT if and only if along the curve $x = x(t)$, it holds that

$$\frac{\mathrm{d}u_l}{\mathrm{d}t}\Big|_{t=0} \le \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}\Big|_{t=0}. \tag{3.13}$$

Differentiating the first equation in (3.12), we can get

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}u_r} = -\frac{f(u^i) - f(\widetilde{u}) - \lambda(u_r)(u^i + q - \widetilde{u})}{f(u_r) - f(u^i) - \lambda(\widetilde{u})(u_r - u^i)},$$

which, by letting $u_r = u^+$, becomes

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}u_r}\Big|_{u_r=u^+} = \frac{(w(u^i,u^+) - \lambda(u^+))(u^i + q - u_2)}{(\lambda(u_2) - w(u^i,u^+))(u^+ - u^i)}. \tag{3.14}$$

Using the condition $\dot{x}(0) = w(u^+, u^i)$, we have

$$\frac{\mathrm{d}u_l}{\mathrm{d}t}\Big|_{t=0} = (w(u^i,u^+) - \lambda(u_2))\dot{u}_0^-(0), \tag{3.15}$$

$$\frac{\mathrm{d}u_r}{\mathrm{d}t}\Big|_{t=0} = (w(u^i,u^+) - \lambda(u^+))\dot{u}_0^+(0), \tag{3.16}$$

so that

$$\frac{\mathrm{d}\widetilde{u}}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}\widetilde{u}}{\mathrm{d}u_r}\Big|_{u_r=u^+}\frac{\mathrm{d}u_r}{\mathrm{d}t}\Big|_{t=0} = \frac{(w(u^i,u^+)-\lambda(u^+))^2(u^i+q-u_2)}{(\lambda(u_2)-w(u^i,u^+))(u^+-u^i)}\dot{u}_0^+(0). \tag{3.17}$$

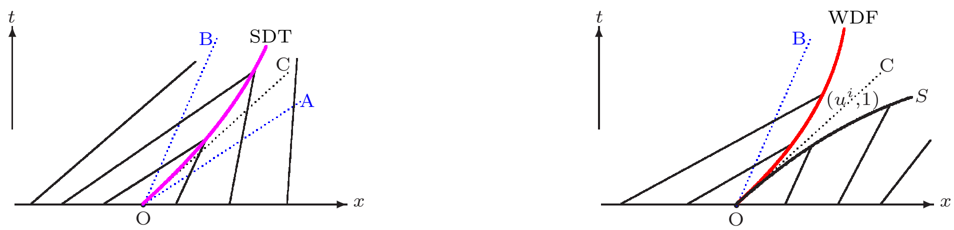
In view of (3.15) and (3.17), we see that (3.13) can be written as

$$\frac{(w(u^i,u^+)-\lambda(u_2))^2}{u^i+q-u_2}\dot{u}_0^-(0) \geq \frac{(w(u^i,u^+)-\lambda(u^+))^2}{u^i-u^+}\dot{u}_0^+(0). \tag{3.18}$$

(i) When $\dot{u}_0^+(0) < 0$ and $\dot{u}_0^-(0) > 0$, (3.18) obviously holds. Moreover, we can derive $\ddot{x}(0) < 0$ for the SDT by noticing that $u^+ + q < u_2$ and

$$(u^+ + q - u_2)\ddot{x}(0) = (w(u^i,u^+)-\lambda(u^-))^2\dot{u}_0^-(0) - (w(u^i,u^+)-\lambda(u^+))^2\dot{u}_0^+(0) > 0 \tag{3.19}$$

(see Figure 4(a)).



(a) $\dot{u}_0^+(0)<0,\ \dot{u}_0^-(0)>0$ (b) $\dot{u}_0^+(0)>0,\ \dot{u}_0^-(0)<0$

Figure 4 The perturbed solutions for Case 2, where OC: $x = w(u^i,u^+)t$.

(ii) When $\dot{u}_0^+(0) > 0$ and $\dot{u}_0^-(0) < 0$, (3.18) certainly fails. Therefore, we should not expect the combustion jump to be SDT. By the entropy conditions, a WDF : $x = x_1(t)$ should be constructed in the domain $\lambda(u^+)t < x_1(t) < w(u^i,u^+)t$ near the origin, which is determined by

$$\begin{cases} \dfrac{\mathrm{d}x_1(t)}{\mathrm{d}t} = \dfrac{f(u^i)-f(u_l(x,t))}{(u^i+q)-u_l(x,t)}, \\ x_1(0) = 0. \end{cases} \tag{3.20}$$

Then we construct a shock wave S : $x = x_2(t)$ in front of the WDF to connect u^i and $u_r(x,t)$, i.e.,

$$\begin{cases} \dfrac{\mathrm{d}x_2(t)}{\mathrm{d}t} = w(u^i,u_r(x,t)), \\ x_2(0) = 0, \end{cases} \tag{3.21}$$

which lies in the domain $w(u^i,u^+)t \leq x_2(t) < +\infty$. We now determine the sign of $\ddot{x}_1(0)$ and $\ddot{x}_2(0)$. With a similar calculation, it can be obtained from (3.20)–(3.21) that

$$\ddot{x}_1(0) = \frac{(w(u^i,u^+)-\lambda(u_2))^2}{u^i+q-u_2}\dot{u}_0^-(0), \quad \ddot{x}_2(0) = \frac{(w(u^i,u^+)-\lambda(u^+))^2}{u^i-u^+}\dot{u}_0^+(0), \tag{3.22}$$

where the condition $\dot{x}_1(0) = \dot{x}_2(0) = w(u^i,u^+)$ is used. Since $u^i + q > u_2$ and $u^i > u^+$, we see from (3.22) that $\ddot{x}_1(0) < 0$ and $\ddot{x}_2(0) > 0$. The solution is illustrated in Figure 4(b).

(iii) It remains to discuss the cases: $\dot{u}_0^+(0) < 0$ and $\dot{u}_0^-(0) < 0$; $\dot{u}_0^+(0) > 0$ and $\dot{u}_0^-(0) > 0$. Since the discussions for these two cases are similar, we here just give the discussion for $\dot{u}_0^+(0) < 0$ and $\dot{u}_0^-(0) < 0$ in detail.

If (3.18) is satisfied, the perturbed solution contains an SDT: $x = x(t)$ satisfying (3.2), and it can be claimed that $\ddot{x}(0) < 0$. In fact, note that $0 < u^i + q - u_2 < u^i - u^+$ which, together with $\dot{u}_0^+(0) < 0$ gives that

$$\frac{u^i + q - u_2}{u^i - u^+}(w(u^i, u^+) - \lambda(u^+))^2 \dot{u}_0^+(0) > (w(u^i, u^+) - \lambda(u^+))^2 \dot{u}_0^+(0). \tag{3.23}$$

Combining (3.18) and (3.23), we have

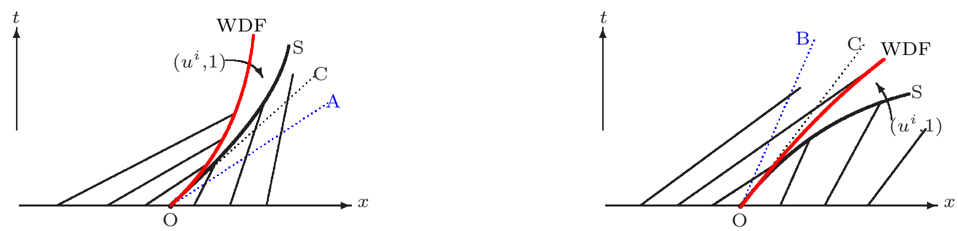
$$(w(u^i, u^+) - \lambda(u_2))^2 \dot{u}_0^-(0) > (w(u^i, u^+) - \lambda(u^+))^2 \dot{u}_0^+(0),$$

so that (3.19) holds. Thus the statement $\ddot{x}(0) < 0$ follows.

If (3.18) is invalid, a WDF: $x = x_1(t)$ following a shock wave S: $x = x_2(t)$ appears near the origin by the entropy conditions, where the WDF and S are given by (3.20)–(3.21), respectively. In order to show the existence of the shock $x = x_2(t)$, we need the priori estimate: $\ddot{x}_1(0) < \ddot{x}_2(0) < 0$. To prove this, we should notice (3.22) and the contrary of (3.18) which give that $\ddot{x}_1(0) < 0$, $\ddot{x}_2(0) < 0$ and

$$\ddot{x}_1(0) - \ddot{x}_2(0) = \frac{(w(u^i, u^+) - \lambda(u_2))^2}{u^i + q - u_2} \dot{u}_0^-(0) - \frac{(w(u^i, u^+) - \lambda(u^+))^2}{u^i - u^+} \dot{u}_0^+(0) < 0.$$

Hence there exists a solution $x = x_2(t)$ of (3.21) by virtue of $x_1(0) = x_2(0) = 0$, $\dot{x}_1(0) = \dot{x}_2(0) = w(u^i, u^+)$, $\ddot{x}_1(0) < \ddot{x}_2(0) < 0$ (see Figure 5(a)). We remark that in the case of $\dot{u}_0^+(0) > 0$ and $\dot{u}_0^-(0) > 0$, $0 < \ddot{x}_1(0) < \ddot{x}_2(0)$ can be obtained in the same way (see Figure 5(b)).



(a) $\dot{u}_0^+(0)<0$, $\dot{u}_0^-(0)<0$ and (3.18) fails. (b) $\dot{u}_0^+(0)>0$, $\dot{u}_0^-(0)>0$ and (3.18) fails.

Figure 5 The perturbed solutions for Case 2, where OC: $x = w(u^i, u^+)t$.

Thus we have finished the discussion for Case 2. In brief, the solution of (1.7)–(1.8) may contain an SDT if (3.18) holds; otherwise a WDF following a shock may appear.

Case 3 A combustion wave WDF appears in the corresponding Riemann solution.

In this case, the initial data have two possibilities: $u^+ \le u_i$, $u_2 < u^- < u^i$ and $q \le q^*$; $u^+ \le u_i$, $\widehat{u} < u^- < u^i$ and $q > q^*$ (see Figures 2.3 and 2.5 in [18]).

It is easy to find that the perturbed solution for this case is governed by the corresponding Riemann solution at the origin. That is, a WDF: $x = x_1(t)$ with $\dot{x_1}(0) = \frac{f(u^i) - f(u_-)}{u^i + q - u_-}$ satisfying (3.20) should be constructed near the origin due to the entropy conditions, ahead of which is a shock $x = x_2(t)$ with $\dot{x_2}(0) = w(u^i, u^+)$ satisfying (3.21).

Case 4 A combustion wave CJDF appears in the corresponding Riemann solution.

The appearance of this case depends on the condition $u^+ \le u_i$, $u^i < u^- \le \widehat{u}$ and $q > q^*$ (see Figure 2.5 in [18]).

We can see that, for this case, the perturbation has no influence on the combustion wave CJDF : $x = \frac{f(u^i) - f(\widehat{u})}{u^i + q - \widehat{u}} t$, which propagates with the speed $\lambda(\widehat{u})$ in the neighborhood of the origin. Furthermore, it can be derived similarly as before that $\dot{u}_0^+(0) < 0$ (resp. $\dot{u}_0^+(0) > 0$), corresponds to the shock S_1 (resp. S_2), which connects u^i and $u_r(x,t)$ (see Figure 6(a)).

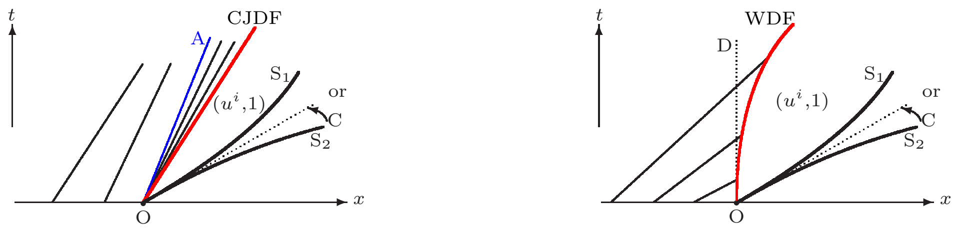
Case 5 The corresponding Riemann solution is $\mathrm{J}+\mathrm{S}$.

This case occurs when $u^+ \le u_i$, $u^- \ge u^i$ are satisfied (see Figure 2.1).

It is obvious that the structure for the perturbed solution is still $\mathrm{J}+\mathrm{S}$ if $u^- > u^i$. For the boundary case $u^- = u^i$, there exist two possibilities as follows.

(i) When $\dot{u}_0^-(0) < 0$ which indicates that $u_l(x,t) > u^i$ as $t > 0$, burning does not happen, and the solution J+S can be constructed in the neighborhood of the origin.

(ii) When $\dot{u}_0^-(0) > 0$, the perturbation gives birth to a combustion wave WDF by the entropy conditions. This phenomenon may owe to the instability of the unburnt state. The combustion wave WDF: $x = x_1(t)$ with $\dot{x}_1(0) = 0$ satisfies (3.20), from which it can be derived that $q\ddot{x}_1(0) = (\lambda(u_i))^2\dot{u}_0^-(0) > 0$, so $\ddot{x}_1(0) > 0$ (see Figure 6(b)).



(a) Solutions for Case 4.

(b) Solutions for Case 5, when $u^-=u^i$ and $\dot{u}_0^-(0)>0$.

Figure 6 The perturbed solutions for Cases 4–5, where OC: $x = w(u^i, u^+)t$, OD: $x = 0$, S_1 corresponds to $\dot{u}_0^+(0) < 0$, S_2 corresponds to $\dot{u}_0^+(0) > 0$.

4 Conclusion

So far we have finished constructing the solution of (1.7)–(1.8) completely. Our result can be summarized as follows.

Under the pointwise and global entropy conditions, there exists the unique solution of the perturbed Riemann problem (1.7)–(1.8) near the origin. We can see that the combustion waves WDF and CJDF are stable since they can retain their forms after perturbation. The combustion wave CJDT is unstable if $\dot{u}_0^+(0) < 0$ under which the perturbation transforms it into an SDT, whereas the CJDT is stable if $\dot{u}_0^+(0) > 0$. As for an SDT, it is unstable just for the boundary case: $u^- = u_2$ and (3.18) fails, under which the perturbation transforms an SDT into a WDF following a shock wave. Especially, we can observe the interesting phenomenon that burning happens although the corresponding Riemann solution does not contain combustion waves, which exhibits the instability of the unburnt state.

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