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Derivations of the Even Part of Finite-Dimensional Simple Modular Lie Superalgebra $\mathcal{M}$*

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Abstract Let $\mathbb{F}$ be the underlying base field of characteristic $p > 3$ and denote by $\mathfrak{M}$ the even part of the finite-dimensional simple modular Lie superalgebra $\mathcal{M}$. In this paper, the generator sets of the Lie algebra $\mathfrak{M}$ which will be heavily used to consider the derivation algebra $\mathrm{Der}(\mathfrak{M})$ are given. Furthermore, the derivation algebra of $\mathfrak{M}$ is determined by reducing derivations and a torus of $\mathfrak{M}$, i.e.,

$$\mathrm{Der}(\mathfrak{M}) = \mathrm{ad}(\mathfrak{M}) \oplus \mathrm{span}_{\mathbb{F}}\Big\{\prod_l \mathrm{ad}(\xi_{r+1}\xi_l)\Big\} \oplus \mathrm{span}_{\mathbb{F}}\Big\{\mathrm{ad}x_{i'}\,\mathrm{ad}(x_i\xi^v)\prod_l \mathrm{ad}(\xi_{r+1}\xi_l)\Big\}.$$

As a result, the derivation algebra of the even part of $\mathcal{M}$ does not equal the even part of the derivation superalgebra of $\mathcal{M}$.

Keywords Modular Lie superalgebra, Derivation algebra, Generator sets
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1 Introduction

In this paper, we pay our main attention to finite-dimensional Lie superalgebras over a field of prime characteristic. As is well-known, the theory of Lie superalgebras over a field of characteristic zero (see [1–4]) has experienced a vigorous development. For example, the classification by Kac of finite-dimensional simple Lie superalgebras over algebraically closed fields of characteristic zero was completely obtained (see [4]). However, that is not the case for modular Lie superalgebras, that is, Lie superalgebras of prime characteristic. The early work about modular Lie superalgebras was reported in [5]. Recently, eight families of finite-dimensional Cartan-type modular Lie superalgebras $X(m, n, \underline{t})$ were defined and the derivation superalgebras of $X(m, n, \underline{t})$ were discussed, where $X = W, S, H, K, HO, KO, \Omega$ and Γ (see [7–8, 11, 13–17]). The finite-dimensional simple modular Lie superalgebra $\mathcal{M}$ and its derivation superalgebra were investigated in [9]. This paper is a continuation of [9].

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Given a Lie superalgebra $\mathcal{M} = \mathcal{M}_{\overline{0}} \oplus \mathcal{M}_{\overline{1}}$, it is clear that the derivation superalgebra of $\mathcal{M}$ is still a Lie superalgebra, denoted by $\mathrm{Der}(\mathcal{M}) = \mathrm{Der}_{\overline{0}}(\mathcal{M}) \oplus \mathrm{Der}_{\overline{1}}(\mathcal{M})$. An interesting question naturally arises: Whether the derivation algebra of Lie algebra $\mathcal{M}_{\overline{0}}$ equals the even part of derivation superalgebra $\mathrm{Der}(\mathcal{M})$ or not, namely, $\mathrm{Der}(\mathcal{M}_{\overline{0}}) = \mathrm{Der}(\mathcal{M})_{\overline{0}}$? In this paper, this question will be answered for the finite-dimensional simple modular Lie superalgebra $\mathcal{M}$ over a field of prime characteristic.

Let $\mathfrak{M}$ denote the even part of the Lie superalgebra $\mathcal{M}$. Since $\mathfrak{M}$ is a Lie algebra, one may describe the derivation algebra $\mathrm{Der}(\mathfrak{M})$ in a systematic way. However, in contrast to the case of Lie superalgebra $\mathcal{M}$, one can not obtain directly the structures of $\mathrm{Der}(\mathfrak{M})$, since $\mathfrak{M}$ is neither transitive nor admissibly graded. [6, 12] may inspire our work on modular Lie algebra $\mathfrak{M}$, the even part of the Lie superalgebra $\mathcal{M}$, in which the even parts of Lie superalgebras W, S, and Ω were discussed.

The organization of the rest of this paper is as follows. In Section 2, it is necessary to recall notions concerning Lie algebras, Lie superalgebras and the modular Lie superalgebra $\mathcal{M}$. In Section 3, the generator sets of $\mathfrak{M}$ are discussed in order to investigate the derivation algebra $\mathrm{Der}(\mathfrak{M})$. In Section 4, the derivation algebra $\mathrm{Der}(\mathfrak{M})$ is explicitly described by using the method of reduction on $\mathbb{Z}$-gradations.

2 Preliminaries

Throughout, the ground field $\mathbb{F}$ is assumed to be of characteristic $p > 3$ and $\mathbb{F}$ is not equal to its prime field Π. Let $\mathbb{N}$ be the set of positive integers and $\mathbb{N}_0$ be the set of non-negative integers. Given $n \in \mathbb{N}$, let $r = 2n + 2$. Suppose that $\mu_1, \cdots, \mu_{r-1} \in \mathbb{F}$, such that $\mu_1 = 0$, $\mu_j + \mu_{n+j} = 1$, $j = 2, \cdots, n+1$. Set $M = \{1, \cdots, r-1\}$. Assume that $s_i \in \mathbb{N}_0$, $i = 1, \cdots, r-1$, and $\underline{s} = (s_1 + 1, \cdots, s_{r-1} + 1) \in \mathbb{N}^{r-1}$. We define a truncated polynomial algebra

$$A = \mathbb{F}[x_{10}, x_{11}, \cdots, x_{1s_1}, \cdots, x_{(r-1)0}, x_{(r-1)1}, \cdots, x_{(r-1)s_{(r-1)}}],$$

such that

$$x_{ij}^p = 0, \quad i \in M,\ j = 0, 1, \cdots, s_i.$$

For $i \in M$, we let $\pi_i = p^{s_i+1} - 1$. If $k_i \in \mathbb{N}_0$ such that $0 \leq k_i \leq \pi_i$, then k_i can be uniquely expressed in a p-adic form as follows:

$$k_i = \sum_{v=0}^{s_i} \varepsilon_v(k_i) p^v, \quad \text{where } 0 \leq \varepsilon_v(k_i) < p.$$

We set $x_i^{k_i} = \prod\limits_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i)}$. For $0 \leq k_i, k_i' \leq \pi_i$ and $x_i^{k_i'} = \prod\limits_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i')}$, it is easy to verify that

$$x_i^{k_i} x_i^{k_i'} = x_i^{k_i + k_i'} \neq 0 \Leftrightarrow \varepsilon_v(k_i) + \varepsilon_v(k_i') < p, \quad v = 0, 1, \cdots, s_i\,.$$

Let $Q = \{(k_1, \cdots, k_{r-1}) \mid 0 \leq k_i \leq \pi_i, i \in M\}$. If $k = (k_1, \cdots, k_{r-1}) \in Q$, then let $x^k = x_1^{k_1} \cdots x_{r-1}^{k_{r-1}}$.

Given $q \in \mathbb{N} \setminus \{1\}$. Let $\Lambda(q)$ be the Grassmann superalgebra over $\mathbb{F}$ in q variables $\xi_{r+1}, \cdots, \xi_{r+q}$. Denote the tensor product by $\widetilde{\mathcal{M}}(r, q, \underline{s}) := A \otimes \Lambda(q)$. For convenience, $\widetilde{\mathcal{M}}(r, q, \underline{s})$ will be denoted by $\widetilde{\mathcal{M}}$. Let $\mathbb{Z}_2 := \{\overline{0}, \overline{1}\}$ denote the ring of integers modulo 2. Obviously, $\widetilde{\mathcal{M}}$ is an associative superalgebra with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of A and the natural $\mathbb{Z}_2$-gradation of $\Lambda(q)$ as follows:

$$\widetilde{\mathcal{M}}_{\overline{0}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\overline{0}}, \quad \widetilde{\mathcal{M}}_{\overline{1}} = A \otimes_{\mathbb{F}} \Lambda(q)_{\overline{1}}.$$

If $f \in A$, $g \in \Lambda(q)$, then we simply write $f \otimes g$ as fg. For $k \in \{1, \cdots, q\}$, we set

$$\mathbb{B}_k = \{\langle i_1, i_2, \cdots, i_k\rangle \mid r+1 \le i_1 < i_2 < \cdots < i_k \le r+q\}$$

and $\mathbb{B}(q) = \bigcup_{k=0}^{q} \mathbb{B}_k$, where $\mathbb{B}_0 = \emptyset$. If $u = \langle i_1, \cdots, i_k\rangle \in \mathbb{B}_k$, we let $|u| = k, \{u\} = \{i_1, \cdots, i_k\}$ and $\xi^u = \xi_{i_1} \cdots \xi_{i_k}$. Put $|\emptyset| = 0$ and $\xi^{\emptyset} = 1$. Then $\{x^k \xi^u \mid k \in Q,\ u \in \mathbb{B}(q)\}$ is an $\mathbb{F}$-basis of $\widetilde{\mathcal{M}}$.

If L is a Lie superalgebra, then $h(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of L, i.e., $h(L) = L_{\overline{0}} \cup L_{\overline{1}}$. If $|x|$ occurs in some expression in this paper, we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of x.

Set $s = r+q$ and $T = \{r+1, \cdots, s\}$. Put $M_1 = \{2, \cdots, r-1\}$. Define $\widetilde{i} = \overline{0}$, if $i \in M_1$; and $\widetilde{i} = \overline{1}$, if $i \in T$.

Let

$$i' = \begin{cases} i+n, & \text{if } 2 \le i \le n+1, \\ i-n, & \text{if } n+2 \le i \le r-1, \\ i, & \text{if } r+1 \le i \le s, \end{cases} \qquad [i] = \begin{cases} 1, & \text{if } 2 \le i \le n+1, \\ -1, & \text{if } n+2 \le i \le r-1, \\ 1, & \text{if } r+1 \le i \le s. \end{cases}$$

Put $e_i = (\delta_{i1}, \cdots, \delta_{i(r-1)})$, $i = 1, \cdots, r-1$, and $\mathbb{R} = M \cup T$. If $i \in \mathbb{R}$, then we let D_i be the linear transformations of $\widetilde{\mathcal{M}}$, such that

$$D_i(x^k \xi^u) = \begin{cases} k_i^* x^{k-e_i} \xi^u, & \text{if } i \in M, \\ x^k (\frac{\partial \xi^u}{\partial \xi_i}), & \text{if } i \in T, \end{cases}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \cdots, \varepsilon_{s_i}(k_i)$. Then D_i is an even derivation of $\widetilde{\mathcal{M}}$ for $i \in M$, and D_j is an odd derivation for $j \in T$. Set

$$\overline{\partial} = I - \sum_{j \in M_1} \mu_j x_{j\,0} \frac{\partial}{\partial x_{j\,0}} - 2^{-1} \sum_{j \in T} \xi_j \frac{\partial}{\partial \xi_j},$$

where I is the identity mapping of $\widetilde{\mathcal{M}}$.

For $f \in h(\widetilde{\mathcal{M}})$, $g \in \widetilde{\mathcal{M}}$, we define a bilinear operation $[\,,\,]$ in $\widetilde{\mathcal{M}}$, such that

$$[f, g] = D_1(f)\overline{\partial}(g) - \overline{\partial}(f) D_1(g) + \sum_{i \in M_1 \cup T} [i](-1)^{\widetilde{i}|f|} D_i(f) D_{i'}(g).$$

Then $\widetilde{\mathcal{M}}$ becomes a finite-dimensional Lie superalgebra for the operation $[\,,\,]$ defined above. Let $x_i = x_i^1 = x_{i0}$, $i \in M$. Set $\pi = (\pi_1, \cdots, \pi_{r-1})$. Define $\mathcal{M} = [\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}]$. If $2^{-1}q - n - 2 \not\equiv 0 \pmod{p}$, we obtain that $\mathcal{M} = \widetilde{\mathcal{M}}$.

Recall that $\mathcal{M}$ is a $\mathbb{Z}_2$-graded Lie superalgebra: $\mathcal{M} = \bigoplus_{\alpha \in \mathbb{Z}_2} \mathcal{M}_\alpha$ by $\mathcal{M}_\alpha = \mathrm{span}_{\mathbb{F}}\{x^k \xi^u \mid k \in Q,\ u \in \mathbb{B}(q),\ \overline{|u|} = \alpha\}$.

For $i \in \mathbb{Z}$, we let

$$\mathcal{M}_i := \mathrm{span}_{\mathbb{F}} \Big\{ x^k \xi^u \Big| \sum_{i \in M_1} k_i + 2k_1 + |u| - 2 = i \Big\}.$$

Then $\mathcal{M}=\bigoplus\limits_{i\in X}\mathcal{M}_i$, where $X=\{-2,-1,\cdots,\tau\}$ and $\tau=\sum\limits_{i\in M_1}\pi_i+2\pi_1+q-2$ is a $\mathbb{Z}$-graded Lie superalgebra. Let $f\in\mathcal{M}$. If $f\in\mathcal{M}_i$, then f is called a $\mathbb{Z}$-homogeneous element and i is the $\mathbb{Z}$-degree of f which is denoted by $\mathrm{zd}(f)$.

We shall obtain the following fact. The proof is straightforward and is therefore a simple statement.

Let $\mathfrak{B}:=\mathrm{span}_{\mathbb{F}}\{\xi^u\mid u\in\mathbb{B}(q),\,|u|\,\mathrm{even}\}$, and then $\mathfrak{B}=C_{\mathfrak{M}}(\mathfrak{M}_{-1})$, the centralizer of $\mathfrak{M}_{-1}$ in $\mathfrak{M}$. In fact, for any $x^k\xi^u\in C_{\mathfrak{M}}(\mathfrak{M}_{-1})$, from $[x^k\xi^u,x_i]=0$, $i\in M_1$, it follows that

$$\mu_{i'}k_1^*x^{k-e_1+e_i}\xi^u+[i']k_{i'}^*x^{k-e_{i'}}\xi^u=0. \tag{2.1}$$

From (2.1), this shows that

$$\mu_{i'}k_1^*x^{k-e_1+e_i}\xi^u=0, \tag{2.2}$$

$$[i']k_{i'}^*x^{k-e_{i'}}\xi^u=0. \tag{2.3}$$

According to (2.2)–(2.3), we have

$$C_{\mathfrak{M}}(\mathfrak{M}_{-1})=\mathrm{span}_{\mathbb{F}}\{\xi^u\mid u\in\mathbb{B}(q),|u|\,\mathrm{even}\}.$$

Therefore, $\mathfrak{B}$ is a $\mathbb{Z}$-graded subalgebra of $\mathfrak{M}$.

Put $\mathfrak{B}_i=\mathfrak{B}\cap\mathfrak{M}_i$ and

$$E(\mathfrak{B}):=\bigoplus_{i\in\mathbb{Z}}\mathfrak{B}_{2r},\quad O(\mathfrak{B}):=\bigoplus_{i\in\mathbb{Z}}\mathfrak{B}_{2r+1}.$$

Since $[O(\mathfrak{B}),O(\mathfrak{B})]=0$, $O(\mathfrak{B})$ is an ideal of $\mathfrak{M}$. It is easily seen that

$$\mathfrak{B}O/(\mathfrak{B})\simeq E(\mathfrak{B})\simeq\mathcal{M}(q)_{\overline{0}},$$

where $\mathcal{M}(q)\subset\mathcal{M}$ satisfying $\mathcal{M}(q)=\mathrm{span}_{\mathbb{F}}\{\xi_j\mid j\in T\}$, and $\mathcal{M}(q)_{\overline{0}}$ is the even part of $\mathcal{M}(q)$.

Let $\mathcal{G}=\bigoplus\limits_{q=-r}^{s}\mathcal{G}_q$ be a $\mathbb{Z}$-graded Lie algebra. Recall that $\mathcal{G}$ is called transitive (with respect to $\mathbb{Z}$-gradation) provided that $\{x\in\mathcal{G}_q\mid[x,\mathcal{G}_{-1}]=0\}=0$ for all $q\in\mathbb{N}_0$. We say that $\mathcal{G}$ is admissibly graded if $C_{\mathcal{G}}(\mathcal{G}_{-1})=\mathcal{G}_{-r}$.

By the remarks above, $\mathfrak{M}$ is neither transitive nor admissibly graded. In particular, $\mathfrak{M}$ is not a simple Lie algebra.

3 Generator Sets of $\mathfrak{M}$

In this section, the generator sets of $\mathfrak{M}$, which will be applied to determine the derivation superalgebra of $\mathfrak{M}$, are given.

Put

$$\mathcal{P}:=\{x_i^{k_i}\mid 0\le k_i\le\pi_i,i\in M\},$$
$$\mathcal{Q}:=\{x_1\xi_{r+1}\xi_l\mid l\ne r+1\in T\}.$$

For $u,v\in\mathbb{B}(q)$ with $u\cap v=\emptyset$, define $u+v$ to be $w\in\mathbb{B}(q)$, such that $w=u\cup v$. If $\max u<\min v$, then we denote $u\oplus v=w$.

Theorem 3.1 *$\mathfrak{M}$ is generated by $\mathcal{P} \cup \mathcal{Q}$.*

Proof Let Y be the subalgebra of $\mathfrak{M}$ generated by $\mathcal{P} \cup \mathcal{Q}$. We first prove the following statements.

(i) Assert that $x_i\xi^u \in Y$, $\xi^u \in Y$ for all $i \in M$, $u \in \mathbb{B}(q)$.

In fact, by computation, we know that for $i \in M_1$,

$$x_i\xi_{r+1}\xi_l = \mu_{i'}^{-1}[x_1\xi_{r+1}\xi_l, x_i].$$

If $\mu_{i'} \neq 0$, then $x_i\xi_{r+1}\xi_l \in Y$. Otherwise, if $\mu_{i'} = 0$, observing that

$$x_i\xi_{r+1}\xi_l = 2^{-1}[i][x_i^2, [x_1\xi_{r+1}\xi_l, x_{i'}]],$$

we get $x_i\xi_{r+1}\xi_l \in Y$. It follows that $x_i\xi_{r+1}\xi_l \in Y$ for $i \in M_1$.

Clearly, for $t \in T$ and $i \in M_1$,

$$\xi_{r+1}\xi_t = [i'][x_{i'}, x_i\xi_{r+1}\xi_t] \in Y.$$

For $|u| = 2$,

$$x_i\xi_l\xi_t = [x_i\xi_{r+1}\xi_l, \xi_{r+1}\xi_t] \in Y,$$

where $i \in M$ and $l, t \in T$.

For $|u| > 2$, we can write $u = v \oplus w$, where $v, w \in \mathbb{B}(q)$, such that $|w| = 2$. By the inductive hypothesis, we know $x_i\xi^v \in Y$ and $x_i\xi^w \in Y$ for $i \in M_1$.

Case 1 $|v| \not\equiv 0 \pmod{p}$. For $\mu_{i'} \neq 0$, we get

$$x_i\xi^u = [i]|v|^{-1}\mu_{i'}^{-1}[x_i, [x_1^2, [x_i\xi^v, x_{i'}\xi^w]]] \in Y.$$

For $\mu_{i'} = 0$, we have

$$x_{i'}\xi^u = [i]|v|^{-1}[x_{i'}, [x_1^2, [x_i\xi^v, x_{i'}\xi^w]]] \in Y.$$

Thus,

$$x_i\xi^u = [i]2^{-1}[x_i^2, x_{i'}\xi^u] \in Y.$$

Moreover,

$$x_1\xi^u = -[i]|v|^{-1}[x_1^2, [x_i\xi^v, x_{i'}\xi^w]] \in Y.$$

Case 2 $|v| \equiv 0 \pmod{p}$. For $\mu_i \neq 0$, we obtain

$$x_i\xi^u = -2^{-1}\mu_i^{-1}[[x_1^2, \xi^v], x_i\xi^w] \in Y.$$

For $\mu_i = 0$, we can get $x_{i'}\xi^u \in Y$. Then

$$x_i\xi^u = 2^{-1}[i][x_i^2, x_{i'}\xi^u] \in Y.$$

It remains to show that

$$x_1\xi^u = -2^{-1}[[x_1^2, \xi^v], x_1\xi^w] \in Y.$$

In view of $x_i\xi^u \in Y$ for $i \in M_1$, $u \in \mathbb{B}(q)$ and $|u|$ even, we get

$$\xi^u = [i'][x_{i'}, x_i\xi^u] \in Y$$

as desired.

(ii) Assert that $x_i^m x_{i'}^{m'} \in Y$ for $m < \pi_i$, $m' < \pi_{i'}$, $i \in M_1$.

Clearly,

$$x_i^m x_{i'}^{m'} = [i]((m+1)^*)^{-1}((m'+1)^*)^{-1}[x_i^{m+1}, x_{i'}^{m'+1}] \in Y.$$

(iii) Assert that $x_1 x_i x_{i'} \in Y$ for all $i \in M_1$.

For $u_{i'} \neq 0$, it follows that $x_1 x_i = 2^{-1}\mu_{i'}^{-1}[x_1^2, x_i] \in Y$. If $u_{i'} = 0$, then

$$x_1 x_i = 4^{-1}[i]([x_i^2, [x_1^2, x_{i'}]] - x_i^2 x_{i'}) \in Y.$$

Thus, $x_1 x_i \in Y$ for all $i \in M_1$.

If $1 - 2\mu_{i'} \neq 0$, then

$$x_1 x_i x_{i'} = (1 - 2\mu_{i'})^{-1}([x_1 x_i, x_1 x_{i'}] - [i]x_1^2) \in Y.$$

If $1 - 2\mu_{i'} = 0$, then

$$x_1 x_i x_{i'} = (2(1-\mu_i))^{-1}(((3(1-\mu_{i'}))^{-1}[[x_1^3, x_{i'}], x_i]) - [i']x_1^2) \in Y.$$

(iv) Assert that $x_i^{\pi_i} x_{i'}^{\pi_{i'}} \in Y$ for all $i \in M_1$.

Using (ii)–(iii), it is clear that

$$x_i^{\pi_i} x_{i'}^{\pi_{i'}} = 3^{-1}[x_1 x_i x_{i'}, x_i^{\pi_i - 1} x_{i'}^{\pi_{i'}-1}] \in Y.$$

(v) Assert that $x_1 x_i \xi^u \in Y$ for all $i \in M_1$, $u \in \mathbb{B}(q)$.

Case 1 $|u| \not\equiv 1 \pmod{p}$. For our purpose, by (i), we first show that

$$x_1^2 \xi^u = (1 - |u|)^{-1}[x_1^2, x_1 \xi^u] \in Y.$$

For $\mu_i \neq 1$, we know

$$x_1 x_i \xi^u = (2(1-\mu_i))^{-1}[x_1^2 \xi^u, x_i] \in Y.$$

For $\mu_i = 1$, by (ii), we have

$$x_1 x_i \xi^u = 2^{-1}[i'](2^{-1}[[x_1^2 \xi^u, x_{i'}], x_i^2] - [x_1 \xi^u, x_i^2 x_{i'}]) \in Y.$$

Case 2 $|u| \equiv 1 \pmod{p}$.

Discussing just as in (iii), we obtain $x_1^2 x_i \in Y$. It is easy to verify the following fact:

$$x_1 x_i \xi^u = [x_1^2 x_i, \xi^u] \in Y.$$

(vi) Assert that $x_1^{k_1} \xi^u \in Y$ for $0 \leq k_1 \leq \pi_1$.

For $|u| \equiv 2 \pmod{p}$, it follows that $x_1^{k_1} \xi^u = -[x_1^{k_1}, x_1 \xi^u] \in Y$.

For $|u| \not\equiv 2 \pmod{p}$, $k_1 < \pi_1$, we get

$$x_1^{k_1} \xi^u = ((k_1+1)^*)^{-1}(1 - 2^{-1}|u|)^{-1}[x_1^{k_1+1}, \xi^u] \in Y.$$

If $k_1 = \pi_1$, the following equation holds:

$$x_1^{\pi_1} \xi^u = (2^{-1}|u| - 2)^{-1}[x_1^{\pi_1}, x_1 \xi^u].$$

For $2^{-1}|u|-2 \not\equiv 0 \pmod{p}$, it is easy to see that $x_1^{\pi_1}\xi^u \in Y$. Suppose that $2^{-1}|u|-2 \equiv 0 \pmod{p}$, namely, $|u| \equiv 4 \pmod{p}$. In view of (iii) and $x_1^{k_1}\xi^u \in Y$ for $|u| \neq 2$, $k_1 < \pi_1$, it follows that

$$x_1^{\pi_1-1}x_ix_{i'}\xi^u = [x_1^{\pi_1-1}\xi^u, x_1x_ix_{i'}] \in Y.$$

Let $\mu_i \neq 0$ (otherwise, $\mu_{i'} \neq 0$, the proof is similar). By virtue of (v), we get

$$x_1^{\pi_1}x_i\xi^u = \mu_i^{-1}[x_1^{\pi_1}, x_1x_i\xi^u] \in Y.$$

Then

$$x_1^{\pi_1}\xi^u = [i][x_1^{\pi_1}x_i\xi^u, x_{i'}] + [i]\mu_i x_1^{\pi_1-1}x_ix_{i'}\xi^u \in Y.$$

(vii) Let

$$\rho_1(k_1,u) = x_1^{k_1}\xi^u, \quad \rho_i(k_1,u) = x_1^{k_1}\Big(\prod_{j=2}^{i} x_j^{\pi_j}x_{j'}^{\pi_{j'}}\Big)\xi^u,$$

where $i \geq 2$. Then $\rho_i(k_1,u) \in Y$.

Let us complete the proof of (vii) by induction on i. Using (vi), it is clear that $\rho_1(k_1,u) \in Y$.

For $k_1 < \pi_1$, we show that $\rho_i(k_1,u) \in Y$ by the inductive hypothesis and (iv). In fact,

$$\rho_i(k_1,u) = (2(k_1+1)^*)^{-1}[\rho_{i-1}(k_1+1,u), x_i^{\pi_i}x_{i'}^{\pi_{i'}}] \in Y.$$

Suppose that $k_1 = \pi_1$. We obtain

$$\rho_i(\pi_1,u) = 2^{-1}(i+1-2^{-1}|u|)^{-1}[x_1^2, \rho_i(\pi_1-1,u)].$$

If $i+1-2^{-1}|u| \not\equiv 0 \pmod{p}$, then $\rho_i(\pi,u) \in Y$. Let $i+1-2^{-1}|u| \equiv 0 \pmod{p}$, $\mu_{i+1} \neq 0$ (otherwise, $\mu_{i+1} = 0$, the proof is similar and is therefore omitted). It is easy to verify the following equations:

$$\begin{aligned}
&\rho_i(\pi_1,u)x_{i+1} = -2^{-1}\mu_{i+1}^{-1}[x_1^2x_{i+1}, \rho_i(\pi_1-1,u)] \in Y,\\
&\rho_i(\pi_1-1,u)x_{i+1}x_{(i+1)'} = [\,[x_1x_{i+1}x_{(i+1)'}, x_1^{\pi_1}], \rho_i(0,u)] \in Y,\\
&\rho_i(\pi_1,u) = [i+1][\,\rho_i(\pi_1,u)x_{i+1}, x_{(i+1)'}] + [i+1]\mu_{i+1}\rho_i(\pi_1-1,u)x_{i+1}x_{(i+1)'} \in Y.
\end{aligned}$$

Now, we prove the theorem by using (i)–(vii). Let $z := x^k\xi^u$ be any basis element of $\mathfrak{M}$. Set $l_z := \sum_{i\in M_1}\pi_i - \sum_{i\in M_1}k_i$. We propose to prove that $x^k\xi^u \in Y$ by induction on l_z. For $l_z = 0$, it follows that $z = \rho_{n+1}(k_1,u) \in Y$. Let $l_z > 0$. Then there is $i \in M_1$, such that $k_i < \pi_i$. By the assumption of induction, we have $z' := x^{k+e_i}\xi^u \in Y$ and $g := x^{k-e_1+e_i}x_{i'}\xi^u \in Y$. Thus,

$$x^k\xi^u = [i]((k_i+1)^*)^{-1}([z', x_{i'}] - k_1^*\mu_i g) \in Y.$$

The induction is complete.

4 Derivations of $\mathfrak{M}$

In this section, we shall describe the structure of the derivation algebra of $\mathfrak{M}$. Since $\mathfrak{M}$ is not a simple modular Lie algebra, we can not obtain the corresponding conclusion for a simple modular Lie superalgebra $\mathcal{M}$ directly. This observation motivates us to pay our attention to the gradation component of zero $\mathbb{Z}$-degree. Then, using the generator sets of $\mathfrak{M}$ which have been obtained in Section 3, we decompose the derivations of nonzero $\mathbb{Z}$-degree and zero $\mathbb{Z}$-degree. As the final results, the derivations of the even part for the simple modular Lie superalgebra $\mathcal{M}$ are investigated.

Lemma 4.1 *Let $\varphi \in \mathrm{Der}(\mathfrak{M})$, $f \in \mathfrak{M}$. If $\varphi(x_i) = \varphi[f, x_i] = 0$, $\forall i \in M_1$, then $D_i(\varphi(f)) = 0, \forall i \in M$, that is, $\varphi(f) = \sum_u a_u \xi^u$, where $a_u \in \mathbb{F}$, $u \in \mathbb{B}(q)$.*

Proof Noting that $\varphi(x_i) = \varphi[f, x_i] = 0$, it yields that $[\varphi(f), x_i] = 0$, $\forall i \in M_1$. Thus,

$$[\varphi(f), 1] = [i][\varphi(f), [x_i, x_{i'}]] = 0, \quad \forall i \in M_1.$$

It follows that $D_1(\varphi(f)) = 0$ and $D_i(\varphi(f)) = [i][\varphi(f), x_{i'}] = 0$, $\forall i \in M_1$.

An element f of $\mathfrak{M}$ is called $\tau(i)$-truncated if $D_i^{\tau(i)}(f) = 0$, where

$$\tau(i) = \begin{cases} \pi_i, & \text{if } i \in M, \\ 1, & \text{if } i \in T. \end{cases}$$

For $i \in M$, we define a linear mapping ρ_i, such that

$$\rho_i(x^k \xi^u) = ((k_i + 1)^*)^{-1} x^{k+e_i} \xi^u,$$

where $x^{k+e_i} = 0$ for $k + e_i \notin Q$.

From the definitions above, we can get the following result directly.

Lemma 4.2 *The following statements hold:*

(i) *If $f \in \mathfrak{M}$ is $\tau(i)$-truncated, then $D_i \rho_i(f) = f$ for all $i \in M$.*

(ii) *$D_i \rho_j = \rho_j D_i$, where $i \neq j \in M$.*

Lemma 4.3 *Let $f_{t_1}, \cdots, f_{t_k} \in \mathfrak{M}$, where $t_1, \cdots, t_k \in M$. If f_i is $\tau(i)$-truncated, $i = t_1, \cdots, t_k$ and $D_i(f_j) = D_j(f_i)$, $i, j = t_1, \cdots, t_k \in M$, then there exists $f \in \mathfrak{M}$, such that $D_i(f) = f_i$, $i = t_1, \cdots, t_k \in M$.*

Proof Induction on k. If $k = 1$, then let $f = \rho_{t_1}(f_{t_1})$. In view of Lemma 4.2(i), we have $D_{t_1}(f) = D_{t_1}\rho_{t_1}(f_{t_1}) = f_{t_1}$. Assume that there exists $g \in \mathfrak{M}$, such that $D_i(g) = f_i$, $i = t_1, \cdots, t_{k-1}$. Let $f = g + \rho_{t_k}(f_{t_k} - D_{t_k}(g))$. For $i = t_1, \cdots, t_{k-1}$, utilizing Lemma 4.2(ii) and the hypothesis of this lemma, we have

$$\begin{aligned} D_i(f) &= f_i + D_i \rho_{t_k}(f_{t_k} - D_{t_k}(g)) \\ &= f_i + \rho_{t_k}(D_i(f_{t_k}) - D_i D_{t_k}(g)) \\ &= f_i + \rho_{t_k}(D_{t_k}(f_i) - D_{t_k} D_i(g)) \\ &= f_i. \end{aligned}$$

Since $f_{t_k} - D_{t_k}(g)$ is $\tau(t_k)$-truncated, by virtue of Lemma 4.2, we have

$$\begin{aligned} D_{t_k}(f) &= D_{t_k}(g) + D_{t_k}\rho_{t_k}(f_{t_k} - D_{t_k}(g)) \\ &= D_{t_k}(g) + f_{t_k} - D_{t_k}(g) \\ &= f_{t_k}. \end{aligned}$$

Our assertion follows and this completes the proof.

Lemma 4.4 *Suppose that $\varphi \in \mathrm{Der}(\mathfrak{M})$. Let $f_1 = \varphi(1)$ and $f_i = [i]\varphi(x_{i'}) - [i]\,\mu_i f_1 x_{i'}$ for $i \in M_1$. Then there exists $f \in \mathfrak{M}$, such that $D_i(f) = f_i$ for $i \in M$.*

Proof Considering the definition of $\tau(t_i)$-truncated, it is clear that f_i $(i \in M)$ are $\tau(t_i)$-truncated. It remains to verify that f_i $(i \in M)$ satisfy the condition of Lemma 4.3, that is, $D_i(f_j) = D_j(f_i),\ i, j \in M$.

By the assumption, we have

$$\varphi(x_{i'}) = [i]f_i + \mu_i f_1 x_{i'} \quad \text{for all } i \in M_1. \tag{4.1}$$

Applying φ to the equation $[1, x_{i'}] = 0$, we obtain that

$$\begin{aligned}\varphi([1, x_{i'}]) &= [f_1, x_{i'}] + [1, [i]f_i + [i]\mu_i f_1 x_{i'}]\\ &= D_1(f_1)\mu_i x_{i'} + [i]D_i(f_1) - [i]D_1(f_i) - \mu_i D_1(f_1 x_{i'})\\ &= [i]D_i(f_1) - [i]D_1(f_i)\\ &= 0,\end{aligned}$$

that is, $D_i(f_1) = D_1(f_i)$, $i \in M_1$. Applying φ to the equation $[x_{i'}, x_{j'}] = \delta_{ji'}[i']$, it follows that

$$\varphi([x_{i'}, x_{j'}]) = [i']\delta_{ji'}\varphi(1).$$

Moreover, $D_i(f_j) = D_j(f_i)$, $i, j \in M_1$. Consequently, there exists $f \in \mathfrak{M}$, such that $D_i(f) = f_i$, $i \in M$.

Proposition 4.1 *Let $\varphi \in \mathrm{Der}(\mathfrak{M})$. Then there exists $f \in \mathfrak{M}$, such that $(\varphi - \mathrm{ad}\, f)(\mathfrak{M}_{-1}) = 0$.*

Proof It is easily seen that $\mathfrak{M}_{-1} = \mathrm{span}_{\mathbb{F}}\{x_i \mid i \in M_1\}$. Let f_i be defined as in Lemma 4.4. Then there exists $f \in \mathfrak{M}$, such that $D_i(f) = f_i$, $i \in M$. Put $\varphi_1 := \varphi - \mathrm{ad}\, f$. It follows from (4.1) that

$$\begin{aligned}\varphi_1(x_i) &= \varphi(x_i) - [f, x_i]\\ &= \varphi(x_i) - (D_1(f)\mu_{i'} x_i + [i']D_{i'}(f))\\ &= \varphi(x_i) - (\mu_{i'} f_1 x_i + [i']f_{i'})\\ &= 0,\end{aligned}$$

where $i \in M_1$.

Proposition 4.2 *Let $\varphi \in \mathrm{Der}(\mathfrak{M})$. If $\varphi|_{\mathfrak{M}_{-1}} = 0$, then $\varphi(x_i x_{i'}) = 0$, $i \in M_1$.*

Proof Using Proposition 4.1, we have $\varphi(x_i) = 0$ for $i \in M_1$, since $x_i \in \mathfrak{M}_{-1}$. Applying φ to the following equations respectively:

$$[x_i^2, x_j] = 0, \quad [x_i^2, x_{i'}] = 2[i]x_i,$$

where $j \neq i' \in M_1$, we obtain that

$$\varphi([x_i^2, x_j]) = 0, \quad \varphi([x_i^2, x_{i'}]) = 2[i]\varphi(x_i) = 0,$$

where $j \neq i' \in M_1$. In view of Lemma 4.1, we know $\varphi(x_i^2) = \sum\limits_u a_u \xi^u$, where $a_u \in \mathbb{F}$, $u \in \mathbb{B}(q)$. Applying φ to the equation $[x_i^2, x_{i'}^2] = 4[i]x_i x_{i'}$, it follows that

$$\begin{aligned}\varphi(x_i x_{i'}) &= 4^{-1}[i]\varphi([x_i^2, x_{i'}^2]) \\ &= 4^{-1}[i]([\varphi(x_i^2), x_{i'}^2] + [x_i^2, \varphi(x_{i'}^2)]) \\ &= 0,\end{aligned}$$

as desired.

The following lemma is from [10, Proposition 8.4].

Lemma 4.5 *Let $\mathcal{G} = \bigoplus\limits_{i=-r}^{s} \mathcal{G}_i$ be a $\mathbb{Z}$-graded centerless Lie algebra, and let $\mathfrak{T} \subset \mathcal{G}_0$ be a torus of $\mathcal{G}$. If $\varphi \in \mathrm{Der}(\mathcal{G})$ is homogeneous of $\mathbb{Z}$-degree t, then there exists $e \in \mathcal{G}_t$, such that $(\varphi - \mathrm{ad}\, e)|_{\mathfrak{T}} = 0$.*

It is easily verified that $\mathcal{T} := \mathrm{span}_{\mathbb{F}}\{x_1\}$ is a torus of $\mathfrak{M}$.

Lemma 4.6 *Let $\varphi \in \mathrm{Der}_t(\mathfrak{M})$, and suppose that $\varphi(\mathfrak{M}_{-1}) = 0$. Then there exists $g \in (C_{\mathfrak{M}}(\mathfrak{M}_{-1}))_t$, such that $(\varphi - \mathrm{ad}\, g)|_{\mathcal{T}} = 0$.*

Proof Recall that

$$C_{\mathfrak{M}}(\mathfrak{M}_{-1}) = \mathrm{span}_{\mathbb{F}}\{\xi^u \mid u \in \mathbb{B}(q),\ |u|\, \mathrm{even}\}.$$

Noting that $x_1 \in \mathfrak{M}_0$, we get $[x_1, \mathfrak{M}_{-1}] \subseteq \mathfrak{M}_{-1}$. Thus, $\varphi(x_1) \in C_{\mathfrak{M}}(\mathfrak{M}_{-1})$. By Lemma 4.5, there is $e \in \mathfrak{M}_t$, such that $(\varphi - \mathrm{ad}\, e)|_{\mathcal{T}} = 0$. Thus, $\varphi(x_1) = [e, x_1] \in (C_{\mathfrak{M}}(\mathfrak{M}_{-1}))_t$. Noticing that $[x_1, \xi^u] = (1 - 2^{-1}|u|)\xi^u$, then there exists an element $g \in (C_{\mathfrak{M}}(\mathfrak{M}_{-1}))_t$, such that $\varphi(x_1) = [g, x_1]$. Consequently, $(\varphi - \mathrm{ad}\, g)|_{\mathcal{T}} = 0$.

Theorem 4.1 *Let $\varphi \in \mathrm{Der}_t(\mathfrak{M})$, $t \neq 0$. Then there exists $f \in \mathfrak{M}$, such that $\varphi = \mathrm{ad}\, f$.*

Proof By Proposition 4.1, there exists $f' \in \mathfrak{M}$, such that $(\varphi - \mathrm{ad}\, f')(\mathfrak{M}_{-1}) = 0$. In view of Lemma 4.6, there exists $g \in (C_{\mathfrak{M}}(\mathfrak{M}_{-1}))_t$, such that $(\varphi - \mathrm{ad}\, f' - \mathrm{ad}\, g)|_{\mathcal{T}} = 0$. Recall that $\mathcal{T} = \mathrm{span}_{\mathbb{F}}\{x_1\}$, and it follows that $(\varphi - \mathrm{ad}\, f' - \mathrm{ad}\, g)(x_1) = 0$.

Set

$$\sigma := \varphi - \mathrm{ad}\, f' - \mathrm{ad}\, g.$$

It is clear that $\sigma(\mathfrak{M}_{-1}) = 0$.

In the following, recall that $\mathcal{P} = \{x_i^{k_i} \mid 0 \leq k_i \leq \pi_i,\ i \in M\}$ and $\mathcal{Q} = \{x_1 \xi_{r+1} \xi_l \mid l \neq r+1 \in T\}$, and we proceed in several steps to show that $\sigma(\mathcal{P}) = 0$, $\sigma(\mathcal{Q}) = 0$, respectively.

First, it will be proved that $\sigma(x_i^{k_i}) = 0$ for all $i \in M$, by induction on k_i. For $k_i = 0$, from $1 \in \mathfrak{M}_{-2}$, $\sigma(1) = 0$ holds. Using Proposition 4.2, we get $\sigma(x_i x_{i'}) = 0$. Suppose that $k_i > 0$. Then $\sigma(x_i^{k_i - 1}) = 0$ by the inductive hypothesis. Applying σ to $[x_{i'}, x_i^{k_i}] = [i']k_i^* x_i^{k_i - 1}$, we obtain

$$[x_{i'}, \sigma(x_i^{k_i})] = [i']k_i^* \sigma(x_i^{k_i - 1}) = 0.$$

In view of Lemma 4.1, we have $D_j(\sigma(x_i^{k_i})) = 0$ for $j \in M$. Applying σ to

$$[x_i x_{i'}, x_i^{k_i}] = [i']k_i^* x_i^{k_i},$$

it follows that $\sigma(x_i^{k_i}) = 0$ (for $i \in M_1$) from $[x_i x_{i'}, \sigma(x_i^{k_i})] = [i']k_i^* \sigma(x_i^{k_i})$.

Next, we prove that $\sigma(x_1^{k_1}) = 0$ for $0 \leq k_1 \leq \pi_1$ by induction on k_1. To do that, we assert that $\sigma(x_1^2) = 0$, $\sigma(x_1^3) = 0$. To simplify the proof, we only verify $\sigma(x_1^2) = 0$ as the proof of $\sigma(x_1^3) = 0$ is similar. For this propose, we may assume that $\sigma(x_1^2) := \sum_{k,u} a_{k,u} x^k \xi^u$, where $k \in Q,\ u \in \mathbb{B}(q)$. One may check that $D_1(\sigma(x_1^2)) = 0$ by applying σ to $[x_1^2, 1] = 2x_1$. Similarly, applying σ to $[x_1^2, x_i x_{i'}] = 0$, we can obtain

$$\sum_{k,u} a_{k,u}([i]k_i^* + [i']k_{i'}^*) x^k \xi^u = 0.$$

For $a_{k,u} \neq 0$, it is easily verified that $k_i^* = k_{i'}^*$. Applying σ to the following equations:

$$\begin{aligned}[x_i, x_1 x_i] &= -\mu_{i'} x_i^2, \\ [x_{i'}, x_1 x_i] &= -\mu_i x_i x_{i'} + [i'] x_1, \\ [1, x_1 x_i] &= -x_i,\end{aligned}$$

respectively, from Proposition 4.2, we have

$$[x_i, \sigma(x_1 x_i)] = 0, \quad [x_{i'}, \sigma(x_1 x_i)] = 0, \quad [1, \sigma(x_1 x_i)] = 0.$$

Moreover, the following equations hold by direct computations:

$$D_{i'}(\sigma(x_1 x_i)) = 0, \quad D_i(\sigma(x_1 x_i)) = 0, \quad D_1(\sigma(x_1 x_i)) = 0.$$

Applying σ to the equation $[x_1^2, x_i] = 2\mu_{i'} x_1 x_i$, we get

$$[i'] D_{i'} \sigma(x_1^2) = 2\mu_{i'} \sigma(x_1 x_i). \tag{4.2}$$

Using (4.2), we may obtain by comparing coefficients that $k_i^* = k_{i'}^* = 0$. Then $D_j(\sigma(x_1^2)) = 0$, $\forall j \in M_1$. So far, we may assume that $\sigma(x_1^2) = \sum_u a_u \xi^u$.

In the following, we will show that $\sigma(\xi_{r+1}\xi_l) = 0$, $l \in T$. Using Lemma 4.1, it follows that $D_i(\sigma(\xi_{r+1}\xi_l)) = 0$, $i \in M$. We may assume that $\sigma(\xi_{r+1}\xi_l) = \sum_u a_u \xi^u$, where $a_u \in \mathbb{F}$, $u \in \mathbb{B}(q)$, and $|u|$ is even.

Noticing that $\mathrm{zd}(\sigma(\xi_{r+1}\xi_l)) = t > 0$ or $\mathrm{zd}(\sigma(\xi_{r+1}\xi_l)) = t < -1$ and applying σ to $[x_1, \xi_{r+1}\xi_l] = 0$, we obtain $\overline{\partial}(\sigma(\xi_{r+1}\xi_l)) = 0$. Moreover, $\sum_u a_u(1 - 2^{-1}|u|)\xi^u = 0$. For $a_u \neq 0$, we get $|u| = 2$. Note that $\mathrm{zd}(\sigma(\xi_{r+1}\xi_l)) = \mathrm{zd}\big(\sum_{|u|=2} a_u \xi^u\big) = 0$ and $\mathrm{zd}(\sigma(\xi_{r+1}\xi_l)) = t \neq 0$. This shows $\sigma(\xi_{r+1}\xi_l) = 0$. Applying σ to $[x_1^2, \xi_{r+1}\xi_l] = 0$, from $\sigma(\xi_{r+1}\xi_l) = 0$, we have

$$[\sigma(x_1^2), \xi_{r+1}\xi_l] = 0. \tag{4.3}$$

(4.3) yields that $\sigma(x_1^2) = c$, $c \in \mathbb{F}$. Applying σ to $[x_1^2, x_1] = x_1^2$, we obtain $\sigma(x_1^2) = 0$.

Assume that $k_1 > 3$. Applying σ to the following equations:

$$[x_1^3, x_1^{k_1 - 2}] = (3^* - (k_1 - 2)^*) x_1^{k_1}, \tag{4.4}$$

$$[x_1^2, x_1^{k_1 - 1}] = (2^* - (k_1 - 1)^*) x_1^{k_1}, \tag{4.5}$$

respectively, we get

$$[x_1^3, \sigma(x_1^{k_1-2})] = (3^* - (k_1-2)^*)\sigma(x_1^{k_1}), \quad (4.6)$$

$$[x_1^2, \sigma(x_1^{k_1-1})] = (2^* - (k_1-1)^*)\sigma(x_1^{k_1}). \quad (4.7)$$

For $2^* - (k_1-1)^* \not\equiv 0 \pmod{p}$, by (4.7), it is clear that $\sigma(x_1^{k_1}) = 0$ holds. For $2^* - (k_1-1)^* \equiv 0 \pmod{p}$, that is, $(k_1-1)^* = 2$, we have $3 - (k_1-2)^* \not\equiv 0 \pmod{p}$. Using (4.6), we obtain $\sigma(x_1^{k_1}) = 0$. To sum up, $\sigma(x_1^{k_1}) = 0$ holds.

Now, let us prove that $\sigma(x_1\xi_{r+1}\xi_l) = 0$, $l \in T$. Apply σ to $[x_i, x_{i'}\xi_{r+1}\xi_l] = [i]\xi_{r+1}\xi_l$ and $[x_j, x_{i'}\xi_{r+1}\xi_l] = 0$, $j \neq i \in M_1$. Using Lemma 4.1, we get $D_j(\sigma(x_{i'}\xi_{r+1}\xi_l)) = 0$, $j \in M$. Applying σ to

$$[x_i x_{i'}, x_{i'}\xi_{r+1}\xi_l] = [i]x_{i'}\xi_{r+1}\xi_l,$$

we have $\sigma(x_{i'}\xi_{r+1}\xi_l) = 0$, $i \in M_1$. Noting that the following equation holds by a direct computation:

$$x_1\xi_{r+1}\xi_l = [i]([x_1x_i, x_{i'}\xi_{r+1}\xi_l] - 4^{-1}3^{-1}\mu_{i'}[x_i^2, [x_{i'}^3, x_i\xi_{r+1}\xi_l]]),$$

we have $\sigma(x_1\xi_{r+1}\xi_l) = 0$, $l \in T$. The proof is complete.

Theorem 4.2 *Let $\varphi \in \mathrm{Der}_t(\mathfrak{M})$, $t = 0$. Then there exists $f \in \mathfrak{M}$, such that*

$$\varphi = \mathrm{ad}\, f + \sum_{|u|=2} a_u \prod_l \mathrm{ad}(\xi_{r+1}\xi_l) + \sum_{|u|=4} b_u \mathrm{ad}\, x_{i'}\, \mathrm{ad}\,(x_i\xi^{v(u)}) \prod_l \mathrm{ad}\,(\xi_{r+1}\xi_l),$$

where $|v(u)| = 2$, $u_{i'} \neq 0$, $i \in M_1$, $a_u, b_u \in \mathbb{F}$, and $l \in T$.

Proof In view of the proof of Theorem 4.1, there exists $\sigma = \varphi - \mathrm{ad}\, f' - \mathrm{ad}\, g$, where $f' \in \mathfrak{M}$, $g \in C_{\mathfrak{M}}(\mathfrak{M}_{-1})$, such that $\sigma(\mathfrak{M}_{-1}) = 0$, $\sigma(\mathcal{T}) = 0$.

We obtain the following results:

$$\sigma(\mathcal{P}) = 0, \quad \sigma(\mathcal{Q}) = 0, \quad \sigma(\xi_{r+1}\xi_l) = \sum_{|u|=2} c_u\xi^u, \quad c_u \in \mathbb{F}.$$

Set

$$\sigma(x_1\xi_{r+1}\xi_l) := \sum_{k,u} a_{k,u}x^k\xi^u. \quad (4.8)$$

Applying σ to $[x_ix_{i'}, x_1\xi_{r+1}\xi_l] = 0$, we obtain

$$[i]x_{i'}\Big(\sum_{k,u} a_{k,u}k_{i'}^* x^{k-e_{i'}}\xi^u\Big) + [i']x_i\Big(\sum_{k,u} a_{k,u}k_i^* x^{k-e_i}\xi^u\Big) = 0. \quad (4.9)$$

For $a_{k,u} \neq 0$, by (4.8)–(4.9), we have $k_i^* = k_{i'}^*$. Applying σ to

$$[1, x_i\xi_{r+1}\xi_l] = 0$$

and

$$[x_j, x_i\xi_{r+1}\xi_l] = 0, \quad j \neq i' \in M_1,$$

we get

$$D_j(\sigma(x_i\xi_{r+1}\xi_l)) = 0, \quad j \neq i \in M_1. \quad (4.10)$$

Applying σ to $[x_{i'}, x_i\xi_{r+1}\xi_l] = [i']\xi_{r+1}\xi_l$, it follows that

$$D_i(\sigma(x_i\xi_{r+1}\xi_l)) = \sigma(\xi_{r+1}\xi_l). \tag{4.11}$$

Using (4.10)–(4.11), we have

$$\sigma(x_i\xi_{r+1}\xi_l) = x_i\sigma(\xi_{r+1}\xi_l). \tag{4.12}$$

Applying σ to $[1, x_1\xi_{r+1}\xi_l] = -\xi_{r+1}\xi_l$, we get

$$D_1(\sigma(x_1\xi_{r+1}\xi_l)) = \sigma(\xi_{r+1}\xi_l). \tag{4.13}$$

Applying σ to $[x_i, x_1\xi_{r+1}\xi_l] = -\mu_{i'}x_i\xi_{r+1}\xi_l$, it follows that

$$-\mu_{i'}x_iD_1(\sigma(x_1\xi_{r+1}\xi_l)) + [i]D_{i'}(\sigma(x_1\xi_{r+1}\xi_l)) = -\mu_{i'}\sigma(x_i\xi_{r+1}\xi_l). \tag{4.14}$$

Moreover, $k^*_{i'} = k^*_i = 0$.

It is easily verified that each term $x_1\xi^u$ is generated by some $f_m \in \mathfrak{M}$ and $x_1\xi_{r+1}\xi_l$. Consequently, combining (4.12)–(4.14), we have

$$\begin{aligned}\sigma(x_1\xi_{r+1}\xi_l) &= x_1\sigma(\xi_{r+1}\xi_l) + \sum_{|u|=4} d_u\xi^u \\ &= \sum_{|u|=2} c_u x_1\xi^u + \sum_{|u|=4} d_u\xi^u \\ &= \sum_{|u|=2} a_u\Big(\prod_{m(u)} \operatorname{ad} f_{m(u)}(x_1\xi_{r+1}\xi_l)\Big) \\ &\quad + \sum_{|u|=4} b_u\Big(\operatorname{ad} x_{i'} \operatorname{ad} x_i\xi^{v(u)} \prod_{n(u)} \operatorname{ad} f_{n(u)}(x_1\xi_{r+1}\xi_l)\Big),\end{aligned}$$

where $a_u, b_u, c_u, d_u \in \mathbb{F}$, $f_{m(u)}, f_{n(u)} \in \mathfrak{M}$.

Let

$$\Delta := \sigma + \sum_{|u|=2} a_u \prod_{m(u)} \operatorname{ad} f_{m(u)} + \sum_{|u|=4} b_u\Big(\operatorname{ad} x_{i'} \operatorname{ad} x_i\xi^{v(u)} \prod_{n(u)} \operatorname{ad} f_{n(u)}\Big),$$

where $a_u, b_u \in \mathbb{F}$.

Noticing that the following equations hold by Theorem 3.1:

$$\Delta(\mathcal{P}) = 0, \quad \Delta(\mathcal{Q}) = 0,$$

we have $\Delta = 0$. Hence, there exists $f \in \mathfrak{M}$, such that

$$\varphi = \operatorname{ad} f + \sum_{|u|=2} a_u \prod_l \operatorname{ad}(\xi_{r+1}\xi_l) + \sum_{|u|=4} b_u \operatorname{ad} x_{i'} \operatorname{ad}(x_i\xi^{v(u)}) \prod_l \operatorname{ad}(\xi_{r+1}\xi_l),$$

where $|v(u)| = 2$, $i \in M_1$, $l \in T$ and $a_u, b_u \in \mathbb{F}$.

Theorem 4.3

$$\operatorname{Der}(\mathfrak{M}) = \operatorname{ad}(\mathfrak{M}) \oplus \operatorname{span}_{\mathbb{F}}\Big\{\prod_l \operatorname{ad}(\xi_{r+1}\xi_l)\Big\} \oplus \operatorname{span}_{\mathbb{F}}\Big\{\operatorname{ad} x_{i'} \operatorname{ad}(x_i\xi^v) \prod_l \operatorname{ad}(\xi_{r+1}\xi_l)\Big\},$$

where $i \in M_1$, $l \in T$, and $|v| = 2$.

Proof This is a direct consequence of Theorems 4.1–4.2.

Remark 4.1 Now we can answer the question mentioned in the introduction for the simple modular Lie superalgebra $\mathcal{M}$. By Theorem 4.3 and [9, Theorem 3], it is easy to see that the derivation algebra of the even part of $\mathcal{M}$ does not equal the even part of the derivation superalgebra of $\mathcal{M}$, that is, $\mathrm{Der}(\mathcal{M}_{\overline{0}}) \neq \mathrm{Der}(\mathcal{M})_{\overline{0}}$.

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