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# Hypercube and Tetrahedron Algebra*

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**Abstract** Let $D$ be an integer at least 3 and let $H(D,2)$ denote the hypercube. It is known that $H(D,2)$ is a $Q$-polynomial distance-regular graph with diameter $D$, and its eigenvalue sequence and its dual eigenvalue sequence are all $\{D-2i\}_{i=0}^{D}$. Suppose that $\boxtimes$ denotes the tetrahedron algebra. In this paper, the authors display an action of $\boxtimes$ on the standard module $V$ of $H(D,2)$. To describe this action, the authors define six matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A,\ A^*,\ B,\ B^*,\ K,\ K^*.$$

Moreover, for each matrix above, the authors compute the transpose and then compute the transpose of each generator of $\boxtimes$ on $V$.

**Keywords** Tetrahedron algebra, Hypercube, Distance-regular graph, Onsager algebra
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## 1 Introduction

Throughout this paper, $\mathbb{C}$ denotes the field of complex numbers and $\mathbb{R}$ denotes the field of real numbers.

In [20], Hartwig and Terwilliger found a presentation for the three-point $sl_2$ loop algebra via generators and relations. To obtain this presentation, they defined a Lie algebra $\boxtimes$ by generators and relations, and displayed an isomorphism from $\boxtimes$ to the three-point $sl_2$ loop algebra. In [15], Elduque found an attractive decomposition of $\boxtimes$ into a direct sum of three abelian subalgebras, and showed how these subalgebras are related to the Onsager subalgebras. In [19], Hartwig classified the finite-dimensional irreducible $\boxtimes$-modules over an algebraically closed field $\mathbb{F}$ with characteristic 0. In [22], Itô and Terwilliger described the finite-dimensional irreducible $\boxtimes$-modules from multiple points of view.

Let $D$ be an integer at least 3 and let $H(D,2)$ denote the hypercube. It is known that $H(D,2)$ is a $Q$-polynomial distance-regular graph with diameter $D$, and its eigenvalue sequence and its dual eigenvalue sequence are all $\{D-2i\}_{i=0}^{D}$. In this paper, we display an action of $\boxtimes$ on the standard module $V$ of $H(D,2)$. To describe this action we define six matrices in $\mathrm{Mat}_X(\mathbb{C})$, called

$$A, A^*, B, B^*, K, K^*.$$

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Moreover, for each matrix above we compute the transpose and then compute the transpose of each generator of $\boxtimes$ on $V$.

## 2 Tetrahedron Algebra $\boxtimes$ and Onsager Algebra $\mathcal{O}$

In this section, we recall the definitions of the tetrahedron algebra $\boxtimes$ and the Onsager algebra $\mathcal{O}$ and show how the finite-dimensional irreducible modules for $\boxtimes$ and $\mathcal{O}$ are related.

**Definition 2.1** (see [20, Definition 1.1]) *Let $\boxtimes$ denote the Lie algebra over $\mathbb{C}$ with generators*

$$\{x_{rs} \mid r, s \in \mathrm{I},\ r \neq s\}, \quad \mathrm{I} = \{0, 1, 2, 3\}$$

*and the following relations:*

(i) *For all distinct $r, s \in \mathrm{I}$,*

$$x_{rs} + x_{sr} = 0. \tag{2.1}$$

(ii) *For all mutually distinct $r, s, t \in \mathrm{I}$,*

$$[x_{rs}, x_{st}] = 2x_{rs} + 2x_{st}. \tag{2.2}$$

(iii) *For all mutually distinct $r, s, t, u \in \mathrm{I}$,*

$$[x_{rs}, [x_{rs}, [x_{rs}, x_{tu}]]] = 4[x_{rs}, x_{tu}]. \tag{2.3}$$

*We call $\boxtimes$ the tetrahedron algebra.*

**Remark 2.1** (2.3) is the Dolan-Grady relation.

**Definition 2.2** (see [19, Definition 1.2]) *Let $\mathcal{O}$ denote the Lie algebra over $\mathbb{C}$ with generators $X, Y$ satisfying relations*

$$[X, [X, [X, Y]]] = 4[X, Y], \tag{2.4}$$

$$[Y, [Y, [Y, X]]] = 4[Y, X]. \tag{2.5}$$

*We call $\mathcal{O}$ the Onsager algebra. We call $X, Y$ the standard generators for $\mathcal{O}$.*

**Proposition 2.1** (see [20, Proposition 4.7]) *Let $r, s, t, u$ denote mutually distinct elements of $\mathrm{I}$. Then there exists a unique Lie algebra homomorphism from $\mathcal{O}$ to $\boxtimes$ that sends*

$$X \to x_{rs}, \quad Y \to x_{tu}.$$

**Note 2.1** (see [20, Note 4.8]) The homomorphism in Proposition 2.1 is an injection.

Let $V$ denote a finite-dimensional irreducible $\mathcal{O}$-module. Then by [19, Theorem 2.4], the standard generators $X, Y$ are diagonalizable on $V$. Moreover, there exist an integer $d \geq 0$ and scalars $\alpha, \alpha^* \in \mathbb{C}$ such that the set of distinct eigenvalues of $X$ (resp. $Y$) on $V$ is $\{d - 2i + \alpha \mid 0 \leq i \leq d\}$ (resp. $\{d - 2i + \alpha^* \mid 0 \leq i \leq d\}$). We call the ordered pair $(\alpha, \alpha^*)$ the type of $V$. Replacing $X$, $Y$ by $X - \alpha I$, $Y - \alpha^* I$, respectively, the type becomes $(0, 0)$. Let $V$ denote a finite-dimensional irreducible $\boxtimes$-module. Then by [19, Theorem 3.8], each generator

$x_{rs}$ of $\boxtimes$ is diagonalizable on $V$. Moreover, there exists an integer $d \geq 0$ such that the set of distinct eigenvalues of $x_{rs}$ on $V$ is $\{d-2i \mid 0 \leq i \leq d\}$. We call $d$ the diameter of $V$. The finite-dimensional irreducible modules for $\boxtimes$ and $\mathcal{O}$ are related according to the following two propositions and the subsequent remark.

**Proposition 2.2** (see [19, Theorem 1.7]) *Let $V$ denote a finite-dimensional irreducible $\boxtimes$-module. Then there exists a unique $\mathcal{O}$-module structure on $V$ such that the standard generators $X$, $Y$ act on $V$ as $x_{01}$, $x_{23}$ respectively. This $\mathcal{O}$-module structure is irreducible and has type* $(0,0)$.

**Proposition 2.3** (see [19, Theorem 1.8]) *Let $V$ denote a finite-dimensional irreducible $\mathcal{O}$-module of type* $(0,0)$*. Then there exists a unique $\boxtimes$-module structure on $V$ such that the standard generators $X$, $Y$ act on $V$ as $x_{01}$, $x_{23}$ respectively. This $\boxtimes$-module structure is irreducible.*

**Remark 2.2** (see [19, Remark 1.9]) Combining the previous two propositions, we obtain a bijection between the following two sets:

(i) The isomorphism classes of finite-dimensional irreducible $\mathcal{O}$-modules of type $(0,0)$.

(ii) The isomorphism classes of finite-dimensional irreducible $\boxtimes$-modules.

## 3 Terwilliger Algebra of a Distance-Regular Graph

In this section, we review some definitions and basic results concerning the distance-regular graphs. For more background information, we refer the readers to [1, 3, 18, 29].

Let $X$ denote a nonempty finite set. Let $\mathrm{Mat}_X(\mathbb{C})$ denote the $\mathbb{C}$-algebra consisting of all matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{C}$. Let $V = \mathbb{C}^X$ denote the vector space over $\mathbb{C}$ consisting of column vectors whose coordinates are indexed by $X$ and whose entries are in $\mathbb{C}$. We observe that $\mathrm{Mat}_X(\mathbb{C})$ acts on $V$ by left multiplication. We call $V$ the standard module. We endow $V$ with the Hermitian inner product $\langle\ ,\ \rangle$ that satisfies $\langle u, v\rangle = u^t\overline{v}$ for $u, v \in V$, where $t$ denotes transpose and $-$ denotes complex conjugation. For all $y \in X$, let $\widehat{y}$ denote the element of $V$ with 1 in $y$ coordinate and 0 in all other coordinates. We observe that $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for $V$.

Let $\Gamma = (X, R)$ denote a finite, undirected, connected graph, without loops or multiple edges, but with a vertex set $X$ and an edge set $R$. Let $\partial$ denote the path-length distance function for $\Gamma$, and set $D := \max\{\partial(x,y) \mid x, y \in X\}$. We call $D$ the diameter of $\Gamma$. We say $\Gamma$ is distance-regular whenever for all integers $h, i, j$ $(0 \leq h, i, j \leq D)$ and for all vertices $x, y \in X$ with $\partial(x, y) = h$, the number

$$p^h_{ij} = |\{z \in X \mid \partial(x,z) = i, \partial(z,y) = j\}|$$

is independent of $x$ and $y$. The $p^h_{ij}$ are called the intersection numbers of $\Gamma$.

For the rest of this paper, we assume that $\Gamma$ is a distance-regular graph with diameter $D \geq 3$.

We mention a fact for later use. By the triangle inequality, for $0 \leq h, i, j \leq D$, we have $p^h_{ij} = 0$ (resp. $p^h_{ij} \neq 0$), whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two.

We recall the Bose-Mesner algebra of $\Gamma$. For $0 \leq i \leq D$, let $A_i$ denote the matrix in $\mathrm{Mat}_X(\mathbb{C})$

with the $(x,y)$-entry:

$$(A_i)_{xy} = \begin{cases} 1, & \text{if } \partial(x,y) = i, \\ 0, & \text{if } \partial(x,y) \neq i, \end{cases} \quad x, y \in X. \tag{3.1}$$

We call $A_i$ the $i$th distance matrix of $\Gamma$. The matrix $A_1$ is often called the adjacency matrix of $\Gamma$. We observe that (i) $A_0 = I$; (ii) $\sum_{i=0}^{D} A_i = J$; (iii) $\overline{A_i} = A_i$ $(0 \le i \le D)$; (iv) $A_i^t = A_i$ $(0 \le i \le D)$; (v) $A_iA_j = \sum_{h=0}^{D} p_{ij}^h A_h$ $(0 \le i, j \le D)$, where $I$ (resp. $J$) denotes the identity matrix (resp. all 1s matrix) in $\mathrm{Mat}_X(\mathbb{C})$. Using these facts, we find that $A_0, A_1, \cdots, A_D$ form a basis for a commutative subalgebra $M$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M$ the Bose-Mesner algebra of $\Gamma$. It turns out that $A_1$ generates $M$ (see [1, p. 190]). By [3, p. 45], $M$ has a second basis $E_0, E_1, \cdots, E_D$ such that (i) $E_0 = |X|^{-1}J$; (ii) $\sum_{i=0}^{D} E_i = I$; (iii) $\overline{E_i} = E_i$ $(0 \le i \le D)$; (iv) $E_i^t = E_i$ $(0 \le i \le D)$; (v) $E_iE_j = \delta_{ij}E_i$ $(0 \le i, j \le D)$. We call $E_0, E_1, \cdots, E_D$ the primitive idempotents of $\Gamma$.

We recall the eigenvalues of $\Gamma$. Since $E_0, E_1, \cdots, E_D$ form a basis for $M$, there exist complex scalars $\theta_0, \theta_1, \cdots, \theta_D$ such that $A_1 = \sum_{i=0}^{D} \theta_i E_i$. Observe that $A_1E_i = E_iA_1 = \theta_iE_i$ for $0 \le i \le D$. By [1, p. 197], the scalars $\theta_0, \theta_1, \cdots, \theta_D$ are in $\mathbb{R}$. Observe that $\theta_0, \theta_1, \cdots, \theta_D$ are mutually distinct since $A_1$ generates $M$. We call $\theta_i$ the eigenvalue of $\Gamma$ associated with $E_i$ $(0 \le i \le D)$. Observe that

$$V = E_0V + E_1V + \cdots + E_DV \quad \text{(an orthogonal direct sum)}.$$

For $0 \le i \le D$, the space $E_iV$ is the eigenspace of $A_1$ associated with $\theta_i$.

We now recall the Krein parameters. Let $\circ$ denote the entrywise product in $\mathrm{Mat}_X(\mathbb{C})$. Observe that $A_i \circ A_j = \delta_{ij}A_i$ for $0 \le i, j \le D$, so $M$ is closed under $\circ$. Thus, there exist complex scalars $q_{ij}^h$ $(0 \le h, i, j \le D)$ such that

$$E_i \circ E_j = |X|^{-1} \sum_{h=0}^{D} q_{ij}^h E_h, \quad 0 \le i, j \le D.$$

By [2, p. 170], $q_{ij}^h$ is real and nonnegative for $0 \le h, i, j \le D$. The $q_{ij}^h$ are called the Krein parameters of $\Gamma$. The graph $\Gamma$ is said to be $Q$-polynomial (with respect to the given ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents) whenever for $0 \le h, i, j \le D$, $q_{ij}^h = 0$ (resp. $q_{ij}^h \neq 0$), whenever one of $h, i, j$ is greater than (resp. equal to) the sum of the other two (see [4, p. 235]). See [3, 5–7, 10, 12–13, 26] for the background information on the $Q$-polynomial property.

For the rest of this section, we assume $\Gamma$ is a $Q$-polynomial distance-regular graph with respect to $E_0, E_1, \cdots, E_D$.

We recall the dual Bose-Mesner algebra of $\Gamma$. For the rest of this paper, we fix a vertex $x \in X$. We view $x$ as a "base vertex". For $0 \le i \le D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with the $(y,y)$-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } \partial(x,y) = i, \\ 0, & \text{if } \partial(x,y) \neq i, \end{cases} \quad y \in X. \tag{3.2}$$

We call $E_i^*$ the $i$th dual idempotent of $\Gamma$ with respect to $x$ (see [28, p. 378]). We observe that (i) $\sum_{i=0}^{D} E_i^* = I$; (ii) $\overline{E_i^*} = E_i^*$ $(0 \le i \le D)$; (iii) $E_i^{*t} = E_i^*$ $(0 \le i \le D)$; (iv) $E_i^* E_j^* = \delta_{ij} E_i^*$ $(0 \le i, j \le D)$. By these facts, $E_0^*, E_1^*, \cdots, E_D^*$ form a basis for a commutative subalgebra $M^*$ of $\mathrm{Mat}_X(\mathbb{C})$. We call $M^*$ the dual Bose-Mesner algebra of $\Gamma$ with respect to $x$ (see [28, p. 378]). For $0 \le i \le D$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $\mathrm{Mat}_X(\mathbb{C})$ with $(y, y)$-entry $(A_i^*)_{yy} = |X|(E_i)_{xy}$ for $y \in X$. Then $A_0^*, A_1^*, \cdots, A_D^*$ is a basis for $M^*$ (see [28, p. 379]). Moreover, (i) $A_0^* = I$; (ii) $\overline{A_i^*} = A_i^*$ $(0 \le i \le D)$; (iii) $A_i^{*t} = A_i^*$ $(0 \le i \le D)$; (iv) $A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^*$ $(0 \le i, j \le D)$ (see [28, p. 379]). We call $A_0^*, A_1^*, \cdots, A_D^*$ the dual distance matrices of $\Gamma$ with respect to $x$. The matrix $A_1^*$ is often called the dual adjacency matrix of $\Gamma$ with respect to $x$. The matrix $A_1^*$ generates $M^*$ (see [28, Lemma 3.11]).

We recall the dual eigenvalues of $\Gamma$. Since $E_0^*, E_1^*, \cdots, E_D^*$ form a basis for $M^*$, there exist complex scalars $\theta_0^*, \theta_1^*, \cdots, \theta_D^*$ such that $A_1^* = \sum_{i=0}^{D} \theta_i^* E_i^*$. Observe that $A_1^* E_i^* = E_i^* A_1^* = \theta_i^* E_i^*$ for $0 \le i \le D$. By [28, Lemma 3.11], the scalars $\theta_0^*, \theta_1^*, \cdots, \theta_D^*$ are in $\mathbb{R}$. Observe that $\theta_0^*, \theta_1^*, \cdots, \theta_D^*$ are mutually distinct since $A_1^*$ generates $M^*$. We call $\theta_i^*$ the dual eigenvalue of $\Gamma$ associated with $E_i^*$ $(0 \le i \le D)$.

We recall the subconstituents of $\Gamma$. From (3.2) we find

$$E_i^* V = \mathrm{span}\{\widehat{y} \mid y \in X,\ \partial(x, y) = i\}, \quad 0 \le i \le D. \tag{3.3}$$

By (3.3) and since $\{\widehat{y} \mid y \in X\}$ is an orthonormal basis for $V$, we find

$$V = E_0^* V + E_1^* V + \cdots + E_D^* V \quad \text{(the orthogonal direct sum)}.$$

For $0 \le i \le D$, the space $E_i^* V$ is the eigenspace of $A_1^*$ associated with $\theta_i^*$. We call $E_i^* V$ the $i$th subconstituent of $\Gamma$ with respect to $x$.

We recall the Terwilliger algebra of $\Gamma$. Let $T = T(x)$ denote the subalgebra of $\mathrm{Mat}_X(\mathbb{C})$ generated by $M$ and $M^*$. We call $T$ the Terwilliger algebra (or the subconstituent algebra) of $\Gamma$ with respect to $x$ (see [29, Definition 3.3]). We observe that $T$ is generated by $A_1, A_1^*$ and has finite dimension. Moreover, $T$ is semisimple since it is closed under the conjugate transpose map (see [11, p. 157]). By [29, Lemma 3.2], the following are relations in $T$:

$$E_h^* A_i E_j^* = 0 \quad \text{if and only if} \quad p_{ij}^h = 0, \quad 0 \le h, i, j \le D, \tag{3.4}$$
$$E_h A_i^* E_j = 0 \quad \text{if and only if} \quad q_{ij}^h = 0, \quad 0 \le h, i, j \le D. \tag{3.5}$$

See [8–10, 14, 16–17, 21, 27–30] for more information on the Terwilliger algebra.

For the rest of this paper, we adopt the following notation convention.

**Notation 3.1** Assume that $\Gamma = (X; R)$ is a distance-regular graph with diameter $D \ge 3$ and has a $Q$-polynomial structure with respect to the ordering $E_0, E_1, \cdots, E_D$ of the primitive idempotents. We fix $x \in X$ and write $A_1^* = A_1^*(x)$, $E_i^* = E_i^*(x)$ $(0 \le i \le D)$, $T = T(x)$. We use the abbreviation $V = \mathbb{C}^X$.

With reference to Notation 3.1, we recall some useful results on $T$-modules. By a $T$-module, we mean a subspace $W \subseteq V$ such that $BW \subseteq W$ for all $B \in T$. Let $W$ denote a $T$-module and let $W'$ denote a $T$-module contained in $W$. Then the orthogonal complement of $W'$ in $W$ is a $T$-module (see [17, p. 802]). It follows that each $T$-module is an orthogonal direct sum of irreducible $T$-modules. In particular, $V$ is an orthogonal direct sum of irreducible $T$-modules.

Let $W$ denote an irreducible $T$-module. Observe that $W$ is the direct sum of the nonzero spaces among $E_0^*W, \cdots, E_D^*W$. Similarly, $W$ is the direct sum of the nonzero spaces among $E_0W, \cdots, E_DW$. By the endpoint of $W$, we mean $\min\{i \mid 0 \le i \le D, E_i^*W \ne 0\}$. By the diameter of $W$, we mean $|\{i \mid 0 \le i \le D, E_i^*W \ne 0\}| - 1$. By the dual endpoint of $W$, we mean $\min\{i \mid 0 \le i \le D, E_iW \ne 0\}$. By the dual diameter of $W$, we mean $|\{i \mid 0 \le i \le D, E_iW \ne 0\}| - 1$. It turns out that the diameter of $W$ is equal to the dual diameter of $W$ (see [26, Corollary 3.3]).

**Lemma 3.1** (see [28, Lemmas 3.4, 3.9, 3.12]) *With reference to Notation* 3.1*, let $W$ denote an irreducible $T$-module with endpoint $\rho$, dual endpoint $\tau$, and diameter $d$. Then $\rho, \tau, d$ are nonnegative integers such that $\rho + d \le D$ and $\tau + d \le D$. Moreover, the following* (i)–(iv) *hold:*

(i) $E_i^*W \ne 0$ *if and only if* $\rho \le i \le \rho + d$ $(0 \le i \le D)$.

(ii) $W = \sum_{h=0}^{d} E_{\rho+h}^*W$ (*the orthogonal direct sum*).

(iii) $E_iW \ne 0$ *if and only if* $\tau \le i \le \tau + d$ $(0 \le i \le D)$.

(iv) $W = \sum_{h=0}^{d} E_{\tau+h}W$ (*the orthogonal direct sum*).

We finish this section with a comment.

**Lemma 3.2** (see [23, Lemma 12.1]) *With reference to Notation* 3.1*, for $Y \in \mathrm{Mat}_X(\mathbb{C})$, the following are equivalent:*

(i) $Y \in T$.

(ii) $YW \subseteq W$ *for all irreducible $T$-modules $W$.*

## 4 Split Decompositions of Standard Module

In this section, we recall the split decompositions for the standard module and define some useful matrices by using these decompositions.

**Definition 4.1** (see [23, Definition 10.1]) *With reference to Notation* 3.1*, for $-1 \le i, j \le D$, we define*

$$
\begin{aligned}
V_{i,j}^{\downarrow\downarrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_0V + \cdots + E_jV),\\
V_{i,j}^{\uparrow\downarrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_0V + \cdots + E_jV),\\
V_{i,j}^{\downarrow\uparrow} &= (E_0^*V + \cdots + E_i^*V) \cap (E_DV + \cdots + E_{D-j}V),\\
V_{i,j}^{\uparrow\uparrow} &= (E_D^*V + \cdots + E_{D-i}^*V) \cap (E_DV + \cdots + E_{D-j}V).
\end{aligned}
$$

*In each of the above four equations, we interpret the right-hand side as being* 0 *if* $i = -1$ *or* $j = -1$.

**Definition 4.2** (see [23, Definition 10.2]) *With reference to Notation* 3.1 *and Definition*

4.1, *for* $\mu, \nu \in \{\downarrow, \uparrow\}$ *and* $0 \leq i, j \leq D$, *we have* $V^{\mu\nu}_{i-1,j} \subseteq V^{\mu\nu}_{i,j}$ *and* $V^{\mu\nu}_{i,j-1} \subseteq V^{\mu\nu}_{i,j}$. *Therefore,*

$$V^{\mu\nu}_{i-1,j} + V^{\mu\nu}_{i,j-1} \subseteq V^{\mu\nu}_{i,j}.$$

*Referring to the above inclusion, we define* $\widetilde{V}^{\mu\nu}_{i,j}$ *to be the orthogonal complement of the left-hand side in the right-hand side; that is,*

$$\widetilde{V}^{\mu\nu}_{i,j} = (V^{\mu\nu}_{i-1,j} + V^{\mu\nu}_{i,j-1})^{\perp} \cap V^{\mu\nu}_{i,j}.$$

**Lemma 4.1** (see [23, Definition 10.3]) *With reference to Notation* 3.1 *and Definition* 4.2, *we have that for* $\mu, \nu \in \{\downarrow, \uparrow\}$,

$$V = \sum_{i=0}^{D} \sum_{j=0}^{D} \widetilde{V}^{\mu\nu}_{i,j} \quad (\textit{the direct sum}). \tag{4.1}$$

**Definition 4.3** (see [25, Definition 6.4]) *We call the sum* (4.1) *the* $(\mu, \nu)$*-split decomposition of* $V$ *with respect to* $x$. *This decomposition is not orthogonal in general.*

**Definition 4.4** (see [24, Definition 4.1]) *With reference to Notation* 3.1 *and Definition* 4.2, *for* $\mu, \nu \in \{\downarrow, \uparrow\}$ *and* $0 \leq i, j \leq D$, *we define* $E^{\mu\nu}_{i,j} \in \mathrm{Mat}_X(\mathbb{C})$ *so that*

$$(E^{\mu\nu}_{i,j} - I)\widetilde{V}^{\mu\nu}_{i,j} = 0,$$
$$E^{\mu\nu}_{i,j}\widetilde{V}^{\mu\nu}_{r,s} = 0, \quad \textit{if } (i,j) \neq (r,s),\ 0 \leq r, s \leq D.$$

**Lemma 4.2** (see [24, Theorem 4.7]) *With reference to Notation* 3.1 *and Definition* 4.4, *for* $0 \leq i, j \leq D$,

(i) $(E^{\downarrow\downarrow}_{i,j})^t = E^{\uparrow\uparrow}_{D-i,D-j}$.

(ii) $(E^{\uparrow\downarrow}_{i,j})^t = E^{\downarrow\uparrow}_{D-i,D-j}$.

The following result on irreducible $T$-modules is a mild generalization of [31, Lemma 6.1].

**Lemma 4.3** (see [23, Lemma 11.4]) *With reference to Notation* 3.1 *and Definition* 4.2, *let* $W$ *denote an irreducible* $T$*-module with the endpoint* $\rho$, *the dual endpoint* $\tau$, *and the diameter* $d$. *Then the following* (i)–(iv) *hold for* $0 \leq i, j \leq d$:

(i) *The space*

$$(E^*_{\rho}W + \cdots + E^*_{\rho+d-i}W) \cap (E_{\tau+d-i}W + \cdots + E_{\tau+d}W)$$

*is contained in* $\widetilde{V}^{\downarrow\uparrow}_{\rho+d-i,D-d-\tau+i}$.

(ii) *The space*

$$(E^*_{\rho+d-i}W + \cdots + E^*_{\rho+d}W) \cap (E_{\tau}W + \cdots + E_{\tau+d-i}W)$$

*is contained in* $\widetilde{V}^{\uparrow\downarrow}_{D-d-\rho+i,\tau+d-i}$.

(iii) *The space*

$$(E^*_{\rho}W + \cdots + E^*_{\rho+d-i}W) \cap (E_{\tau}W + \cdots + E_{\tau+i}W)$$

*is contained in* $\widetilde{V}^{\downarrow\downarrow}_{\rho+d-i,\tau+i}$.

(iv) *The space*

$$(E^*_{\rho+i}W + \cdots + E^*_{\rho+d}W) \cap (E_{\tau+d-i}W + \cdots + E_{\tau+d}W)$$

*is contained in* $\widetilde{V}^{\uparrow\uparrow}_{D-\rho-i,D-d-\tau+i}$.

## 5 Displacement Decompositions of Standard Module

In this section, we recall the displacement decompositions for the standard module and discuss their basic properties.

**Definition 5.1** (see [31, Definition 4.1]) *With reference to Notation* 3.1*, let* $W$ *denote an irreducible* $T$*-module with the endpoint* $\rho$*, the dual endpoint* $\tau$*, and the diameter* $d$*. By the displacement of* $W$ *of the first kind* (*resp. the second kind*)*, we mean the integer* $\rho+\tau+d-D$ (*resp.* $\rho-\tau$).

**Lemma 5.1** (see [24, Corollary 3.2]) *With reference to Notation* 3.1*, let* $W$ *denote an irreducible* $T$*-module. Then the following hold:*

(i) *Let* $\eta$ *denote the displacement of* $W$ *of the first kind. Then* $0 \le \eta \le D$.

(ii) *Let* $\zeta$ *denote the displacement of* $W$ *of the second kind. Then* $-D \le \zeta \le D$.

**Definition 5.2** (see [31, Definitions 4.3, 4.5]) *With reference to Notation* 3.1*, for* $0 \le \eta \le D$*, let* $V_\eta$ *denote the subspace of* $V$ *spanned by the irreducible* $T$*-modules, for which* $\eta$ *is the displacement of the first kind. Observe that* $V_\eta$ *is a* $T$*-module. By* [31, *Lemma* 4.4]*, we have*

$$V = \sum_{\eta=0}^{D} V_\eta \quad (\textit{the direct sum}). \tag{5.1}$$

*We call the sum* (5.1) *the displacement decomposition of* $V$ *of the first kind with respect to* $x$.

**Definition 5.3** (see [24, Definitions 3.7, 3.9]) *With reference to Notation* 3.1*, for* $-D \le \zeta \le D$*, let* $V_\zeta$ *denote the subspace of* $V$ *spanned by the irreducible* $T$*-modules, for which* $\zeta$ *is the displacement of the second kind. Observe that* $V_\zeta$ *is a* $T$*-module. By* [24, *Lemma* 3.8]*, we have*

$$V = \sum_{\zeta=-D}^{D} V_\zeta \quad (\textit{the direct sum}). \tag{5.2}$$

*We call the sum* (5.2) *the displacement decomposition of* $V$ *of the second kind with respect to* $x$.

**Lemma 5.2** (see [24, Theorem 3.20]) *With reference to Notation* 3.1 *and Definitions* 4.2 *and* 5.2*, the following hold for* $0 \le \eta \le D$*:*

(i) $V_\eta = \sum \widetilde{V}^{\downarrow\downarrow}_{i,j}$*, where the sum is over all ordered pairs* $i, j$ *such that* $0 \le i, j \le D$ *and* $i+j = D+\eta$.

(ii) $V_\eta = \sum \widetilde{V}^{\uparrow\uparrow}_{i,j}$*, where the sum is over all ordered pairs* $i, j$ *such that* $0 \le i, j \le D$ *and* $i+j = D-\eta$.

**Lemma 5.3** (see [24, Theorem 3.21]) *With reference to Notation* 3.1 *and Definitions* 4.2 *and* 5.3*, the following hold for* $-D \le \zeta \le D$*:*

(i) $V_\zeta = \sum \widetilde{V}_{i,j}^{\downarrow\uparrow}$, *where the sum is over all ordered pairs* $i, j$ *such that* $0 \le i, j \le D$ *and* $i + j = D + \zeta$.

(ii) $V_\zeta = \sum \widetilde{V}_{i,j}^{\uparrow\downarrow}$, *where the sum is over all ordered pairs* $i, j$ *such that* $0 \le i, j \le D$ *and* $i + j = D - \zeta$.

## 6 Hypercube $H(D, 2)$ and Matrices $A, A^*, B, B^*, K, K^*$

In this section, we recall some facts concerning the hypercube, and define some useful matrices by using its split decompositions.

**Definition 6.1** *Let* $D$ *denote a positive integer, and let* $\{1, -1\}^D$ *denote the set of sequences* $\epsilon_1\epsilon_2 \cdots \epsilon_D$, *where* $\epsilon_i \in \{1, -1\}$ *for* $1 \le i \le D$. *We let* $H(D, 2)$ *denote the graph with a vertex set*

$$X = \{1, -1\}^D$$

*and an edge set*

$$R = \{xy \mid x, y \in X,\ x, y \text{ differ in exactly one coordinate}\}.$$

*We refer to* $H(D, 2)$ *the hypercube.* $H(D, 2)$ *is also known as a* $D$*-cube or a Hamming cube.*

For the rest of this paper, we always assume that the diameter $D$ of the hypercube $H(D, 2)$ is at least 3.

**Definition 6.2** *For the hypercube* $H(D, 2)$, *let* $E_0, E_1, \cdots, E_D$ *denote the primitive idempotents and let* $A$ *be the adjacency matrix. Let* $V$ *be the standard module. Fix a vertex* $x \in X$ *of* $H(D, 2)$. *Let* $E_0^*, E_1^*, \cdots, E_D^*$ *denote the dual primitive idempotents with respect to* $x$ *and let* $A^*$ *be the dual adjacency matrix with respect to* $x$. *Let* $T$ *be the Terwilliger algebra with respect to* $x$.

**Lemma 6.1** *With reference to Definition* 6.2, *the hypercube* $H(D, 2)$ *is a* $Q$*-polynomial distance-regular graph whose eigenvalue sequence and dual eigenvalue sequence are all* $\{D - 2i\}_{i=0}^D$. *Moreover, the space* $E_iV$ *(resp.* $E_i^*V$*) is the eigenspace of* $A$ *(resp.* $A^*$*) associated with the eigenvalue* $D - 2i$ *for* $0 \le i \le D$.

**Proof** Immediate from [3, p. 261] and [16, Theorems 3.7, 12.1].

**Lemma 6.2** *With reference to Definition* 6.2, *the matrices* $A$ *and* $A^*$ *satisfy the Dolan-Grady relations*

$$[A, [A, [A, A^*]]] - 4[A, A^*] = 0, \tag{6.1}$$

$$[A^*, [A^*, [A^*, A]]] - 4[A^*, A] = 0. \tag{6.2}$$

**Proof** Immediate from [16, Theorem 4.2].

**Lemma 6.3** (see [16, Theorems 6.1, 8.1]) *With reference to Definition* 6.2, *let* $W$ *denote an irreducible* $T$*-module with the endpoint* $\rho$, *the dual endpoint* $\tau$, *and the diameter* $d$. *Then the endpoint and the dual endpoint are equal. Moreover, we have*

$$d = D - 2\rho = D - 2\tau.$$

**Corollary 6.1** *With reference to Definition* 6.2*, let $W$ denote an irreducible $T$-module. Then the displacements of $W$ of the first kind and the second kind are zero.*

**Proof** Immediate from Definition 5.1 and Lemma 6.3.

**Lemma 6.4** *With reference to Definitions* 4.2, 4.4 *and* 6.2*, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$, we have $\widetilde{V}^{\mu\nu}_{i,j} = 0$ and $E^{\mu\nu}_{i,j} = 0$ unless $i + j = D$.*

**Proof** From Lemmas 5.2 and 5.3 and Corollary 6.1, for $\mu, \nu \in \{\downarrow, \uparrow\}$ and $0 \leq i, j \leq D$, we have $\widetilde{V}^{\mu\nu}_{i,j} = 0$ unless $i + j = D$. Then by Definition 4.4, we have $E^{\mu\nu}_{i,j} = 0$ unless $i + j = D$.

**Corollary 6.2** *With reference to Definitions* 4.2 *and* 6.2*, the following holds for $\mu, \nu \in \{\downarrow, \uparrow\}$:*

$$V = \sum_{i=0}^{D} \widetilde{V}^{\mu\nu}_{D-i,i} \quad (\textit{the direct sum}).$$

**Proof** Immediate from Lemmas 4.1 and 6.4.

**Definition 6.3** *With reference to Definitions* 4.2 *and* 6.2*, by Corollary* 6.2 *we define $B, B^*, K, K^*$ to be the unique matrices in $\mathrm{Mat}_X(\mathbb{C})$, which satisfy the requirements of the following Table* 1 *for $0 \leq i \leq D$.*

Table 1

| *The matrix* | *is* 0 *on* |
|---|---|
| $B - (D - 2i)I$ | $\widetilde{V}^{\downarrow\uparrow}_{D-i,i}$ |
| $B^* + (D - 2i)I$ | $\widetilde{V}^{\uparrow\downarrow}_{D-i,i}$ |
| $K - (D - 2i)I$ | $\widetilde{V}^{\downarrow\downarrow}_{D-i,i}$ |
| $K^* - (D - 2i)I$ | $\widetilde{V}^{\uparrow\uparrow}_{D-i,i}$ |

**Lemma 6.5** *With reference to Definitions* 4.4 *and* 6.3*, the following* (i)–(iv) *hold:*

(i) $B = \sum_{i=0}^{D} (D - 2i) E^{\downarrow\uparrow}_{D-i,i}$.

(ii) $B^* = -\sum_{i=0}^{D} (D - 2i) E^{\uparrow\downarrow}_{D-i,i}$.

(iii) $K = \sum_{i=0}^{D} (D - 2i) E^{\downarrow\downarrow}_{D-i,i}$.

(iv) $K^* = \sum_{i=0}^{D} (D - 2i) E^{\uparrow\uparrow}_{D-i,i}$.

**Proof** Immediate from Definitions 4.4 and 6.3 and Corollary 6.2.

**Lemma 6.6** *With reference to Definitions* 6.2 *and* 6.3*, the following* (i)–(iv) *hold:*

(i) *$A$ is symmetric.*

(ii) *$A^*$ is symmetric.*

(iii) $B^t = B^*$.

(iv) $K^t = -K^*$.

**Proof** (i)–(ii) are from the definitions of $A$ and $A^*$.

(iii) Combining Lemma 4.2(ii) and Lemma 6.5(i)–(ii), we have

$$B^t = \sum_{i=0}^{D}(D-2i)(E^{\downarrow\uparrow}_{D-i,i})^t = \sum_{i=0}^{D}(D-2i)E^{\uparrow\downarrow}_{i,D-i} = \sum_{i=0}^{D}(2i-D)E^{\uparrow\downarrow}_{D-i,i} = B^*.$$

(iv) Combining Lemma 4.2(i) and Lemma 6.5(iii)–(iv), we have

$$K^t = \sum_{i=0}^{D}(D-2i)(E^{\downarrow\downarrow}_{D-i,i})^t = \sum_{i=0}^{D}(D-2i)E^{\uparrow\uparrow}_{i,D-i} = \sum_{i=0}^{D}(2i-D)E^{\uparrow\uparrow}_{D-i,i} = -K^*.$$

## 7 An Action of ⊠ on the Standard Module of $H(D,2)$

In this section, we continue our discussion for the hypercube $H(d,2)$, and state our main result of this paper, in which we will display an action of ⊠ on the standard module $V$ of $H(D,2)$.

**Lemma 7.1** *With reference to Definitions* 2.1 *and* 6.2*, let $W$ denote an irreducible $T$-module with the endpoint $\rho$, and recall that $W$ has diameter $d = D - 2\rho$. Then there exists a unique ⊠-module structure on $W$ such that the generators $x_{01}$ and $x_{23}$ act as $A$ and $A^*$, respectively. This ⊠-module structure is irreducible.*

**Proof** The matrices $A$ and $A^*$ satisfy the Dolan-Grady relations (6.1) and (6.2) by Lemma 6.2. Therefore, there exists an $\mathcal{O}$-module structure on $W$ such that the standard generators act as $A$ and $A^*$, respectively. The $\mathcal{O}$-module $W$ is irreducible since $A$ and $A^*$ generate $T$ and the $T$-module $W$ is irreducible. By Lemmas 3.1 and 6.1, and since the endpoint and the dual endpoint of $W$ are equal, the actions of $A$ and $A^*$ on $W$ are semisimple with the same eigenvalues $D - 2\rho - 2i$ $(0 \le i \le d)$. Therefore, by Lemma 6.3, the actions of $A$ and $A^*$ on $W$ are semisimple with the same eigenvalues $d - 2i$ $(0 \le i \le d)$. Thus, the $\mathcal{O}$-module $W$ has type $(0,0)$. So far we have shown that the $\mathcal{O}$-module $W$ is irreducible and has type $(0,0)$. Combining this with Proposition 2.3, we obtain the result.

**Lemma 7.2** *With reference to Definitions* 2.1 *and* 6.2*, let $W$ denote an irreducible $T$-module with the endpoint $\rho$, and recall that $W$ has diameter $d = D-2\rho$. Consider the ⊠-module structure on $W$ from Lemma* 7.1*. For each generator $x_{rs}$ of ⊠ and for $0 \le i \le d$, the eigenspace of $x_{rs}$ on $W$ associated with the eigenvalue $d - 2i$ is given in the following Table* 2.

Table 2

| $r$ | $s$ | *the eigenspace of $x_{rs}$ for the eigenvalue $d-2i$* |
|---|---|---|
| 0 | 1 | $E_{\rho+i}W$ |
| 1 | 2 | $(E^*_\rho W + \cdots + E^*_{\rho+d-i}W) \cap (E_{\rho+d-i}W + \cdots + E_{\rho+d}W)$ |
| 2 | 3 | $E^*_{\rho+i}W$ |
| 3 | 0 | $(E^*_{\rho+d-i}W + \cdots + E^*_{\rho+d}W) \cap (E_\rho W + \cdots + E_{\rho+d-i}W)$ |
| 0 | 2 | $(E^*_\rho W + \cdots + E^*_{\rho+d-i}W) \cap (E_\rho W + \cdots + E_{\rho+i}W)$ |
| 1 | 3 | $(E^*_{\rho+i}W + \cdots + E^*_{\rho+d}W) \cap (E_{\rho+d-i}W + \cdots + E_{\rho+d}W)$ |

**Proof** Referring to the table, we first verify row $(r,s) = (0,1)$. By Lemma 7.1, the generator $x_{01}$ acts on $W$ as $A$. By Lemma 3.1(iii)–(iv) and Lemma 6.1, the space $E_{\rho+i}W$ is the eigenspace of $A$ on $W$ for the eigenvalue $D - 2\rho - 2i$. By these comments and Lemma 6.3, the space $E_{\rho+i}W$ is the eigenspace of $x_{01}$ on $W$ for the eigenvalue $d-2i$. We have verified row $(r,s)=(0,1)$. Next, we verify row $(r,s)=(2,3)$. By Lemma 7.1, the generator $x_{23}$ acts on $W$ as $A^*$. By Lemma 3.1(i)–(ii) and Lemma 6.1 the space $E^*_{\rho+i}W$ is the eigenspace of $A^*$ on $W$ for the eigenvalue $D-2\rho-2i$. By these comments and Lemma 6.3, the space $E^*_{\rho+i}W$ is the eigenspace of $x_{23}$ on $W$ for the eigenvalue $d-2i$. We have verified row $(r,s)=(2,3)$. The remaining rows are valid by [19, Lemma 5.7].

**Lemma 7.3** *With reference to Definitions* 2.1 *and* 6.2–6.3, *let $W$ denote an irreducible $T$-module with the endpoint $\rho$, and recall that $W$ has diameter $d=D-2\rho$. Consider the $\boxtimes$-module structure on $W$ from Lemma* 7.1. *In Table* 3 *below, each row contains an element of $\boxtimes$ and a matrix in* $\mathrm{Mat}_X(\mathbb{C})$. *The actions of these two objects on $W$ coincide.*

Table 3

| *matrix* | *element of* $\boxtimes$ |
|---|---|
| $x_{01}$ | $A$ |
| $x_{12}$ | $B$ |
| $x_{23}$ | $A^*$ |
| $x_{30}$ | $B^*$ |
| $x_{02}$ | $K$ |
| $x_{13}$ | $K^*$ |

**Proof** By Lemma 7.1, the expressions $A-x_{01}$ and $A^*-x_{23}$ are all 0 on $W$. Next, we show that $B-x_{12}$ is 0 on $W$. To this end, we pick $w\in W$ and show

$$Bw=x_{12}w.$$

Recall that $x_{12}$ is semisimple on $W$ with eigenvalues $d-2i$ $(0\le i\le d)$. Therefore, without loss of generality, we may assume that there exists an integer $i$ $(0\le i\le d)$ such that $x_{12}w=(d-2i)w$. By row $(r,s)=(1,2)$ in Table 2 of Lemma 7.2 and by Lemma 4.3(i) and Lemma 6.3, we find $w\in\widetilde{V}^{\downarrow\uparrow}_{\rho+d-i,D-d-\rho+i}$. By this and the first row in the table of Definition 6.3 we find

$$Bw=(2(\rho+d-i)-D)w=(d-2i)w.$$

So we find $Bw=x_{12}w$ as desired. Similarly, by rows $(r,s)=(3,0),(0,2),(1,3)$ in Table 2 of Lemma 7.2 and by Lemma 4.3(ii)–(iv), one can show that each of $B^*-x_{30}$, $K-x_{02}$ and $K^*-x_{13}$ is 0 on $W$. The results follow.

**Theorem 7.1** *With reference to Definitions* 2.1 *and* 6.2–6.3, *there exists a $\boxtimes$-module structure on $V$ such that the generators $x_{ij}$ act as follows* (*see Table* 4).

Table 4

| *generator* | *action on V* |
|---|---|
| $x_{01}$ | $A$ |
| $x_{12}$ | $B$ |
| $x_{23}$ | $A^*$ |
| $x_{30}$ | $B^*$ |
| $x_{02}$ | $K$ |
| $x_{13}$ | $K^*$ |

**Proof** Note that the standard module $V$ decomposes into a direct sum of irreducible $T$-modules. Since each irreducible $T$-module in this decomposition supports a $\boxtimes$-module structure from Lemma 7.1, the assertion holds by Lemma 7.3.

**Theorem 7.2** *With reference to Definitions* 2.1 *and* 6.2*, the following hold on* $V$

$$\begin{aligned} x^t_{01} = x_{01}, \quad x^t_{12} = x_{30}, \quad x^t_{23} = x_{23}, \\ x^t_{30} = x_{12}, \quad x^t_{02} = x_{31}, \quad x^t_{13} = x_{20}. \end{aligned}$$

**Proof** Immediate from Definition 2.1, Lemma 6.6 and Theorem 7.1.

Let $U(\boxtimes)$ denote the universal enveloping algebra of $\boxtimes$. In Theorem 7.1 we displayed an action of $\boxtimes$ on the standard module of $V$; observe that this action induces a $\mathbb{C}$-algebra homomorphism from $U(\boxtimes)$ to $\mathrm{Mat}_X(\mathbb{C})$ which we will denote by $\vartheta$. Now we clarify how the image $\vartheta(U(\boxtimes))$ is related to the Terwilliger algebra $T$.

**Theorem 7.3** *With reference to Definition* 6.2*, then $T$ is equal to the image $\vartheta(U(\boxtimes))$.*

**Proof** Note that $T$ is generated by $A, A^*$ and $\vartheta(U(\boxtimes))$ is generated by $A, A^*, B, B^*, K, K^*$. To prove that the two subalgebras of $\mathrm{Mat}_X(\mathbb{C})$ are equal, it suffices to verify that each of $B, B^*, K, K^*$ is contained in $T$. Clearly those follow from Lemma 3.2 and Lemma 7.3.

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