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Asymptotic Stability of Equilibrium State to the Mixed Initial-Boundary Value Problem for Quasilinear Hyperbolic Systems*

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Abstract Under the internal dissipative condition, the Cauchy problem for inhomogeneous quasilinear hyperbolic systems with small initial data admits a unique global C^1 solution, which exponentially decays to zero as $t \to +\infty$, while if the coefficient matrix Θ of boundary conditions satisfies the boundary dissipative condition, the mixed initial-boundary value problem with small initial data for quasilinear hyperbolic systems with nonlinear terms of at least second order admits a unique global C^1 solution, which also exponentially decays to zero as $t \to +\infty$. In this paper, under more general conditions, the authors investigate the combined effect of the internal dissipative condition and the boundary dissipative condition, and prove the global existence and exponential decay of the C^1 solution to the mixed initial-boundary value problem for quasilinear hyperbolic systems with small initial data. This stability result is applied to a kind of models, and an example is given to show the possible exponential instability if the corresponding conditions are not satisfied.

Keywords Quasilinear hyperbolic system, Mixed initial-boundary value problem, Classical solution, Asymptotic stability
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1 Introduction and Main Results

Consider the following first-order quasilinear hyperbolic system:

$$\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = F(u), \tag{1.1}$$

where $u = (u_1, \cdots, u_n)^{\mathrm{T}}$ is the unknown vector function of (t, x), $A(u) = (a_{ij}(u))_{n\times n}$ is a C^2 matrix function of u, and $F(u) = (F_1(u), \cdots, F_n(u))^{\mathrm{T}}$ is a C^2 vector function of u with

$$F(0) = 0. \tag{1.2}$$

By strict hyperbolicity, for any given u on the domain under consideration, $A(u)$ has n distinct real eigenvalues $\lambda_1(u) < \cdots < \lambda_n(u)$ and a complete set of left (resp. right) eigenvectors

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$l_i(u) = (l_{i1}(u), \cdots, l_{in}(u))$ (resp. $r_i(u) = (r_{1i}(u), \cdots, r_{ni}(u))^{\mathrm{T}}$) as follows:

$$l_i(u)A(u) = \lambda_i(u)l_i(u), \quad \forall i \in \mathscr{N}, \tag{1.3}$$

$$A(u)r_i(u) = \lambda_i(u)r_i(u), \quad \forall i \in \mathscr{N}, \tag{1.4}$$

where

$$\mathscr{N} = \{1, \cdots, n\}. \tag{1.5}$$

Assume that $\lambda_i(u), l_i(u)$ and $r_i(u)$ $(i \in \mathscr{N})$ have the same regularity as $A(u)$. Without loss of generality, we assume that

$$l_i(u)r_j(u) \equiv \delta_{ij}, \quad \forall i, j \in \mathscr{N}, \tag{1.6}$$

$$r_i^{\mathrm{T}}(u)r_i(u) \equiv 1, \qquad \forall i \in \mathscr{N}, \tag{1.7}$$

where δ_{ij} stands for the Kronecker symbol. Let $L(u)$ and $R(u)$ be the matrices composed of the left and right eigenvectors:

$$L(u) = \begin{pmatrix} l_1(u) \\ \vdots \\ l_n(u) \end{pmatrix}, \tag{1.8}$$

$$R(u) = (r_1(u), \cdots, r_n(u)), \tag{1.9}$$

respectively.

In order to consider the mixed initial-boundary value problem for (1.1), assume that there exists an index $m \in \{1, \cdots, n-1\}$, such that

$$\lambda_r(0) < 0 < \lambda_s(0), \quad \forall r = 1, \cdots, m, \ s = m+1, \cdots, n. \tag{1.10}$$

Remark 1.1 For $u = (u_1, \cdots, u_n)^{\mathrm{T}} \in \mathbb{R}^n$, denote

$$|u| = \sum_{i \in \mathscr{N}} |u_i|. \tag{1.11}$$

Let $\mathscr{M}^{n,n}$ be the set of $n \times n$ real matrices, and $\mathscr{D}^{n,n}$ be the set of $n \times n$ real diagonal matrices with strictly positive diagonal elements. For any given $B = (b_{ij}) \in \mathscr{M}^{n,n}$, we define

$$|B| = \max_{i \in \mathscr{N}} \Big\{ \sum_{j \in \mathscr{N}} |b_{ij}| \Big\}. \tag{1.12}$$

If (1.1) satisfies the following internal dissipative condition: There exists $\Lambda \in \mathscr{D}^{n,n}$, such that

$$-G^{\Lambda} \overset{\text{def.}}{=} -\Lambda L(0)\nabla F(0)R(0)\Lambda^{-1} \tag{1.13}$$

is a strictly row-diagonal dominant matrix as follows:

$$-G_{ii}^{\Lambda}(0) > \sum_{\substack{j \in \mathscr{N} \\ j \neq i}} |G_{ij}^{\Lambda}(0)|, \quad \forall i \in \mathscr{N}, \tag{1.14}$$

then the Cauchy problem for (1.1) admits a unique global C^1 solution $u = u(t,x)$ on $t \geq 0$, and the C^1 norm of u decays exponentially to zero as $t \to +\infty$, provided that the C^1 norm of the initial data is small enough (see [6–7]). By the method of energy integration, [5] gave the global existence and uniqueness of H^2 solutions to the hyperbolic system of conservation laws with small initial data under the Shizuta-Kawashima condition and the entropy dissipative condition, and then this result was reproved in [16] in a different way under slightly different hypotheses. The generalization to the higher dimensional case can be found in [15], and [1–2] gave the corresponding asymptotic behavior. Moreover, this result was generalized to some systems without the Shizuta-Kawashima condition in [11–14].

Consider the mixed initial-boundary value problem (1.1) with the initial condition

$$t = 0: \quad u = u_0(x), \quad x \in [0,1] \tag{1.15}$$

and the following boundary conditions:

$$\begin{cases} x = 0: \quad v_s = H_s(v_1, \cdots, v_m), \quad s = m+1, \cdots, n, & (1.16) \\ x = 1: \quad v_r = H_r(v_{m+1}, \cdots, v_n), \quad r = 1, \cdots, m, & (1.17) \end{cases}$$

where $v_i = l_i(u)u$ $(i \in \mathscr{N})$, H_r and H_s $(r = 1, \cdots, m,\ s = m+1, \cdots, n)$ are C^2 functions of $(v_{m+1}, \cdots, v_n)^{\mathrm{T}}$ and $(v_1, \cdots, v_m)^{\mathrm{T}}$, respectively, and

$$H_i(0) = 0, \quad \forall i \in \mathscr{N}. \tag{1.18}$$

Assume that the conditions of C^1 compatibility hold at the points $(0,0)$ and $(0,1)$, respectively.

If $F(u)$ satisfies (1.2) and

$$\nabla F(0) = 0, \tag{1.19}$$

and the matrix

$$\Theta \stackrel{\text{def.}}{=} \begin{pmatrix} 0 & \dfrac{\partial(H_1, \cdots, H_m)}{\partial(v_{m+1}, \cdots, v_n)}\Big|_{(v_{m=1}, \cdots, v_n) = (0, \cdots, 0)} \\ \dfrac{\partial(H_{m+1}, \cdots, H_n)}{\partial(v_1, \cdots, v_m)}\Big|_{(v_1, \cdots, v_m) = (0, \cdots, 0)} & 0 \end{pmatrix} \tag{1.20}$$

satisfies the following boundary dissipative condition:

$$\|\Theta\|_1 \stackrel{\text{def.}}{=} \inf_{\Lambda \in \mathscr{D}^{n,n}} |\Lambda \Theta \Lambda^{-1}| < 1, \tag{1.21}$$

then the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) with small initial data admits a unique global C^1 solution $u = u(t,x)$ on the domain $\{(t,x) \mid t \geq 0,\ 0 \leq x \leq 1\}$, and the C^1 norm of the solution decays exponentially as $t \to +\infty$ (see [7]). In the case $F(u) \equiv 0$, by constructing a Lyapunov function and considering the problem in H^2 space, the condition (1.21) is weakened in [3]. For linear hyperbolic systems, the exponential stability for the mixed initial-boundary value problem was established in L^2 space in [4].

The results mentioned above inspire us to consider the following problem: In the case $\nabla F(0) \neq 0$, under what conditions on $\nabla F(0)$ and Θ, i.e., under which kind of combined effect of internal dissipation and boundary dissipation, we can get the global existence and the exponential decay of the C^1 solution to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17)?

Let

$$G = L(0)\nabla F(0)R(0), \tag{1.22}$$

$$T_i = \frac{1}{|\lambda_i(0)|}, \quad \forall i \in \mathscr{N}, \tag{1.23}$$

$$\Theta^\lambda = \mathrm{diag}\{\lambda_1(0), \cdots, \lambda_n(0)\}^{-1}\Theta \mathrm{diag}\{\lambda_1(0), \cdots, \lambda_n(0)\} \tag{1.24}$$

and

$$\varepsilon = \|u_0\|_{C^1[0,1]}. \tag{1.25}$$

Our main result is as follows.

Theorem 1.1 *Under the hypotheses* (1.2), (1.10) *and* (1.18), *assume furthermore that* $G_{ii} \neq 0$ $(i \in \mathscr{N})$. *If* $G, \Theta, \Theta^\lambda$ *satisfy*

$$\max_{i\in\mathscr{N}}\Big\{\max_{t\in[0,T_i]}\sum_{j\neq i}\Big(|\Theta_{ij}| + \frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{G_{ii}t} - \sum_{j\neq i}\frac{|G_{ij}|}{G_{ii}}\Big\} < 1, \tag{1.26}$$

$$\max_{i\in\mathscr{N}}\Big\{\max_{t\in[0,T_i]}\sum_{j\neq i}\Big(|\Theta^\lambda_{ij}| + \frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{G_{ii}t} - \sum_{j\neq i}\frac{|G_{ij}|}{G_{ii}}\Big\} < 1, \tag{1.27}$$

then there exists $\varepsilon_0 > 0$ *so small that for any given* $\varepsilon \in [0, \varepsilon_0]$ *and the initial data* $u_0(x)$ *satisfying* (1.25), *the mixed initial-boundary value problem* (1.1) *and* (1.15)–(1.17) *admits a unique global* C^1 *solution* $u = u(t, x)$ *on the domain* $\{(t, x) \mid t \geq 0, 0 \leq x \leq 1\}$, *and there exists a number* $\alpha > 0$, *such that for any given* $t \geq 0$, *we have the following uniform a priori estimate:*

$$\|u(t,\cdot)\|_{C^1[0,1]} \leq C\varepsilon \mathrm{e}^{-\alpha t}, \quad \forall t \geq 0, \tag{1.28}$$

where C *stands for a positive constant independent of* ε *and* t.

Remark 1.2 If $G_{ii} = 0$ $(i \in \mathscr{N})$, (1.26)–(1.27) should be replaced by

$$\max_{i\mathscr{N}}\sum_{j\neq i}(|\Theta_{ij}| + |G_{ij}|T_i) < 1, \tag{1.29}$$

$$\max_{i\mathscr{N}}\sum_{j\neq i}(|\Theta^\lambda_{ij}| + |G_{ij}|T_i) < 1. \tag{1.30}$$

Through the proof in Section 3, we can see that Theorem 1.1 still holds in this case. In order to get (1.29)–(1.30), we may use the Taylor expansion of $\mathrm{e}^{G_{ii}t}$ in (1.26)–(1.27), and then ask G_{ii} to tend to 0.

Generally speaking, if there exists a set $\mathscr{P}_1 \subseteq \mathscr{N}$, such that $G_{kk} = 0$, $\forall k \in \mathscr{P}_1$, then Theorem 1.1 still holds, provided that (1.26)–(1.27) are replaced by

$$\max\Big\{ \max_{\substack{t\in[0,T_i]\\ i\in\mathscr{N}\setminus\mathscr{P}_1}} \Big(\sum_{j\neq i} \Big(|\Theta_{ij}| + \frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{G_{ii}t} - \sum_{j\neq i} \frac{|G_{ij}|}{G_{ii}}\Big),\ \max_{i\in\mathscr{P}_1} \sum_{j\neq i}(|\Theta_{ij}| + |G_{ij}|T_i)\Big\} < 1, \tag{1.31}$$

$$\max\Big\{ \max_{\substack{t\in[0,T_i]\\ i\in\mathscr{N}\setminus\mathscr{P}_1}} \Big(\sum_{j\neq i} \Big(|\Theta^{\lambda}_{ij}| + \frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{G_{ii}t} - \sum_{j\neq i} \frac{|G_{ij}|}{G_{ii}}\Big),\ \max_{i\in\mathscr{P}_1} \sum_{j\neq i}(|\Theta^{\lambda}_{ij}| + |G_{ij}|T_i)\Big\} < 1. \tag{1.32}$$

Remark 1.3 If there exist Λ and $\Delta \in \mathscr{D}^{n,n}$, such that

$$G^{\Lambda} \overset{\text{def.}}{=} \Lambda G \Lambda^{-1} \tag{1.33}$$

and

$$\Theta^{\Lambda} \overset{\text{def.}}{=} \Lambda \Theta \Lambda^{-1} \tag{1.34}$$

satisfy (1.26), while

$$G^{\Delta} \overset{\text{def.}}{=} \Delta G \Delta^{-1} \tag{1.35}$$

and

$$\Theta^{\Delta,\lambda} \overset{\text{def.}}{=} \Delta \Theta^{\lambda} \Delta^{-1} \tag{1.36}$$

satisfy (1.27), then the conclusion of Theorem 1.1 still holds.

Remark 1.4 When

$$F(0) = 0, \quad \nabla F(0) = 0,$$

namely, there do not exist any linear internal dissipative terms, through the proof and analysis in Section 3, (1.26)–(1.27) can be reduced to (1.21), and then the result in [7] can be obtained from the conclusion of Theorem 1.1.

When

$$\Theta = 0, \tag{1.37}$$

namely, there do not exist any linear boundary dissipative terms, in order to get (1.26)–(1.27), the nonlinear term $F(u)$ of (1.1) might have a growth effect on the solution $u = u(t,x)$ (i.e., $G_{ii} > 0$). Specially, taking

$$G_{ij} = 0, \quad \forall i \neq j,\ i,j \in \mathscr{N}, \tag{1.38}$$

we can see it.

Remark 1.5 (1.14) fails for some systems in physics, however, it is possible to get

$$-G_{ii} = \sum_{\substack{j\in\mathscr{N}\\ j\neq i}} |G_{ij}|, \quad \forall i \in \mathscr{N}. \tag{1.39}$$

If

$$\max_{i\in\mathscr{N}}\sum_{j\in\mathscr{N}}|\Theta_{ij}|<1 \tag{1.40}$$

and

$$\max_{i\in\mathscr{N}}\sum_{j\in\mathscr{M}}|\Theta_{ij}^{\lambda}|<1, \tag{1.41}$$

then (1.26)–(1.27) still hold. By Theorem 1.1, the C^1 solution to the corresponding mixed initial-boundary value problem decays exponentially.

In Section 2, we give the semi-global existence of a C^1 solution to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) (see [8]), and a kind of formulas of wave decomposition. By using these formulas, we give the proof of Theorem 1.1 in Section 3, and we apply our result to a kind of models in Section 4. In Section 5, one example is given to show that the conclusion of Theorem 1.1 may fail if (1.26)–(1.27) do not hold, and some further discussions about our main results are carried out in Section 6.

2 Preliminaries

Under hypotheses (1.2), (1.10) and (1.18), there exists $t^*>0$, such that the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) admits a unique C^1 solution $u=u(t,x)$ on the domain $\{(t,x)\mid 0\le t\le t^*,\ 0\le x\le 1\}$ (see [10]). In order to prove Theorem 1.1, we first give the semi-global existence of a C^1 solution to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) (see [8]) and some formulas of wave decomposition in this section.

Lemma 2.1 *Suppose that* (1.2), (1.10) *and* (1.18) *hold. For any given $T>0$, there exists an $\varepsilon_0>0$ so small that for any given $\varepsilon\in[0,\varepsilon_0]$ and $u_0(x)$ satisfying* (1.25), *the mixed initial-boundary value problem* (1.1) *and* (1.15)–(1.17) *admits a unique C^1 solution $u=u(t,x)$ on the domain*

$$D(T)=\{(t,x)\mid 0\le t\le T,\ 0\le x\le 1\}, \tag{2.1}$$

and satisfies

$$\|u(t,\cdot)\|_{C^1[0,1]}\le C\varepsilon,\quad \forall 0\le t\le T, \tag{2.2}$$

where C is a positive constant independent of $t\in[0,T]$ and $\varepsilon\in[0,\varepsilon_0]$.

2.1 Formulas of wave decomposition

In order to prove Theorem 1.1, we introduce some formulas of wave decomposition.

Let

$$v_i=l_i(u)u,\quad i\in\mathscr{N}, \tag{2.3}$$

$$w_i = l_i(u)u_x, \quad i \in \mathscr{N}. \tag{2.4}$$

Then

$$u = \sum_{j\in\mathscr{N}} v_j r_j(u), \tag{2.5}$$

$$\partial_x u = \sum_{j\in\mathscr{N}} w_j r_j(u), \tag{2.6}$$

$$\frac{\partial u}{\partial v}\Big|_{v=0} = R(0). \tag{2.7}$$

By (1.1), we have

$$\frac{\mathrm{d}v_i}{\mathrm{d}_i t} = l_i(u)F(u) + \sum_{j\in\mathscr{N}} F^{\mathrm{T}}(u)\nabla l_i(u)r_j(u)v_j + \sum_{j,k\in\mathscr{N}} \beta_{ijk}(u)v_j w_k, \quad i \in \mathscr{N}, \tag{2.8}$$

where

$$\beta_{ijk}(u) = (\lambda_i(u) - \lambda_k(u))r_k^{\mathrm{T}}\nabla l_i(u)r_j(u), \quad \forall i,j,k \in \mathscr{N}, \tag{2.9}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}_i t} = \frac{\partial}{\partial t} + \lambda_i(u)\frac{\partial}{\partial x}$$

denotes the directional derivative with respect to t along the ith characteristic curve (see [7]).

Similarly, by (1.1) we have

$$\begin{aligned}\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = &\sum_{j\in\mathscr{N}} l_i(u)\nabla F(u)r_j(u)w_j - \sum_{k\in\mathscr{N}} l_i(u)\nabla r_k(u)F(u)w_k \\ &+ \sum_{j,k\in\mathscr{N}} \gamma_{ijk}(u)w_j w_k, \quad i \in \mathscr{N},\end{aligned} \tag{2.10}$$

where

$$\gamma_{ijk}(u) = (\lambda_j(u) - \lambda_k(u))l_i(u)\nabla r_k(u)r_j(u) + \delta_{ij}\nabla\lambda_j(u)r_k(u), \quad \forall i,j,k \in \mathscr{N} \tag{2.11}$$

(see [7]).

Noting (1.2) and (2.3), we have

$$\frac{\partial}{\partial v_j}(l_i(u)F(u))\Big|_{v=0} = l_i(0)\nabla F(0)r_j(0) = G_{ij}. \tag{2.12}$$

Thus, using Hadamard's formula and Taylor expansion for functions $l_i(u)F(u)$, we have

$$l_i(u)F(u) = \sum_{j\in\mathscr{N}} G_{ij}v_j + \sum_{j,k\in\mathscr{N}} \xi_{ijk}(u)v_j v_k, \quad i \in \mathscr{N}, \tag{2.13}$$

where $\xi_{ijk}(u) \in C^0$ $(i,j,k \in \mathscr{N})$. In a similar way, by (1.2), for functions $F^{\mathrm{T}}(u)\nabla l_i(u)r_j(u)$ $(i,j \in \mathscr{N})$, we have

$$F^{\mathrm{T}}(u)\nabla l_i(u)r_j(u)v_j = \sum_{k\in\mathscr{N}} \eta_{ijk}(u)v_j v_k, \quad \forall i,j \in \mathscr{N}, \tag{2.14}$$

where $\eta_{ijk}(u) \in C^0\ (i,j,k \in \mathscr{N})$.

Similarly, we have

$$l_i(u)\nabla F(u) r_j(u) w_j = G_{ij} w_j + \sum_{k\in\mathscr{N}} \varphi_{ijk}(u) w_j v_k, \quad i,j \in \mathscr{N}, \tag{2.15}$$

$$-l_i(u)\nabla r_k(u) F(u) w_k = \sum_{j\in\mathscr{N}} \psi_{ijk}(u) w_k v_j, \quad i,k \in \mathscr{N}, \tag{2.16}$$

where $\varphi_{ijk}(u), \psi_{ijk}(u) \in C^0\ (i,j,k \in \mathscr{N})$. Substituting (2.13)–(2.14) and (2.15)–(2.16) into (2.8) and (2.10), respectively, we get

$$\frac{\mathrm{d}v_i}{\mathrm{d}_i t} = \sum_{j\in\mathscr{N}} G_{ij} v_j + \sum_{j,k\in\mathscr{N}} \Phi_{ijk}(u) v_j v_k + \sum_{j,k\in\mathscr{N}} \beta_{ijk}(u) v_j w_k, \quad i \in \mathscr{N}, \tag{2.17}$$

$$\frac{\mathrm{d}w_i}{\mathrm{d}_i t} = \sum_{j\in\mathscr{N}} G_{ij} w_j + \sum_{j,k\in\mathscr{N}} \Xi_{ijk}(u) v_j w_k + \sum_{j,k\in\mathscr{N}} \gamma_{ijk}(u) w_j w_k, \quad i \in \mathscr{N}, \tag{2.18}$$

where $\Phi_{ijk}(u)$ and $\Xi_{ijk}(u) \in C^0\ (i,j,k \in \mathscr{N})$ are given by

$$\Phi_{ijk}(u) = \xi_{ijk}(u) + \eta_{ijk}(u), \tag{2.19}$$

$$\Xi_{ijk}(u) = \varphi_{ikj}(u) + \psi_{ijk}(u). \tag{2.20}$$

2.2 Representation of v and w on the boundaries $x = 0, 1$

Noting (1.18) and (1.20), it follows from boundary conditions (1.16)–(1.17) that

$$\begin{cases} x = 0: \quad v_s = \displaystyle\sum_{j=1}^{m} \Theta_{sj} v_j + \sum_{j,k\in\mathscr{N}} \chi_{sjk}(u) v_j v_k, \quad s = m+1, \cdots, n, \\ x = 1: \quad v_r = \displaystyle\sum_{j=m+1}^{n} \Theta_{rj} v_j + \sum_{j,k\in\mathscr{N}} \chi_{rjk}(u) v_j v_k, \quad r = 1, \cdots, m, \end{cases} \tag{2.21}$$

where $\chi_{ijk}(u) \in C^0\ (i,j,k \in \mathscr{N})$.

Differentiating the boundary conditions (1.16)–(1.17) with respect to t yields

$$\begin{cases} x = 0: \quad \dfrac{\partial v_s}{\partial t} = \displaystyle\sum_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \frac{\partial v_r}{\partial t}, \quad s = m+1, \cdots, n, \\ x = 1: \quad \dfrac{\partial v_r}{\partial t} = \displaystyle\sum_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \frac{\partial v_s}{\partial t}, \quad r = 1, \cdots, m. \end{cases} \tag{2.22}$$

For any given $i \in \mathscr{N}$, by (2.3)–(2.4) and (2.13)–(2.14), we have

$$\begin{aligned} \frac{\partial v_i}{\partial t} &= \frac{\partial (l_i(u) u)}{\partial t} = l_i(u) \frac{\partial u}{\partial t} + \Big(\frac{\partial u}{\partial t}\Big)^{\mathrm{T}} \nabla l_i(u) u \\ &= -\lambda_i(u) w_i + l_i(u) F(u) + \Big(\sum_{k\in\mathscr{N}} -\lambda_k(u) r_k(u) w_k + F(u)\Big)^{\mathrm{T}} \nabla l_i(u) u \\ &= -\lambda_i(u) w_i + \sum_{j\in\mathscr{N}} G_{ij} v_j + \sum_{j,k\in\mathscr{N}} \xi_{ijk}(u) v_j v_k \end{aligned}$$

$$
\begin{aligned}
&- \sum_{j,k\in\mathscr{N}} \lambda_k(u) r_k^{\mathrm{T}}(u)\nabla l_i(u) r_j(u) v_j w_k + \sum_{j,k\in\mathscr{N}} \eta_{ijk}(u) v_j v_k \\
&= -\lambda_i(u) w_i + \sum_{j\in\mathscr{N}} G_{ij} v_j + \sum_{j,k\in\mathscr{N}} \Phi_{ijk}(u) v_j v_k + \sum_{j,k\in\mathscr{N}} \widetilde{\phi}_{ijk}(u) v_j w_k,
\end{aligned} \tag{2.23}
$$

where

$$
\widetilde{\phi}_{ijk}(u) = -\lambda_k(u) r_k^{\mathrm{T}}(u) \nabla l_i(u) r_j(u), \quad i,j,k \in \mathscr{N}.
$$

Substituting (2.23) into (2.22), we get

$$
\begin{cases}
x=0: \quad w_s = \displaystyle\sum_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \frac{\lambda_r(u)}{\lambda_s(u)} w_r + \sum_{j\in\mathscr{N}} \frac{G_{sj} - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} G_{rj}}{\lambda_s(u)} v_j \\
\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \frac{\Phi_{sjk}(u) - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \Phi_{rjk}(u)}{\lambda_s(u)} v_j v_k \\
\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \frac{\widetilde{\phi}_{sjk}(u) - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \widetilde{\phi}_{rjk}(u)}{\lambda_s(u)} v_j w_k, \quad s = m+1, \cdots, n, \\
x=1: \quad w_r = \displaystyle\sum_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \frac{\lambda_s(u)}{\lambda_r(u)} w_s + \sum_{j\in\mathscr{N}} \frac{G_{rj} - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} G_{sj}}{\lambda_r(u)} v_j \\
\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \frac{\Phi_{rjk}(u) - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \Phi_{sjk}(u)}{\lambda_r(u)} v_j v_k \\
\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \frac{\widetilde{\phi}_{rjk}(u) - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \widetilde{\phi}_{sjk}(u)}{\lambda_r(u)} v_j w_k, \quad r = 1, \cdots, m.
\end{cases} \tag{2.24}
$$

For the functions $\frac{\partial H_s}{\partial v_r}\frac{\lambda_r(u)}{\lambda_s(u)}$ $(s = m+1, \cdots, n;\ r = 1, \cdots, m)$ and $\frac{\partial H_r}{\partial v_s}\frac{\lambda_s(u)}{\lambda_r(u)}$ $(s = m+1, \cdots, n;\ r = 1, \cdots, m)$ appearing on the right-hand side of (2.24), using Taylor expansion, we obtain

$$
\frac{\partial H_s}{\partial v_r} \frac{\lambda_r(u)}{\lambda_s(u)} w_r = \Theta_{sr}^{\lambda} w_r + \sum_{j\in\mathscr{N}} \Upsilon_{sjr}(u) v_j w_r, \tag{2.25}
$$

$$
\frac{\partial H_r}{\partial v_s} \frac{\lambda_s(u)}{\lambda_r(u)} w_s = \Theta_{rs}^{\lambda} w_s + \sum_{j\in\mathscr{N}} \Upsilon_{rjs}(u) v_j w_s, \tag{2.26}
$$

where $\Upsilon_{sjr}(u), \Upsilon_{rjs}(u) \in C^0$, and Θ^{λ} is given by (1.24). Substituting (2.25)–(2.26) into (2.24), we get

$$\begin{cases} x=0: \quad w_s = \sum_{r=1}^{m} \Theta_{sr}^{\lambda} w_r + \sum_{j\in\mathscr{N}} G_{sj}^{\flat}(u)v_j + \sum_{j,k\in\mathscr{N}} \phi_{sjk}^{\flat}(u)v_jv_k \\ \qquad\qquad + \sum_{j,k\in\mathscr{N}} \widetilde{\phi}_{sjk}^{\flat}(u)v_jw_k, \quad s=m+1,\cdots,n, \\ x=1: \quad w_r = \sum_{s=m+1}^{n} \Theta_{rs}^{\lambda} w_s + \sum_{j\in\mathscr{N}} G_{rj}^{\flat}(u)v_j + \sum_{j,k\in\mathscr{N}} \phi_{sjk}^{\flat}(u)v_jv_k \\ \qquad\qquad + \sum_{j,k\in\mathscr{N}} \widetilde{\phi}_{rjk}^{\flat}(u)v_jw_k, \quad r=1,\cdots,m, \end{cases} \tag{2.27}$$

where

$$G_{sj}^{\flat}(u) = \frac{G_{sj} - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} G_{rj}}{\lambda_s(u)}, \quad s=m+1,\cdots,n,\ j\in\mathscr{N}, \tag{2.28}$$

$$G_{rj}^{\flat}(u) = \frac{G_{rj} - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} G_{sj}}{\lambda_r(u)}, \quad r=1,\cdots,m,\ j\in\mathscr{N}, \tag{2.29}$$

$$\phi_{sjk}^{\flat}(u) = \frac{\Phi_{sjk}(u) - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \Phi_{rjk}(u)}{\lambda_s(u)}, \quad s=m+1,\cdots,n,\ j,k\in\mathscr{N}, \tag{2.30}$$

$$\phi_{rjk}^{\flat}(u) = \frac{\Phi_{rjk}(u) - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \Phi_{sjk}(u)}{\lambda_r(u)}, \quad r=1,\cdots,m,\ j,k\in\mathscr{N}, \tag{2.31}$$

$$\widetilde{\phi}_{sjk}^{\flat}(u) = \frac{\widetilde{\phi}_{sjk}(u) - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \widetilde{\phi}_{rjk}(u)}{\lambda_s(u)} + \Upsilon_{sjk}(u), \quad k=1,\cdots,m, \\ s=m+1,\cdots,n\ j\in\mathscr{N}; \tag{2.32}$$

$$\widetilde{\phi}_{sjk}^{\flat}(u) = \frac{\widetilde{\phi}_{sjk}(u) - \sum\limits_{r=1}^{m} \frac{\partial H_s}{\partial v_r} \widetilde{\phi}_{rjk}(u)}{\lambda_s(u)}, \quad s,k=m+1,\cdots,n,\ j\in\mathscr{N}, \tag{2.33}$$

$$\widetilde{\phi}_{rjk}^{\flat}(u) = \frac{\widetilde{\phi}_{rjk}(u) - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \widetilde{\phi}_{sjk}(u)}{\lambda_r(u)} + \Upsilon_{rjk}(u), \quad k=m+1,\cdots,n, \\ r=1,\cdots,m,\ j\in\mathscr{N}; \tag{2.34}$$

$$\widetilde{\phi}_{rjk}^{\flat}(u) = \frac{\widetilde{\phi}_{rjk}(u) - \sum\limits_{s=m+1}^{n} \frac{\partial H_r}{\partial v_s} \widetilde{\phi}_{sjk}(u)}{\lambda_r(u)}, \quad r,k=1,\cdots,m,\ j\in\mathscr{N}. \tag{2.35}$$

Remark 2.1 Noting that the coefficient matrix G appearing on the right-hand side of (2.17) for $v=(v_1,\cdots,v_n)^{\mathrm{T}}$ is the same as that of (2.18) for $w=(w_1,\cdots,w_n)^{\mathrm{T}}$, but the coefficient matrix Θ appearing on the right-hand side of (2.21) for $v=(v_1,\cdots,v_n)^{\mathrm{T}}$ is different from Θ^{λ} of (2.27) for $w=(w_1,\cdots,w_n)^{\mathrm{T}}$. Thus, for any given $\Lambda=\mathrm{diag}\{\Lambda_{11},\cdots,\Lambda_{nn}\}\in\mathscr{D}^{n,n}$ and $\Delta=\mathrm{diag}\{\Delta_{11},\cdots,\Delta_{nn}\}\in\mathscr{D}^{n,n}$, using the following linear transformations for $v=(v_1,\cdots,v_n)^{\mathrm{T}}$

and $w = (w_1, \cdots, w_n)^{\mathrm{T}}$:

$$\widetilde{v}_i = \Lambda_{ii} v_i, \quad i \in \mathscr{N}, \tag{2.36}$$

$$\widetilde{w}_i = \Delta_{ii} w_i, \quad i \in \mathscr{N}, \tag{2.37}$$

the corresponding coefficient matrices of (2.17)–(2.18) are

$$G^{\Lambda} = \Lambda G \Lambda^{-1}, \tag{2.38}$$

$$G^{\Delta} = \Delta G \Delta^{-1}, \tag{2.39}$$

respectively, while the corresponding coefficient matrices of (2.21) and (2.27) are

$$\Theta^{\Lambda} = \Lambda \Theta \Lambda^{-1}, \tag{2.40}$$

$$\Theta^{\Delta,\lambda} = \Delta \Theta^{\lambda} \Delta^{-1}, \tag{2.41}$$

respectively. Therefore, (1.26)–(1.27) in Theorem 1.1 should be satisfied at the same time.

3 Proof of Theorem 1.1

On any given existence domain $D(T)$ of the C^1 solution $u = u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17), assume that

$$|u(t,x)| \le \delta \tag{3.1}$$

and

$$\lambda_r(u) < -\delta_0 < 0 < \delta_0 < \lambda_s(u), \quad \forall r = 1, \cdots, m, \ s = m+1, \cdots, n, \tag{3.2}$$

where δ and δ_0 are positive constants independent of ε and T. Noting (1.26)–(1.27) and the continuity, there exists $\alpha > 0$, such that

$$\Gamma_\alpha \stackrel{\text{def.}}{=} \max_{i \in \mathscr{N}} \Big\{ \max_{t \in [0,T_i]} \sum_{j \ne i} \Big(|\Theta_{ij}| + \frac{|G_{ij}|}{G_{ii} + \alpha} \Big) \mathrm{e}^{(G_{ii}+\alpha)t} - \sum_{j \ne i} \frac{|G_{ij}|}{G_{ii} + \alpha} \Big\} < 1, \tag{3.3}$$

$$\Gamma_\alpha^{\lambda} \stackrel{\text{def.}}{=} \max_{i \in \mathscr{N}} \Big\{ \max_{t \in [0,T_i]} \sum_{j \ne i} \Big(|\Theta_{ij}^{\lambda}| + \frac{|G_{ij}|}{G_{ii} + \alpha} \Big) \mathrm{e}^{(G_{ii}+\alpha)t} - \sum_{j \ne i} \frac{|G_{ij}|}{G_{ii} + \alpha} \Big\} < 1. \tag{3.4}$$

Noting (3.1), the functions $\Phi_{ijk}(u)$, $\beta_{ijk}(u)$, $\Xi_{ijk}(u)$, $\Pi_{ijk}(u)$, $\chi_{ijk}(u)$, $G_{ij}^{\flat}(u)$, $\phi_{ijk}^{\flat}(u)$, $\widetilde{\phi}_{ijk}^{\flat}(u)$ $(i,j,k \in \mathscr{N})$ appearing on the right-hand sides of (2.17)–(2.18) and (2.27) are all bounded, so then there exists a constant $M \gg 1$, such that

$$\begin{aligned} &\max_{|u| \le \delta} \sum_{i,j,k \in \mathscr{N}} \{ |\Phi_{ijk}(u)| + |\beta_{ijk}(u)| + |\Xi_{ijk}(u)| + |\Pi_{ijk}(u)| + |\chi_{ijk}(u)| \\ &+ |G_{ij}^{\flat}(u)| + |\phi_{ijk}^{\flat}(u)| + |\widetilde{\phi}_{ijk}^{\flat}(u)| \} \le M. \end{aligned} \tag{3.5}$$

Let

$$V(T)=\sup_{t\in[0,T]}\max_{i\in\mathcal{N}}\|\mathrm{e}^{\alpha t}v_i(t,\cdot)\|_{C^0[0,1]},\tag{3.6}$$

$$W(T)=\sup_{t\in[0,T]}\max_{i\in\mathcal{N}}\|\mathrm{e}^{\alpha t}w_i(t,\cdot)\|_{C^0[0,1]}.\tag{3.7}$$

To prove Theorem 1.1, we only need to prove that on any given existence domain $D(T)$ of the C^1 solution $u=u(t,x)$ to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17), we have

$$V(T)\le C_1\varepsilon,\tag{3.8}$$

$$W(T)\le C_2\varepsilon,\tag{3.9}$$

where C_1 and C_2 are positive constants large enough, independent of ε and T, to be specified later on.

By Lemma 2.1, for

$$T^*\gg T_{\max}\overset{\text{def.}}{=}\max_{i\in\mathcal{N}}T_i,$$

there exists $\varepsilon_0>0$ so small that for any given $\varepsilon\in[0,\varepsilon_0]$ and $u_0(x)$ satisfying (1.25), the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) admits a unique C^1 solution $u=u(t,x)$ on the domain $D(T^*)$, and

$$V(T^*),W(T^*)\le C^*\varepsilon,\tag{3.10}$$

where C^* is a positive constant.

In what follows, we use a bootstrap argument to prove (3.8)–(3.9), namely, under the assumptions (3.8)–(3.9), we will prove that there exists $T_0>0$ independent of T, such that

$$V(T+T_0)\le C_1\varepsilon,\tag{3.11}$$

$$W(T+T_0)\le C_2\varepsilon.\tag{3.12}$$

By (3.3)–(3.4), we can take $\gamma>0$ so small that

$$\frac{(C_1+\gamma)}{C_1}\Gamma_\alpha<1,\tag{3.13}$$

$$\frac{(C_2+\gamma)}{C_2}\Gamma_\alpha^\lambda<1.\tag{3.14}$$

Noting (3.1), by the local well-poseness of the C^1 solution to the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) (see [10]), there exists $T_0>0$, such that the mixed initial-boundary value problem (1.1) and (1.15)–(1.17) admits a unique C^1 solution $u=u(t,x)$ on the domain $D(T+T_0)$, and we have

$$V(T+T_0)\le(C_1+\gamma)\varepsilon,\tag{3.15}$$

$$W(T+T_0)\le(C_2+\gamma)\varepsilon.\tag{3.16}$$

To prove (3.11), we rewrite (2.17) as

$$\frac{\mathrm{d}\big(\mathrm{e}^{-G_{ii}t}|v_i|\big)}{\mathrm{d}_i t} = \sum_{j\neq i} G_{ij}\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}t}v_j + \sum_{j,k\in\mathscr{N}} \Phi_{ijk}(u)\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}t}v_jv_k + \sum_{j,k\in\mathscr{N}} \beta_{ijk}(u)\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}t}v_jw_k, \quad i\in\mathscr{N}. \tag{3.17}$$

For each $s = m+1,\cdots,n$ and any given point $(t,x)\in D(T+T_0)\setminus D(T)$, draw the sth characteristic curve $\mathfrak{C}_s : x = x_s(\tau;t_0)$ passing through (t,x):

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}\tau}x_s(\tau;t_0) = \lambda_s(u(\tau,x_s(\tau;t_0))), \\ x_s(t;t_0) = x, \end{cases}$$

which intersects $x = 0$ at the point $(t_0,0)$ (it can always be realized when $T > 0$ is large enough). Integrating (3.17) along the characteristic curve $\mathfrak{C}_s$ with respect to τ from t_0 to t, we get

$$\begin{aligned} \mathrm{e}^{-G_{ss}t}|v_s(t,x)| = \mathrm{e}^{-G_{ss}t_0}|v_s(t_0,0)| &+ \sum_{j\neq i}\int_{\mathfrak{C}_s} G_{ij}\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}\tau}v_j\mathrm{d}\tau \\ &+ \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Phi_{ijk}(u)\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}\tau}v_jv_k\mathrm{d}\tau \\ &+ \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\beta_{ijk}(u)\mathrm{sig}(v_i)\mathrm{e}^{-G_{ii}t}v_jw_k\mathrm{d}\tau, \end{aligned} \tag{3.18}$$

and then

$$\begin{aligned} \mathrm{e}^{\alpha t}|v_s(t,x)| &= \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}|v_s(t_0,0)| \\ &\quad + \mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\neq s}\int_{\mathfrak{C}_s} G_{sj}\mathrm{sig}(v_i)\mathrm{e}^{-G_{ss}\tau}v_j\mathrm{d}\tau\Big) \\ &\quad + \mathrm{e}^{(G_{ss}+\alpha)t}\Big\{\sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Phi_{sjk}(u)\mathrm{sig}(v_s)\mathrm{e}^{-G_{ss}\tau}v_jv_k\mathrm{d}\tau \\ &\quad + \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\beta_{sjk}(u)\mathrm{sig}(v_s)\mathrm{e}^{-G_{ss}t}v_jw_k\mathrm{d}\tau\Big\} \\ &\overset{\text{def.}}{=} \mathcal{T}_{11}+\mathcal{T}_{12}+\mathcal{T}_{13}, \end{aligned} \tag{3.19}$$

where

$$\begin{aligned} \mathcal{T}_{11} &= \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}|v_s(t_0,0)|, \\ \mathcal{T}_{12} &= \mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\neq s}\int_{\mathfrak{C}_s} G_{sj}\mathrm{sig}(v_i)\mathrm{e}^{-G_{ss}\tau}v_j\mathrm{d}\tau\Big), \\ \mathcal{T}_{13} &= \mathrm{e}^{(G_{ss}+\alpha)t}\Big\{\sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Phi_{sjk}(u)\mathrm{sig}(v_s)\mathrm{e}^{-G_{ss}\tau}v_jv_k\mathrm{d}\tau \\ &\quad + \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\beta_{sjk}(u)\mathrm{sig}(v_s)\mathrm{e}^{-G_{ss}\tau}v_jw_k\mathrm{d}\tau\Big\}. \end{aligned}$$

Using the boundary condition (2.21), we have

$$\mathcal{T}_{11} \le \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}\Big(\sum_{r=1}^{m}|\Theta_{sr}||v_r(t_0,0)|+\sum_{j,k\in\mathcal{N}}|\chi_{sjk}(u)||v_jv_k|\Big).$$

Noting (3.5) and $t-t_0\le T_s$, by (3.15)–(3.16), we get

$$\begin{aligned}\mathcal{T}_{11} &\le (C_1+\gamma)\varepsilon\Big(\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\sum_{r=1}^{m}|\Theta_{sr}|+M(C_1+\gamma)\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)-\alpha t_0}\varepsilon\Big)\\ &\le (C_1+\gamma)\varepsilon\Big(\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\sum_{r=1}^{m}|\Theta_{sr}|+M_1(C_1+\gamma)\varepsilon\Big).\end{aligned} \tag{3.20}$$

Similarly, we have

$$\begin{aligned}\mathcal{T}_{12} &\le (C_1+\gamma)\varepsilon\mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\ne s}\int_{t_0}^{t}|G_{sj}|\mathrm{e}^{-(G_{ss}+\alpha)\tau}\mathrm{d}\tau\Big)\\ &= (C_1+\gamma)\varepsilon\Big(\sum_{j\ne s}\frac{|G_{sj}|}{G_{ss}+\alpha}\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}-\sum_{j\ne s}\frac{|G_{sj}|}{G_{ss}+\alpha}\Big)\end{aligned} \tag{3.21}$$

and

$$\mathcal{T}_{13} \le M_1(C_1+\gamma)(C_1+C_2+2\gamma)\varepsilon^2, \tag{3.22}$$

where M_1 is a positive constant satisfying

$$M_1 \ge M\max_{1\le k\le n}\mathrm{e}^{(|G_{kk}|+\alpha)T_k}(1+T_k). \tag{3.23}$$

Substituting (3.20)–(3.22) into (3.19) and noting (3.3), we get

$$\begin{aligned}\mathrm{e}^{\alpha t}|v_s(t,x)| &\le (C_1+\gamma)\varepsilon\Big\{\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\Big(\sum_{r=1}^{m}|\Theta_{sr}|+\sum_{j\ne s}\frac{|G_{sj}|}{G_{ss}+\alpha}\Big)\\ &\qquad -\sum_{j\ne s}\frac{|G_{sj}|}{G_{ss}+\alpha}+M_1(C_1+C_2+2\gamma)\varepsilon\Big\}\\ &\le (C_1+\gamma)\varepsilon\Gamma_\alpha+M_1(C_1+\gamma)(C_1+C_2+2\gamma)\varepsilon^2\\ &= C_1\varepsilon\Big\{\frac{C_1+\gamma}{C_1}\Gamma_\alpha+\frac{M_1(C_1+\gamma)(C_1+C_2+2\gamma)}{C_1}\varepsilon\Big\}.\end{aligned} \tag{3.24}$$

Noting (3.13) and that $\varepsilon_0>0$ is small enough, we have

$$\frac{C_1+\gamma}{C_1}\Gamma_\alpha+\frac{M(C_1+\gamma)(C_1+C_2+2\gamma)}{C_1}\varepsilon<1, \tag{3.25}$$

so

$$|v_s(t,x)|\le C_1\varepsilon\mathrm{e}^{-\alpha t},\quad \forall s=m+1,\cdots,n,\ (t,x)\in D(T+T_0). \tag{3.26}$$

For $r=1,\cdots,m$, similar estimates hold. Thus, we get (3.11).

To prove (3.12), we rewrite (2.18) as

$$\frac{\mathrm{d}\big(\mathrm{e}^{-G_{ii}t}|w_i|\big)}{\mathrm{d}_i t} = \sum_{j\neq i} G_{ij}\mathrm{sig}(w_i)\mathrm{e}^{-G_{ii}t}w_j + \sum_{j,k\in\mathscr{N}} \Xi_{ijk}(u)\mathrm{sig}(w_i)\mathrm{e}^{-G_{ii}t}v_j w_k \\ + \sum_{j,k\in\mathscr{N}} \gamma_{ijk}(u)\mathrm{sig}(w_i)\mathrm{e}^{-G_{ii}t}w_j w_k, \quad i\in\mathscr{N}. \tag{3.27}$$

For each $s=m+1,\cdots,n$ and any given $(t,x)\in D(T+T_0)\setminus D(T)$, draw the sth characteristic curve $\mathfrak{C}_s$ passing through the point (t,x), which intersects $x=0$ at the point $(t_0,0)$. Integrating (3.27) along the characteristic curve $\mathfrak{C}_s$ with respect to τ from t_0 to t, we get

$$\begin{aligned}\mathrm{e}^{\alpha t}|w_s(t,x)| &= \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}|w_s(t_0,0)| \\ &\quad + \mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\neq s}\int_{\mathfrak{C}_s} G_{sj}\mathrm{sig}(w_i)\mathrm{e}^{-G_{ss}\tau}v_j\mathrm{d}\tau\Big) \\ &\quad + \mathrm{e}^{(G_{ss}+\alpha)t}\Big\{\sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Xi_{sjk}(u)\mathrm{sig}(w_s)\mathrm{e}^{-G_{ss}\tau}v_j w_k\mathrm{d}\tau \\ &\quad + \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Pi_{sjk}(u)\mathrm{sig}(w_s)\mathrm{e}^{-G_{ss}\tau}w_j w_k\mathrm{d}\tau\Big\} \\ &\overset{\text{def.}}{=} \mathcal{T}_{21}+\mathcal{T}_{22}+\mathcal{T}_{23},\end{aligned}\tag{3.28}$$

where

$$\begin{aligned}\mathcal{T}_{21} &= \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}|w_s(t_0,0)|, \\ \mathcal{T}_{22} &= \mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\neq s}\int_{\mathfrak{C}_s} G_{sj}\mathrm{sig}(w_i)\mathrm{e}^{-G_{ss}\tau}v_j\mathrm{d}\tau\Big), \\ \mathcal{T}_{23} &= \mathrm{e}^{(G_{ss}+\alpha)t}\Big\{\sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Xi_{sjk}(u)\mathrm{sig}(w_s)\mathrm{e}^{-G_{ss}\tau}v_j w_k\mathrm{d}\tau \\ &\quad + \sum_{j,k\in\mathscr{N}}\int_{\mathfrak{C}_s}\Pi_{sjk}(u)\mathrm{sig}(w_s)\mathrm{e}^{-G_{ss}\tau}w_j w_k\mathrm{d}\tau\Big\}.\end{aligned}$$

By the boundary condition (2.27), we have

$$\mathcal{T}_{21} \le \mathrm{e}^{(G_{ss}+\alpha)t-G_{ss}t_0}\Big\{\sum_{r=1}^m |\Theta_{sr}^{\lambda}||w_r(t_0,0)| + \sum_{j\in\mathscr{N}}|G_{sj}^{\flat}(u)v_j| \\ + \sum_{j,k\in\mathscr{N}}|\phi_{sjk}^{\flat}(u)v_j v_k| + \sum_{j,k\in\mathscr{N}}|\widetilde{\phi}_{sjk}^{\flat}(u)v_j w_k|\Big\}. \tag{3.29}$$

Noting (3.5), (3.11) and $t-t_0\le T_s$, by (3.15)–(3.16), we have

$$\begin{aligned}\mathcal{T}_{21} &\le (C_2+\gamma)\varepsilon\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\sum_{r=1}^m|\Theta_{sr}^{\lambda}| + M_1C_1\varepsilon + M_1C_1^2\varepsilon^2 + M_1C_1(C_2+\gamma)\varepsilon^2 \\ &\le (C_2+\gamma)\varepsilon\Big(\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\sum_{r=1}^m|\Theta_{sr}^{\lambda}| + \frac{M_1C_1}{C_2+\gamma} + \frac{M_1C_1^2}{C_2+\gamma}\varepsilon + M_1C_1\varepsilon\Big).\end{aligned}\tag{3.30}$$

Similarly, we have

$$\begin{aligned}\mathcal{T}_{22} &\le (C_2+\gamma)\varepsilon \mathrm{e}^{(G_{ss}+\alpha)t}\Big(\sum_{j\neq s}\int_{t_0}^t |G_{sj}|\mathrm{e}^{-(G_{ss}+\alpha)\tau}\mathrm{d}\tau\Big)\\ &= (C_2+\gamma)\varepsilon\Big(\sum_{j\neq s}\frac{|G_{sj}|}{G_{ss}+\alpha}\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)} - \sum_{j\neq s}\frac{|G_{sj}|}{G_{ss}+\alpha}\Big)\end{aligned} \tag{3.31}$$

and

$$\mathcal{T}_{23} \le M_1(C_2+\gamma)(C_1+C_2+\gamma)\varepsilon^2, \tag{3.32}$$

where M_1 is given by (3.23). Substituting (3.30)–(3.32) into (3.28) and noting (3.4), we get

$$\begin{aligned}\mathrm{e}^{\alpha t}|w_s(t,x)| &\le (C_2+\gamma)\varepsilon\Big\{\mathrm{e}^{(G_{ss}+\alpha)(t-t_0)}\Big(\sum_{r=1}^m|\Theta_{sr}^\lambda| + \sum_{j\neq s}\frac{|G_{sj}|}{G_{ss}+\alpha}\Big)\\ &\quad - \sum_{j\neq s}\frac{|G_{sj}|}{G_{ss}+\alpha} + \frac{M_1C_1}{C_2+\gamma} + M_1\Big(2C_1+\frac{C_1^2}{C_2+\gamma}+C_2+\gamma\Big)\varepsilon\Big\}\\ &\le (C_2+\gamma)\varepsilon\Gamma_\alpha^\lambda + M_1C_1\varepsilon + M_1(C_2+\gamma)\Big(2C_1+\frac{C_1^2}{C_2+\gamma}+C_2+\gamma\Big)\varepsilon^2\\ &= C_2\varepsilon\Big\{\frac{C_2+\gamma}{C_2}\Gamma_\alpha^\lambda + \frac{M_1C_1}{C_2} + M_1\frac{C_2+\gamma}{C_2}\Big(2C_1+\frac{C_1^2}{C_2+\gamma}+C_2+\gamma\Big)\varepsilon\Big\}.\end{aligned} \tag{3.33}$$

Noting (3.4) and $0<\varepsilon_0\ll 1$, we can take $C_2 \gg MC_1$, such that

$$\frac{C_2+\gamma}{C_2}\Gamma_\alpha + \frac{M_1C_1}{C_2} + M_1\frac{C_2+\gamma}{C_2}\Big(2C_1+\frac{C_1^2}{C_2+\gamma}+C_2+\gamma\Big)\varepsilon < 1, \tag{3.34}$$

and then

$$|w_s(t,x)| \le C_2\varepsilon\mathrm{e}^{-\alpha t}, \quad \forall s=m+1,\cdots,n,\ (t,x)\in D(T+T_0). \tag{3.35}$$

For $r=1,\cdots,m$, similar estimates hold. Hence, we get (3.12) and complete the proof of Theorem 1.1.

Remark 3.1 When $G_{ss}=0$, for the term $\mathcal{T}_{12}$ in (3.19), the estimate (3.21) for $\mathcal{T}_{12}$ should be replaced by

$$\begin{aligned}\mathcal{T}_{12} &\le (C_1+\gamma)\varepsilon\mathrm{e}^{\alpha t}\Big(\sum_{j\neq s}\int_{t_0}^t|G_{sj}|\mathrm{e}^{-\alpha\tau}\mathrm{d}\tau\Big)\\ &= (C_1+\gamma)\varepsilon\mathrm{e}^{\alpha(t-t_0)}\sum_{j\neq s}|G_{sj}|\Big(\frac{1-\mathrm{e}^{-\alpha(t-t_0)}}{\alpha}\Big)\\ &\le (C_1+\gamma)\varepsilon\mathrm{e}^{\alpha(t-t_0)}\sum_{j\neq s}|G_{sj}|(t-t_0),\end{aligned} \tag{3.36}$$

while the estimates (3.20) and (3.22) for $\mathcal{T}_{11}$ and $\mathcal{T}_{13}$ still hold. Noting (1.31), the sth term at the right-hand side of (3.3) should be replaced by

$$\mathrm{e}^{\alpha t}\sum_{j\neq s}(|\theta_{sj}|+|G_{sj}|T_i). \tag{3.37}$$

Similar treatments can be done for $\mathcal{T}_{21}, \mathcal{T}_{23}$ and $\mathcal{T}_{22}$. Hence, Theorem 1.1 still holds, provided that (1.31)–(1.32) hold.

Remark 3.2 If there exist $\Lambda = \mathrm{diag}\{\Lambda_{11}, \cdots, \Lambda_{nn}\}$ and $\Delta = \mathrm{diag}\{\Delta_{11}, \cdots, \Delta_{nn}\} \in \mathscr{D}^{n,n}$, such that

$$G^{\Lambda} \overset{\text{def.}}{=} \Lambda G \Lambda^{-1}, \tag{3.38}$$

$$\Theta^{\Lambda} \overset{\text{def.}}{=} \Lambda \Theta \Lambda^{-1} \tag{3.39}$$

satisfy (1.26), and

$$G^{\Delta} \overset{\text{def.}}{=} \Delta G \Delta^{-1}, \tag{3.40}$$

$$\Theta^{\Delta,\lambda} \overset{\text{def.}}{=} \Delta \Theta^{\lambda} \Delta^{-1} \tag{3.41}$$

satisfy (1.27), we can replace the variables v and w in the proof of Theorem 1.1 by

$$\widetilde{v} = \Lambda v, \tag{3.42}$$

$$\widetilde{w} = \Delta w, \tag{3.43}$$

respectively, and get the conclusion of Theorem 1.1.

Remark 3.3 Suppose that the boundary conditions are given by

$$\begin{cases} x = 0: \quad v_s = H_s(t, v_1, \cdots, v_m), \quad s = m+1, \cdots, n, & (3.44) \\ x = 1: \quad v_r = H_r(t, v_{m+1}, \cdots, v_n), \quad r = 1, \cdots, m, & (3.45) \end{cases}$$

where H_r and H_s $(r = 1, \cdots, m,\ s = m+1, \cdots, n)$ are C^2 functions of $(t, v_{m+1}, \cdots, v_n)^{\mathrm{T}}$ and $(t, v_1, \cdots, v_m)^{\mathrm{T}}$, respectively, and

$$H_i(t, 0) \equiv 0, \quad \forall t \geq 0,\ \forall i \in \mathscr{N}. \tag{3.46}$$

Let

$$\begin{aligned} \overline{\Theta}(t) &= (\overline{\theta}_{ij}(t)) \\ &\overset{\text{def.}}{=} \begin{pmatrix} 0 & \left.\frac{\partial(H_1, \cdots, H_m)}{\partial(t, v_{m+1}, \cdots, v_n)}\right|_{(t, v_{m+1}, \cdots, v_n) = (t, 0, \cdots, 0)} \\ \left.\frac{\partial(H_{m+1}, \cdots, H_n)}{\partial(t, v_1, \cdots, v_m)}\right|_{(t, v_1, \cdots, v_m) = (t, 0, \cdots, 0)} & 0 \end{pmatrix}. \end{aligned} \tag{3.47}$$

Then, boundary conditions (1.16)–(1.17) can be rewritten as

$$\begin{cases} x = 0: \quad v_s = \displaystyle\sum_{j=1}^{m} \overline{\theta}_{sj}(t) v_j + \sum_{j,k=1}^{m} \overline{\chi}_{sjk}(t, u) v_j v_k, \quad s = m+1, \cdots, n, \\ x = l: \quad v_r = \displaystyle\sum_{j=m+1}^{n} \overline{\theta}_{rj}(t) v_j + \sum_{j,k=m+1}^{n} \overline{\chi}_{rjk}(t, u) v_j v_k, \quad r = 1, \cdots, m, \end{cases} \tag{3.48}$$

where $\overline{\chi}_{ijk} \in C^0$ $(i,j,k \in \mathscr{N})$. Similarly, we have

$$\begin{cases} x=0: \quad w_s = \displaystyle\sum_{r=1}^{m} \overline{\Theta}^{\lambda}_{sr}(t) w_r + \sum_{j\in\mathscr{N}} \overline{G^{\flat}}_{sj}(t,u) v_j + \sum_{j,k\in\mathscr{N}} \overline{\phi^{\flat}}_{sjk}(t,u) v_j v_k \\ \qquad\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \overline{\widetilde{\phi^{\flat}}}_{sjk}(t,u) v_j w_k, \quad s = m+1,\cdots,n, \\ x=1: \quad w_r = \displaystyle\sum_{s=m+1}^{n} \overline{\Theta}^{\lambda}_{rs}(t) w_s + \sum_{j\in\mathscr{N}} \overline{G^{\flat}}_{rj}(t,u) v_j + \sum_{j,k\in\mathscr{N}} \overline{\phi^{\flat}}_{sjk}(t,u) v_j v_k \\ \qquad\qquad + \displaystyle\sum_{j,k\in\mathscr{N}} \overline{\widetilde{\phi^{\flat}}}_{rjk}(t,u) v_j w_k, \quad r = 1,\cdots,m, \end{cases} \tag{3.49}$$

where $\overline{\chi}_{ijk}(t,u), \overline{G^{\flat}}_{ij}(t,u), \overline{\phi^{\flat}}_{ijk}(t,u)$ and $\overline{\widetilde{\phi^{\flat}}}_{ijk}(t,u)$ $(i,j,k \in \mathscr{N})$ are bounded continuous functions.

Comparing (3.48)–(3.49) with (2.21) and (2.27), through the procedure of the proof of Theorem 1.1, we can get the following theorem.

Theorem 3.1 *Under hypotheses* (1.2), (1.10) *and* (3.46)*, if* (1.26)–(1.27) *hold, then there exists* $\varepsilon_0 > 0$ *so small that for any given* $\varepsilon \in [0, \varepsilon_0]$*, and any given initial data* $u_0(x)$ *satisfying* (1.25) *and* $H_i(t,v)$ $(i = 1,\cdots,n)$ *satisfying* $|\overline{\theta}_{ij}(t) - \theta_{ij}(0)| \le \varepsilon$ $(i,j = 1,\cdots,n,\ t > 0)$*, the conclusion of Theorem* 1.1 *is still valid.*

4 Application

In this section, we give a kind of models to illustrate the application of Theorem 1.1. We consider the following mixed initial-boundary value problem for a system composed of two equations:

$$\begin{cases} \dfrac{\partial Z}{\partial t} - \lambda(Z,W)\dfrac{\partial Z}{\partial x} = -\kappa Z - \kappa W, & (4.1) \\ \dfrac{\partial W}{\partial t} + \lambda(Z,W)\dfrac{\partial W}{\partial x} = -\kappa Z - \kappa W, & (4.2) \\ t = 0: \quad (Z,W)^{\mathrm{T}} = (Z_0(x), W_0(x))^{\mathrm{T}}, \quad x \in [0,1], & (4.3) \\ x = 0: \quad W = \beta Z, & (4.4) \\ x = 1: \quad Z = \alpha W, & (4.5) \end{cases}$$

where $\lambda(Z,W)$ is a C^2 function of (Z,W), satisfying

$$\lambda(0,0) > 0, \tag{4.6}$$

$\kappa > 0$, and α, β are constants. By Theorem 1.1, we have the following theorem.

Theorem 4.1 *Suppose that* (4.6) *holds. If* $|\alpha| < 1$ *and* $|\beta| < 1$*, then there exists* $\theta_0 > 0$ *so small that for any given* $\theta \in [0, \theta_0]$ *and any given initial data* $(Z_0(x), W_0(x))$ *satisfying*

$$\|(Z_0(\cdot), W_0(\cdot))\|_{C^1[0,1]} \le \theta, \tag{4.7}$$

the mixed initial-boundary value problem (4.1)–(4.5) *admits a unique global* C^1 *solution* $(Z, W) = (Z(t,x), W(t,x))$ *on the domain* $\{(t,x) \mid t \geq 0, 0 \leq x \leq 1\}$*, and there exists* $\alpha > 0$*, such that we have the following uniform a priori estimate:*

$$\|(Z(t,\cdot), W(t,\cdot))\|_{C^1[0,1]} \leq C\theta \mathrm{e}^{-\alpha t}, \quad \forall t \geq 0, \tag{4.8}$$

where C *is a positive constant independent of* θ *and* t.

Remark 4.1 Some physical models can be written in the form of (4.1)–(4.2), for instance, the p-system with damping

$$\begin{cases} \dfrac{\partial u}{\partial t} - \dfrac{\partial v}{\partial x} = 0, & (4.9) \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial p(u)}{\partial x} = -v, & (4.10) \end{cases}$$

where u and v stand for the specific volume and the velocity of the fluid. For polytropic gases, the pressure p is given by the following thermodynamic state equation (see [7]):

$$p = p(u) = \kappa u^{-\gamma_0},$$

where $\kappa > 0$ and $\gamma_0 > 1$ are constants.

For any given $u^* > 0$, $(u^*, 0)$ is an equilibrium state of system (4.9)–(4.10). Using the Riemann invariants

$$r = \frac{v}{2} - \frac{\sqrt{\kappa\gamma_0}}{\gamma_0 - 1} u^{-\frac{\gamma_0 - 1}{2}} = \frac{v}{2} - \frac{\sqrt{\kappa\gamma_0}}{\gamma_0 - 1} (\widetilde{u} + u^*)^{-\frac{\gamma_0 - 1}{2}}, \tag{4.11}$$

$$s = \frac{v}{2} + \frac{\sqrt{\kappa\gamma_0}}{\gamma_0 - 1} u^{-\frac{\gamma_0 - 1}{2}} = \frac{v}{2} + \frac{\sqrt{\kappa\gamma_0}}{\gamma_0 - 1} (\widetilde{u} + u^*)^{-\frac{\gamma_0 - 1}{2}}, \tag{4.12}$$

(4.9)–(4.10) can be rewritten as

$$\frac{\partial r}{\partial t} - c\frac{\partial r}{\partial x} = -\frac{1}{2}r - \frac{1}{2}s, \tag{4.13}$$

$$\frac{\partial s}{\partial t} + c\frac{\partial s}{\partial x} = -\frac{1}{2}r - \frac{1}{2}s, \tag{4.14}$$

where

$$c = \sqrt{-p'(u + u^*)} > 0$$

(see [7]).

Moreover, for 1D linear wave equation

$$\Box u = -2u_t, \tag{4.15}$$

by the following transformation of variables:

$$Z = (\partial_t + \partial_x)u, \tag{4.16}$$

$$W = (\partial_t - \partial_x)u, \tag{4.17}$$

we get

$$\begin{cases}\dfrac{\partial Z}{\partial t}-\dfrac{\partial Z}{\partial x}=-Z-W, & (4.18)\\ \dfrac{\partial W}{\partial t}+\dfrac{\partial W}{\partial x}=-Z-W. & (4.19)\end{cases}$$

5 A Counterexample

In this section, we give an example to show that the conclusion of Theorem 1.1 may fail if (1.26)–(1.27) do not hold.

Example 5.1

$$\begin{cases}\dfrac{\partial u_1}{\partial t}-\dfrac{\partial u_1}{\partial x}=u_1, & (5.1)\\ \dfrac{\partial u_2}{\partial t}+\dfrac{\partial u_2}{\partial x}=u_2, & (5.2)\\ x=0: \quad u_2=\mathrm{e}^{-1}u_1, & (5.3)\\ x=1: \quad u_1=\mathrm{e}^{-1}u_2, & (5.4)\\ t=0: \quad u=(u_{01}(x),u_{02}(x))^{\mathrm{T}}, \quad x\in[0,1]. & (5.5)\end{cases}$$

The coefficient matrix $A(u)$ of (5.1)–(5.2) is

$$A(u)=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}, \tag{5.6}$$

whose eigenvalues are

$$\lambda_1=-1, \quad \lambda_2=1. \tag{5.7}$$

Then

$$T_1=T_2=1. \tag{5.8}$$

The inhomogeneous term $F(u)$ is

$$F(u)=\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \tag{5.9}$$

and then

$$G=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}. \tag{5.10}$$

Moreover, the coefficient matrix Θ of boundary conditions (5.3)–(5.4) is

$$\Theta=\begin{pmatrix}0 & \mathrm{e}^{-1}\\ \mathrm{e}^{-1} & 0\end{pmatrix}. \tag{5.11}$$

Thus,

$$\max_{i\in\{1,2\}}\Big\{\max_{t\in[0,1]}\sum_{j\neq i}\Big(|\Theta_{ij}|+\frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{G_{ii}t}-\sum_{j\neq i}\frac{|G_{ij}|}{G_{ii}}\Big\}=1. \tag{5.12}$$

It is easy to see that the hypothesis (1.26) in Theorem 1.1 does not hold.

Let the $C^1[0,1]$ initial data $u_{0i}(x)$ $(i=1,2)$ satisfy

$$u_{0i}\Big(\frac{1}{2}\Big)=\varepsilon, \tag{5.13}$$

$$\|u_{0i}(\cdot)\|_{C^1[0,1]}\le 2\varepsilon, \tag{5.14}$$

where $\varepsilon>0$ is a small parameter. By the conclusion in [9], the mixed initial-boundary value problem (5.1)–(5.5) admits a global C^1 solution. In particular, we have

$$\begin{cases} x=0: \quad u_1\Big(\dfrac{3}{2},0\Big)=\cdots=u_1\Big(\dfrac{4K+3}{2},0\Big)=\varepsilon \mathrm{e}^{\frac{1}{2}}, \quad \forall K=0,1,2,3,\cdots, \\ x=0: \quad u_2\Big(\dfrac{1}{2},0\Big)=\cdots=u_2\Big(\dfrac{4K+1}{2},0\Big)=\varepsilon \mathrm{e}^{-\frac{1}{2}}, \quad \forall K=0,1,2,3,\cdots. \end{cases}$$

Hence, the C^1 solution to the mixed initial-boundary value problem (5.1)–(5.5) does not decay with respect to t.

6 Further Discussion on Theorem 1.1

First of all, the hypotheses (1.26)–(1.27) in Theorem 1.1 can be rewritten as

$$\sum_{j\ne i}|\Theta_{ij}|<\Big(1+\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{-G_{ii}t}-\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}, \quad \forall t\in[0,T_i],\ \forall i\in\mathscr{N} \tag{6.1}$$

and

$$\sum_{j\ne i}|\Theta^{\lambda}_{ij}|<\Big(1+\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{-G_{ii}t}-\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}, \quad \forall t\in[0,T_i],\ \forall i\in\mathscr{N}. \tag{6.2}$$

Case 1 If $G_{ii}=0$ $(i\in\mathscr{N})$, note that (1.29)–(1.30), (6.1)–(6.2) should be replaced by

$$\sum_{j\ne i}|\Theta_{ij}|+\sum_{j\ne i}|G_{ij}|t<1, \quad \forall t\in[0,T_i],\ \forall i\in\mathscr{N}, \tag{6.3}$$

$$\sum_{j\ne i}|\Theta^{\lambda}_{ij}|+\sum_{j\ne i}|G_{ij}|t<1, \quad \forall t\in[0,T_i],\ \forall i\in\mathscr{N}, \tag{6.4}$$

respectively. Specially, if $G_{ij}=0$ $(i,j\in\mathscr{N})$, (6.3)–(6.4) can be simplified to

$$\max_{i\in\mathscr{N}}\sum_{j\ne i}|\Theta_{ij}|<1, \tag{6.5}$$

$$\max_{i\in\mathscr{N}}\sum_{j\ne i}|\Theta^{\lambda}_{ij}|<1, \tag{6.6}$$

respectively, which are the boundary dissipative condition for the quasilinear hyperbolic system without internal dissipative terms (see [6–7]).

Case 2 If $G_{ii}>0$, then

$$\Big(1+\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}\Big)\mathrm{e}^{-G_{ii}t}-\sum_{j\ne i}\frac{|G_{ij}|}{G_{ii}}<1. \tag{6.7}$$

In this case, (6.1)–(6.2) indicate that if the inhomogeneous term $F(u)$ has a growth effect for v_i, a stronger boundary dissipative condition is needed to guarantee the exponential decay.

Case 3 If $G_{ii} < 0$, then

$$\Big(1 + \sum_{j \neq i} \frac{|G_{ij}|}{G_{ii}}\Big) \mathrm{e}^{-G_{ii} T_i} - \sum_{j \neq i} \frac{|G_{ij}|}{G_{ii}} > 1. \tag{6.8}$$

In this case, (6.1)–(6.2) indicate that if the inhomogeneous term $F(u)$ has a reduced effect for v_i, by a combination effect of the weaker boundary dissipative condition and the internal dissipative condition, the C^1 solution still decays exponentially.

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