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On Deformed Riemannian Extensions Associated with Twin Norden Metrics*

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Abstract The main purpose of this paper is to study the deformed Riemannian extension ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$ in the cotangent bundle, where G is a twin Norden metric on the base manifold.

Keywords Riemannian extension, Cotangent bundle, Vertical and complete lifts, Horizontal lifts, Deformed metric, Twin Norden metric
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1 Introduction

Let M_n be an n-dimensional C^∞-manifold with torsion-free connection ∇, ${}^{C}T(M_n)$ be its cotangent bundle, and π be the natural projection ${}^{C}T(M_n) \to M_n$. A system of local coordinates (U, x^i), $i = 1, \cdots, n$ in M_n induces on ${}^{C}T(M_n)$ a system of local coordinates $(\pi^{-1}(U),\ x^i,\ x^{\overline{i}} = p_i)$, $\overline{i} = n + i = n + 1, \cdots, 2n$, where $x^{\overline{i}} = p_i$ are components of covectors p in each cotangent space ${}^{C}T_x(M_n)$, $x \in U$ with respect to the natural coframe $\{\mathrm{d}x^i\}$.

We denote by $\Im^r_s(M_n)(\Im^r_s({}^{C}T(M_n)))$ the module over $F(M_n)(F({}^{C}T(M_n)))$ of C^∞ tensor fields of type (r, s), where $F(M_n)(F({}^{C}T(M_n)))$ is the ring of real-valued C^∞ functions on $M_n({}^{C}T(M_n))$.

Let $X = X^i\partial_i$ and $\xi = \xi_i \mathrm{d}x^i$ be the local expressions in $U \subset M_n$ of vector and covector (1-form) fields $X \in \Im^1_0(M_n)$ and $\xi \in \Im^0_1(M_n)$, respectively. Then the complete and horizontal lifts ${}^{C}X$, ${}^{H}X \in \Im^1_0({}^{C}T(M_n))$ of $X \in \Im^1_0(M_n)$ and the vertical lift ${}^{V}\xi \in \Im^1_0({}^{C}T(M_n))$ of $\xi \in \Im^0_1(M_n)$ are given, respectively, by

$$ {}^{C}X = X^i\partial_i - \sum_i p_h \partial_i X^h \partial_{\overline{i}}, \tag{1.1}$$

$$ {}^{H}X = X^i\partial_i + \sum_i p_h \Gamma^h_{ij} X^j \partial_{\overline{i}}, \tag{1.2}$$

$$ {}^{V}\xi = \sum_i \xi_i \partial_{\overline{i}} \tag{1.3}$$

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with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, where Γ_{ij}^h are components of the torsion-free connection ∇ on M_n.

A new (pseudo) Riemannian metric ${}^\nabla g \in \Im_2^0({}^C T(M_n))$ on ${}^C T(M_n)$ is defined by the equation (see [7, p. 268])

$$ {}^\nabla g({}^C X, {}^C Y) = -\gamma(\nabla_X Y + \nabla_Y X) $$

for any $X, Y \in \Im_0^1(M_n)$, where $\gamma(\nabla_X Y + \nabla_Y X)$ is a function in $\pi^{-1}(U) \subset {}^C T(M_n)$ with a local expression $\gamma(\nabla_X Y + \nabla_Y X) = p_h(X^i \nabla_i Y^h + Y^i \nabla_i X^h)$. We call ${}^\nabla g$ the Riemannian extension of the symmetric connection ∇ to ${}^C T(M_n)$. The Riemannian extension ${}^\nabla g$ has components of the form

$$ {}^\nabla g = ({}^\nabla g_{IJ}) = \begin{pmatrix} -2p_m \Gamma_{ij}^m & \delta_i^j \\ \delta_j^i & 0 \end{pmatrix} \tag{1.4} $$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, where δ_j^i is the Kronecker delta.

On the other hand, the vector fields ${}^H X$ and ${}^V \xi$ span the module $\Im_0^1({}^C T(M_n))$. Hence the tensor field ${}^\nabla g$ is also determined by its action of ${}^H X$ and ${}^V \xi$. From (1.2)–(1.4), we have

$$ {}^\nabla g({}^V \xi,\ {}^V \theta) = 0, \tag{1.5} $$

$$ {}^\nabla g({}^V \xi,\ {}^H X) = {}^V (\xi(X)) = (\xi(X)) \circ \pi, \tag{1.6} $$

$$ {}^\nabla g({}^H X, {}^H Y) = 0 \tag{1.7} $$

for any $X, Y \in \Im_0^1(M_n)$ and $\xi, \theta \in \Im_1^0(M_n)$. Thus ${}^\nabla g$ is completely determined by the conditions (1.5)–(1.7).

It is well known that ${}^C T(M_n)$ has a canonical symplectic structure $\omega = \mathrm{d}p$, where p is a basic 1-form in $\pi^{-1}(U) \subset {}^C T(M_n)$. The symplectic 2-form has components of the form

$$ \omega = (\omega_{IJ}) = \begin{pmatrix} 0 & \delta_j^i \\ -\delta_i^j & 0 \end{pmatrix} \tag{1.8} $$

with respect to the natural frame $\{\partial_i, \partial_{\bar{i}}\}$, where $I = (i, \bar{i}),\ J = (j, \bar{j})$ and $I, J = 1, \cdots, 2n$.

2 The Metric ${}^\nabla g$ as a Pullback of ${}^C g$

Let now M_n be a Riemannian manifold with metric g and $\nabla = \nabla_g$ be the Levi-Civita connection of g. We denote by $T(M_n)$ the tangent bundle over M_n with local coordinates $(x^i, \widetilde{x}^{\bar{i}}) = (x^i, y^i)$, where $y_x = y^i \partial_i \in T_x(M_n)$, $\forall x \in M_n$. Let ${}^C g$ be a complete lift of a Riemannian metric g to $T(M_n)$ with components

$$ {}^C g = ({}^C g_{IJ}) = \begin{pmatrix} y^s \partial_s g_{ij} & g_{ij} \\ g_{ij} & 0 \end{pmatrix}. \tag{2.1} $$

A very important feature of any Riemannian metric g is that it provides a musical (natural) isomorphism $g^\sharp : {}^C T(M_n) \to T(M_n)$ between the cotangent and tangent bundles. The musical isomorphism $g^\sharp$ is expressed by $g^\sharp : x^K = (x^k, p_k) \to \widetilde{x}^I = (x^i, \widetilde{x}^{\bar{i}}) = (x^i, y^i = g^{is}p_s)$ with respect to the local coordinates. The Jacobian matrix of $g^\sharp$ is given by

$$(A_K^I) = \Big(\frac{\partial \widetilde{x}^I}{\partial x^K}\Big) = \begin{pmatrix} \delta_k^i & 0 \\ p_s\partial_k g^{is} & g^{ik} \end{pmatrix}. \tag{2.2}$$

Using (2.1)–(2.2) we see that the pullback of ${}^Cg$ by $g^\sharp$ is the $(0,2)$-tensor field $(g^\sharp)^* {}^Cg$ on ${}^CT(M_n)$ and has components

$$\begin{aligned}(((g^\sharp)^* {}^Cg)_{KL}) &= (A_K^I A_L^J {}^Cg_{IJ}) \\ &= \begin{pmatrix} A_k^i A_l^j {}^Cg_{ij} + A_k^{\bar{i}} A_l^j {}^Cg_{\bar{i}j} + A_k^i A_l^{\bar{j}} {}^Cg_{i\bar{j}} & A_k^i A_{\bar{l}}^{\bar{j}} {}^Cg_{i\bar{j}} \\ A_{\bar{k}}^{\bar{i}} A_l^j {}^Cg_{\bar{i}j} & 0 \end{pmatrix} \\ &= \begin{pmatrix} \delta_k^i\delta_l^j y^s\partial_s g_{ij} + p_s(\partial_k g^{is})\delta_l^j g_{ij} + \delta_k^i p_s(\partial_l g^{js})g_{ij} & \delta_k^l \\ g^{ik}\delta_l^j g_{ij} & 0 \end{pmatrix} \\ &= \begin{pmatrix} y^s\partial_s g_{kl} + p_s((\partial_k g^{is})g_{il} + (\partial_l g^{js})g_{kj}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\ &= \begin{pmatrix} p_t g^{st}\partial_s g_{kl} - p_s(g^{is}\partial_k g_{il} + g^{js}\partial_l g_{kj}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\ &= \begin{pmatrix} -p_s g^{ts}(\partial_l g_{tk} + \partial_k g_{lt} - \partial_t g_{kl}) & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix} \\ &= \begin{pmatrix} -2p_s\Gamma_{kl}^s & \delta_k^l \\ \delta_l^k & 0 \end{pmatrix}. \end{aligned} \tag{2.3}$$

Thus, from (1.4) and (2.3), we have $(g^\sharp)^* {}^Cg = {}^\nabla g$, i.e., the Riemannian extension ${}^\nabla g \in \Im_2^0({}^CT(M_n))$ should be considered as a pullback of the complete lift ${}^Cg \in \Im_2^0(T(M_n))$.

3 The Deformed Metrics in the Cotangent Bundle

It is well known that the deformed complete lift of Riemannian metric g to the tangent bundle is defined by

$${}^{Syn}g = {}^Cg + {}^Va,$$

where a is a symmetric tensor field on M_n, and ${}^Va \in \Im_2^0(T(M_n))$ has components

$${}^Va = ({}^Va_{IJ}) = \begin{pmatrix} a_{ij} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the natural frame in $\pi^{-1}(U) \subset T(M_n)$. Lifts of this kind have also been studied under the name: The synectic lift of metrics (see [1, 4, p. 88, 5, 6, p. 165]). Also we note that, if (M_n, g) is flat, then $(T(M_n), {}^{Syn}g)$ is not necessarily flat, but $(T(M_n), {}^Cg)$ is necessarily flat. If $a = g$, then we have ${}^{Syn}g = {}^Cg + {}^Vg$. The metric ${}^Cg + {}^Vg$ is called a metric of type $\mathrm{I} + \mathrm{II}$. The metric $\mathrm{I} + \mathrm{II}$ was used by Yano and Ishihara [7] to study the geometry of tangent bundles.

The pullback of ${}^Va$ has components

$$(g^\sharp)^* {}^Va = (((g^\sharp)^* {}^Va)_{IJ}) = \begin{pmatrix} a_{ij} & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.1}$$

Using (1.8), (2.3) and (3.1) we have

$$(g^{\sharp})^{*}\left({}^{C}g + {}^{V}a\right) = \left(\left((g^{\sharp})^{*}\left({}^{C}g + {}^{V}a\right)_{IJ}\right) = \begin{pmatrix} -2p_s\Gamma^{s}_{kl} + a_{kl} & \delta^{l}_{k} \\ \delta^{k}_{l} & 0 \end{pmatrix}\right.$$

$$= {}^{\nabla}g + \begin{pmatrix} a_{kl} & 0 \\ 0 & 0 \end{pmatrix} = {}^{\nabla}g + \left(\omega_{IK}\omega_{JL}{}^{V}a^{IJ}\right) = {}^{\nabla}g + \left({}^{V}a^{\bullet\bullet}_{KL}\right) = {}^{\nabla}g + {}^{V}a^{\bullet\bullet},$$

where

$${}^{V}a = \left({}^{V}a^{JL}\right) = \begin{pmatrix} 0 & 0 \\ 0 & a_{ij} \end{pmatrix}$$

is a tensor field of type (2, 0) in $\pi^{-1}(U) \subset {}^{C}T(M_n)$ (see [7, p. 230]). The tensor field

$$(g^{\sharp})^{*}\left({}^{C}g + {}^{V}a\right) = {}^{\nabla}g + {}^{V}a^{\bullet\bullet}$$

is non-singular and can be regarded as a new metric on the cotangent bundle ${}^{C}T(M_n)$.

4 The Deformed Metrics of Type ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$

Let (M_n, J) be a $2n$-dimensional almost complex manifold, where J denotes its almost complex structure. A semi-Riemannian metric g of the natural signature (n, n) is a Norden (anti-Hermitian) metric if (see [3])

$$g(JX, Y) = g(X, JY)$$

for any $X, Y \in \Im^{1}_{0}(M_n)$. An almost complex manifold (M_n, J) with a Norden metric is referred to as an almost Norden manifold. Structures of this kind have also been studied under the name: Almost complex structures with pure (or B-) metrics. A Kähler-Norden (anti-Kähler) manifold can be defined as a triple (M_n, g, J) which consists of a smooth manifold M_n endowed with an almost complex structure J and a Norden metric g such that $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. It is well known that the condition $\nabla J = 0$ is equivalent to $\mathbb{C}$-holomorphicity (analyticity) of the Norden metric g (see [4]), i.e., $\Phi_J g = 0$, where $(\Phi_J g)(X, Y, Z) = (L_{JX}g - L_X G)(Y, Z)$ and $G(Y, Z) = (g \circ J)(Y, Z) = g(JY, Z)$ is the twin Norden metric. It is a remarkable fact that (M_n, g, J) is Kähler-Norden if and only if the twin Norden structure (M_n, G, J) is Kähler-Norden. This is of special significance for Kähler-Norden metrics since in such case g and G share the same Levi-Civita connection ($\nabla g = \nabla G = 0$). Since in dimension 2, a Kähler-Norden manifold is flat, we assume in the sequel that $\dim M \geq 4$.

Let now (M_n, g, J) be an almost Norden manifold. If $a = G$ (see Section 3), where G is a twin Norden metric, then we have a metric

$$(g^{\sharp})^{*}\left({}^{C}g + {}^{V}G\right) = {}^{\nabla}g + {}^{V}G^{\bullet\bullet}.$$

The metric ${}^{\nabla}g + {}^{V}G^{\bullet\bullet} = \widetilde{g}$ has components

$$(\widetilde{g}_{IJ}) = \begin{pmatrix} -2p_s\Gamma^{s}_{ij} + G_{ij} & \delta^{j}_{i} \\ \delta^{i}_{j} & 0 \end{pmatrix} \tag{4.1}$$

with respect to the induced coordinates (x^i, p_i).

The main purpose of the next sections is to study the metric ${}^\nabla g + {}^V G^{\bullet\bullet}$ in the cotangent bundle and also the metric connection with respect to this metric.

The line element of (4.1) is given by

$$\begin{aligned}\mathrm{d}s^2 &= \widetilde{g}_{IJ}\mathrm{d}x^I\mathrm{d}x^J = \widetilde{g}_{ij}\mathrm{d}x^i\mathrm{d}x^j + \widetilde{g}_{\overline{i}\,j}\mathrm{d}x^{\overline{i}}\mathrm{d}x^j + \widetilde{g}_{i\,\overline{j}}\mathrm{d}x^i\mathrm{d}x^{\overline{j}} + \widetilde{g}_{\overline{i}\,\overline{j}}\mathrm{d}x^{\overline{i}}\mathrm{d}x^{\overline{j}}\\ &= (-2p_s\Gamma^s_{ij} + G_{ij})\mathrm{d}x^i\mathrm{d}x^j + \delta^i_j\mathrm{d}p_i\mathrm{d}x^j + \delta^j_i\mathrm{d}x^i\mathrm{d}p_j = G_{ij}\mathrm{d}x^i\mathrm{d}x^j + 2\mathrm{d}x^i\delta p_i\,,\end{aligned}$$

where $\delta p_j = \mathrm{d}p_j - p_s\Gamma^s_{ij}\mathrm{d}x^i$. From here, we have the following theorem.

Theorem 4.1 *Let (M_n, g, J) be an almost Norden manifold. Then the fibre represented by $\mathrm{d}x^i = 0$ is a null submanifold in ${}^CT(M_n)$ with a deformed Riemann extension metric ${}^\nabla g + {}^V G^{\bullet\bullet}$, but the horizontal distribution defined by $\delta p_i = 0$ is not null.*

Let ${}^C\nabla$ be the Levi-Civita connection determined by ${}^\nabla g$, i.e., ${}^C\nabla({}^\nabla g) = 0$ (${}^C\nabla$ is called the complete lift of ∇_g to ${}^CT(M_n)$). The components of ${}^C\nabla$ in $\pi^{-1}(U) \subset {}^CT(M_n)$ are given by

$$\begin{aligned}&{}^C\Gamma^h_{ji} = \Gamma^h_{ji}, \quad {}^C\Gamma^h_{\overline{j}\,i} = {}^C\Gamma^h_{\overline{j}\overline{i}} = {}^C\Gamma^{\overline{h}}_{\overline{j}\overline{i}} = 0, \quad {}^C\Gamma^{\overline{h}}_{j\overline{i}} = \Gamma^i_{jh},\\ &{}^C\Gamma^{\overline{h}}_{ij} = p_a(\partial_h\Gamma^a_{ji} - \partial_j\Gamma^a_{ih} - \partial_i\Gamma^a_{jh} + 2\Gamma^a_{ht}\Gamma^t_{ji})\end{aligned} \tag{4.2}$$

with respect to induced coordinates in $\pi^{-1}(U) \subset {}^CT(M_n)$. If (M_n, g, J) is Kähler-Norden ($\nabla g = \nabla G = 0$), then using the expression

$${}^V G^{\bullet\bullet} = \begin{pmatrix} G_{kl} & 0\\ 0 & 0\end{pmatrix},$$

from (1.1) and (4.2) we find ${}^C\nabla_{{}^CX}{}^V G^{\bullet\bullet} = {}^V(\nabla_X G)^{\bullet\bullet} = 0$ for any $X \in \Im^1_0(M_n)$. Then we have

$${}^C\nabla_{{}^CX}({}^\nabla g + {}^V G^{\bullet\bullet}) = 0.$$

On the other hand, ${}^C\nabla$ is torsion-free, so we have the following theorem.

Theorem 4.2 *Let (M_n, J, g) be a Kähler-Norden manifold. Then the Levi-Civita connection of ${}^\nabla g$ coincides with the Levi-Civita connection of ${}^\nabla g + {}^V G^{\bullet\bullet}$.*

Using (1.1) and (4.1), we easily see that the inner product of the complete lifts ${}^CX$ and ${}^CY$ of vector fields X, $Y \in M_n$ with respect to the metric ${}^\nabla g + {}^V G^{\bullet\bullet}$ is given by

$$({}^\nabla g + {}^V G^{\bullet\bullet})({}^CX, {}^CY) = G(X, Y) - \gamma(\nabla_X Y + \nabla_Y X).$$

From this equation, we have the following theorem.

Theorem 4.3 *Let (M_n, J, g) be a Kähler-Norden manifold. Then the complete lifts ${}^CX, {}^CY$ of two vector fields X, Y to ${}^CT(M_n)$ with a metric ${}^\nabla g + {}^V G^{\bullet\bullet}$ are orthogonal if X, Y are orthogonal with respect to G and are parallel.*

5 The Metric Connection with Respect to $^{\nabla}g + {}^{V}G^{\bullet\bullet}$

Let ∇_g be the Levi-Civita connection on M_n. In $U \subset M_n$, we put

$$X_{(i)} = \frac{\partial}{\partial x^i}, \quad \theta^{(i)} = \mathrm{d}x^i, \quad i = 1, \cdots, n.$$

Then from (1.2)–(1.3), we see that ${}^{H}X_{(i)}$ and ${}^{V}\theta^{(i)}$ have respectively local expressions of the form

$$^{H}X_{(i)} = \frac{\partial}{\partial x^i} + \sum_h p_a \Gamma^a_{hi} \frac{\partial}{\partial x^{\overline{h}}}, \tag{5.1}$$

$$^{V}\theta^{(i)} = \frac{\partial}{\partial x^{\overline{i}}}. \tag{5.2}$$

We call the set $\{{}^{H}X_{(i)}, {}^{V}\theta^{(i)}\} = \{\widetilde{e}_{(i)}, \widetilde{e}_{(\overline{i})}\} = \{\widetilde{e}_{(\alpha)}\}$ the frame adapted to the connection ∇_g. The indices $\alpha, \beta, \gamma, \cdots = 1, \cdots, 2n$ indicate the indices with respect to the adapted frame.

From equations (1.2)–(1.3) and (5.1)–(5.2), we see that the lifts ${}^{H}X$ and ${}^{V}\omega$ have respectively components

$$^{H}X = X^i \widetilde{e}_{(i)}, \quad {}^{H}X = \begin{pmatrix} X^i \\ 0 \end{pmatrix}, \tag{5.3}$$

$$^{V}\xi = \sum_i \xi_i \widetilde{e}_{(\overline{i})}, \quad {}^{V}\xi = \begin{pmatrix} 0 \\ \xi^i \end{pmatrix} \tag{5.4}$$

with respect to the adapted frame $\{\widetilde{e}_{(\alpha)}\}$, where $X \in \Im^1_0(M_n)$, $\xi \in \Im^0_1(M_n)$, and X^i and ω_i are local components of X and ω, respectively. Also from (1.5)–(1.7), we see that

$$\begin{aligned}
&{}^{\nabla}g({}^{V}\xi^{(i)}, {}^{V}\theta^{(j)}) = {}^{\nabla}g(\widetilde{e}_{(\overline{i})}, \widetilde{e}_{(\overline{j})}) = {}^{\nabla}\widetilde{g}_{\overline{i}\overline{j}} = 0,\\
&{}^{\nabla}g({}^{H}X_{(i)}, {}^{H}Y_{(j)}) = {}^{\nabla}g(\widetilde{e}_{(i)}, \widetilde{e}_{(j)}) = {}^{\nabla}\widetilde{g}_{ij} = 0,\\
&{}^{\nabla}g({}^{V}\xi^{(i)}, {}^{H}X^{(j)}) = {}^{\nabla}g(\widetilde{e}_{(\overline{i})}, \widetilde{e}_{(j)}) = {}^{\nabla}\widetilde{g}_{\overline{i}j} = {}^{\nabla}\widetilde{g}_{j\overline{i}} = (\mathrm{d}x^i)\Big(\frac{\partial}{\partial x^j}\Big) = \delta^i_j,\\
&{}^{\nabla}g({}^{H}X_{(i)}, {}^{V}\xi^{(j)}) = {}^{\nabla}g(\widetilde{e}_{(i)}, \widetilde{e}_{(\overline{j})}) = {}^{\nabla}\widetilde{g}_{i\overline{j}} = {}^{\nabla}\widetilde{g}_{\overline{j}i} = (\mathrm{d}x^j)\Big(\frac{\partial}{\partial x^i}\Big) = \delta^j_i,
\end{aligned}$$

i.e., ${}^{\nabla}g$ has components

$$^{\nabla}g = ({}^{\nabla}\widetilde{g}_{\beta\alpha}) = \begin{pmatrix} {}^{\nabla}\widetilde{g}_{ji} & {}^{\nabla}\widetilde{g}_{j\overline{i}} \\ {}^{\nabla}\widetilde{g}_{\overline{j}i} & {}^{\nabla}\widetilde{g}_{\overline{j}\overline{i}} \end{pmatrix} = \begin{pmatrix} 0 & \delta^i_j \\ \delta^j_i & 0 \end{pmatrix} \tag{5.5}$$

with respect to the adapted frame $\{\widetilde{e}_{(\alpha)}\}$.

Using (5.1)–(5.2), we now consider local vector fields $\widetilde{e}_\beta$ and 1-forms $\widetilde{\xi}^\alpha$ in $\pi^{-1}(U) \subset {}^{C}T(M_n)$ defined by

$$\widetilde{e}_\beta = A_\beta{}^A\, \partial_A, \quad \widetilde{\xi}^\alpha = \overline{A}^\alpha_B\, \mathrm{d}x^B,$$

where

$$A = (A^A_\beta) = \begin{pmatrix} A_j{}^i & A_{\overline{j}}{}^i \\ A_j{}^{\overline{i}} & A_{\overline{j}}{}^{\overline{i}} \end{pmatrix} = \begin{pmatrix} \delta^i_j & 0 \\ p_a \Gamma^a_{ij} & \delta^j_i \end{pmatrix}, \tag{5.6}$$

$$A^{-1} = (\overline{A}^{\alpha}_{B}) = \begin{pmatrix} \overline{A}^{i}_{j} & \overline{A}^{i}_{\overline{j}} \\ \overline{A}^{\overline{i}}_{j} & \overline{A}^{\overline{i}}_{\overline{j}} \end{pmatrix} = \begin{pmatrix} \delta^{i}_{j} & 0 \\ -p_a \Gamma^{a}_{ij} & \delta^{j}_{i} \end{pmatrix}. \tag{5.7}$$

We easily see that the set $\{\widetilde{\xi}^{\ \alpha}\}$ is the coframe dual to the adapted frame $\{\widetilde{e}_{\beta}\}$, i.e., $\widetilde{\xi}^{\alpha}\widetilde{e}_{\beta} = \overline{A}^{\alpha}_{B} A^{B}_{\beta} = \delta^{\alpha}_{\beta}$.

Using (3.1) and (5.6), we see that ${}^{V}G^{\bullet\bullet}$ has components

$$(({}^{V}G^{\bullet\bullet})_{\beta\alpha}) = \begin{pmatrix} G_{ji} & 0 \\ 0 & 0 \end{pmatrix}$$

with respect to the adapted frame $\{\widetilde{e}_{\alpha}\}$. Thus ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$ has components

$$(({}^{\nabla}g + {}^{V}G^{\bullet\bullet})_{\beta\alpha}) = \begin{pmatrix} G_{ji} & \delta^{i}_{j} \\ \delta^{j}_{i} & 0 \end{pmatrix} \tag{5.8}$$

with respect to the adapted frame $\{\widetilde{e}_{\alpha}\}$.

Since the adapted frame $\{\widetilde{e}_{\beta}\}$ is non-holonomic, we put

$$[\widetilde{e}_{\gamma}, \widetilde{e}_{\beta}] = \Omega^{\alpha}_{\gamma\beta}\widetilde{e}_{\alpha}$$

from which we have

$$\Omega^{\alpha}_{\gamma\beta} = \left(\widetilde{e}_{\gamma} A^{A}_{\beta} - \widetilde{e}_{\beta} A^{A}_{\gamma}\right) \overline{A}^{\alpha}_{A}.$$

According to (4.2), (5.1) and (5.6)–(5.7), the components of the non-holonomic object $\Omega^{\alpha}_{\gamma\beta}$ are given by

$$\begin{cases} \Omega^{\overline{i}}_{l\overline{j}} = -\Omega^{\overline{i}}_{\overline{j}l} = -\Gamma^{j}_{li}, \\ \Omega^{\overline{i}}_{lj} = p_a R^{a}_{lji}, \end{cases} \tag{5.9}$$

all the others being zero, where R^{h}_{ijk} are local components of the curvature tensor R of ∇.

Let now (M_n, J, g) be a Kähler-Norden manifold, and let ${}^{C}\nabla$ be the Levi-Civita connection determined by the Riemannian extension ${}^{\nabla}g$ or by the deformed Riemannian extension ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$ (see Theorem 4.2). We put

$${}^{C}\nabla_{\widetilde{e}_{\gamma}}\widetilde{e}_{\beta} = {}^{C}\Gamma^{\alpha}_{\gamma\beta}\widetilde{e}_{\alpha}.$$

From ${}^{C}\nabla_{\widetilde{X}}\widetilde{Y} - {}^{C}\nabla_{\widetilde{Y}}\widetilde{X} = [\widetilde{X}, \widetilde{Y}],\ \forall \widetilde{X}, \widetilde{Y} \in \Im^{1}_{0}({}^{C}T(M_n))$, we have

$${}^{C}\Gamma^{\alpha}_{\gamma\beta} - {}^{C}\Gamma^{\alpha}_{\beta\gamma} = \Omega^{\alpha}_{\gamma\beta}\,. \tag{5.10}$$

The equation $({}^{C}\nabla_{\widetilde{X}}{}^{\nabla}g)(\widetilde{Y}, \widetilde{Z}) = 0$ has the form

$$\widetilde{e}_{\delta}{}^{\nabla}g_{\gamma\beta} - {}^{C}\Gamma^{\varepsilon}_{\delta\gamma}{}^{\nabla}g_{\varepsilon\beta} - {}^{C}\Gamma^{\varepsilon}_{\delta\beta}{}^{\nabla}g_{\gamma\varepsilon} = 0 \tag{5.11}$$

with respect to the adapted frame $\{\widetilde{e}_{\beta}\}$. We have from (5.10)–(5.11) that

$${}^{C}\Gamma^{\alpha}_{\gamma\beta} = \frac{1}{2}\,{}^{\nabla}g^{\alpha\varepsilon}(\widetilde{e}_{\gamma}{}^{\nabla}g_{\varepsilon\beta} + \widetilde{e}_{\beta}{}^{\nabla}g_{\gamma\varepsilon} - \widetilde{e}_{\varepsilon}{}^{\nabla}g_{\gamma\beta}) + \frac{1}{2}\,(\Omega^{\alpha}_{\gamma\beta} + \Omega^{\alpha}_{\gamma\beta} + \Omega^{\alpha}_{\beta\gamma}),$$

where $\Omega^{\alpha}_{\gamma\beta} = {}^{\nabla}g^{\alpha\varepsilon}\ {}^{\nabla}g_{\delta\beta}\Omega^{\delta}_{\varepsilon\gamma}$ and $({}^{\nabla}g^{\alpha\varepsilon}) = \begin{pmatrix} 0 & \delta^i_m \\ \delta^m_i & 0 \end{pmatrix}$.

Taking account of (5.3)–(5.5) and (5.9), we obtain (see [2])

$$\begin{cases} {}^C\Gamma^{i}_{\overline{k}j} = {}^C\Gamma^{i}_{k\overline{j}} = {}^C\Gamma^{i}_{\overline{k}\overline{j}} = {}^C\Gamma^{\overline{i}}_{\overline{k}\overline{j}} = {}^C\Gamma^{\overline{i}}_{\overline{k}j} = 0, \\ {}^C\Gamma^{i}_{kj} = \Gamma^{i}_{kj}, \quad {}^C\Gamma^{\overline{i}}_{k\overline{j}} = -\Gamma^{j}_{ki}, \\ {}^C\Gamma^{\overline{i}}_{kj} = \dfrac{1}{2}p_a(R^a_{kji} - R^a_{jik} + R^a_{ikj}) \end{cases}$$

with respect to the adapted frame $\{\widetilde{e}_\beta\}$.

Untill now, we have given the metric ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$ to the cotangent bundle ${}^{C}T(M_n)$ and considered the Levi-Civita connection${}^C\nabla$ of ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$. This is the unique connection which satisfies ${}^C\nabla({}^{\nabla}g + {}^{V}G^{\bullet\bullet}) = 0$, and has no torsion. But there exists another connection $\widetilde{\nabla}$ which satisfies $\widetilde{\nabla}({}^{\nabla}g + {}^{V}G^{\bullet\bullet}) = 0$, and has the non-trivial torsion tensor. We call this connection the metric connection of ${}^{\nabla}g + {}^{V}G^{\bullet\bullet}$.

The horizontal lift ${}^H\nabla$ of the torsion-free connection ∇ to the cotangent bundle ${}^{C}T(M_n)$ is defined by

$$\begin{cases} {}^H\nabla_{{}^V\theta}{}^V\omega = 0, \quad {}^H\nabla_{{}^V\theta}{}^HY = 0, \\ {}^H\nabla_{{}^HX}{}^V\omega = {}^V(\nabla_X\omega), \quad {}^H\nabla^H_{{}^HX}Y = {}^H(\nabla_XY) \end{cases} \tag{5.12}$$

for any $X, Y \in \Im^1_0(M_n)$ and $\omega, \theta \in \Im^0_1(M_n)$.

We now put ${}^H\nabla_\alpha = {}^H\nabla_{\widetilde{e}_{(\alpha)}}$, where $\{\widetilde{e}_{(\alpha)}\} = \{\widetilde{e}_{(i)}, \widetilde{e}_{(\overline{i})}\}$-adapted frame. Then taking account of ${}^C\nabla_\alpha\widetilde{e}_{(\beta)} = {}^H\Gamma^{\gamma}_{\alpha\beta}\widetilde{e}_{(\gamma)}$ and writing ${}^H\widetilde{\Gamma}^{\gamma}_{\alpha\beta}$ for the different indices, from (5.12) we have

$$\begin{cases} {}^H\widetilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij}, \quad {}^H\widetilde{\Gamma}^{\overline{k}}_{i\overline{j}} = -\Gamma^{j}_{ik}, \\ {}^H\widetilde{\Gamma}^{k}_{\overline{i}j} = {}^H\widetilde{\Gamma}^{k}_{i\overline{j}} = {}^H\widetilde{\Gamma}^{k}_{\overline{i}\overline{j}} = {}^H\widetilde{\Gamma}^{\overline{k}}_{\overline{i}j} = {}^H\widetilde{\Gamma}^{\overline{k}}_{ij} = {}^H\widetilde{\Gamma}^{\overline{k}}_{\overline{i}\overline{j}} = 0. \end{cases} \tag{5.13}$$

Let T be the torsion tensor of the horizontal lift ${}^H\nabla$. Then T is the skew-symmetric tensor field of type $(1,2)$ in ${}^{C}T(M_n)$ determined by [7, p. 287]

$$T({}^V\omega, {}^V\theta) = 0, \quad T({}^HX, {}^V\theta) = 0, \quad T({}^HX, {}^HY) = -\gamma R(X,Y),$$

where R is the curvature tensor of ∇ and $\gamma R(X,Y) = \sum\limits_i p_h R^h_{kli} X^k Y^l \frac{\partial}{\partial x^{\overline{i}}}$. Thus the connection ${}^H\nabla$ has non-trivial torsion even for Levi-Civita connection $\nabla = \nabla_g$ determined by g, unless g is locally flat.

Since $\nabla g = \nabla G = 0$, by virtue of (1.5)–(1.7) and (5.8), we have

$$\begin{aligned} &({}^H\nabla_{{}^V\omega}({}^{\nabla}g + {}^{V}G^{\bullet\bullet}))({}^V\theta, {}^V\varepsilon) = 0, \\ &({}^H\nabla_{{}^HX}({}^{\nabla}g + {}^{V}G^{\bullet\bullet}))({}^V\theta, {}^V\varepsilon) \\ = &-({}^{\nabla}g + {}^{V}G^{\bullet\bullet})({}^V(\nabla_X\theta), {}^V\varepsilon) - ({}^{\nabla}g + {}^{V}G^{\bullet\bullet})({}^V\theta, {}^V(\nabla_X\varepsilon)) = 0, \\ &({}^H\nabla_{{}^V\omega}({}^{\nabla}g + {}^{V}G^{\bullet\bullet}))({}^V\theta, {}^HZ) = {}^V\omega\ {}^V(\theta(Z)) = 0, \\ &({}^H\nabla_{{}^HX}({}^{\nabla}g + {}^{V}G^{\bullet\bullet}))({}^V\theta, {}^HZ) \end{aligned}$$

$$
\begin{aligned}
&= {}^HX\ {}^V(\theta(Z)) - ({}^\nabla g + {}^VG^{\bullet\bullet})({}^V(\nabla_X\theta), {}^HZ) - ({}^\nabla g + {}^VG^{\bullet\bullet})({}^V\theta, {}^H(\nabla_XZ)) \\
&= {}^V(\nabla_X\theta(Z) - (\nabla_X\theta)Z - \theta\nabla_XZ) = 0, \\
&\quad ({}^H\nabla_{{}^V\omega}({}^\nabla g + {}^VG^{\bullet\bullet}))({}^HY, {}^V\varepsilon) = {}^V\omega\ {}^V(\varepsilon(Y)) = 0, \\
&\quad ({}^H\nabla_{{}^HX}({}^\nabla g + {}^VG^{\bullet\bullet}))({}^HY, {}^V\varepsilon) \\
&= {}^HX\ {}^V(\varepsilon(Y)) - ({}^\nabla g + {}^VG^{\bullet\bullet})({}^V(\nabla_XY), {}^V\varepsilon) - ({}^\nabla g + {}^VG^{\bullet\bullet})({}^HY, {}^V(\nabla_X\varepsilon)) \\
&= {}^V(\nabla_X\varepsilon(Y) - \varepsilon(\nabla_XY) - (\nabla_X\varepsilon)Y) = 0, \\
&\quad ({}^H\nabla_{{}^V\omega}({}^\nabla g + {}^VG^{\bullet\bullet}))({}^HY, {}^HZ) = {}^V\omega {}^V(G(Y,Z)) = 0, \\
&\quad ({}^H\nabla_{{}^HX}({}^\nabla g + {}^VG^{\bullet\bullet}))({}^HY, {}^HZ) = {}^V((\nabla_XG)(Y,Z)) = 0
\end{aligned}
$$

for any $X, Y, Z \in \Im_0^1(M_n)$ and $\omega, \theta, \varepsilon \in \Im_1^0(M_n)$, i.e., the horizontal lift ${}^H(\nabla_g)$ is a metric connection of ${}^\nabla g + {}^VG^{\bullet\bullet}$. Thus we have the following theorem.

Theorem 5.1 *Let (M_n, J, g) be a Kähler-Norden manifold, and let ∇_g be the Levi-Civita connection of g. Then the horizontal lift ${}^H(\nabla_g)$ is a metric connection of the deformed Riemannian extension ${}^\nabla g + {}^VG^{\bullet\bullet}$.*

Let now ${}^HR$ be a curvature tensor field of ${}^H(\nabla_g)$. The curvature tensor ${}^HR$ of the metric connection ${}^H(\nabla_g)$ has components

$$
{}^H\widetilde{R}_{\delta\gamma\beta}^{\alpha} = \widetilde{e}_{(\delta)}^H\widetilde{\Gamma}_{\gamma\beta}^{\alpha} - \widetilde{e}_{(\gamma)}^H\widetilde{\Gamma}_{\delta\beta}^{\alpha} + {}^H\widetilde{\Gamma}_{\delta\varepsilon}^{\alpha}\,{}^H\widetilde{\Gamma}_{\gamma\beta}^{\varepsilon} - {}^H\widetilde{\Gamma}_{\gamma\varepsilon}^{\alpha}\,{}^H\widetilde{\Gamma}_{\delta\beta}^{\varepsilon} - \Omega_{\delta\gamma}^{\varepsilon}\,{}^H\widetilde{\Gamma}_{\varepsilon\beta}^{\alpha} \tag{5.14}
$$

with respect to the adapted frame. Using (5.3)–(5.4), (5.9), (5.13)–(5.14) and computing components of the contracted curvature tensor field (the Ricci tensor field) ${}^H\widetilde{R}_{\gamma\beta} = {}^H\widetilde{R}_{\alpha\gamma\beta}^{\alpha}$, we obtain

$$
\begin{cases}
{}^H\widetilde{R}_{kj} = {}^H\widetilde{R}_{\alpha kj}^{\alpha} = {}^H\widetilde{R}_{ikj}^{i} + {}^H\widetilde{R}_{\bar{i}kj}^{\bar{i}} = R_{ikj}^{i} = R_{kj}, \\
{}^H\widetilde{R}_{\bar{k}\,j} = 0, \quad {}^H\widetilde{R}_{k\,\bar{j}} = 0, \quad {}^H\widetilde{R}_{\bar{k}\,\bar{j}} = 0,
\end{cases} \tag{5.15}
$$

where R_{kj} is the Ricci tensor field of ∇_g in M_n.

By virtue of (5.15), for the scalar curvature $\widetilde{r}$ of ${}^CT(M_n)$ with the metric connection ${}^H(\nabla_g)$, we have

$$
\widetilde{r} = ({}^\nabla g + {}^VG^{\bullet\bullet})^{\gamma\beta}\,{}^H\widetilde{R}_{\gamma\beta} = 0,
$$

where

$$
(({}^\nabla g + {}^VG^{\bullet\bullet})^{\gamma\beta}) = \begin{pmatrix} 0 & \delta_j^k \\ \delta_k^j & -G_{jk} \end{pmatrix}.
$$

Thus we have the following theorem.

Theorem 5.2 *Let (M_n, J, g) be a Kähler-Norden manifold. Then the cotangent bundle ${}^CT(M_n)$ with the metric connection ${}^H(\nabla_g)$ has vanishing scalar curvature with respect to the deformed Riemannian extension ${}^\nabla g + {}^VG^{\bullet\bullet}$.*

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