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Almost Linear Nash Groups*

Binyong SUN[1]

Abstract A Nash group is said to be almost linear if it has a Nash representation with a finite kernel. Structures and basic properties of these groups are studied.

Keywords Nash manifold, Nash group, Nash representation, Jordan decomposition, Levi decomposition, Cartan decomposition, Iwasawa decomposition
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1 Introduction

Basic notions and properties concerning Nash manifolds are reviewed in Section 2. In this introduction, we introduce some basic notions concerning Nash groups. See Section 3 for more details.

A Nash group is a group which is simultaneously a Nash manifold so that all group operations are Nash maps. A Nash homomorphism is a group homomorphism between two Nash groups which is simultaneously a Nash map. If a Nash homomorphism is bijective, then its inverse is also a Nash homomorphism. In this case, we say that the Nash homomorphism is a Nash isomorphism. Two Nash groups are said to be Nash isomorphic to each other if there exists a Nash isomorphism between them.

Given a subgroup of a Nash group G, if it is semialgebraic, then it is automatically a closed Nash submanifold of G (see Proposition 3.1). In this case, we call it a Nash subgroup of G. A Nash subgroup is canonically a Nash group.

As usual, all finite-dimensional real representations of Lie groups are assumed to be continuous. A Nash representation is a finite-dimensional real representation of a Nash group such that the action map is a Nash map.

Definition 1.1 *A Nash group is said to be almost linear if it has a Nash representation with a finite kernel.*

Almost linear Nash groups form a nice class of mathematical objects. Their structures are simpler than those of general Lie groups, and in the study of infinite-dimensional representation theory, they are more flexible than linear algebraic groups. Although there is a vast literature on Lie groups and linear algebraic groups, it seems that almost linear Nash groups have not been systematically studied (see [17] for a brief introduction to Nash groups). The goal of this

Manuscript received October 30, 2013.
[1]Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, China.
E-mail: sun@math.ac.cn
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article is to provide a detailed study of structures of almost linear Nash groups, for possible later reference. The structure theory of almost linear Nash groups is similar to that of linear algebraic groups. However, we try to avoid the language of algebraic geometry to keep the article as elementary as possible. In what follows, we summarize some basic results about almost linear Nash groups which are either well-known or will be proved in this article.

It is clear that a Nash subgroup of an almost linear Nash group is an almost linear Nash group. The product of two almost linear Nash groups is an almost linear Nash group. By the following proposition, the quotient of an almost linear Nash group by a Nash subgroup is canonically an affine Nash manifold, and the quotient by a normal Nash subgroup is canonically an almost linear Nash group.

Proposition 1.1 *Let G be an almost linear Nash group, and H be a Nash subgroup of it. Then there exists a unique Nash structure on the quotient topological space G/H which makes the quotient map $G \to G/H$ a submersive Nash map. With this Nash structure, G/H becomes an affine Nash manifold, and the left translation map $G\times G/H \to G/H$ is a Nash map. Furthermore, if H is a normal Nash subgroup of G, then the topological group G/H becomes an almost linear Nash group under this Nash structure.*

For each normal Nash subgroup H of an almost linear Nash group G, the Nash group G/H is called a Nash quotient group of G.

There are three classes of almost linear Nash groups which are basic to the general structure theory, namely, elliptic Nash groups, hyperbolic Nash groups and unipotent Nash groups.

Definition 1.2 *A Nash group is said to be elliptic if it is almost linear and compact. It is said to be hyperbolic if it is Nash isomorphic to $(\mathbb{R}_+^\times)^n$ for some $n \geq 0$. It is said to be unipotent if it has a faithful Nash representation such that all group elements act as unipotent linear operators.*

Here and as usual, $\mathbb{R}_+^\times$ denotes the set of positive real numbers. It is a Nash group in the obvious way. Recall that a linear operator x on a finite-dimensional vector space is said to be unipotent if $x-1$ is nilpotent.

There is no need to say that a Nash group is almost linear if it is elliptic, hyperbolic or unipotent.

Definition 1.3 *An element of an almost linear Nash group G is said to be elliptic, hyperbolic, or unipotent if it is contained in a Nash subgroup of G which is elliptic, hyperbolic, or unipotent, respectively.*

Definitions 1.2–1.3 are related as follows.

Proposition 1.2 *An almost linear Nash group is elliptic, hyperbolic, or unipotent if and only if all of its elements are elliptic, hyperbolic, or unipotent, respectively.*

In general, we have the following proposition.

Proposition 1.3 *Let G be an almost linear Nash group. If G is elliptic, hyperbolic or unipotent, then all Nash subgroups and all Nash quotient groups of G are elliptic, hyperbolic or unipotent, respectively. If G has a normal Nash subgroup H so that H and G/H are both elliptic,* *both hyperbolic or both unipotent, then G is elliptic, hyperbolic or unipotent, respectively.*

Concerning elliptic Nash groups, we have the following theorem.

Theorem 1.1 *The followings hold true:*

(1) *Every compact Lie group has a unique Nash structure on its underlying topological space which makes it an almost linear Nash group.*

(2) *Every continuous homomorphism from an elliptic Nash group to an almost linear Nash group is a Nash homomorphism.*

(3) *Every compact subgroup of an almost linear Nash group is a Nash subgroup.*

Theorem 1.1 implies that the category of elliptic Nash groups is isomorphic to the category of compact Lie groups.

Recall that a subgroup of a Lie group G is said to be analytic if it is equal to the image of an injective Lie group homomorphism from a connected Lie group to G. Every analytic subgroup is canonically a Lie group (this is implied by [18, Theorem 1.62]).

For unipotent Nash groups, we have the following theorem.

Theorem 1.2 *The followings hold true:*

(1) *As a Lie group, every unipotent Nash group is connected, simply connected and nilpotent.*

(2) *Every connected, simply connected, nilpotent Lie group has a unique Nash structure on its underlying topological space which makes it a unipotent Nash group.*

(3) *Every continuous homomorphism between two unipotent Nash groups is a Nash homomorphism.*

(4) *Every analytic subgroup of a unipotent Nash group is a Nash subgroup.*

Theorem 1.2 implies that the category of unipotent Nash groups is isomorphic to the category of connected, simply connected, nilpotent Lie groups. Recall that the later category is equivalent to the category of finite-dimensional nilpotent real Lie algebras.

For every $r \in \mathbb{Q}$, the map

$$\mathbb{R}_+^\times \to \mathbb{R}_+^\times, \quad x \mapsto x^r$$

is a Nash homomorphism from $\mathbb{R}_+^\times$ to itself. Conversely, all Nash homomorphisms from $\mathbb{R}_+^\times$ to itself are of this form. We view the abelian group $\mathbb{R}_+^\times$ as a right $\mathbb{Q}$-vector space so that

$$\text{the scalar multiplication } x \cdot r := x^r$$

for all $x \in \mathbb{R}_+^\times$ and $r \in \mathbb{Q}$. Note that for every finite-dimensional left $\mathbb{Q}$-vector space E, $\mathbb{R}_+^\times \otimes_{\mathbb{Q}} E$ is obviously a hyperbolic Nash group. Moreover, we have the following theorem.

Theorem 1.3 *The functor*

$$A \mapsto \mathrm{Hom}(\mathbb{R}_+^\times, A)$$

establishes an equivalence from the category of hyperbolic Nash groups to the category of finite-dimensional left $\mathbb{Q}$-vector spaces. It has a quasi-inverse

$$E \mapsto \mathbb{R}_+^\times \otimes_{\mathbb{Q}} E.$$

Here and henceforth, for any two Nash groups G_1 and G_2, $\mathrm{Hom}(G_1, G_2)$ denotes the set of all Nash homomorphisms from G_1 to G_2. It is obviously an abelian group when G_2 is abelian. The abelian group $\mathrm{Hom}(\mathbb{R}_+^\times, A)$ of Theorem 1.3 is a left $\mathbb{Q}$-vector space as follows:

$$(r \cdot \phi)(x) := \phi(x^r), \quad r \in \mathbb{Q},\, \phi \in \mathrm{Hom}(\mathbb{R}_+^\times, A),\, x \in \mathbb{R}_+^\times.$$

In a way similar to Jordan decompositions for linear algebraic groups, we have the following theorem.

Theorem 1.4 *Every element x of an almost linear Nash group G is uniquely of the form $x = ehu$ such that $e \in G$ is elliptic, $h \in G$ is hyperbolic, $u \in G$ is unipotent, and they pairwise commute with each other.*

We call the equality $x = ehu$ of Theorem 1.4 the Jordan decomposition of x. In Section 8, Jordan decompositions at the Lie algebra level are also discussed.

Besides elliptic Nash groups, hyperbolic Nash groups and unipotent Nash groups, there are two other classes of Nash groups which are important to the general structure theory, namely, reductive Nash groups and exponential Nash groups.

Definition 1.4 *A Nash group is said to be reductive if it has a completely reducible Nash representation with a finite kernel. It is said to be exponential if it is almost linear and has no non-trivial elliptic element.*

Here and as usual, a representation is said to be completely reducible if it is a direct sum of irreducible subrepresentations, or equivalently, if each subrepresentation of it has a complementary subrepresentation.

A general reductive Nash group is more or less the direct product of two reductive Nash groups of the particular type, namely, a semisimple Nash group and a Nash torus.

Definition 1.5 *A Nash group or a Lie group is said to be semisimple if its Lie algebra is semisimple. A Nash torus is a Nash group which is Nash isomorphic to $\mathbb{S}^m \times (\mathbb{R}_+^\times)^n$ for some $m, n \geq 0$.*

Here $\mathbb{S}$ denotes the Nash group of complex numbers of modulus one.

Concerning semisimple Nash groups, we have the following theorem.

Theorem 1.5 *The followings hold true:*

(1) *Every semisimple Nash group is almost linear.*

(2) *Every semisimple Nash group has finitely many connected components, and its identity-connected component has a finite center.*

(3) *Let G be a semisimple Lie group which has finitely many connected components, and whose identity connected component has a finite center. Then there exists a unique Nash structure on the underlying topological space of G which makes G a Nash group.*

(4) *Every continuous homomorphism from a semisimple Nash group to an almost linear Nash group is a Nash homomorphism.*

(5) *Every semisimple analytic subgroup of an almost linear Nash group is a Nash subgroup.*

Theorem 1.5 implies that the category of semisimple Nash groups is isomorphic to the category of semisimple Lie groups which have finitely many connected components, and whose identity connected components have finite centers.

For every almost linear Nash group G, define its unipotent radical to be

$$\mathfrak{U}_G := \text{the identity connected component of } \bigcap_{\pi} \ker \pi,$$

where π runs through all irreducible Nash representations of G. This is the largest normal unipotent Nash subgroup of G (see Proposition 14.1).

We have the following theorem concerning reductive Nash groups.

Theorem 1.6 *The followings are equivalent for an almost linear Nash group G:*

(a) *It is reductive.*

(b) *All Nash representations of G are completely reducible.*

(c) *The unipotent radical of G is trivial.*

(d) *For some Nash representations of G with a finite kernel, the attached trace form on the Lie algebra of G is non-degenerate.*

(e) *For every Nash representation of G with a finite kernel, the attached trace form on the Lie algebra of G is non-degenerate.*

(f) *The identity connected component of G is reductive.*

(g) *There exists a connected semisimple Nash group H, a Nash torus T, and a Nash homomorphism $H \times T \to G$ with a finite kernel and open image.*

Here, for every Nash representation V of a Nash group G, the attached trace form $\langle\, ,\, \rangle_\phi$ on the Lie algebra $\mathfrak{g}$ of G is defined by

$$\langle x, y \rangle_\phi := \operatorname{tr}(\phi(x)\phi(y)), \quad x, y \in \mathfrak{g},$$

where $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ denotes the differential of the representation V of G. Here and as usual, $\mathfrak{gl}(V)$ denotes the algebra of all linear endomorphisms of V; and as quite often, when no confusion is possible, we do not distinguish a representation with its underlying vector space.

Denote by $\mathrm{B}_n(\mathbb{R})$ the Nash subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting all upper-triangular matrices with positive diagonal entries ($n \geq 0$). It is obviously an exponential Nash group.

Theorem 1.7 *The followings are equivalent for an almost linear Nash group G:*

(a) *It is exponential.*

(b) *It has no non-trivial compact subgroup.*

(c) *It has no proper co-compact Nash subgroup.*

(d) *The quotient $G/\mathfrak{U}_G$ is a hyperbolic Nash group.*

(e) *It is Nash isomorphic to a Nash subgroup of $\mathrm{B}_n(\mathbb{R})$ for some $n \geq 0$.*

(f) *The exponential map from the Lie algebra of G to G is a diffeomorphism.*

(g) *Every Nash action of G on every non-empty compact Nash manifold has a fixed-point.*

Here a Nash action means an action of a Nash group on a Nash manifold such that the action map is Nash.

The following theorem makes the structure theory of almost linear Nash groups extremely pleasant.

Theorem 1.8 *Let G be an almost linear Nash group. Then every elliptic* (*hyperbolic, unipotent, reductive or exponential*) *Nash subgroup of G is contained in a maximal one, and all maximal elliptic* (*hyperbolic, unipotent, reductive or exponential*) *Nash subgroups of G are conjugate to each other in G.*

A maximal reductive Nash subgroup of an almost linear Nash group G is called a Levi component of G.

Theorem 1.9 *Let L be a Levi component of an almost linear Nash group G. Then $G = L \ltimes \mathfrak{U}_G$.*

The equality $G = L \ltimes \mathfrak{U}_G$ of Theorem 1.9 is called a Levi decomposition of G.

Theorem 1.10 *Let G be an almost linear Nash group, K be a maximal elliptic Nash subgroup of G, and B be a maximal exponential Nash subgroup of G. Then the multiplication map $K \times B \to G$ is a Nash diffeomorphism.*

Let G, K and B be as in Theorem 1.10. Let A be a Levi component of B, which is a hyperbolic Nash group. Denote by N the unipotent radical of B. Then by Theorems 1.9–1.10, we have $G = KAN$. This is called an Iwasawa decomposition of G.

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2 Nash Manifolds

We begin with a review of basic concepts and properties of Nash manifolds which are necessary for this article (see [2, 16] for more details). Recall that a subset of $\mathbb{R}^n$ $(n \geq 0)$ is said to be semialgebraic if it is a finite union of the sets of the form

$$\{x \in \mathbb{R}^n \mid f_1(x) > 0,\ f_2(x) > 0, \cdots, f_r(x) > 0,\ g_1(x) = g_2(x) = \cdots = g_s(x) = 0\},$$

where $r, s \geq 0$, $f_1, f_2, \cdots, f_r$ and $g_1, g_2, \cdots, g_s$ are real polynomial functions on $\mathbb{R}^n$. For $n = -\infty$, we define $\mathbb{R}^n$ to be the empty set, and its only subset is defined to be semialgebraic. It is clear that the collection of semialgebraic sets in $\mathbb{R}^n$ ($n \geq 0$ or $n = -\infty$) is closed under taking finite union, finite intersection, and complement.

A map $\varphi : X \to X'$ from a semialgebraic set $X \subset \mathbb{R}^n$ to a semialgebraic set $X' \subset \mathbb{R}^m$ ($m \geq 0$ or $m = -\infty$) is said to be semialgebraic if its graph is semialgebraic in $\mathbb{R}^{n+m}$. Tarski-Seidenberg theorem asserts that the image of a semialgebraic set under a semialgebraic map is semialgebraic: If $\varphi : X \to X'$ is semialgebraic, then $\varphi(X_0)$ is semialgebraic for each semialgebraic set $X_0 \subset X$. As an easy consequence of Tarski-Seidenberg theorem, we know that the composition of two semialgebraic maps is also semialgebraic; and the inverse image of a semialgebraic set under a semialgebraic map is semialgebraic.

Definition 2.1 *A Nash structure on a topological space M is an element $n \in \{-\infty, 0, 1, 2, \cdots\}$ together with a set $\mathcal{N}$ with the following properties:*

(a) *The set $\mathcal{N}$ is contained in $\mathrm{N}(\mathbb{R}^n, M)$, where $\mathrm{N}(\mathbb{R}^n, M)$ denotes the set of all triples (ϕ, U, U') such that U is an open semialgebraic subset of $\mathbb{R}^n$, U' is an open subset of M, and $\phi : U \to U'$ is a homeomorphism.*

(b) *Every two elements* (ϕ_1, U_1, U_1') *and* (ϕ_2, U_2, U_2') *of* $\mathcal{N}$ *are Nash compatible, namely, the homeomorphism*

$$\phi_2^{-1} \circ \phi_1 : \phi_1^{-1}(U_1' \cap U_2') \to \phi_2^{-1}(U_1' \cap U_2')$$

has a semialgebraic domain and a codomain, and is semialgebraic and smooth.

(c) *There are finitely many elements* (ϕ_i, U_i, U_i') *of* $\mathcal{N}$, $i = 1, 2, \cdots, r$ $(r \geq 0)$, *such that*

$$M = U_1' \cup U_2' \cup \cdots \cup U_r'.$$

(d) *For every element of* $\mathrm{N}(\mathbb{R}^n, M)$, *if it is Nash compatible with all elements of* $\mathcal{N}$, *then it is an element of* $\mathcal{N}$.

(e) *If* M *is empty, then* $n = -\infty$.

The following lemma is routine to check.

Lemma 2.1 *With the notation as in Definition* 2.1, *let*

$$\mathcal{N}_0 = \{(\phi_i, U_i, U_i') \mid i = 1, 2, \cdots, r\}$$

be a finite subset of $\mathrm{N}(\mathbb{R}^n, M)$, *whose elements are pairwise Nash compatible with each other. If* M *is non-empty and*

$$M = U_1' \cup U_2' \cup \cdots \cup U_r',$$

then together with n, *the set*

$$\{(\phi, U, U') \in \mathrm{N}(\mathbb{R}^n, M) \mid (\phi, U, U') \text{ is Nash compatible with all elements of } \mathcal{N}_0\}$$

is a Nash structure on M.

A Nash manifold is defined to be a Hausdorff topological space together with a Nash structure on it. The element n in Definition 2.1 of the Nash structure is called the dimension of the Nash manifold, and an element of $\mathcal{N}$ in Definition 2.1 of the Nash structure is called a Nash chart of the Nash manifold.

Definition 2.2 *A continuous map* $\varphi : M \to N$ *between Nash manifolds is called a Nash map if for all Nash charts* (ϕ, U, U') *of* M *and* (ψ, V, V') *of* N, *the set* $\phi^{-1}(U' \cap \varphi^{-1}(V'))$ *is semialgebraic, and the map*

$$\psi^{-1} \circ \varphi \circ \phi : \phi^{-1}(U' \cap \varphi^{-1}(V')) \to V$$

is semialgebraic and smooth.

It is clear that every Nash manifold is a smooth manifold, and every Nash map is a smooth map. The composition of two Nash maps is certainly a Nash map.

Definition 2.3 *A subset* X *of a Nash manifold* M *is said to be semialgebraic if* $\phi^{-1}(X \cap U')$ *is semialgebraic for every Nash chart* (ϕ, U, U') *of* M.

As in the case of $\mathbb{R}^n$, the collection of semialgebraic sets in a Nash manifold is closed under taking finite union, finite intersection, and complement. Tarski-Seidenberg theorem easily implies the following lemma.

Lemma 2.2 *Let $\varphi : M \to N$ be a Nash map of Nash manifolds. Then for each semialgebraic subset X of M, the image $\varphi(X)$ is a semialgebraic subset of N; and for each semialgebraic subset Y of N, the inverse image $\varphi^{-1}(Y)$ is a semialgebraic subset of M.*

The following proposition is a useful criterion for a continuous map to be a Nash map.

Proposition 2.1 *Let $\varphi : M \to N$ be a continuous map of Nash manifolds. Then φ is a Nash map if and only if*

(1) *for each semialgebraic open subset Y of N, the inverse image $\varphi^{-1}(Y)$ is semialgebraic in M,*

(2) *for every $x \in M$, there are Nash charts (ϕ, U, U') of M and (ψ, V, V') of N such that $x \in U'$, $\varphi(U') \subset V'$, and the map*

$$\psi^{-1} \circ \varphi \circ \phi : U \to V$$

is semialgebraic and smooth.

Proof This is an easy consequence of [2, Proposition 8.1.8].

The following lemma will be used for several times.

Lemma 2.3 *Let $\varphi : M \to M'$ be a surjective submersive Nash map of Nash manifolds. Let N be a Nash manifold and let $\psi : M' \to N$ be a map. Then ψ is a Nash map if and only if $\psi \circ \varphi$ is a Nash map.*

Proof The "only if" part of the lemma is obvious. Using Proposition 2.1 and Lemma 2.2, the "if" part holds because the map φ has local Nash sections.

Given a Nash map, if it is a diffeomorphism as a map of smooth manifolds, then its inverse is also a Nash map. In this case, we call the Nash map a Nash diffeomorphism. Two Nash manifolds are said to be Nash diffeomorphic to each other if there exists a Nash diffeomorphism between them.

Definition 2.4 *A semialgebraic locally closed submanifold of a Nash manifold M is called a Nash submanifold of M.*

In this article, all locally closed submanifolds of a smooth manifold are assumed to be equidimensional. By the following proposition, every Nash submanifold is automatically a Nash manifold.

Proposition 2.2 *Let X be a Nash submanifold of a Nash manifold M. Then there exists a unique Nash structure on the topological space X which makes the inclusion $X \hookrightarrow M$ an immersive Nash map.*

We say that a Nash map $\varphi : M \to N$ is a Nash embedding if $\varphi(M)$ is a Nash submanifold of N, and the induced map $\varphi : M \to \varphi(M)$ is a Nash diffeomorphism.

By the following proposition, the product of two Nash manifolds is again a Nash manifold.

Proposition 2.3 *Let M and N be two Nash manifolds. Then there exists a unique Nash structure on the topological space $M \times N$ which makes the projections $M \times N \to M$ and $M \times N \to N$ submersive Nash maps.*

Both Propositions 2.2–2.3 are standard. We shall not go to their proofs.

The following lemma is obvious.

Lemma 2.4 *Let $\varphi : M \to N$ be a smooth map of Nash manifolds. Then φ is a Nash map if and only if its graph is semialgebraic in $M \times N$.*

Lemmas 2.2 and 2.4, and the following basic result will be used without further explicit mention.

Lemma 2.5 (see [7, Theorem 2.23]) *Every semialgebraic subset of a Nash manifold has only finitely many connected components and each of them is semialgebraic.*

Recall the following lemma.

Lemma 2.6 (see [16, Remark I.5.12]) *Every Nash manifold of dimension n ($n \geq 0$) is covered by finitely many open Nash submanifolds which are Nash diffeomorphic to $\mathbb{R}^n$.*

Using Lemmas 2.5–2.6, it is easy to prove the following lemma.

Lemma 2.7 *Let $\varphi : M \to M'$ be a submersive Nash map of Nash manifolds. Assume that φ is a finite-fold covering map as a map of topological spaces. Let N be a Nash manifold, and $\psi : N \to M$ be a continuous map. Then ψ is a Nash map if and only if $\varphi \circ \psi$ is a Nash map.*

By the following proposition, a finite-fold cover of a Nash manifold is a Nash manifold.

Proposition 2.4 *Let N be a Nash manifold, M be a topological space, and $\varphi : M \to N$ be a finite-fold covering map of topological spaces. Then there exists a unique Nash structure on M which makes φ a submersive Nash map.*

Proof This is known to experts. We sketch a proof for the lack of reference. First note that M is Hausdorff, since N is Hausdorff. Since this proposition is trivial when M is an empty set, we assume that M is non-empty. The uniqueness assertion of this proposition is a direct consequence of Lemma 2.7. In what follows, we construct a Nash structure on M which makes φ a submersive Nash map.

Write $(n, \mathcal{N}_N)$ for the Nash structure on N. Put

$$\mathcal{N}'_M := \bigcup_{\substack{(\phi,U,U')\in\mathcal{N}_N, \\ \text{with } U \text{ connected and simply connected}}} \mathcal{N}_\phi,$$

where

$$\mathcal{N}_\phi := \{(\psi, U, U'') \in \mathrm{N}(\mathbb{R}^n, M) \mid \psi \text{ lifts the homeomorphism } \phi : U \to U'\}.$$

One checks that all elements in $\mathcal{N}'_M$ are pairwise Nash compatible. Lemma 2.6 implies that the set $\mathcal{N}'_M$ has property (c) of Definition 2.1. Denote by $\mathcal{N}_M$ the set of all elements in $\mathrm{N}(\mathbb{R}^n, M)$ which are Nash compatible with all elements of $\mathcal{N}'_M$. Lemma 2.1 implies that $(n, \mathcal{N}_M)$ is a Nash structure on M. With this Nash structure, φ is clearly a submersive Nash map.

Every finite-dimensional real vector space is obviously a Nash manifold. A Nash manifold is said to be affine if it is Nash diffeomorphic to a Nash submanifold of some finite-dimensional real vector spaces. It is known that every affine Nash manifold is actually Nash diffeomorphic to a closed Nash submanifold of some finite-dimensional real vector spaces (see [17, Section 2.22]).

It is clear that a Nash submanifold of an affine Nash manifold is an affine Nash manifold; the product of two affine Nash manifolds is an affine Nash manifold. The following criterion implies that a finite-fold cover of an affine Nash manifold is an affine Nash manifold.

Proposition 2.5 (see [16, Proposition III.1.7]) *Let M be a Nash manifold of dimension $n \geq 0$. Then M is affine if and only if for every $x \in M$, there is a Nash map $M \to \mathbb{R}^n$ which is submersive at x.*

Projective spaces form an important family of affine Nash manifolds: For each finite-dimensional real vector space V, the set $\mathrm{P}(V)$ of all one-dimensional subspaces of V is naturally an affine Nash manifold (see [2, Theorem 3.4.4]).

For each semialgebraic subset X of a Nash manifold M, define its dimension

$$\dim X := \max \left\{ d \in \{-\infty, 0, 1, 2, \cdots\} \,\middle|\, \begin{array}{l} X \text{ contains a Nash submanifold} \\ \text{of } M \text{ of dimension } d \end{array} \right\}.$$

The following properties of the dimensions of semialgebraic sets are obvious (see [7, p. 56]): The dimension of the union of finitely many semialgebraic sets is the maximum of the dimensions of these semialgebraic sets, and the dimension of a finite product of semialgebraic sets is the sum of their dimensions. The following basic facts concerning dimensions of semialgebraic sets are well-known.

Proposition 2.6 (see [7, Proposition 3.16 and Theorem 3.20]) *The followings hold true:*

(1) *The closure $\overline{X}$ of a semialgebraic set X in a Nash manifold is semialgebraic. Moreover, $\dim \overline{X} = \dim X$; and $\dim \overline{X} \setminus X < \dim X$ whenever X is non-empty.*

(2) *Each semialgebraic subset of a finite-dimensional real vector space has the same dimension as its Zariski closure.*

Note that all Zariski closed subsets of a finite-dimensional real vector space are semialgebraic.

For a semialgebraic set X of a Nash manifold M, an element $x \in X$ is said to be smooth of dimension $d \geq 0$ if there is a semialgebraic open neighborhood U of x in M such that $X \cap U$ is a d-dimensional Nash submanifold of M. Note that X is a Nash submanifold of M if and only if all points of it are smooth of dimension $\dim X$.

Lemma 2.8 (see [1, Proposition 5.53]) *Let X be a non-empty semialgebraic subset of a Nash manifold M. Then X has a point which is smooth of dimension $\dim X$.*

3 Nash Groups and Almost Linear Nash Groups

In this section, we introduce some generalities on Nash groups and almost linear Nash groups.

Definition 3.1 *A Nash group is a Hausdorff topological group G, equipped with a Nash structure on its underlying topological space so that both the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are Nash maps between Nash manifolds. A Nash homomorphism between two Nash groups is a group homomorphism between them which is simultaneously a Nash map.*

The following basic result will be used freely without further explicit mention.

Proposition 3.1 *Every semialgebraic subgroup of a Nash group G is a closed Nash submanifold of G.*

Proof This is well-known. We sketch a proof for convenience of the reader. Let H be a semialgebraic subgroup of G. Lemma 2.8 implies that H is a Nash submanifold of G. In particular, H is locally closed, and thus is an open subgroup of its closure $\overline{H}$. Therefore, H is also closed in $\overline{H}$. This proves this proposition (recall that every closed subgroup of a Lie group is a submanifold).

In view of Proposition 3.1, a semialgebraic subgroup of a Nash group G is also called a Nash subgroup of G.

By Lemma 2.2, we have the following proposition.

Proposition 3.2 *The image of a Nash homomorphism $\varphi : G \to G'$ is a Nash subgroup of G'. In particular, it is closed in G'.*

It is clear that a Nash subgroup of a Nash group is a Nash group, and the product of two Nash groups is a Nash group. Proposition 2.4 implies that a finite-fold covering group of a Nash group is a Nash group.

Proposition 3.3 *Let G be a topological group, G' be a Nash group, and $G \to G'$ be a group homomorphism which is simultaneously a finite-fold topological covering map. Equip on G the Nash structure which makes $G \to G'$ a submersive Nash map. Then G becomes a Nash group.*

Proof Using Lemma 2.7, this is routine to check.

Note that there is no strictly decreasing infinite sequence of Nash subgroups of a Nash group. Consequently, we have the following proposition.

Proposition 3.4 *Let G be a Nash group, and $\{G_i\}_{i\in I}$ be a family of Nash subgroups of G. Then*

$$\bigcap_{i\in I} G_i = \bigcap_{i\in I_0} G_i$$

for some finite subset I_0 of I. Consequently, the intersection of an arbitrary family of Nash subgroups of G is again a Nash subgroup of G.

By a Nash action of a Nash group G on a Nash manifold M, we mean a group action $G \times M \to M$ which is simultaneously a Nash map. Using Lemma 2.8, we know that each G-orbit of a Nash action $G \times M \to M$ is a Nash submanifold of M.

The analog of the following proposition for algebraic groups is proved in [3, Chapter I, Proposition 1.8].

Proposition 3.5 *Let G be a Nash group with a Nash action on a non-empty Nash manifold M. Then each G-orbit in M of the minimal dimension is closed.*

Proof For each non-closed G-orbit O in M, there is an orbit O' in $\overline{O} \setminus O$, where $\overline{O}$ denotes the closure of O in M. Then $\dim O' < \dim O$ by the first assertion of Proposition 2.6. Therefore, O is not of the minimal dimension.

A finite-dimensional real representation V of a Nash group G is called a Nash representation if the action map $G\times V \to V$ is a Nash map. This is equivalent to saying that the corresponding homomorphism $G \to \mathrm{GL}(V)$ is a Nash homomorphism. Recall from the introduction that a Nash group is said to be almost linear if it has a Nash representation with a finite kernel.

In this article, we use a superscript " $\circ$ " to indicate the identity connected component of a Nash group.

Proposition 3.6 *A Nash group G is almost linear if and only if G° is so.*

Proof The "only if" part is trivial. Assume that G° is almost linear. Let V_0 be a Nash representation of G° with a finite kernel. Put

$$V := \mathrm{Ind}_{G^\circ}^{G} V_0 := \{f : G \to V_0 \mid f(g_0 g) = g_0.f(g),\, g_0 \in G^\circ,\, g \in G\}.$$

Under right translations, this is a Nash representation of G with a finite kernel.

To treat quotient spaces of almost linear Nash groups, recall the following proposition.

Proposition 3.7 *Let G be an almost linear Nash group, and H be a Nash subgroup of it.*

(1) *There exists a Nash representation V of G, and a one-dimensional subspace $V_1 \subset V$ such that the stabilizer*

$$\{g \in G \mid g.V_1 = V_1\}$$

contains H as an open subgroup.

(2) *If H is normal, then there exists a Nash representation of G whose kernel contains H as an open subgroup.*

Proof Using the second assertion of Proposition 2.6, this is an easy consequence of Chevalley's theorem (see [9, Theorem 11.1.13]).

Also recall the following well-known lemma.

Lemma 3.1 (see [18, Theorem 3.62]) *Let $G \times M \to M$ be a transitive smooth action of a Lie group G on a smooth manifold M. Then for each $x \in M$, the map*

$$G/G_x \to M, \quad g \mapsto g.x$$

is a diffeomorphism. Here $G_x := \{g \in G \mid g.x = x\}$, and the quotient topological space G/G_x is equipped with the manifold structure so that the quotient map $G \to G/G_x$ is smooth and submersive. Consequently, all surjective Lie group homomorphisms are submersive.

Here and as usual, all Lie groups and smooth manifolds are assumed to be Hausdorff and second countable as topological spaces.

Proposition 3.8 *Let G be an almost linear Nash group, and H be a Nash subgroup of it. Then there exists a unique Nash structure on the quotient topological space G/H which makes the quotient map $G \to G/H$ a submersive Nash map. With this Nash structure, G/H becomes an affine Nash manifold, and the left translation map $G\times G/H \to G/H$ is a Nash map. Furthermore, if H is a normal Nash subgroup of G, then the topological group G/H becomes an almost linear Nash group under this Nash structure.*

Proof Uniqueness of such Nash structures is implied by Lemma 2.3. Let V and V_1 be as in the first assertion of Proposition 3.7. The projective space $\mathrm{P}(V)$, which is naturally a Nash manifold, carries the induced Nash action of G. The image of the map

$$\varphi : G/H \to \mathrm{P}(V), \quad gH \mapsto g.V_1$$

is a G-orbit in $\mathrm{P}(V)$, and thus is a Nash submanifold of $\mathrm{P}(V)$. It is affine, since $\mathrm{P}(V)$ is an affine Nash manifold. Lemma 3.1 implies that the map

$$\varphi : G/H \to \varphi(G/H), \quad gH \mapsto g.V_1 \tag{3.1}$$

is a finite-fold topological covering map. Using Proposition 2.4, we equip on G/H the Nash structure which makes the map (3.1) a submersive Nash map. Then by Proposition 2.5, G/H is an affine Nash manifold, and Lemma 2.7 implies that the left translation map $G\times G/H \to G/H$ is a Nash map.

Now assume that H is normal. Using the second assertion of Proposition 3.7, we get a Nash homomorphism

$$\psi : G \to \mathrm{GL}_n(\mathbb{R}), \quad n \geq 0, \tag{3.2}$$

whose kernel contains H as an open subgroup. Equip on G/H the aforementioned Nash structure. Then by Lemmas 2.3 and 3.1, the map (3.2) descends to a submersive Nash map

$$G/H \to \psi(G). \tag{3.3}$$

Since (3.3) is a group homomorphism as well as a finite-fold covering map of topological spaces, Proposition 3.3 implies that G/H is a Nash group, which is obviously almost linear.

4 Elliptic Nash Groups

We first observe that every compact subgroup of an almost linear Nash group is a Nash subgroup.

Lemma 4.1 *Let G be an almost linear Nash group, and let K be a compact subgroup of it. Then K is a Nash subgroup of G.*

Proof Fix a Nash homomorphism $\varphi : G \to \mathrm{GL}_n(\mathbb{R})$ with a finite kernel. Write $K' := \varphi(K)$, which is a compact subgroup of $\mathrm{GL}_n(\mathbb{R})$. It is well-known that K' is semialgebraic in $\mathrm{GL}_n(\mathbb{R})$ (it is actually Zariski closed in $\mathrm{GL}_n(\mathbb{R})$ (see [6, Lemma 3.3.1])). Note that $\varphi^{-1}(K')$ is a Nash subgroup of G, and has the same dimension as that of K. Therefore, K is an open subgroup of $\varphi^{-1}(K')$, and thus is semialgebraic in G.

Recall from Section 1 that an elliptic Nash group is defined to be an almost linear Nash group which is compact as a topological space.

Lemma 4.2 *Let G be an almost linear Nash group, and K be an elliptic Nash group. Then every Lie group homomorphism $\varphi : K \to G$ is a Nash homomorphism. In particular, every finite-dimensional real representation of K is a Nash representation.*

Proof The graph of φ is a compact subgroup of the almost linear Nash group $K \times G$. Therefore, it is semialgebraic by Lemma 4.1.

Lemma 4.3 *Let K be a compact Lie group. Then there is a unique Nash structure on the underlying topological space of K which makes K an almost linear Nash group.*

Proof Uniqueness follows from Lemma 4.2. To prove the existence, fix an injective Lie group homomorphism $\varphi : K \hookrightarrow \mathrm{GL}_n(\mathbb{R})$ (such a homomorphism always exists, see [6, Section 3.3.C]). By Lemma 4.1, $\varphi(K)$ is a Nash group. The existence then follows by transferring the Nash structure on $\varphi(K)$ to K, through the topological group isomorphism $\varphi : K \xrightarrow{\sim} \varphi(K)$.

Combining Lemmas 4.1–4.3, we get Theorem 1.1. Moreover, we have proved the following theorem.

Theorem 4.1 *The category of elliptic Nash groups is isomorphic to the category of compact Lie groups.*

The following proposition is obvious.

Proposition 4.1 *All Nash subgroups and Nash quotient groups of all elliptic Nash groups are elliptic as Nash groups.*

Recall that a linear operator x on a finite-dimensional vector space V is said to be semisimple if every x-stable subspace of V has a complementary x-stable subspace. If V is defined over a field k of characteristic zero, then for each field extension k' of k, x is semisimple if and only if the k'-linear operator

$$\mathrm{k}' \otimes_{\mathrm{k}} V \to \mathrm{k}' \otimes_{\mathrm{k}} V, \quad a \otimes v \mapsto a \otimes x(v)$$

is semisimple. If V is defined over an algebraically closed field, then x is semisimple if and only if it is diagonalizable.

The following result concerning representations of compact Lie groups is well-known. We provide a proof for completeness.

Proposition 4.2 *Let V be a Nash representation of an elliptic Nash group G. Then each element of G acts as a semisimple linear operator on V, and all its eigenvalues are complex numbers of modulus* 1.

Proof Since every element of G is contained in a compact abelian subgroup of G, we assume without loss of generality that G is abelian. Then the complexification $V_{\mathbb{C}}$ of V is a direct sum of one-dimensional subrepresentations. By choosing an appropriate basis of $V_{\mathbb{C}}$, the representation corresponds to a Nash homomorphism

$$G \to (\mathbb{C}^\times)^n, \quad \text{where } n := \dim V. \tag{4.1}$$

Compactness of G implies that the image of (4.1) is contained in $\mathbb{S}^n$ (recall from the Introduction that $\mathbb{S}$ denotes the Nash group of complex numbers of modulus one). This proves this proposition.

The following important result is due to Cartan, Malcev and Iwasawa. For a proof, see [4, Theorem 1.2] for example.

Theorem 4.2 *Let G be a Lie group with finitely many connected components. Then every compact subgroup of G is contained in a maximal one, and all maximal compact subgroups of G are conjugate to each other. Moreover, for each maximal compact subgroup K of G, there exists a closed submanifold X of G which is diffeomorhpic to $\mathbb{R}^{\dim G - \dim K}$ such that the multiplication map $K \times X \to G$ is a diffeomorphism.*

5 Unipotent Nash Groups

We say that a Nash group is unipotent if it has a faithful Nash representation so that all group elements act as unipotent linear operators. It is obvious that each Nash subgroup of a unipotent Nash group is a unipotent Nash group.

First recall the following well-known result, which is basic to the study of unipotent Nash groups.

Lemma 5.1 (see [8]) *For each connected, simply connected, nilpotent Lie group N, the exponential map*

$$\exp : \mathrm{Lie}\, N \to N$$

is a diffeomorphism.

Here and henceforth, "Lie" indicates the Lie algebra of a Lie group.

Recall from Section 1 that a subgroup of a Lie group G is said to be analytic if it equals the image of an injective Lie group homomorphism from a connected Lie group to G. Every analytic subgroup is canonically a connected Lie group. We remark that in general, the topology on a non-closed analytic subgroup does not coincide with the subspace topology. The set of all analytic subgroups of G is in one-to-one correspondence with the set of all Lie subalgebras of $\mathrm{Lie}\, G$.

Lemma 5.2 *Let N be a connected, simply connected, nilpotent Lie group. Then each analytic subgroup of N is closed in N, and is a connected, simply connected, nilpotent Lie group.*

Proof Let $\mathfrak{n}_0$ be a Lie subalgebra of $\mathrm{Lie}\, N$. Let N_0 be a connected, simply connected Lie group with Lie algebra $\mathfrak{n}_0$. Then N_0 is nilpotent, and there is a commutative diagram

$$\begin{array}{ccc} \mathfrak{n}_0 & \xrightarrow{\subset} & \mathrm{Lie}\, N \\ \simeq \Big\downarrow \exp & & \simeq \Big\downarrow \exp \\ N_0 & \xrightarrow{\varphi} & N \end{array}$$

where φ is the Lie group homomorphism whose differential is the inclusion map $\mathfrak{n}_0 \hookrightarrow \mathrm{Lie}\, N$. By Lemma 5.1, the two vertical arrows are diffeomorphisms. Since the top horizontal arrow is a closed embedding, φ is also a closed embedding. Then this lemma follows, since $\varphi(N_0)$ is the analytic subgroup of N corresponding to $\mathfrak{n}_0$.

Lemma 5.3 *Let V be a finite-dimensional real representation of a connected Lie group G. If all elements of G act as unipotent linear operators on V, then G kills a full flag of V, namely, there exists a sequence*

$$V_0 \subset V_1 \subset V_2 \subset \cdots \subset V_n \quad (n := \dim V)$$

of subspaces of V*, such that* $\dim V_i = i$ $(i = 0, 1, 2, \cdots, n)$*, and*

$$(g-1).V_i \subset V_{i-1} \quad \text{for all } g \in G \text{ and all } i = 1, 2, \cdots, n.$$

Proof Taking the differential of the representation, $\mathrm{Lie}\, G$ acts as nilpotent linear operators on V. Therefore, this lemma is a direct consequence of Engel's theorem.

Now we come to the study of unipotent Nash groups.

Lemma 5.4 *Every unipotent Nash group is connected.*

Proof Proposition 4.2 implies that a maximal compact subgroup of a unipotent Nash group is trivial. Therefore, this lemma follows by Theorem 4.2.

Denote by $\mathrm{U}_n(\mathbb{R})$ $(n \geq 0)$ the subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting all unipotent upper-triangular matrices. This is a unipotent Nash group. As a Lie group, it is connected, simply connected, and nilpotent. The Lie algebra $\mathfrak{u}_n(\mathbb{R})$ of $\mathrm{U}_n(\mathbb{R})$ consists all nilpotent upper-triangular matrices in $\mathfrak{gl}_n(\mathbb{R})$.

Lemma 5.5 *Every unipotent Nash group is Nash isomorphic to a Nash subgroup of* $\mathrm{U}_n(\mathbb{R})$ *for some* $n \geq 0$*.*

Proof This is a direct consequence of Lemmas 5.3–5.4.

Proposition 5.1 *Every unipotent Nash group is connected, simply connected and nilpotent.*

Proof This is implied by Lemmas 5.2 and 5.4–5.5.

Proposition 5.2 *For each unipotent Nash group* N*, the exponential map*

$$\exp : \mathrm{Lie}\, N \to N \tag{5.1}$$

is a Nash diffeomorphism.

Proof Proposition 5.1 and Lemma 5.1 imply that (5.1) is a diffeomorphism. Using Lemma 5.5, we fix an injective Nash homomorphism $\varphi : N \to \mathrm{U}_n(\mathbb{R})$. Then we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Lie}\, N & \xrightarrow{\phi} & \mathfrak{u}_n(\mathbb{R}) \\
\simeq \big\downarrow \exp & & \simeq \big\downarrow \exp \\
N & \xrightarrow{\varphi} & \mathrm{U}_n(\mathbb{R})
\end{array}$$

where ϕ denotes the differential of φ. Note that in the above diagram, the right vertical arrow is a Nash diffeomorphism, and the two horizontal arrows are Nash embeddings. Therefore, the left vertical arrow is a Nash diffeomorphism.

Proposition 5.3 *Every analytic subgroup of a unipotent Nash group is a Nash subgroup.*

Proof Let N_0 be an analytic subgroup of a unipotent Nash group N. Then we have a commutative diagram

$$\begin{array}{ccc} \operatorname{Lie} N_0 & \xrightarrow{\subset} & \operatorname{Lie} N \\ \simeq \downarrow \exp & & \simeq \downarrow \exp \\ N_0 & \xrightarrow{\subset} & N \end{array} \tag{5.2}$$

By Proposition 5.2, the right vertical arrow of (5.2) is a Nash diffeomorphism. By Lemmas 5.1–5.2, the left vertical arrow of (5.2) is a diffeomorphism. Therefore, N_0 is semialgebraic in N, since $\operatorname{Lie} N_0$ is semialgebraic in $\operatorname{Lie} N$.

Proposition 5.4 *Let N, N' be two unipotent Nash groups. Then every Lie group homomorphism $\varphi : N \to N'$ is a Nash homomorphism.*

Proof It easily follows from Proposition 5.2.

Proposition 5.5 *Let N be a connected, simply connected, nilpotent Lie group. Then there exists a unique Nash structure on the underlying topological space of N which makes N a unipotent Nash group.*

Proof The uniqueness assertion follows from Proposition 5.4. To prove the existence, it suffices to show that there exists a unipotent Nash group which is isomorphic to N as a Lie group. Since all unipotent Nash groups are connected and simply connected (see Proposition 5.1), it suffices to show that there exists a unipotent Nash group whose Lie algebra is isomorphic to $\operatorname{Lie} N$.

As a special case of Ado's theorem, $\operatorname{Lie} N$ is isomorphic to a Lie subalgebra of $\mathfrak{u}_n(\mathbb{R})$ for some $n \geq 0$ (see [14, Theorem 7.19]). Identify $\operatorname{Lie} N$ with a Lie subalgebra of $\mathfrak{u}_n(\mathbb{R})$. By Proposition 5.3, the corresponding analytic subgroup of $\mathrm{U}_n(\mathbb{R})$ is a unipotent Nash group. Therefore, this proposition follows.

Combining Propositions 5.1 and 5.3–5.5, we get Theorem 1.2. We have also proved the following theorem.

Theorem 5.1 *The category of unipotent almost linear Nash groups is isomorphic to the category of connected, simply connected, nilpotent Lie groups.*

The following lemma will be used later.

Lemma 5.6 *Let N be a connected Nash group. If N has a Nash representation with a finite kernel so that all group elements act as unipotent linear operators, then N is unipotent.*

Proof Using Lemma 5.3, we get a Nash homomorphism $\varphi : N \to \mathrm{U}_n(\mathbb{R})$ with finite kernel. By Proposition 5.1, $\varphi(N)$ is connected and simply connected. This implies that $\varphi : N \to \varphi(N)$ is a Nash isomorphism, and this lemma follows.

6 Hyperbolic Nash Groups

The group $\mathbb{R}_+^\times$ of positive real numbers is an almost linear Nash group in the obvious way. We define a hyperbolic Nash group to be a Nash group which is Nash isomorphic to $(\mathbb{R}_+^\times)^n$ for some $n \geq 0$.

As in Section 1, for every two Nash groups G and G', write $\mathrm{Hom}(G, G')$ for the set of all Nash homomorphisms from G to G'. It is obviously an abelian group when G' is abelian, and is a ring when $G = G'$ and G' is abelian.

We leave the proof of the following lemma to the interested reader.

Lemma 6.1 *The map*

$$\mathbb{Q} \to \mathrm{Hom}(\mathbb{R}_+^\times, \mathbb{R}_+^\times), \quad r \mapsto (x \mapsto x^r) \tag{6.1}$$

is a ring isomorphism.

Using the isomorphism (6.1), we view $\mathbb{R}_+^\times$ as a right $\mathbb{Q}$-vector space as in Section 1. Then for every hyperbolic Nash group A, the abelian group $\mathrm{Hom}(\mathbb{R}_+^\times, A)$ is a left $\mathbb{Q}$-vector space:

$$r \cdot \varphi := \varphi \circ (\cdot)^r, \quad r \in \mathbb{Q}, \varphi \in \mathrm{Hom}(\mathbb{R}_+^\times, A),$$

where $(\cdot)^r$ denotes the endomorphism $x \mapsto x^r$ of $\mathbb{R}_+^\times$. By Lemma 6.1, the dimension of $\mathrm{Hom}(\mathbb{R}_+^\times, A)$ equals that of A.

On the other hand, given a $\mathbb{Q}$-vector space E of finite dimension k, the tensor product

$$\mathbb{R}_+^\times \otimes_{\mathbb{Q}} E$$

is obviously a hyperbolic Nash group of dimension k.

Proof of Theorem 1.3 This is an obvious consequence of Lemma 6.1.

Moreover, we have the following proposition.

Proposition 6.1 *Every Nash subgroup of a hyperbolic Nash group is a hyperbolic Nash group.*

Proof Let H be a Nash subgroup of $A_1 \times A_2 \times \cdots \times A_k$, where $k \geq 0$, and each A_i is a Nash group which is Nash isomorphic to $\mathbb{R}_+^\times$ $(i = 1, 2, \cdots, k)$. We want to show that H is a hyperbolic Nash group. Assume without loss of generality that $H \neq A$. Then A_i is not contained in H for some i. Then $A_i \cap H = \{1\}$, and we get an injective Nash homomorphism $H \hookrightarrow A/A_i$. The lemma then follows by an inductive argument.

Similarly, we have the following proposition.

Proposition 6.2 *Every Nash quotient group of a hyperbolic Nash group is a hyperbolic Nash group.*

Proof Let A_0 be a Nash subgroup of a hyperbolic Nash group A. By Proposition 6.1, A_0 is also a hyperbolic Nash group. Using Theorem 1.3, we get an exact sequence

$$0 \to \mathrm{Hom}(\mathbb{R}_+^\times, A_0) \to \mathrm{Hom}(\mathbb{R}_+^\times, A) \to \mathrm{Hom}(\mathbb{R}_+^\times, A)/\mathrm{Hom}(\mathbb{R}_+^\times, A_0) \to 0$$

of left $\mathbb{Q}$-vector spaces. Tensoring with $\mathbb{R}_+^\times$, we get an exact sequence

$$1 \to A_0 \to A \to \mathbb{R}_+^\times \otimes_{\mathbb{Q}} (\mathrm{Hom}(\mathbb{R}_+^\times, A)/\mathrm{Hom}(\mathbb{R}_+^\times, A_0)) \to 1$$

of hyperbolic Nash groups and Nash homomorphisms. Therefore, the proposition follows.

7 Disjointness of Elliptic, Hyperbolic and Unipotent Nash Groups

First recall the following well-known fact.

Lemma 7.1 *The hyperbolic Nash group $\mathbb{R}_+^\times$ is not Nash isomorphic to the unipotent Nash group $\mathbb{R}$.*

Proof Note that the Nash endomorphism ring $\mathrm{Hom}(\mathbb{R}, \mathbb{R})$ is isomorphic to $\mathbb{R}$. By Lemma 6.1, the Nash endomorphism ring $\mathrm{Hom}(\mathbb{R}_+^\times, \mathbb{R}_+^\times)$ is isomorphic to $\mathbb{Q}$. Therefore, this lemma holds.

The following is a useful fact about unipotent Nash groups.

Lemma 7.2 *Every non-trivial element of a unipotent Nash group is contained in a Nash subgroup which is Nash isomorphic to $\mathbb{R}$.*

Proof This is directly implied by Proposition 5.2.

Elliptic Nash groups, hyperbolic Nash groups and unipotent Nash groups are disjoint to each other in the following sense.

Proposition 7.1 *Let G and G' be two Nash groups. If G is elliptic, and G' is hyperbolic or unipotent, then*

$$\mathrm{Hom}(G, G') = \{1\}.$$

The same holds if G is hyperbolic, and G' is elliptic or unipotent; or if G is unipotent, and G' is elliptic or hyperbolic.

Proof Note that all hyperbolic Nash groups and all unipotent Nash groups have no non-trivial compact subgroups. Therefore, $\mathrm{Hom}(G, G') = \{1\}$ if G is elliptic, and G' is hyperbolic or unipotent.

Note that if G' is elliptic, then

$$\mathrm{Hom}(\mathbb{R}_+^\times, G') = \{1\}, \quad \mathrm{Hom}(\mathbb{R}, G') = \{1\}. \tag{7.1}$$

The first equality of (7.1) implies that $\mathrm{Hom}(G, G') = \{1\}$, if G is hyperbolic and G' is elliptic. By Lemma 7.2, the second equality of (7.1) implies that $\mathrm{Hom}(G, G') = \{1\}$, if G is unipotent and G' is elliptic.

Lemma 7.1 implies that

$$\mathrm{Hom}(\mathbb{R}_+^\times, G') = \{1\},$$

if G' is unipotent. Therefore, $\mathrm{Hom}(G, G') = \{1\}$, if G is hyperbolic and G' is unipotent. Lemma 7.1 also implies that

$$\mathrm{Hom}(\mathbb{R}, G') = \{1\}, \tag{7.2}$$

if G' is hyperbolic. By Lemma 7.2, (7.2) implies that $\mathrm{Hom}(G, G') = \{1\}$, if G is unipotent and G' is hyperbolic. This finishes the proof of this proposition.

Proposition 7.2 *Let H_1, H_2, H_3 be three Nash subgroups of a Nash group G. If they are respectively elliptic, hyperbolic and unipotent, then they have pairwise trivial intersections.*

Proof The Nash group $H_1 \cap H_2$ is elliptic and hyperbolic, and is hence trivial by Proposition 7.1. Similarly, $H_1 \cap H_3$ and $H_2 \cap H_3$ are trivial.

Lemma 7.3 *Let G_1 be an elliptic Nash group, and let G_2 be a hyperbolic Nash group. Then all Nash subgroups of $G_1 \times G_2$ are of the form $H_1 \times H_2$, where H_i is a Nash subgroup of G_i, $i = 1, 2$.*

Proof Let H be a Nash subgroup of $G_1 \times G_2$. We first claim that

$$\text{if} \quad H \cap G_2 = \{1\}, \quad \text{then} \quad H \subset G_1. \tag{7.3}$$

Consider the restriction to H of the projection map

$$G_1 \times G_2 \to G_1.$$

The condition $H \cap G_2 = \{1\}$ implies that H is an elliptic Nash group. Then by Proposition 7.1, the projection map

$$G_1 \times G_2 \to G_2$$

has trivial restriction to H. Therefore, $H \subset G_1$, and the claim is proved.

In general, put $G_2' := G_2/(G_2 \cap H)$, which is a hyperbolic Nash group. Write H' for the image of H under the Nash homomorphism

$$p : G_1 \times G_2 \to G_1 \times G_2', \quad (g_1, g_2) \mapsto (g_1, g_2(G_2 \cap H)).$$

Inside the group $G_1 \times G_2'$, we have

$$H' \cap G_2' = \{1\},$$

and then (7.3) implies

$$H' \subset G_1.$$

This lemma then follows as $H = p^{-1}(H')$.

Lemma 7.4 *Let G_1 be the direct product of an elliptic Nash group and a hyperbolic Nash group, and let G_2 be a unipotent Nash group. Then all Nash subgroups of $G_1 \times G_2$ are of the form $H_1 \times H_2$, where H_i is a Nash subgroup of G_i, $i = 1, 2$.*

Proof Lemma 7.3 and its proof show Lemma 7.4 when G_2 is abelian. In general, let H be a Nash subgroup of $G_1 \times G_2$, and let $xy \in H$, where $x \in G_1$ and $y \in G_2$. It suffices to show that $y \in G_2$. Replacing G_2 by an abelian Nash subgroup G_2' of G containing y, and replacing H by $H \cap (G_1 \times G_2')$, this lemma is reduced to the case when G_2 is abelian.

Combining Lemmas 7.3–7.4, we get the following proposition.

Proposition 7.3 *Let G_1, G_2, G_3 be three Nash groups which are respectively elliptic, hyperbolic and unipotent. Then every Nash subgroup of $G_1 \times G_2 \times G_3$ is of the form $H_1 \times H_2 \times H_3$, where H_i is a Nash subgroup of G_i, $i = 1, 2, 3$.*

As a direct consequence of Proposition 7.3, we have the following proposition.

Proposition 7.4 *Let H_1, H_2, H_3 be three Nash subgroups of a Nash group G which are respectively elliptic, hyperbolic and unipotent. If they pairwise commute with each other, then the multiplication map $H_1 \times H_2 \times H_3 \to G$ is an injective Nash homomorphism.*

In the rest of this section, we draw some consequences of Proposition 7.1 on unipotent Nash groups and hyperbolic Nash groups.

Proposition 7.5 *Let V be a Nash representation of a unipotent Nash group G. Then each element of G acts as a unipotent linear operator on V.*

Proof Using Lemma 7.2, we assume without loss of generality that $G = \mathbb{R}$. Let V_1 be an irreducible subquotient representatoin of the complexification $V_{\mathbb{C}}$ of V. Since G is abelian, it is one-dimensional and corresponds to a Nash homomorphism

$$G \to \mathbb{C}^{\times}.$$

This homomorphism is trivial by Proposition 7.1. Therefore, this proposition follows.

As a consequence of Proposition 7.5, we have the following proposition.

Proposition 7.6 *Every Nash quotient group of a unipotent Nash group is unipotent.*

Proof Let N be a unipotent Nash group, and N' be a Nash quotient group of it. Fix a Nash representation V of N' with finite kernel. Applying Proposition 7.5 to the inflation of the representation V to N, we know that N' acts on V as unipotent linear operators. Then this proposition follows by Lemma 5.6.

Proposition 7.7 *All irreducible Nash representations of all unipotent Nash groups are trivial.*

Proof This is implied by Proposition 7.5 and Lemma 5.3.

Now we consider Nash representations of hyperbolic Nash groups.

Lemma 7.5 *Let V be a Nash representation of a hyperbolic Nash group G. If each element of G acts as a unipotent linear operator on V, then the representation V is trivial.*

Proof By Lemma 5.3, the image of the attached homomorphism $G \to \mathrm{GL}(V)$ is contained in a unipotent Nash subgroup of $\mathrm{GL}(V)$. Therefore, the homomorphism is trivial by Proposition 7.1.

Proposition 7.8 *Let V be a Nash representation of a hyperbolic Nash group G. Then each element of G acts as a semisimple linear operator on V, and all its eigenvalues are positive real numbers.*

Proof By Proposition 7.1, the image of every Nash homomorphism from G to $\mathbb{C}^{\times}$ is contained in $\mathbb{R}_+^{\times}$. This implies that for every $g \in G$, all eigenvalues of $\varphi(g)$ are positive real numbers, where $\varphi : G \to \mathrm{GL}(V)$ denotes the Nash homomorphism attached to the representation. Using the generalized eigenspace decomposition, we assume without loss of generality that there is a Nash homomorphism $\chi : G \to \mathbb{R}_+^{\times}$ such that for every $g \in G$, all eigenvalues of $\varphi(g)$ are equal to $\chi(g)$. Then G acts on $V \otimes \chi^{-1}$ by unipotent linear operators. This action is trivial by Lemma 7.5. Therefore, G acts on V via the character χ, and this proposition is proved.

Proposition 7.8 clearly implies the following result.

Proposition 7.9 *Every Nash representation of a hyperbolic Nash group is a direct sum of one-dimensional subrepresentations.*

8 Jordan Decompositions

Let G be an almost linear Nash group throughout this section. For every $x \in G$, define its replica $\langle x \rangle$ to be the smallest Nash subgroup of G containing x. This is well-defined by Proposition 3.4. It is easy to see that $\langle x \rangle$ is abelian.

We say that $x \in G$ is elliptic, hyperbolic or unipotent if it is contained in a Nash subgroup of G which is elliptic, hyperbolic or unipotent, respectively. This is equivalent to saying that the abelian Nash group $\langle x \rangle$ is respectively elliptic, hyperbolic or unipotent. Respectively write G_{e}, G_{h} and G_{u} for the sets of all elliptic, hyperbolic and unipotent elements in G.

Lemma 8.1 *An element in* $\mathrm{GL}_n(\mathbb{R})$ $(n \geq 0)$ *is elliptic if and only if it is semisimple and all its eigenvalues are complex numbers of modulus one; it is hyperbolic if and only if it is semisimple and all its eigenvalues are positive real numbers; it is unipotent if and only if all its eigenvalues are equal to* 1.

Proof The "if" parts of the three assertions of this lemma are obvious. The "only if" parts are implied by Propositions 4.2, 7.5 and 7.8.

Lemma 8.2 *Let* $e, h, u \in G$. *Assume that they are respectively elliptic, hyperbolic and unipotent, and they pairwise commute with each other. Then*

$$\langle ehu \rangle \supset \langle e \rangle,\ \langle h \rangle,\ \langle u \rangle,$$

and the multiplication map

$$\langle e \rangle \times \langle h \rangle \times \langle u \rangle \to \langle ehu \rangle$$

is an isomorphism of Nash groups.

Proof First note that the subgroups $\langle e \rangle$, $\langle h \rangle$, $\langle u \rangle$ are pairwise commutative to each other. Using Proposition 7.4, we view $\langle e \rangle \times \langle h \rangle \times \langle u \rangle$ as a Nash subgroup G. Then $\langle ehu \rangle$ is a Nash subgroup of $\langle e \rangle \times \langle h \rangle \times \langle u \rangle$, and Proposition 7.3 implies that

$$\langle ehu \rangle = \langle e \rangle \times \langle h \rangle \times \langle u \rangle.$$

This proves this lemma.

Here is the Jordan decomposition theorem for almost linear Nash groups as follows.

Theorem 8.1 *Every element* x *of an almost linear Nash group* G *is uniquely of the form* $x = ehu$ *such that* $e \in G_{\mathrm{e}}$, $h \in G_{\mathrm{h}}$, $u \in G_{\mathrm{u}}$, *and they pairwise commute with each other.*

Proof Fix a Nash homomorphism $\varphi : G \to \mathrm{GL}_n(\mathbb{R})$ with a finite kernel. Put $y := \varphi(x)$ and write $y = y_{\mathrm{e}} y_{\mathrm{h}} y_{\mathrm{u}}$ for the usual Jordan decomposition of y in $\mathrm{GL}_n(\mathbb{R})$, where y_{e} is elliptic, y_{h} is hyperbolic, y_{u} is unipotent and they pairwise commute with each other (see [10, pp. 430–431]). Then Lemma 8.2 implies

$$y_{\mathrm{e}}, y_{\mathrm{h}}, y_{\mathrm{u}} \in \varphi(G).$$

Denote by h the unique element in the identity connected component of $\varphi^{-1}(\langle y_{\mathrm{h}}\rangle)$ which lifts y_{h}. Define u similarly, and put $e := xu^{-1}h^{-1}$. Then it is routine to check that (e, h, u) is the unique triple which fulfills all the requirements of the theorem.

The equality $x = ehu$ of Theorem 8.1 is called the Jordan decomposition of $x \in G$. We respectively use x_{e}, x_{h} and x_{u} to denote the elements e, h and u. They are respectively called the elliptic, hyperbolic and unipotent parts of $x \in G$.

Proposition 8.1 *Let $\varphi : G \to G'$ be a Nash homomorphism of almost linear Nash groups. Then*

$$\varphi(G_{\mathrm{e}}) \subset G'_{\mathrm{e}}, \quad \varphi(G_{\mathrm{h}}) \subset G'_{\mathrm{h}}, \quad \varphi(G_{\mathrm{u}}) \subset G'_{\mathrm{u}}. \tag{8.1}$$

If φ is surjective, then the three inclusions in (8.1) *become equalities.*

Proof The three inclusions are respectively implied by Propositions 4.1, 6.2 and 7.6.

Now assume that φ is surjective, and let $y \in G'_{\mathrm{e}}$. Pick $x \in G$ so that $\varphi(x) = y$. Then $\varphi(x_{\mathrm{e}})\varphi(x_{\mathrm{h}})\varphi(x_{\mathrm{u}}) = y$. By (8.1), the uniqueness of Jordan decompositions implies that $\varphi(x_{\mathrm{e}}) = y$. This proves that $\varphi(G_{\mathrm{e}}) = G'_{\mathrm{e}}$. The same argument proves the other two equalities.

Proposition 8.1 obviously implies that Nash homomorphisms preserve Jordan decompositions:

Proposition 8.2 *Let $\varphi : G \to G'$ be a Nash homomorphism of almost linear Nash groups. Then for every $x \in G$, one has*

$$(\varphi(x))_{\mathrm{e}} = \varphi(x_{\mathrm{e}}), \quad (\varphi(x))_{\mathrm{h}} = \varphi(x_{\mathrm{h}}), \quad (\varphi(x))_{\mathrm{u}} = \varphi(x_{\mathrm{u}}).$$

As one application of Jordan decompositions, we get the following result about structures of abelian almost linear Nash groups.

Proposition 8.3 *Let G be an abelian almost linear Nash group. Then G_{e} is an elliptic Nash subgroup of G, G_{h} is a hyperbolic Nash subgroup of G, and G_{u} is a unipotent Nash subgroup of G. Moreover, the multiplication map*

$$G_{\mathrm{e}} \times G_{\mathrm{h}} \times G_{\mathrm{u}} \to G$$

is a Nash isomorphism.

Proof Let K be a maximal compact subgroup of G, which is unique since G is abelian. Then clearly $K = G_{\mathrm{e}}$. Let A be a hyperbolic Nash subgroup of G of the maximal dimension. Then clearly $A = G_{\mathrm{h}}$. Likewise, let U be a unipotent Nash subgroup of G of the maximal dimension. Then $U = G_{\mathrm{u}}$. The last assertion follows from Theorem 8.1.

In the rest of this section, denote by $\mathfrak{g}$ the Lie algebra of the almost linear Nash group G. For every $x \in \mathfrak{g}$, we define its replica $\langle x\rangle$ to be the smallest Nash subgroup of G containing $\exp(\mathbb{R}x)$. It is connected and abelian. We said that $x \in \mathfrak{g}$ is elliptic, hyperbolic or unipotent, if the Nash group $\langle x\rangle$ is respectively elliptic, hyperbolic or unipotent. As in the group case, respectively write $\mathfrak{g}_{\mathrm{e}}$, $\mathfrak{g}_{\mathrm{h}}$ and $\mathfrak{g}_{\mathrm{u}}$ for the sets of all elliptic, hyperbolic and unipotent elements in $\mathfrak{g}$.

Lemma 8.1 easily implies the following lemma.

Lemma 8.3 *An element in the Lie algebra $\mathfrak{gl}_n(\mathbb{R})$ of $\mathrm{GL}_n(\mathbb{R})$ ($n \geq 0$) is elliptic if and only if it is semisimple and all its eigenvalues are purely imaginary; it is hyperbolic if and only if it is semisimple and all its eigenvalues are real; it is unipotent if and only if it is a nilpotent matrix.*

The same proof as Lemma 8.2 shows the following lemma.

Lemma 8.4 *Let $e, h, u \in \mathfrak{g}$. Assume that they are respectively elliptic, hyperbolic and unipotent, and they pairwise commute with each other. Then*

$$\langle e + h + u \rangle \supset \langle e \rangle,\ \langle h \rangle,\ \langle u \rangle,$$

and the multiplication map

$$\langle e \rangle \times \langle h \rangle \times \langle u \rangle \to \langle e + h + u \rangle$$

is an isomorphism of Nash groups.

The following is the Jordan decomposition theorem at the Lie algebra level.

Theorem 8.2 *Let G be an almost linear Nash group with Lie algebra $\mathfrak{g}$. Then every element $x \in \mathfrak{g}$ is uniquely of the form $x = e + h + u$ such that $e \in \mathfrak{g}_{\mathrm{e}}$, $h \in \mathfrak{g}_{\mathrm{h}}$, $u \in \mathfrak{g}_{\mathrm{u}}$, and they pairwise commute with each other.*

Proof The proof is similar to that of Theorem 8.1. We omit the details.

We also call the equality $x = e + h + u$ of Theorem 8.2 the Jordan decomposition of $x \in \mathfrak{g}$. As in the group case, we respectively use x_{e}, x_{h} and x_{u} to denote the elements e, h and u. They are respectively called the elliptic, hyperbolic and unipotent parts of $x \in \mathfrak{g}$.

The same proof as Proposition 8.1 shows the following proposition.

Proposition 8.4 *Let $\varphi : G \to G'$ be a Nash homomorphism of almost linear Nash groups. Write $\phi : \mathfrak{g} \to \mathfrak{g}'$ for its differential, where $\mathfrak{g}' := \mathrm{Lie}\, G'$. Then*

$$\phi(\mathfrak{g}_{\mathrm{e}}) \subset \mathfrak{g}_{\mathrm{e}}, \quad \phi(\mathfrak{g}_{\mathrm{h}}) \subset \mathfrak{g}'_{\mathrm{h}} \quad \textit{and} \quad \phi(\mathfrak{g}_{\mathrm{u}}) \subset \mathfrak{g}'_{\mathrm{u}}. \tag{8.2}$$

If ϕ is surjective, then the three inclusions in (8.2) become equalities.

Similar to Proposition 8.2, the above proposition implies the following proposition.

Proposition 8.5 *Let $\varphi : G \to G'$ be a Nash homomorphism of almost linear Nash groups. Write $\phi : \mathfrak{g} \to \mathfrak{g}'$ for its differential, where $\mathfrak{g}' := \mathrm{Lie}\, G'$. Then for every $x \in \mathfrak{g}$, one has*

$$(\phi(x))_{\mathrm{e}} = \phi(x_{\mathrm{e}}), \quad (\phi(x))_{\mathrm{h}} = \phi(x_{\mathrm{h}}), \quad (\phi(x))_{\mathrm{u}} = \phi(x_{\mathrm{u}}).$$

9 Exponential Elements

Let G be an almost linear Nash group with Lie algebra $\mathfrak{g}$. The following lemma concerning the exponential map is obvious.

Lemma 9.1 *One has*

$$\exp(\mathfrak{g}_{\mathrm{e}}) \subset G_{\mathrm{e}}, \quad \exp(\mathfrak{g}_{\mathrm{h}}) \subset G_{\mathrm{h}}, \quad \exp(\mathfrak{g}_{\mathrm{u}}) \subset G_{\mathrm{u}}.$$

For each $x \in G_{\mathrm{h}}$ or G_{u}, define $\log(x)$ to be the unique element in the Lie algebra of $\langle x \rangle$ such that $\exp(\log(x)) = x$. Then $\log(x)$ belongs to $\mathfrak{g}_{\mathrm{h}}$ or $\mathfrak{g}_{\mathrm{u}}$, respectively. The maps

$$\exp : \mathfrak{g}_{\mathrm{h}} \to G_{\mathrm{h}} \quad \text{and} \quad \log : G_{\mathrm{h}} \to \mathfrak{g}_{\mathrm{h}}$$

are inverse to each other. Likewise, the maps

$$\exp : \mathfrak{g}_{\mathrm{u}} \to G_{\mathrm{u}} \quad \text{and} \quad \log : G_{\mathrm{u}} \to \mathfrak{g}_{\mathrm{u}}$$

are inverse to each other.

Lemma 9.2 *Let $x \in \mathfrak{g}_{\mathrm{h}}$ or $\mathfrak{g}_{\mathrm{u}}$. Then $\langle x \rangle = \langle \exp(x) \rangle$.*

Proof The Nash subgroup $\langle x \rangle$ of G contains the Nash subgroup $\langle \exp(x) \rangle$. Since $x \in \mathrm{Lie}\,\langle x \rangle$ and $\exp(x) \in \langle \exp(x) \rangle$, using the commutative diagram

$$\begin{tikzcd} \mathrm{Lie}\,\langle \exp(x) \rangle \arrow[r, "\subset"] \arrow[d, "\exp", "\simeq"'] & \mathrm{Lie}\,\langle x \rangle \arrow[d, "\exp", "\simeq"'] \\ \langle \exp(x) \rangle \arrow[r, "\subset"] & \langle x \rangle \end{tikzcd}$$

we know that $x \in \mathrm{Lie}\,\langle \exp(x) \rangle$. Therefore,

$$\langle \exp(x) \rangle \supset \exp(\mathbb{R}x),$$

and hence $\langle \exp(x) \rangle = \langle x \rangle$.

Lemma 9.3 *Let $x \in G_{\mathrm{h}}$ and $y \in G_{\mathrm{u}}$. If they commute with each other in G, then $\log(x)$ and $\log(y)$ commute with each other in $\mathfrak{g}$.*

Proof If x and y commute with each other, then Lemma 9.2 implies that the Nash subgroups $\langle \log(x) \rangle$ and $\langle \log(y) \rangle$ commute with each other. Therefore, $\langle \log(x) \rangle \langle \log(y) \rangle$ is an abelian Nash subgroup of G. Then this lemma follows, since both $\log(x)$ and $\log(y)$ belong to the Lie algebra of $\langle \log(x) \rangle \langle \log(y) \rangle$.

Definition 9.1 *An element of an almost linear Nash group G or its Lie algebra $\mathfrak{g}$ is said to be exponential if its elliptic part is trivial.*

Denote by G_{ex} and $\mathfrak{g}_{\mathrm{ex}}$ the sets of all exponential elements in G and $\mathfrak{g}$, respectively. For every exponential element $x \in G_{\mathrm{ex}}$, define

$$\log(x) := \log(x_{\mathrm{h}}) + \log(x_{\mathrm{u}}).$$

By Lemma 9.3, this is an element of $\mathfrak{g}_{\mathrm{ex}}$.

Proposition 9.1 *The maps*

$$\exp : \mathfrak{g}_{\mathrm{ex}} \to G_{\mathrm{ex}} \quad \textit{and} \quad \log : G_{\mathrm{ex}} \to \mathfrak{g}_{\mathrm{ex}}$$

are inverse to each other.

Proof This is obvious.

Proposition 9.2 *Let $x \in \mathfrak{g}_{\mathrm{ex}}$. Then $\langle x \rangle = \langle \exp(x) \rangle$.*

Proof By Lemmas 9.2, 8.2 and 8.4, we have

$$\langle \exp(x) \rangle = \langle \exp(x_{\mathrm{h}}) \rangle \times \langle \exp(x_{\mathrm{u}}) \rangle = \langle x_{\mathrm{h}} \rangle \times \langle x_{\mathrm{u}} \rangle = \langle x \rangle.$$

10 Semisimple Nash Groups

We say that a Lie group (or a Nash group) is semisimple if its Lie algebra is semisimple.

Lemma 10.1 *Every semisimple Nash group is almost linear.*

Proof Taking the adjoint representation, then this lemma follows.

Recall the following result.

Lemma 10.2 (see [12, Proposition 7.9]) *Every semisimple analytic subgroup of* $\mathrm{GL}_n(\mathbb{R})$ $(n \geq 0)$ *has a finite center.*

Using Cartan decompositions for semisimple Lie groups (see [12, Theorem 7.39]), we easily get the following result.

Lemma 10.3 *Let G be a connected semisimple Lie group with a finite center. Let K be a maximal compact subgroup of G. Then there are analytic subgroups $H_1, H_2, \cdots, H_r$ $(r \geq 0)$ of G such that*

(1) $G = KH_1H_2\cdots H_rK$,

(2) *the Lie algebra of H_i is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$ $(i = 1, 2, \cdots, r)$.*

Recall that every analytic subgroup of $\mathrm{GL}_n(\mathbb{R})$ is isomorphic to either $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$, if its Lie algebra is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. The representation theory of $\mathrm{SL}_2(\mathbb{R})$ implies the following lemma.

Lemma 10.4 *Every finite-dimensional real representation of* $\mathrm{SL}_2(\mathbb{R})$ *or* $\mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ *is a Nash representation.*

As a direct consequence of Lemma 10.4, we have the following lemma.

Lemma 10.5 *An analytic subgroup of* $\mathrm{GL}_n(\mathbb{R})$ *is a Nash subgroup if its Lie algebra is isomorphic to* $\mathfrak{sl}_2(\mathbb{R})$.

Combining Lemmas 4.1, 10.2, 10.3 and 10.5, we have the following lemma.

Lemma 10.6 *Every semisimple analytic subgroup of* $\mathrm{GL}_n(\mathbb{R})$ $(n \geq 0)$ *is a Nash subgroup.*

By Lemma 10.6, the same proof as that of Lemma 4.1 implies the following proposition.

Proposition 10.1 *Every semisimple analytic subgroup of every almost linear Nash group is a Nash subgroup.*

In a way similar to the proof of Lemma 4.2, Proposition 10.1 implies the following proposition.

Proposition 10.2 *Every Lie group homomorphism from a semisimple Nash group to an almost linear Nash group is a Nash homomorphism. In particular, every finite-dimensional representation of a semisimple Nash group is a Nash representation.*

Each semisimple Nash group has finitely many connected components, and its identity connected component has a finite center. Conversely, we have the following proposition.

Proposition 10.3 *Let G be a semisimple Lie group. If it has finitely many connected components, and its identity connected component has a finite center, then there exists a unique Nash structure on the underlying topological space of G which makes G a Nash group.*

Proof Denote by $\mathfrak{g}$ the Lie algebra of G. The automorphism group $\mathrm{Aut}(\mathfrak{g})$ of $\mathfrak{g}$ is obviously a Nash group. The adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$ has an open image and a finite kernel. Therefore, the existence follows by Proposition 3.3. The uniqueness is implied by Proposition 10.2.

In conclusion, we have proved the following theorem.

Theorem 10.1 *The category of semisimple Nash groups is isomorphic to the category of semisimple Lie groups which have finitely many connected components, and whose identity connected components have finite centers.*

Recall the following famous result of Weyl.

Lemma 10.7 (see [11, Theorem 1]) *Let $\mathfrak{g}$ be a semisimple finite-dimensional Lie algebra over a field* k *of characteristic zero. Then all of its finite-dimensional representations over* k *are completely reducible.*

Also recall the following elementary lemma.

Lemma 10.8 (see [15, Lemma 3.1]) *Let H be a normal subgroup of a group G. Let V be a representation of G over a field* k.

(1) *If V is finite-dimensional and completely reducible, then its restriction to H is completely reducible.*

(2) *Assume that H has finite index in G, and k has characteristic zero. Then V is completely reducible if its restriction to H is so.*

Combining Lemma 10.7 and Lemma 10.8(2), we get the following lemma.

Lemma 10.9 *Every Nash representation of a semisimple Nash group is completely reducible.*

11 Reductive Nash Groups

We say that a Nash group is reductive if it has a completely reducible Nash representation with a finite kernel. Using induced representations as in the proof of Proposition 3.6, Lemma 10.8 easily implies the following lemma.

Lemma 11.1 *A Nash group is reductive if and only if its identity connected component is reductive.*

Recall from Introduction that a Nash torus is a Nash group which is Nash isomorphic to $\mathbb{S}^m \times (\mathbb{R}_+^\times)^n$ for some $m, n \geq 0$. Every Nash torus is clearly a reductive Nash group. The main result we will prove in this section is the following theorem.

Theorem 11.1 *A connected Nash group G is reductive if and only if there exists a connected semisimple Nash group H, a Nash torus T, and a surjective Nash homomorphism $H \times T \to G$ with a finite kernel.*

Recall that a finite-dimensional Lie algebra is said to be reductive if its adjoint representation is completely reducible, or equivalently, if it is the direct sum of an abelian Lie algebra and a semisimple Lie algebra. Recall the following result of Jacobson.

Lemma 11.2 ([11, Theorem 1]) *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field k of characteristic zero. If $\mathfrak{g}$ has a faithful completely reducible finite-dimensional representation over k, then $\mathfrak{g}$ is reductive.*

Combining Lemmas 11.1–11.2, we get the following lemma.

Lemma 11.3 *The Lie algebra of every reductive almost linear Nash group is reductive.*

Let G be a connected reductive Nash group with Lie algebra $\mathfrak{g}$. Write

$$\mathfrak{g} = \mathfrak{s} \oplus \mathfrak{z},$$

where $\mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{z}$ denotes the center of $\mathfrak{g}$. Respectively write S and Z for the analytic subgroups of G corresponding to $\mathfrak{s}$ and $\mathfrak{z}$. By Proposition 10.1, S is a Nash subgroup of G. Since Z equals the identity connected component of the center of G, it is also a Nash subgroup of G.

Lemma 11.4 *The Nash group Z is a Nash torus.*

Proof Note that Z is a normal subgroup of G. The first assertion of Lemma 10.8 implies that Z is reductive. Similarly, Z_{u} is reductive (Proposition 8.3 implies that Z_{u} is a unipotent Nash group). Then Proposition 7.7 implies that $Z_{\mathrm{u}} = \{1\}$, and hence Z is a Nash torus by Proposition 8.3.

Since $S \times Z$ is a finite-fold cover of G, we prove the "only if" part of Theorem 11.1.

On the other hand, let G' be a connected Nash group with a surjective Nash group homomorphism $H \times T \to G'$ with a finite kernel, where H is a connected semisimple Nash group, and T is a Nash torus. Then G' is almost linear by Proposition 3.8.

Lemma 11.5 *Every Nash representation of a Nash torus is completely reducible.*

Proof By Weyl's unitary trick, every Nash representation of an elliptic Nash group is completely reducible. In particular, every Nash representation of a compact Nash torus is completely reducible. Together with Proposition 7.9, this implies this lemma.

By Lemmas 11.5 and 10.9, every Nash representation of $H \times T$ is completely reducible. Consequently, every Nash representation of G' is also completely reducible. Therefore, G' is reductive. This proves the "if part" of Theorem 11.1.

By Lemma 10.8, the preceding arguments also show the following theorem.

Theorem 11.2 *Every Nash representation of every reductive almost linear Nash group is completely reducible.*

12 Trace Forms and Reductivity

Let G be an almost linear Nash group with Lie algebra $\mathfrak{g}$. Fix a Nash representation V of G with a finite kernel, and write $\phi : \mathfrak{g} \to \mathfrak{gl}(V)$ for the attached differential. Put

$$\langle x, y\rangle_\phi := \mathrm{tr}(\phi(x)\phi(y)), \quad x, y \in \mathfrak{g}.$$

This defines a G-invariant symmetric bilinear form on $\mathfrak{g}$, which is called the trace form attached to the Nash representation V.

The main result of this section is the following theorem.

Theorem 12.1 *The almost linear Nash group G is reductive if and only if the bilinear form $\langle\, , \rangle_\phi$ is non-degenerate.*

Theorem 12.1 has the following interesting consequence.

Proposition 12.1 *Assume that G is reductive. Then for every reductive Nash subgroup H_1 of G, its centralizer H_2 in G is also a reductive Nash subgroup of G.*

Proof By Theorem 12.1, $\langle\, , \rangle_\phi$ is a non-degenerate symmetric bilinear form on $\mathfrak{g}$. It is G-invariant, and hence H_1-invariant. Since H_1 is reductive, by Theorem 11.2, $\mathfrak{g}$ is completely reducible as a representation of H_1. Taking the isotypic decomposition, we know that the space $\mathfrak{g}^{H_1}$ of H_1-fixed vectors in $\mathfrak{g}$ is non-degenerate with respect to $\langle\, , \rangle_\phi$. Since $\mathfrak{g}^{H_1}$ equals the Lie algebra of H_2, this proposition follows by Theorem 12.1.

The rest of this section is devoted to a proof of Theorem 12.1.

Lemma 12.1 *If G is elliptic, then the bilinear form $\langle\, , \rangle_\phi$ is negative definite. If G is hyperbolic, then the bilinear form $\langle\, , \rangle_\phi$ is positive definite.*

Proof This is implied by Lemma 8.3.

Lemma 12.2 *Let x and y be two commuting elements in the Lie algebra $\mathfrak{gl}(V)$ of $\mathrm{GL}(V)$. If x is elliptic and y is hyperbolic, then $\mathrm{tr}(xy) = 0$.*

Proof Note that all eigenvalues of x are purely imaginary, and all eigenvalues of y are real. Since x and y commute, all eigenvalues of xy are purely imaginary. Therefore, $\mathrm{tr}(xy)$ is purely imaginary. It has to vanish since it is also real.

Lemma 12.3 *If G is a Nash torus, then the bilinear form $\langle\, , \rangle_\phi$ is non-degenerate.*

Proof Write $G = T \times A$, where T is a compact Nash torus, and A is a hyperbolic Nash group. Lemma 12.2 implies that $\mathrm{Lie}\, T$ and $\mathrm{Lie}\, A$ are orthogonal to each other under the symmetric bilinear form $\langle\, , \rangle_\phi$. The lemma then follows by Lemma 12.1.

Lemma 12.4 *If G is a connected semisimple Nash group and $\langle\, , \rangle_\phi$ is zero, then G is trivial.*

Proof Let K be a maximal compact subgroup of G, which is connected since G is connected. Then Lemma 12.1 implies that K is trivial, which further implies that G is trivial (recall that every non-trivial connected semisimple Lie group with a finite center has a non-trivial maximal compact subgroup).

We are now prepared to prove the "only if" part of Theorem 12.1.

Proposition 12.2 *If G is reductive, then the bilinear form $\langle\, ,\, \rangle_\phi$ is non-degenerate.*

Proof Denote by $\mathfrak{n}$ the kernel of the form $\langle\, ,\, \rangle_\phi$. It is an ideal of the reductive Lie algebra $\mathfrak{g}$. Lemma 12.3 implies that $\mathfrak{n} \subset [\mathfrak{g}, \mathfrak{g}]$. Therefore, $\mathfrak{n}$ is semisimple. Denote by N the analytic subgroup of G with Lie algebra $\mathfrak{n}$. It is a connected semisimple Nash subgroup of G by Proposition 10.1. Then Lemma 12.4 implies that N is trivial, and hence $\mathfrak{n} = \{0\}$.

To prove the "if" part of Theorem 12.1, recall the following lemma.

Lemma 12.5 (see [5, Lemma 3.1 and Proposition 3.2]) *Let V_0 be a finite-dimensional vector space over a field of characteristic zero. Let $\mathfrak{g}_0$ be a Lie subalgebra of $\mathfrak{gl}(V_0)$ such that the trace form is non-degenerate on $\mathfrak{g}_0$. Then $\mathfrak{g}_0$ is reductive, and no non-zero element in the center of $\mathfrak{g}_0$ is nilpotent as a linear operator on V_0.*

Now assume that $\langle\, ,\, \rangle_\phi$ is non-degenerate. We want to show that G is reductive. In view of Lemma 11.1, we may (and do) assume that G is connected.

Lemma 12.5 implies that the Lie algebra $\mathfrak{g}$ is reductive. Write

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s},$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$, and $\mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]$. Denote by Z and S the analytic subgroups of G respectively corresponding to $\mathfrak{z}$ and $\mathfrak{s}$. As before, both Z and S are Nash subgroups of G. Using Proposition 8.3, write $Z = Z_{\mathrm{e}} \times Z_{\mathrm{h}} \times Z_{\mathrm{u}}$. Then Lemma 12.5 implies that $Z_{\mathrm{u}} = \{1\}$. Therefore, Z is a Nash torus. Since $Z \times S$ is a finite-fold cover of G, G is reductive by Theorem 11.1. This proves the "if" part of Theorem 12.1.

13 Semisimple Elements

Let G be an almost linear Nash group with Lie algebra $\mathfrak{g}$.

Definition 13.1 *An element of G or $\mathfrak{g}$ is said to be semisimple if its unipotent part is trivial.*

We define a Nash quasi-torus to be an abelian almost linear Nash group without non-trivial unipotent element. All Nash quasi-tori are reductive Nash groups. First, we have the following lemma.

Lemma 13.1 *An element $x \in G$ is semisimple if and only if $\langle x \rangle$ is a Nash quasi-torus. An element $y \in \mathfrak{g}$ is semisimple if and only if $\langle y \rangle$ is a Nash torus.*

Proof The "if" part of the first assertion is obvious. To prove the "only if" part of the first assertion, assume that x is semisimple. Then $\langle x \rangle = \langle x_{\mathrm{e}} \rangle \times \langle x_{\mathrm{h}} \rangle$ by Lemma 8.2. Therefore, $\langle x \rangle$ is a Nash quasi-torus. The proof of the second assertion is similar.

Write G_{ss} and $\mathfrak{g}_{\mathrm{ss}}$ for the sets of all semisimple elements in G and $\mathfrak{g}$, respectively.

Lemma 13.2 *Let $\varphi : G \to G'$ be a Nash homomorphism of almost linear Nash groups. Then*

$$\varphi(G_{\mathrm{ss}}) \subset G'_{\mathrm{ss}},$$

and the inclusion becomes an equality if φ is surjective. Write $\phi : \mathfrak{g} \to \mathfrak{g}'$ for the differential of φ, where $\mathfrak{g}'$ denotes the Lie algebra of G'. Then

$$\phi(\mathfrak{g}_{\mathrm{ss}}) \subset \mathfrak{g}'_{\mathrm{ss}},$$

and the inclusion becomes an equality if ϕ is surjective.

Proof The proof is similar to that of Proposition 8.1.

The rest of this section is to prove the following theorem.

Theorem 13.1 *If G is reductive, then the set G_{ss} is dense in G, and the set $\mathfrak{g}_{\mathrm{ss}}$ is dense in $\mathfrak{g}$.*

We begin with the following lemma.

Lemma 13.3 *If G is connected and $\mathfrak{g}$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, then G_{ss} is dense in G.*

Proof It is elementary to check that this lemma holds when $G = \mathrm{SL}_2(\mathbb{R})$, which implies that this lemma also holds when $G = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$. In general, there is a surjective Nash homomorphism $\varphi : G \to \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ with a finite kernel. The lemma then follows, since

$$G_{\mathrm{ss}} = \varphi^{-1}((\mathrm{SL}_2(\mathbb{R})/\{\pm 1\})_{\mathrm{ss}}).$$

Lemma 13.4 *Let u be a unipotent element of a reductive Nash group G. Then every neighborhood of u in G contains a semisimple element.*

Proof The lemma is trivial when $u = 1$. So assume that $u \neq 1$. Since every element of the center of $\mathfrak{g}$ is semisimple, $\log(u)$ belongs to the semisimple Lie algebra $[\mathfrak{g}, \mathfrak{g}]$. Since $\log(u)$ is unipotent, the linear operator

$$\mathrm{ad}_{\log(u)} : [\mathfrak{g}, \mathfrak{g}] \to [\mathfrak{g}, \mathfrak{g}], \quad x \mapsto [\log(u), x]$$

is nilpotent. Therefore, by the Jacobson-Morozov theorem, there is a Lie subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ containing $\log(u)$ which is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. Denote by G_0 the analytic subgroup of G with Lie algebra $\mathfrak{g}_0$. It is a Nash subgroup by Proposition 10.1. The lemma then follows by Lemma 13.3.

We are now ready to prove Theorem 13.1. Let x be an element of a reductive Nash group G. The centralizer $\mathrm{Z}_G(x_{\mathrm{e}}x_{\mathrm{h}})$ of $x_{\mathrm{e}}x_{\mathrm{h}}$ in G equals the centralizer of the Nash quasi-torus $\langle x_{\mathrm{e}}x_{\mathrm{h}} \rangle$ in G. Therefore, it is a reductive Nash subgroup of G by Proposition 12.1. Note that the product of two commuting semisimple elements in an almost linear Nash group is again semisimple. By Lemma 13.4, every neighborhood of x_{u} in $\mathrm{Z}_G(x_{\mathrm{e}}x_{\mathrm{h}})$ contains a semisimple element. Therefore, every neighborhood of $x = (x_{\mathrm{e}}x_{\mathrm{h}})x_{\mathrm{u}}$ in G contains a semisimple element. This finishes the proof of Theorem 13.1 in the group case. The Lie algebra case is proved similarly.

14 Levi Decompositions

Let G be an almost linear Nash group. Put

$$\mathfrak{U}_G := \text{the identity connected component of } \bigcap_{\pi} \ker \pi,$$

where π runs through all irreducible Nash representations of G. By Proposition 3.4, $\mathfrak{U}_G$ is a Nash subgroup of G.

Proposition 14.1 *The group $\mathfrak{U}_G$ is the largest normal unipotent Nash subgroup of G.*

Proof It is obvious that $\mathfrak{U}_G$ is a normal subgroup of G. Take a Nash representation V of G with a finite kernel. Note that $\mathfrak{U}_G$ acts trivially on all irreducible Nash representations of G. Therefore, by taking a Jordan-Hölder series of V, we know that $\mathfrak{U}_G$ acts on V as unipotent linear operators. Then Lemma 5.6 implies that $\mathfrak{U}_G$ is a unipotent Nash group.

Let U be a normal unipotent Nash subgroup of G. It is connected by Proposition 5.1. For every irreducible Nash representation π of G, the restriction $\pi|_U$ is completely reducible by the first assertion of Lemma 10.8. Since U is unipotent, Proposition 7.7 implies that U acts trivially on π. This shows that $U \subset \mathfrak{U}_G$.

We call $\mathfrak{U}_G$ the unipotent radical of G.

Lemma 14.1 *An almost linear Nash group is reductive if and only if its unipotent radical is trivial.*

Proof The "only if" part of the lemma is obvious. The "if" part is implied by Proposition 3.4.

Proposition 3.4 also implies that $G/\mathfrak{U}_G$ is a reductive Nash group.

Theorem 14.1 *Every reductive Nash subgroup of G is contained in a maximal one, and all maximal reductive Nash subgroups of G are conjugate to each other under $\mathfrak{U}_G$. Moreover, for each maximal reductive Nash subgroup L of G, one has that $G = L \ltimes \mathfrak{U}_G$.*

The equality $G = L \ltimes \mathfrak{U}_G$ of Theorem 14.1 is called a Levi decomposition of G, and a maximal reductive Nash subgroup of G is called a Levi component of G.

The rest of this section is devoted to a proof of Theorem 14.1. We fist recall some results of G. D. Mostow on linear Lie algebras.

For a finite-dimensional Lie algebra $\mathfrak{g}$ over a field of characteristic zero, write $\mathrm{Rad}(\mathfrak{g})$ for its radical, namely, the largest solvable ideal of $\mathfrak{g}$. For a Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, we define $\mathrm{I}_{\mathfrak{g}}(\mathfrak{h})$ to be the subgroup of the automorphism group $\mathrm{Aut}(\mathfrak{g})$ generated by the set

$$\{\exp(\mathrm{ad}_x) \mid x \in \mathfrak{h}, \text{the linear operator } \mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g},\ y \mapsto [x, y] \text{ is nilpotent}\}.$$

Given a finite-dimensional vector space V, we say that a subset $R \subset \mathfrak{gl}(V)$ is fully reducible if each R-stable subspace of V has a complementary R-stable subspace. This generalizes the notion of "semisimple linear operators".

Lemma 14.2 (see [15, Theorems 4.1 and 5.1]) *Let V be a finite-dimensional vector space over a field of characteristic zero, and let $\mathfrak{g}$ be a Lie subalgebra of $\mathfrak{gl}(V)$.*

(1) *All maximal fully reducible Lie subalgebras of* $\mathfrak{g}$ *are conjugate to each other under* $\mathrm{I}_{\mathfrak{g}}(\mathrm{Rad}([\mathfrak{g},\mathfrak{g}]))$.

(2) *Let* R *be a fully reducible subgroup of* $\mathrm{GL}(V)$. *If* R *normalizes* $\mathfrak{g}$, *then* R *normalizes a maximal fully reducible Lie subalgebra of* $\mathfrak{g}$.

Now let G be an almost linear Nash group as before. In the rest of this section, denote by $\mathfrak{g}$ and $\mathfrak{u}$ the Lie algebras of G and $\mathfrak{U}_G$, respectively.

Lemma 14.3 *One has* $\mathrm{Rad}([\mathfrak{g},\mathfrak{g}]) \subset \mathfrak{u}$.

Proof By Lemma 11.3, the Lie algebra $\mathfrak{g}/\mathfrak{u}$ is reductive. Therefore,

$$[\mathfrak{g},\mathfrak{g}]/([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{u}) \cong [\mathfrak{g}/\mathfrak{u},\mathfrak{g}/\mathfrak{u}] \tag{14.1}$$

is semisimple. Note that

$$(\mathrm{Rad}([\mathfrak{g},\mathfrak{g}]) + ([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{u}))/([\mathfrak{g},\mathfrak{g}]\cap\mathfrak{u}) \tag{14.2}$$

is a solvable ideal of the semisimple Lie algebra (14.1). Therefore, (14.2) is the zero ideal, and this lemma follows.

Fix a Nash homomorphism $\varphi : G \to \mathrm{GL}(V)$ with a finite kernel, where V is a finite-dimensional real vector space. Then $\mathfrak{g}$ is identified with a Lie subalgebra of $\mathfrak{gl}(V)$.

Lemma 14.4 *All maximal fully reducible Lie subalgebras of* $\mathfrak{g}$ *are conjugate to each other under* $\mathfrak{U}_G$.

Proof This is a direct consequence of Lemma 14.3 and the first assertion of Lemma 14.2.

Fix a pair $(\mathfrak{l}, K)$, where $\mathfrak{l}$ is a maximal fully reducible Lie subalgebra of $\mathfrak{g}$, and K is a maximal compact subgroup of the normalizer $\widetilde{L}$ of $\mathfrak{l}$ in G. Lemma 14.4 and Theorem 4.2 imply that all such pairs are conjugate to each other under G. Denote by L_0 the analytic subgroup of G with Lie algebra $\mathfrak{l}$.

Lemma 14.5 *The subgroup* L_0 *of* G *is a reductive Nash subgroup of* G.

Proof Denote by L_0' the smallest Nash subgroup of G containing L_0. It is connected, since L_0 is so. Note that the set of L_0-stable subspaces of V is the same as the set of L_0'-stable subspaces. Therefore, V is completely reducible as a representation of L_0'. This implies that L_0' is reductive and its Lie algebra is fully reducible. The maximality of $\mathfrak{l}$ then implies that $L_0 = L_0'$, and this lemma follows.

Put $L := KL_0$, which is a Nash subgroup of G. We want to show that

$$G = L \ltimes \mathfrak{U}_G. \tag{14.3}$$

Lemma 14.6 *One has* $L^\circ = L_0$.

Proof Since L_0 is reductive, the unipotent radical $\mathfrak{U}_L$ of L has trivial intersection with L_0. Then the quotient homomorphism

$$L \to L/L_0 \cong K/(K \cap L_0)$$

is restricted to an injective Nash homomorphism from $\mathfrak{U}_L$ to an elliptic Nash group. Therefore, $\mathfrak{U}_L$ is trivial and L is reductive. Then the Lie algebra of L is fully reducible and contains $\mathfrak{l}$, and hence equals $\mathfrak{l}$ by the maximality of $\mathfrak{l}$. This proves this lemma.

Since we have proved that L is reductive, we know

$$L \cap \mathfrak{U}_G = \{1\}. \tag{14.4}$$

Lemma 14.7 *One has* $\mathfrak{l} + \mathfrak{u} = \mathfrak{g}$.

Proof Let s be a semisimple element of $\mathfrak{g}$. By Lemma 13.1, the replica $\langle s \rangle$ is a Nash torus. Therefore, its Lie algebra Lie $\langle s \rangle$ is fully reducible. By Lemma 14.4, there is an element $u \in \mathfrak{U}_G$ such that

$$s \in \mathrm{Lie}\,\langle s \rangle \subset \mathrm{Ad}_u(\mathfrak{l}) \subset \mathrm{Ad}_u(\mathfrak{l} + \mathfrak{u}) = \mathfrak{l} + \mathfrak{u}.$$

This proves that $\mathfrak{l} + \mathfrak{u} \supset \mathfrak{g}_{\mathrm{ss}}$. Then Lemma 13.2 implies that $(\mathfrak{l} + \mathfrak{u})/\mathfrak{u} \supset (\mathfrak{g}/\mathfrak{u})_{\mathrm{ss}}$. Since $(\mathfrak{g}/\mathfrak{u})_{\mathrm{ss}}$ is dense in $\mathfrak{g}/\mathfrak{u}$ by Theorem 13.1, one knows that $(\mathfrak{l} + \mathfrak{u})/\mathfrak{u} \supset \mathfrak{g}/\mathfrak{u}$. Therefore, $\mathfrak{l} + \mathfrak{u} = \mathfrak{g}$.

Combining (14.4) and Lemma 14.7, we get

$$G^\circ = L_0 \ltimes \mathfrak{U}_G, \quad \mathfrak{g} = \mathfrak{l} \ltimes \mathfrak{u}. \tag{14.5}$$

Recall that $\widetilde{L}$ denotes the normalizer of $\mathfrak{l}$ in G. Write $\widetilde{\mathfrak{l}}$ for its Lie algebra, and put $\mathfrak{u}_0 := \widetilde{\mathfrak{l}} \cap \mathfrak{u}$. Then

$$\widetilde{\mathfrak{l}} = \mathfrak{l} \times \mathfrak{u}_0$$

is a direct product of Lie algebras. Consequently, we have

$$(\widetilde{L})^\circ = L_0 \times U_0, \quad \text{where } U_0 := \widetilde{L} \cap \mathfrak{U}_G. \tag{14.6}$$

Lemma 14.8 *Every connected reductive Nash subgroup of* $\widetilde{L}$ *is contained in* L_0.

Proof In view of (14.6), this lemma holds because every Nash homomorphism from a reductive Nash group to a unipotent Nash group is trivial.

Lemma 14.9 *One has* $G = L \ltimes \mathfrak{U}_G$.

Proof By (14.4)–(14.5), it suffices to show that every connected component of G meets K. Since K meets every connected component of $\widetilde{L}$, it suffices to show that every connected component of G meets $\widetilde{L}$. Let $g \in G$. Then by Lemma 14.4, $\mathrm{Ad}_g(\mathfrak{l}) = \mathrm{Ad}_u(\mathfrak{l})$ for some $u \in \mathfrak{U}_G$. Therefore, $u^{-1}g \in \widetilde{L}$, and this lemma follows.

Lemma 14.9 implies that L is a maximal reductive Nash subgroup of G.

Lemma 14.10 *Every reductive Nash subgroup* R *of* G *is contained in a conjugation of* L.

Proof By Lemma 14.2, we assume without loss of generality that $R \subset \widetilde{L}$. Then Lemma 14.8 implies that $R^\circ \subset L_0$. Let K' be a maximal compact subgroup of R. Then Theorem 4.2 implies that $K' \subset gKg^{-1}$ for some $g \in \widetilde{L}$. Therefore,

$$R = K'R^\circ \subset gKg^{-1}L_0 = gKL_0g^{-1} = gLg^{-1}.$$

Lemma 14.10 implies that all maximal reductive Nash subgroups of G are conjugate to L (since $G = L\mathfrak{U}_G$, they are actually conjugate to L under $\mathfrak{U}_G$). This finishes the proof of Theorem 14.1.

15 Cartan Decompositions and Iwasawa Decompositions

We first recall some basic results concerning Cartan decompositions in the setting of connected semisimple Lie groups with finite centers.

Proposition 15.1 (see [12, Theorems 6.31, 6.51 and Proposition 6.40]) *Let G be a connected semisimple Lie group with a finite center. Denote by $\mathfrak{g}$ its Lie algebra. Let K be a maximal compact subgroup of G. Then the followings hold:*

(1) *There exists a unique continuous involution θ_K of G such that $G^{\theta_K} = K$.*

(2) *Denote by $\mathfrak{p}$ the (-1)-eigenspace in $\mathfrak{g}$ of the differential of θ_K, and then the map*

$$K \times \mathfrak{p} \to G, \quad k, x \mapsto k \exp(x)$$

is a diffeomorphism.

(3) *All maximal abelian subspaces of $\mathfrak{p}$ are conjugate to each other under the adjoint action of K.*

(4) *For every $x \in \mathfrak{p}$, the linear operator*

$$\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}, \quad y \mapsto [x, y]$$

is semisimple and all its eigenvalues are real.

Here "involution" means an automorphism of order 1 or 2; and G^{θ_K} denotes the fixed-point set of θ_K in G (the similar notation will be used without further explanation).

In this section, we investigate Cartan involutions for all reductive Nash groups. Let G be a reductive Nash group in the rest of this section.

Definition 15.1 *A Cartan involution of G is a Nash involution of G whose fixed-point set is a maximal compact subgroup of G.*

Here "Nash involution" means an involution which is simultaneously a Nash map. The first result of this section we intend to prove is the following theorem.

Theorem 15.1 *The map*

$$\begin{gathered}\{\textit{Cartan involution of } G\} \to \{\textit{maximal compact subgroup of } G\},\\ \theta \mapsto G^\theta\end{gathered} \tag{15.1}$$

is bijective.

We begin with the following lemma.

Lemma 15.1 *Theorem* 15.1 *holds if G is a connected semisimple Nash group or a Nash torus.*

Proof If G is a connected semisimple Nash group, then all Lie group automorphisms of G are Nash automorphisms. Therefore, this lemma is implied by the first assertion of Proposition 15.1. If G is a Nash torus, then $G = G_{\mathrm{e}} \times G_{\mathrm{h}}$, and G_{e} is the unique maximal compact subgroup of G. Moreover,

$$G_{\mathrm{e}} \times G_{\mathrm{h}} \to G_{\mathrm{e}} \times G_{\mathrm{h}}, \quad (x, y) \mapsto (x, y^{-1})$$

is the unique Cartan involution of G. Therefore, this lemma also holds.

Denote by $\mathfrak{g}$ the Lie algebra of G, and write

$$\mathfrak{g} = \mathfrak{z} \oplus \mathfrak{s},$$

where $\mathfrak{z}$ denotes the center of $\mathfrak{g}$, and $\mathfrak{s} := [\mathfrak{g}, \mathfrak{g}]$. As before, denote by Z and S the analytic subgroups of G with Lie algebras $\mathfrak{z}$ and $\mathfrak{s}$, respectively. Then Z is a Nash torus, and S is a connected semisimple Nash group. Let K be a maximal compact subgroup of G, and put

$$K_0 := K \cap S.$$

Lemma 15.2 *One has $K \cap Z = Z_{\mathrm{e}}$, which is the unique maximal compact subgroup of Z; K_0 is a maximal compact subgroup of S; and $K^\circ = K_0 Z_{\mathrm{e}}$.*

Proof The equality $K \cap Z = Z_{\mathrm{e}}$ is obvious. Denote by $\varphi : S \times Z \to G^\circ$ the multiplication map. It is a finite-fold covering homomorphism. Note that K° is a maximal compact subgroup of G°. Therefore, $\varphi^{-1}(K^\circ)$ is a maximal compact subgroup of $S \times Z$, which has the form $K_0' \times Z_{\mathrm{e}}$, where K_0' is a maximal compact subgroup of S. We have

$$K^\circ = \varphi(\varphi^{-1}(K^\circ)) = K_0' Z_{\mathrm{e}},$$

which implies that $K_0 \supset K_0'$. Since K_0' is already a maximal compact subgroup of S, we have that $K_0 = K_0'$. This proves this lemma.

Lemma 15.3 *The map* (15.1) *is injective.*

Proof Let θ and θ' be two Cartan involutions of G such that $G^\theta = G^{\theta'}$. Then $S^\theta = S^{\theta'}$ and $Z^\theta = Z^{\theta'}$. Therefore, Lemmas 15.1–15.2 imply that

$$\theta|_S = \theta'|_S \quad \text{and} \quad \theta|_Z = \theta'|_Z.$$

The lemma then follows as $G = G^\theta S Z$.

Using Lemmas 15.1–15.2, write θ_S for the unique Cartan involution of S with a fix-point set K_0. Write θ_Z for the unique Cartan involution of Z.

Lemma 15.4 *There exists a unique Cartan involution of G° extending both θ_S and θ_Z.*

Proof Uniqueness holds as $G^\circ = SZ$. Note that $S \cap Z$ is contained in both K_0 and Z_{e}. Hence θ_S and θ_Z have a common extension to an involution of G°. One checks that this involution has K° as its fixed-point set, and hence it is a Cartan involution.

Denote by θ° the Cartan involution of G° of Lemma 15.4. Define a map

$$G \to G, \quad kg \mapsto k\,\theta^\circ(g) \quad (k \in K,\, g \in G^\circ). \tag{15.2}$$

It is routine to check that (15.2) is a well-defined Nash involution of G whose fixed-point set equals K. This finishes the proof of Theorem 15.1.

Now let θ be the Cartan involution of G so that $G^\theta = K$. Still denote by $\theta : \mathfrak{g} \to \mathfrak{g}$ its differential. Denote by $\mathfrak{p}$ the (-1)-eigenspace of θ in $\mathfrak{g}$.

Proposition 15.2 *The map*

$$\begin{aligned} K \times \mathfrak{p} &\to G, \\ (k, x) &\mapsto k \exp(x) \end{aligned} \tag{15.3}$$

is a diffeomorphism.

Proof Without loss of generality, assume that G is connected. The second assertion of Proposition 15.1 as well as its analog for Nash tori implies that the map

$$\begin{aligned} K_0 \times Z_{\mathrm{e}} \times (\mathfrak{p} \cap \mathfrak{s}) \times (\mathfrak{p} \cap \mathfrak{z}) &\to S \times Z, \\ (k, t, x, y) &\mapsto (k \exp(x), t \exp(y)) \end{aligned} \tag{15.4}$$

is a diffeomorphism. This descends to a deffeomorphism

$$\nabla \backslash (K_0 \times Z_{\mathrm{e}}) \times (\mathfrak{p} \cap \mathfrak{s}) \times (\mathfrak{p} \cap \mathfrak{z}) \to \nabla \backslash (S \times Z), \tag{15.5}$$

where

$$\nabla := \{(t, t^{-1}) \mid t \in K_0 \cap Z_{\mathrm{e}} = S \cap Z\}.$$

The lemma then follows since the smooth map (15.3) is obviously identified with (15.5).

Lemma 15.5 *One has* $\mathfrak{p} \subset \mathfrak{g}_{\mathrm{h}}$.

Proof Without loss of generality, assume that G is connected and semisimple. Let $x \in \mathfrak{p}$. By uniqueness of Jordan decompositions, the quality

$$(-x_{\mathrm{e}}) + (-x_{\mathrm{h}}) + (-x_{\mathrm{u}}) = -x = \theta(x) = \theta(x_{\mathrm{e}}) + \theta(x_{\mathrm{h}}) + \theta(x_{\mathrm{e}})$$

implies that $\theta(x_{\mathrm{e}}) = -x_{\mathrm{e}}$, that is, $x_{\mathrm{e}} \in \mathfrak{p}$. Likewise, $x_{\mathrm{h}} \in \mathfrak{p}$ and $x_{\mathrm{u}} \in \mathfrak{p}$. Therefore, it suffices to show that $\mathfrak{p} \cap \mathfrak{g}_{\mathrm{e}} = \{0\}$ and $\mathfrak{p} \cap \mathfrak{g}_{\mathrm{u}} = \{0\}$. Note that for every $y \in \mathfrak{g}_{\mathrm{e}}$, the linear operator $\mathrm{ad}_y : \mathfrak{g} \to \mathfrak{g}$ is semisimple and all its eigenvalues are purely imaginary. Together with the last assertion of Proposition 15.1, this implies that $\mathfrak{p} \cap \mathfrak{g}_{\mathrm{e}} = \{0\}$. The equality $\mathfrak{p} \cap \mathfrak{g}_{\mathrm{u}} = \{0\}$ is proved similarly.

Proposition 15.3 *Each θ-stable Nash subgroup G_1 of G is reductive and equals $K_1 \exp(\mathfrak{p}_1)$, where*

$$K_1 := G_1 \cap K \quad \textit{and} \quad \mathfrak{p}_1 := (\mathrm{Lie}\, G_1) \cap \mathfrak{p}.$$

Proof Let $g = k\exp(x) \in G_1$, where $k \in K$ and $x \in \mathfrak{p}$. Then

$$\exp(2x) = (\exp(x))^2 = (\exp(x)k^{-1})(k\exp(x)) = \theta(g^{-1})g \in G_1.$$

Then Lemma 15.5 and Proposition 9.2 imply that $\exp(\mathbb{R}x) \subset G_1$. Consequently,

$$x \in \mathrm{Lie}\, G_1, \quad \exp(x) \in G_1 \quad \text{and} \quad k \in G_1.$$

Therefore,

$$G_1 = K_1 \exp(\mathfrak{p}_1).$$

Denote by U_1 the unipotent radical of G_1. Then it is also a θ-stable Nash subgroup of G. Therefore,

$$U_1 = K_1' \exp(\mathfrak{p}_1'), \quad \text{where } K_1' := U_1 \cap K,\ \mathfrak{p}_1' := (\mathrm{Lie}\, U_1) \cap \mathfrak{p}.$$

It is clear that $K_1' = \{1\}$ and $\mathfrak{p}_1' = \{0\}$. Therefore, U_1 is trivial and G_1 is reductive.

The following result is an obvious consequence of the third assertion of Proposition 15.1.

Proposition 15.4 *All maximal abelian subspaces of $\mathfrak{p}$ are conjugate to each other under K.*

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Denote by A the analytic subgroup of G with Lie algebra $\mathfrak{a}$.

Proposition 15.5 *The analytic subgroup A is a hyperbolic Nash subgroup of G.*

Proof Denote by G_1 the centralizer of $\mathfrak{a}$ in G, which is a θ-stable Nash subgroup of G. Note that $(\mathrm{Lie}\, G_1) \cap \mathfrak{p} = \mathfrak{a}$, since $\mathfrak{a}$ is maximal abelian in $\mathfrak{p}$. Therefore, by Proposition 15.3,

$$G_1 = K_1 \exp(\mathfrak{a}) = K_1 \times A, \quad \text{where } K_1 := G_1 \cap K.$$

Denote by Z_1 the center of G_1, and then A equals the identity connected component of the Nash subgroup

$$\{x \in Z_1 \mid \theta(x) = x^{-1}\}.$$

Therefore, A is a Nash subgroup.

Note that A is abelian and all elements of A are hyperbolic. Therefore, A is hyperbolic by Proposition 8.3.

Lemma 15.6 *The set $\exp(\mathfrak{p})$ is a close Nash submanifold of G.*

Proof The set $\exp(\mathfrak{p})$ is a closed submanifold of G by Proposition 15.2. It is semialgebraic, since it is equal to the image of the Nash map

$$K \times A \to G, \quad (k, a) \mapsto kak^{-1}.$$

Combining Propositions 15.2, 15.4 and Lemma 15.6, we obtain the following proposition.

Proposition 15.6 *One has $G = KAK$, and the multiplication map*

$$K \times \exp(\mathfrak{p}) \to G$$

is a Nash diffeomorphism.

Write

$$\mathfrak{g} = \bigoplus_{\alpha\in\mathfrak{a}^*} \mathfrak{g}_\alpha,$$

where $\mathfrak{a}^*$ denotes the space of all real valued linear functionals on $\mathfrak{a}$, and

$$\mathfrak{g}_\alpha := \{x \in \mathfrak{g} \mid [a, x] = \alpha(a)x, \text{ for all } a \in \mathfrak{a}\}.$$

Then the set

$$\Delta(\mathfrak{g}, \mathfrak{a}) := \{\alpha \in \mathfrak{a}^* \mid \alpha \neq 0,\, \mathfrak{g}_\alpha \neq \{0\}\}$$

is a root system in $\mathfrak{a}^*$. Fix a positive system $\Delta(\mathfrak{g}, \mathfrak{a})^+ \subset \Delta(\mathfrak{g}, \mathfrak{a})$, and put

$$\mathfrak{n} := \bigoplus_{\alpha\in\Delta(\mathfrak{g},\mathfrak{a})^+} \mathfrak{g}_\alpha.$$

Then $\mathfrak{n}$ is a Lie subalgebra of $\mathfrak{g}$. Denote by N the analytic subgroup of G with Lie algebra $\mathfrak{n}$.

Proposition 15.7 *The analytic subgroup N is a unipotent Nash subgroup of G.*

Proof Without loss of generality, assume that G is semisimple and connected. Denote by a_0 the element of $\mathfrak{a}$ such that $\alpha(a_0) = 1$ for all simple roots α in $\Delta(\mathfrak{g}, \mathfrak{a})^+$. For every integer i, denote

$$\mathfrak{g}_i := \{x \in \mathfrak{g} \mid [a_0, x] = ix\}.$$

Put

$$\widetilde{\mathfrak{n}} := \Big\{g \in \mathfrak{gl}(\mathfrak{g}) \,\Big|\, g\Big(\bigoplus_{j\geq i} \mathfrak{g}_j\Big) \subset \bigoplus_{j\geq i+1} \mathfrak{g}_j \text{ for all } i \in \mathbb{Z}\Big\},$$
$$\widetilde{N} := \Big\{g \in \mathrm{GL}(\mathfrak{g}) \,\Big|\, (g-1)\Big(\bigoplus_{j\geq i} \mathfrak{g}_j\Big) \subset \bigoplus_{j\geq i+1} \mathfrak{g}_j \text{ for all } i \in \mathbb{Z}\Big\}.$$

Then $\widetilde{N}$ is a unipotent Nash subgroup of $\mathrm{GL}(\mathfrak{g})$ with Lie algebra $\widetilde{\mathfrak{n}}$.

Consider the adjoint representation

$$\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$$

and its differential

$$\mathrm{ad} : \mathfrak{g} \to \mathrm{GL}(\mathfrak{g}).$$

Note that $\mathrm{ad}^{-1}(\widetilde{\mathfrak{n}}) = \mathfrak{n}$. Therefore, the Nash subgroup $\mathrm{Ad}^{-1}(\widetilde{N})$ of G has the Lie algebra $\mathfrak{n}$. Hence N equals the identity connected component of $\mathrm{Ad}^{-1}(\widetilde{N})$, which is a Nash subgroup of G. Since G is assumed to be semisimple, the adjoint representation of N on $\mathfrak{g}$ has a finite kernel. Then Lemma 5.6 implies that N is unipotent.

Theorem 15.2 *The multiplication map*

$$K \times A \times N \to G \tag{15.6}$$

is a Nash diffeomorphism.

Proof Without loss of generality, assume that G is connected. The map (15.6) is clearly a Nash map. We only need to show that it is a diffeomorphism. This is known when G is semisimple (see [12, Theorem 6.46]). The same argument as in Proposition 15.2 reduces the general case to the case when G is semisimple.

As a corollary of Theorem 15.2, we have the following proposition.

Proposition 15.8 *An almost linear Nash group is elliptic if it consists elliptic elements only.*

Proof If an almost linear Nash group consists only elliptic elements, then its unipotent radical is trivial, and is thus reductive. Then Theorem 15.2 implies that it is compact.

The same proof as Proposition 15.8 shows the following proposition.

Proposition 15.9 *An almost linear Nash group is hyperbolic if it consists hyperbolic elements only.*

16 Exponential Nash Groups

Recall from Section 1 that an almost linear Nash group G is said to be exponential if $G_{\mathrm{e}} = \{1\}$. The following lemma is obvious.

Lemma 16.1 *An almost linear Nash group is exponential if and only if all its elements are exponential.*

Proposition 8.1 implies the following lemma.

Lemma 16.2 *All Nash quotient groups of exponential Nash groups are exponential Nash groups.*

Let G be an almost linear Nash group, and let K be a maximal compact subgroup of G.

Lemma 16.3 *The almost linear Nash group G is exponential if and only if K is trivial.*

Proof The lemma is clear, since

$$G_{\mathrm{e}} = \bigcup_{g \in G} gKg^{-1}.$$

Lemma 16.4 *If G is reductive and exponential, then G is hyperbolic.*

Proof The lemma follows by Lemma 16.3 and Proposition 15.6.

Lemma 16.5 *The almost linear Nash group G is exponential if and only if $G/\mathfrak{U}_G$ is a hyperbolic Nash group.*

Proof In view of Lemma 16.2, the "only if" part is implied by Lemma 16.4. To prove the "if" part, assume that $G/\mathfrak{U}_G$ is a hyperbolic Nash group. Then under the quotient map

$$G \to G/\mathfrak{U}_G,$$

the image of G_{e} is contained in

$$(G/\mathfrak{U}_G)_{\mathrm{e}} = \{1\}.$$

Therefore, $G_{\mathrm{e}} \subset \mathfrak{U}_G$, which implies that $G_{\mathrm{e}} = \{1\}$ as $(\mathfrak{U}_G)_{\mathrm{e}} = \{1\}$.

Using Levi decompositions, Lemma 16.5 implies that every exponential Nash group is connected, simply connected and solvable.

Lemma 16.6 *If G is unipotent or hyperbolic, then there is no proper co-compact Nash subgroup of G.*

Proof The hyperbolic case is obvious. Assume that G is unipotent. We prove this lemma by induction on $\dim G$. It is trivial when $\dim G = 0$. Assume that $\dim G > 0$ and this lemma holds for unipotent Nash groups of smaller dimensions.

Let H be a co-compact Nash subgroup of G. Denote by Z the center of G. It is a Nash subgroup of G of positive dimension. Note that ZH is a Nash subgroup of G, and ZH/H is a closed subset of G/H. Therefore,

$$Z/(Z \cap H) = ZH/H$$

is compact. Since the lemma obviously holds for abelian unipotent Nash groups, we have that $Z \cap H = Z$, or equivalently, $H \supset Z$. Then H/Z is a co-compact Nash subgroup of the unipotent Nash group G/Z. Since $\dim G/Z < \dim G$, by the induction hypothesis, we have $H/Z = G/Z$, in other words, $H = G$.

Lemma 16.7 *If G is exponential, then there is no proper co-compact Nash subgroup of G.*

Proof Let H be a co-compact Nash subgroup of G. Using Proposition 3.5, we get a closed orbit $O \subset G/H$ under left translations by $\mathfrak{U}_G$. Since O is compact, Lemma 16.6 implies that O has only one point, say $g_0 H$. Then $\mathfrak{U}_G g_0 H \subset g_0 H$, which implies that $\mathfrak{U}_G \subset H$ as $\mathfrak{U}_G$ is a normal subgroup of G. Now $H/\mathfrak{U}_G$ is a co-compact Nash subgroup of the hyperbolic Nash group $G/\mathfrak{U}_G$. Lemma 16.6 implies that $H/\mathfrak{U}_G = G/\mathfrak{U}_G$, in other words, $H = G$.

The following is the Borel fixed-point theorem in the setting of Nash groups.

Theorem 16.1 *Let $G \times M \to M$ be a Nash action of G on a non-empty Nash manifold M. If G is exponential and M is compact, then the action has a fixed-point.*

Proof Using Proposition 3.5, we get a closed G-orbit $O \subset M$. Then O is compact and Lemma 16.7 implies that O has only one point.

Lemma 16.8 *There exists an exponential Nash subgroup B of G such that the multiplication map $K \times B \to G$ is a Nash diffeomorphism.*

Proof Without loss of generality, assume that G is reductive (otherwise, take a Levi component of G containing K). Then the group AN of Theorem 15.2 fulfills the requirement of this lemma.

Lemma 16.9 *If G is not exponential, then G has a proper co-compact Nash subgroup.*

Proof The group B of Lemma 16.8 is a proper co-compact Nash subgroup of G.

Recall the following lemma.

Lemma 16.10 (see [13, Section I.1, Theorem 1]) *Let H be a connected, simply connected, solvable Lie group with Lie algebra $\mathfrak{h}$. If the exponential map* $\exp : \mathfrak{h} \to H$ *is either injective or surjective, then it is a diffeomorphism.*

Denote by $\mathfrak{g}$ the Lie algebra of G.

Proposition 16.1 *The almost linear Nash group G is exponential if and only if the exponential map*

$$\exp : \mathfrak{g} \to G \tag{16.1}$$

is a diffeomorphism.

Proof The "only if" part is implied by Proposition 9.1 and Lemma 16.10. To prove the "if" part of the proposition, assume that (16.1) is a diffeomorphism. Then G is connected. Therefore, K is connected and the exponential map

$$\exp : \mathfrak{k} \to K \tag{16.2}$$

is injective, where $\mathfrak{k}$ denotes the Lie algebra of K. This forces K to be trivial. Therefore, G is exponential by Lemma 16.3.

Lemma 16.11 *For each co-compact Nash subgroup H of G, one has that* $\dim H \geq \dim G/K$.

Proof Let B be as in Lemma 16.8. By Theorem 16.1, the left translation action of B on G/H has a fixed point, say $g_0 H$. Then $Bg_0H \subset g_0H$, which implies that

$$\dim H \geq \dim B = \dim G/K.$$

Denote by $\mathrm{B}_n(\mathbb{R})$ the Nash subgroup of $\mathrm{GL}_n(\mathbb{R})$ consisting all upper-triangular matrices with positive diagonal entries ($n \geq 0$). Its Lie algebra $\mathfrak{b}_n(\mathbb{R})$ consists all upper-triangular matrices in $\mathfrak{gl}_n(\mathbb{R})$. It is obvious that $\mathrm{B}_n(\mathbb{R})$ and all its Nash subgroups are exponential Nash groups. Conversely, we have the following lemma.

Lemma 16.12 *Every exponential Nash group H is Nash isomorphic to a Nash subgroup of* $\mathrm{B}_n(\mathbb{R})$ *for some* $n \geq 0$.

Proof Fix a Nash representation V of H with a finite kernel. By Lemma 16.3, the representation is actually faithful. Consider the induced action of H on the compact Nash manifold of all full flags in V. Then Theorem 16.1 implies that the action has a fixed-point, that is, H stabilizes a full flag in V. Therefore, there exists an injective Nash homomorphism $\varphi : H \to \mathrm{B}'_n(\mathbb{R})$, where $n := \dim V$, and $\mathrm{B}'_n(\mathbb{R})$ denotes the Nash subgroup of $\mathrm{GL}_n(\mathbb{R})$ of upper-triangular matrices. Since H is connected, $\varphi(H)$ is contained in $\mathrm{B}_n(\mathbb{R})$ and this lemma follows.

Theorem 16.2 *Every exponential Nash subgroup of G is contained in a maximal one, and all maximal exponential Nash subgroups of G are conjugate to each other in G.*

Proof Let B be as in Lemma 16.8. Let H be an exponential Nash subgroup of G. By Theorem 16.1, the left translation action of H on G/B has a fixed-point, say, g_0B. Then $Hg_0B \subset g_0B$, and consequently, H is contained in a conjugation of B. Therefore, B has the largest dimension among all exponential Nash subgroups of G. In particular, B is a maximal exponential Nash subgroup of G (since all exponential Nash groups are connected). This proves this theorem.

Theorem 16.3 *A Nash subgroup B of G is a maximal exponential Nash subgroup if and only if the multiplication map $K \times B \to G$ is a Nash diffeomorphism.*

Proof Let B be a Nash subgroup of G. We first prove the "if" part of the theorem. So assume that the multiplication map $K \times B \to G$ is a Nash diffeomorphism. Then B is connected. Let K' be a maximal compact subgroup of B. Applying Lemma 16.8 to B, we get an exponential Nash subgroup B' of B so that the multiplication map $K' \times B' \to B$ is a Nash diffeomorphism. Then B' is co-compact in G. Hence, by Lemma 16.11,

$$\dim B' \geq \dim G/K = \dim B.$$

Therefore, $B' = B$, and B is an exponential Nash subgroup of G. Then the proof of Theorem 16.2 shows that B is a maximal exponential Nash subgroup of G.

To prove the "only if" part of the theorem, assume that B is a maximal exponential Nash subgroup of G. Using Lemma 16.9, we take an exponential Nash subgroup B_0 of G so that the multiplication map

$$K \times B_0 \to G \quad \text{is a Nash diffeomorphism.} \tag{16.3}$$

The proof of Theorem 16.2 shows that B_0 is a maximal exponential Nash subgroup of G. By Theorem 16.2,

$$B = g_0 B_0 g_0^{-1} \quad \text{for some } g_0 \in G.$$

Write $g_0 = k_0 b_0$, where $k_0 \in K$ and $b_0 \in B_0$. Note that (16.3) implies that the multiplication map

$$K \times k_0 B_0 k_0^{-1} \to G$$

is a Nash diffeomorphism. The "only if" part of the theorem then follows as $k_0 B_0 k_0^{-1} = B$.

Theorem 16.4 *Every hyperbolic Nash subgroup of G is contained in a maximal one, and all maximal hyperbolic Nash subgroups of G are conjugate to each other in G.*

Proof Fix a maximal exponential Nash subgroup B of G, and fix a Levi component A of B. Let H be a hyperbolic Nash subgroup of G. Then by Theorem 16.2, a conjugation of H is contained in B. Theorem 14.1 further implies that a conjugation of H is contained in A. As in the proof of Theorem 16.2, we know that A is a maximal hyperbolic Nash subgroup by reason of dimension. This proves the theorem.

Lemma 16.13 *If G is exponential, then every unipotent Nash subgroup of G is contained in $\mathfrak{U}_G$.*

Proof Let U be a unipotent Nash subgroup of G. Then the quotient homomorphism

$$G \to G/\mathfrak{U}_G$$

has trivial restriction to U. Therefore, $U \subset \mathfrak{U}_G$.

In view of Lemma 16.13, a similar argument as Theorem 16.4 implies the following theorem.

Theorem 16.5 *Every unipotent Nash subgroup of G is contained in a maximal one, and all maximal unipotent Nash subgroups of G are conjugate to each other in G.*

By the preceding arguments, we know that for each maximal exponential Nash subgroup B of G, its unipotent radical $\mathfrak{U}_B$ is a maximal unipotent Nash subgroup of G, and each Levi component of B is a maximal hyperbolic Nash subgroup of G.

17 About Proofs of the Results in Section 1

In this last section, we collect some results of the previous sections to explain the proofs of those propositions and theorems which occur in Section 1.

Proposition 1.1 is a restatement of Proposition 3.8.

Recall that Proposition 1.2 asserts the following: An almost linear Nash group is elliptic, hyperbolic or unipotent if and only if all of its elements are elliptic, hyperbolic or unipotent, respectively. The "only if" part of Proposition 1.2 is trivial. The elliptic case and the hyperbolic case of the "if" part are proved in Propositions 15.8–15.9, respectively. To prove the "if" part in the unipotent case, let G be an almost linear Nash group consisting unipotent elements only. Then G is exponential. Hence, a Levi component of G is a hyperbolic Nash group, which has to be trivial. Therefore, G is unipotent. This finishes the proof of Proposition 1.2.

Proposition 1.3 consists two assertions. The first one is the following proposition.

Proposition 17.1 *Let G be an almost linear Nash group which is elliptic, hyperbolic or unipotent. Then all of its Nash subgroups and Nash quotient groups are elliptic, hyperbolic or unipotent, respectively.*

Proof The assertion for Nash subgroups is obvious. The assertion for Nash quotient groups appears in Propositions 4.1, 6.2 and 7.6.

The second assertion of Proposition 1.3 is the following proposition.

Proposition 17.2 *Let G be an almost linear Nash group. If G has a normal Nash subgroup H so that H and G/H are both elliptic, both hyperbolic or both unipotent, then G is elliptic, hyperbolic or unipotent, respectively.*

Assume that both H and G/H are elliptic. The image of G_{h} under the quotient map

$$G \to G/H$$

is contained in $(G/H)_{\mathrm{h}} = \{1\}$. Therefore, $G_{\mathrm{h}} \subset H$, which implies that $G_{\mathrm{h}} = \{1\}$. Similarly, $G_{\mathrm{u}} = \{1\}$. Therefore, $G = G_{\mathrm{e}}$, and Proposition 1.2 implies that G is elliptic.

The same argument proves this proposition in the hyperbolic and unipotent cases.

As already mentioned, Theorem 1.1 is a combination of Lemmas 4.1–4.3, and Theorem 1.2 is a combination of Propositions 5.1, 5.3–5.5. Theorem 1.3 is a restatement of Theorem 1.3. Theorem 1.4 is the same as Theorem 8.1.

Theorem 1.5 consists of five assertions. The second one is obvious. The others are respectively proved in Lemma 10.1, Propositions 10.3, 10.2 and 10.1.

For Theorem 1.6, it is obvious that (b) $\Rightarrow$ (a), and Theorem 11.2 asserts that (a) $\Rightarrow$ (b). Lemma 14.1 asserts that (a) $\Leftrightarrow$ (c). Theorem 12.1 implies that

$$(\mathrm{d}) \Leftrightarrow (\mathrm{a}) \Leftrightarrow (\mathrm{e}).$$

The equivalence (a) $\Leftrightarrow$ (f) is proved in Lemma 11.1, and (a) $\Leftrightarrow$ (g) is proved in Theorem 11.1. Therefore, Theorem 1.6 holds.

For Theorem 1.7, (a) $\Leftrightarrow$ (b) is implied by Lemma 16.3, and (a) $\Leftrightarrow$ (c) is implied by Lemmas 16.7 and 16.9. The equivalence (a) $\Leftrightarrow$ (d) is proved in Lemma 16.5, (a) $\Leftrightarrow$ (e) is implied by Lemma 16.12, and (a) $\Leftrightarrow$ (f) is proved in Proposition 16.1. By Theorem 16.1, (a) $\Rightarrow$ (g), and by Lemma 16.9, (g) $\Rightarrow$ (a). In conclusion, Theorem 1.7 holds.

Theorem 1.8 is contained in Theorems 4.2, 14.1, 16.2, 16.4–16.5.

Finally, Theorem 1.9 is contained in Theorem 14.1, and Theorem 1.10 is contained in Theorem 16.3.

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