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Spherical Scattered Data Quasi-interpolation by Gaussian Radial Basis Function*

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Abstract Since the spherical Gaussian radial function is strictly positive definite, the authors use the linear combinations of translations of the Gaussian kernel to interpolate the scattered data on spheres in this article. Seeing that target functions are usually outside the native spaces, and that one has to solve a large scaled system of linear equations to obtain combinatorial coefficients of interpolant functions, the authors first probe into some problems about interpolation with Gaussian radial functions. Then they construct quasi-interpolation operators by Gaussian radial function, and get the degrees of approximation. Moreover, they show the error relations between quasi-interpolation and interpolation when they have the same basis functions. Finally, the authors discuss the construction and approximation of the quasi-interpolant with a local support function.

Keywords Scattered data, Approximation, Spherical Gaussian radial basis function, Modulus of continuity
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1 Introduction

Let $\mathbb{S}^2$ be the unit sphere in a Euclidean space $\mathbb{R}^3$ defined by

$$\mathbb{S}^2 := \Big\{x := (x^{(1)}, x^{(2)}, x^{(3)}) \in \mathbb{R}^3 : \|x\|_2 := \sqrt{(x^{(1)})^2 + (x^{(2)})^2 + (x^{(3)})^2} = 1\Big\}.$$

For a target function f defined on the sphere, a set of scattered points $x_i, i = 1, 2, \cdots, n$, lying on $\mathbb{S}^2$, and associated values $f_i, i = 1, 2, \cdots, n$, we try to find a smooth function s defined on $\mathbb{S}^2$ by means of the data (x_i, f_i), such that s can approximate the target function f. This problem is called scattered data fitting on the sphere, and arises in many areas, including geophysics and meteorology, where the sphere $\mathbb{S}^2$ is usually taken as a model of the Earth. To solve the problem, several methods have been proposed (see [7]). In these methods, one of the important methods is the interpolation based on linear combinations of spherical radial basis functions. Up to now, there have been a lot of results on the topic. We refer the readers to [4–5, 9, 12–16, 21–23, 25–28].

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In the Euclidean space $\mathbb{R}^n$, the Gaussian radial basis function defined by

$$g(x) = \mathrm{e}^{-\rho\|x\|_2^2}, \quad \rho > 0,\ x \in \mathbb{R}^n,$$

is usually used to be a tool for constructing approximants to approximate the functions defined on the subset of $\mathbb{R}^n$. Particularly, in the approximation and fitting of scattered data, the Gaussian radial basis function has taken an important role (see [1–3, 18, 20, 26, 28]). On the sphere $\mathbb{S}^2$, the corresponding spherical Gaussian radial basis function (or called the zonal basis function) is defined by

$$g_y(x) = \mathrm{e}^{-2\rho(1-xy)}, \quad x, y \in \mathbb{S}^2, \tag{1.1}$$

where xy denotes the Euclidean inner product of x, y. Usually, $\mathrm{e}^{-2\rho(1-t)}$ $(t \in [-1, 1])$ is called spherical Gaussian kernel (see [5, 14]).

In general, if there exists a reproducing kernel Hilbert space resulted from a kernel, then the space is called a native space. When a target function f is in the native space, the error analysis has been completed. Yet, when f is outside the native space, this time there arises the so called "native space barrier" problem. It is an interesting and important topic, and has been discussed in much literature (see [4, 12–16, 22–23]). For example, in the recent articles (see [12–13, 27]), Le Gia, Sloan, and Wendland constructed approximants of functions outside the native space by means of a kernel.

In this article, we intend to discuss this problem. Our main aim is to study the constructive approximation for scattered data by means of a spherical Gaussian kernel.

The article is organized as follows. In the next section, we will state some preliminary results containing spherical harmonics and the native space. In Section 3, we probe into some problems about the interpolation and approximation by linear combinations of Gaussian radial functions. In Section 4, we will construct quasi-interpolation operators by Gaussian radial functions, and will get the degrees of approximation for continuous functions defined on $\mathbb{S}^2$. In Section 5, we will construct an interpolant to continuous functions, and will obtain the error estimates. Finally, we will briefly discuss the construction and approximation of quasi-interpolation operators with local compact support.

2 Preliminaries

For a function $\phi : [-1, 1] \to \mathbb{R}$, we set

$$K(x, y) := \phi(xy), \quad x, y \in \mathbb{S}^2.$$

A function $K(x, y)$ defined on $\mathbb{S}^2 \times \mathbb{S}^2$ is called a positive definite kernel, if for any finite subset X of $\mathbb{S}^2$, and arbitrary real numbers $C_\xi, \xi \in X$, there holds

$$\sum_{\xi \in X} \sum_{\zeta \in X} C_\xi C_\zeta K(\xi, \zeta) \geq 0. \tag{2.1}$$

If (2.1) is positive whenever the C_ξ are not all zero, then $K(x, y)$ is called strictly positive definite, and we also say ϕ is strictly positive definite.

Positive definite functions on spheres were first introduced and characterized by Schoenberg in [24], where it was showed that the necessary and sufficient condition of a continuous function ϕ being positive definite on $\mathbb{S}^2$ is that ϕ is expressible in the form

$$\phi(t) = \sum_{k=0}^{\infty} a_k P_k(t), \quad a_k \geq 0, \ \sum_{k=0}^{\infty} a_k < \infty, \tag{2.2}$$

where P_k is Legendre polynomial with $P_k(1) = 1$, and in fact, a_k is the Fourier-Legendre coefficient of ϕ. If $a_k > 0$ for all k, then ϕ is strictly positive definite (see [6, 29]). Strictly positive definite functions are both theoretically interesting and practically important, because they are used to reconstruct an unknown function from scattered data by the interpolation of the form $\sum\limits_{x_j \in X} \alpha_j \phi(xx_j)$.

Below, we introduce the native space $\mathcal{N}_\phi$ associated with ϕ. Let $L^2(\mathbb{S}^2)$ be the real Hilbert space equipped with the inner product

$$\langle f, g \rangle := \int_{\mathbb{S}^2} f(x)g(x)\mathrm{d}\omega(x), \tag{2.3}$$

where ω denotes the Lebesgue surface measure on $\mathbb{S}^2$. We will use $Y_{l,m}, m = 1, 2, \cdots, 2l+1$ to denote the usual orthonormal basis of spherical harmonics (see [8, 19, 25]). The class of all spherical harmonics of degree at most n will be denoted by Π_n. If $f \in L^2(\mathbb{S}^2)$, then we may expand it in a series of spherical harmonics,

$$f = \sum_{l=0}^{\infty} \sum_{m=1}^{2l+1} \widehat{f}_{l,m} Y_{l,m}, \quad \text{where } \widehat{f}_{l,m} = \langle f, Y_{l,m} \rangle.$$

Now we set $\phi_\rho(t) := \mathrm{e}^{-2\rho(1-t)}$. Narcowich et al. [21] obtained for $l \geq 0$,

$$\frac{2\rho^l \mathrm{e}^{-2\rho} \pi^{\frac{3}{2}}}{\Gamma\Big(l + \dfrac{3}{2}\Big)} \leq \widehat{\phi}_\rho(l) \leq \frac{2\rho^l \pi^{\frac{3}{2}}}{\Gamma\Big(l + \dfrac{3}{2}\Big)}. \tag{2.4}$$

Then, the native space $\mathcal{N}_{\phi_\rho}$ is defined as

$$\mathcal{N}_{\phi_\rho} := \Big\{ f(x) = \sum_{k=0}^{\infty} \sum_{j=1}^{2k+1} \widehat{f}_{k,j} Y_{k,j}(x) : \sum_{k=0}^{\infty} (\widehat{\phi}_\rho(k))^{-1} \sum_{j=1}^{2k+1} \widehat{f}_{k,j}^2 < \infty \Big\}.$$

The native space $\mathcal{N}_{\phi_\rho}$ is a Hilbert space with the inner product:

$$\langle f, g \rangle_{\mathcal{N}_{\phi_\rho}} := \sum_{k=0}^{\infty} (\widehat{\phi}_\rho(k))^{-1} \sum_{j=1}^{2k+1} \widehat{f}_{k,j} \widehat{g}_{k,j}.$$

Moreover, the space is a reproducing kernel Hilbert space, and the reproducing kernel is $\phi_\rho(xy)$ (see [15, 21, 26]). From [21], we know that the native space $\mathcal{N}_{\phi_\rho}$ is contained in a Sobolev space $H_s(\mathbb{S}^2)$ for all s, which shows that the functions in $\mathcal{N}_{\phi_\rho}$ have sufficient smoothness. However, the target functions usually have less smoothness. So it is important and necessary to discuss further the "native space barrier" problem.

We denote the spherical cap with the center x and the radius r by $C(x,r)$. Given a finite set $X \subset \mathbb{S}^2$, we define its mesh norm (or fill distance) h_X and the separation radius q_X to be

$$h_X := \max_{x\in\mathbb{S}^2}\min_j d(x,x_j), \quad q_X := \frac{1}{2}\min_{j\neq k} d(x_j,x_k),$$

respectively, where $d(x,y)$ is the geodesic distance between the points x and y in $\mathbb{S}^2$. The mesh ratio defined by $\frac{h_X}{q_X}$ measures the uniformity of the distribution of X. Obviously, $\frac{h_X}{q_X} \geq 1$. We say that the point set X is quasi-uniformly distributed, or simply X is quasi-uniform if there exists a constant $c_q > 0$ independent of X such that

$$q_X \leq h_X \leq c_q q_X. \tag{2.5}$$

3 Some Discussions on Approximation by Gaussian Kernel

In this article, we consider that the target function f is in $C(\mathbb{S}^2)$, the continuous function space with uniform norm $\|\cdot\|$. Let $X = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{S}^2$, and $f \in C(\mathbb{S}^2)$. Then we can choose suitable $c_k \in \mathbb{R},\ k = 1, 2, \cdots, N$, such that the function defined by

$$I_X f(x) = \sum_{k=1}^{N} c_k \phi_\rho(xx_k) \tag{3.1}$$

interpolates f on X. Since $I_X f(x) \in \mathcal{N}_\phi$, we use spherical harmonics as an intermediary to estimate the error $f(x) - I_X f(x)$. To show it, we will use a present and classic technique in the following.

It follows from Theorem 3.1 of [23] (taking $\beta = 3$) that there exists a spherical harmonic p_L with the following properties:

(a) $p_L \in \Pi_L$, where $L = \lceil 2Mq_X^{-1}\rceil$ ($L = \lceil a \rceil$ denotes the smallest integer $\geq a$), and M is a constant independent of f, L, and X;

(b) $p_L(x_i) = f(x_i),\ x_i \in X,\ i = 1, 2, \cdots, N$;

(c) $\|f - p_L\| \leq 4\mathrm{dist}(f, \Pi_L)$.

Then

$$|f(x) - I_X f(x)| \leq |f - p_L(x)| + |p_L(x) - I_X p_L(x)| + |I_X p_L(x) - I_X f(x)|. \tag{3.2}$$

From (b) and (c), the inequality (3.2) becomes

$$|f(x) - I_X f(x)| \leq 4\mathrm{dist}(f, \Pi_L) + |p_L(x) - I_X p_L(x)|. \tag{3.3}$$

From Theorem 17 of [27], it follows that

$$|p_L(x) - I_X p_L(x)|^2 \leq \frac{9}{4\pi}\sum_{l=L+1}^{\infty} \widehat{\phi}_\rho(l)(2l+1)\|p_L\|^2_{\mathcal{N}_{\phi_\rho}}.$$

We first estimate $\sum\limits_{l=L+1}^{\infty} \widehat{\phi}_\rho(l)(2l+1)$. From (2.4), we have

$$\sum_{l=L+1}^{\infty} \widehat{\phi}_\rho(l)(2l+1) \leq 2\pi^{\frac{3}{2}} \sum_{l=L+1}^{\infty} (2l+1)\frac{\rho^l}{\Gamma\Big(l+\dfrac{3}{2}\Big)}$$

$$\leq 6\pi^{\frac{3}{2}} \sum_{l=L+1}^{\infty} l \frac{\rho^l}{\Gamma\Big(l+\dfrac{3}{2}\Big)}.$$

Using the asymptotic equation (see [11, p. 400])

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+\frac{1}{2}} \mathrm{e}^{-x+\frac{1}{12(x+\theta)}}, \quad 0 \leq \theta \leq \frac{1}{2}, \tag{3.4}$$

we see that

$$\begin{aligned}\sum_{l=L+1}^{\infty} \widehat{\phi}_\rho(l)(2l+1) &\leq 3\sqrt{2}\pi \sum_{l=L+1}^{\infty} l \frac{\rho^l \mathrm{e}^{l+\frac{1}{2}}}{\Big(l+\dfrac{1}{2}\Big)^{l+1}} \\ &\leq 3\sqrt{2\mathrm{e}}\pi \sum_{l=L+1}^{\infty} \frac{\rho^l \mathrm{e}^l}{l^l} \\ &\leq 3\sqrt{2\mathrm{e}}\pi \sum_{l=L+1}^{\infty} \frac{\rho^l \mathrm{e}^l}{l!} \\ &\leq 3\sqrt{2\mathrm{e}}\pi \mathrm{e}^{\mathrm{e}\rho} \frac{(\mathrm{e}\rho)^{L+1}}{(L+1)!},\end{aligned}$$

where we use the inequality $l^{\frac{1}{2}} < l! < l^l$ (see [11, p. 92]). Differing from the case $f \in \mathcal{N}_\phi$, the value of $\|p_L\|^2_{\mathcal{N}_{\phi_\rho}}$ can change with L, which implies that $\|p_L\|^2_{\mathcal{N}_{\phi_\rho}}$ may be very large. Also,

$$|p_L(x) - I_X p_L(x)| = o(1), \quad L \to \infty, \ x \in \mathbb{S}^2$$

may not hold. In fact, we expand

$$\sum_{l=L+1}^{\infty} \widehat{\phi}_\rho(l)(2l+1) \sum_{l=0}^{L} \frac{1}{\widehat{\phi}_\rho(l)} \sum_{m=1}^{2l+1} \widehat{p_L}^2_{l,m},$$

and we find that

$$\widehat{\phi}_\rho(L)(2L+3) \sum_{m=1}^{2L+1} \frac{\widehat{p_L}^2_{L,m}}{\widehat{\phi}_\rho(L)} = 2\|\mathcal{P}^L_{p_L}\|^2$$

is in expansion, where $\mathcal{P}^L_{p_L}$ denotes the projective of p_L on $\mathbb{H}_L$ (the class of all spherical harmonics with degree L).

So we look forward to the more detailed analysis on the estimates of $|p_L(x) - I_X p_L(x)|$.

On the other hand, we can use the form $\sum\limits_{x_j \in X} \alpha_j \phi(xx_j)$ to interpolate scattered data. To obtain the coefficients $\alpha_1, \alpha_2, \cdots, \alpha_N$, one is required to solve a large scale system of linear equations. The most important advantage of quasi-interpolation is that we can evaluate the approximant directly without the need to solve any linear system of equations.

However, for given ρ, we have

$$\mathrm{e}^{-2\rho(1-x_1x_2)} + \mathrm{e}^{-2\rho(1-x_1x_3)} + \cdots + \mathrm{e}^{-2\rho(1-x_1x_N)} \geq (N-1)\mathrm{e}^{-4\rho},$$

which means that for function $f(x) \equiv 1 \in \mathcal{N}_{\phi_\rho}$, and the error of quasi-interpolation at $x = x_1$,

$$\mathrm{e}^{-2\rho(1-xx_1)} + \mathrm{e}^{-2\rho(1-xx_2)} + \cdots + \mathrm{e}^{-2\rho(1-xx_N)} - 1$$

is not arbitrarily small when N is sufficiently large. So when $f \in \mathcal{N}_{\phi_\rho}$, the form of quasi-interpolation

$$\sum_{i=1}^{N} f(x_i)\mathrm{e}^{-2\rho(1-xx_i)}$$

is invalid. Moreover, the native space $\mathcal{N}_{\phi_\rho}$ is too small. Hence we intend to investigate other forms of the quasi-interpolation approximation on the continuous function space $C(\mathbb{S}^2)$ with the help of the kernel $\mathrm{e}^{-2\rho(1-t)}$.

4 Approximation by Quasi-interpolation Operators

For $f \in C(\mathbb{S}^2)$, we construct the quasi-interpolation operators

$$G_{N,\rho}f(x) = \sum_{j=1}^{N} f(x_j)\frac{\mathrm{e}^{-2\rho(1-xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{-2\rho(1-xx_i)}} = \sum_{j=1}^{N} f(x_j)\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}}. \tag{4.1}$$

Now we prove the error estimates for the quasi-interpolation operators.

Theorem 4.1 *Let $X = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{S}^2$, and $f \in C(\mathbb{S}^2)$. If the mesh norm h_X of X satisfies $h_X < \frac{3}{4}\pi$, then there holds*

$$\|G_{N,\rho}f - f\| \le \omega(f, 2h_X) + 2N\|f\|\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2}, \tag{4.2}$$

where $\omega(f, 2h_X)$ denotes the modulus of continuity of f defined by (see [17, 25])

$$\omega(f,\delta) := \sup_{\substack{x,y\in\mathbb{S}^2\\ d(x,y)\le\delta}} |f(x) - f(y)|.$$

Proof Obviously,

$$\begin{aligned} |G_{N,\rho}f(x) - f(x)| &= \Big|\sum_{j=1}^{N}(f(x_j) - f(x))\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}}\Big| \\ &\le \omega(f, 2h_X) + 2\|f\|\sum_{j:d(x,x_j)>2h_X}\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}}. \end{aligned}$$

For any $x \in \mathbb{S}^2$, there exists $j_0 \in \mathbb{N}$ and $1 \le j_0 \le N$, such that $x \in C(x_{j_0}, h_X)$. Hence,

$$\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}} \le \frac{\mathrm{e}^{2\rho(xx_j)}}{\mathrm{e}^{2\rho(xx_{j_0})}} = \mathrm{e}^{2\rho((xx_j)-(xx_{j_0}))}, \quad j = 1, \cdots, N. \tag{4.3}$$

Since

$$2\rho((xx_j) - (xx_{j_0})) = -4\rho\sin\frac{d(x,x_j)+d(x,x_{j_0})}{2}\sin\frac{d(x,x_j)-d(x,x_{j_0})}{2} \tag{4.4}$$

and

$$h_X < \frac{d(x,x_j)+d(x,x_{j_0})}{2} < \frac{\pi+2h_X}{2} < \pi - h_X,$$

we have

$$\sin\frac{d(x,x_j)+d(x,x_{j_0})}{2} > \frac{2}{\pi}h_X, \quad \sin\frac{d(x,x_j)-d(x,x_{j_0})}{2} > \frac{h_X}{\pi}. \tag{4.5}$$

Combining (4.3)–(4.5) leads to

$$\sum_{j:d(x,x_j)>2h_X}\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}} < N\mathrm{e}^{2\rho((xx_j)-(xx_{j_0}))} < N\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2}.$$

Therefore, for any $x\in\mathbb{S}^2$, one has

$$|G_{N,\rho}f(x)-f(x)| \le \omega(f,2h_X)+2N\|f\|\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2},$$

which shows

$$\|G_{N,\rho}f-f\| \le \omega(f,2h_X)+2N\|f\|\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2}.$$

The proof of Theorem 4.1 is completed.

Clearly, if we take $\rho=\frac{\pi^2}{8h_X^2}N$, then (4.2) becomes

$$\|G_{N,\rho}f-f\| \le \omega(f,2h_X)+2\|f\|N\mathrm{e}^{-N}.$$

5 Approximation by Interpolation Operators

From the strictly positive definiteness of $\mathrm{e}^{-2\rho(1-t)}$, it follows that the matrix

$$\begin{pmatrix} 1 & \mathrm{e}^{-2\rho(1-x_1x_2)} & \cdots & \mathrm{e}^{-2\rho(1-x_1x_N)} \\ \mathrm{e}^{-2\rho(1-x_2x_1)} & 1 & \cdots & \mathrm{e}^{-2\rho(1-x_2x_N)} \\ \vdots & \vdots & & \vdots \\ \mathrm{e}^{-2\rho(1-x_Nx_1)} & \mathrm{e}^{-2\rho(1-x_Nx_2)} & \cdots & 1 \end{pmatrix}$$

is nonsingular. So it is not difficult to see that the matrix

$$G_{\phi_\rho} = \begin{pmatrix} \frac{\mathrm{e}^{2\rho}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_1x_i)}} & \frac{\mathrm{e}^{2\rho(x_1x_2)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_1x_i)}} & \cdots & \frac{\mathrm{e}^{2\rho(x_1x_N)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_1x_i)}} \\ \frac{\mathrm{e}^{2\rho(x_2x_1)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_2x_i)}} & \frac{\mathrm{e}^{2\rho}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_2x_i)}} & \cdots & \frac{\mathrm{e}^{2\rho(x_2x_N)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_2x_i)}} \\ \vdots & \vdots & & \vdots \\ \frac{\mathrm{e}^{2\rho(x_Nx_1)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_Nx_i)}} & \frac{\mathrm{e}^{2\rho(x_Nx_2)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_Nx_i)}} & \cdots & \frac{\mathrm{e}^{2\rho}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(x_Nx_i)}} \end{pmatrix}$$

is also nonsingular. This shows that the operators

$$I_{N,\rho}f(x)=\sum_{j=1}^{N}c_j\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}} \tag{5.1}$$

can be an interpolant for the data points $(x_i,f_i),\ i=1,2,\cdots,N$.

Our target is to estimate the error $f(x)-I_{N,\rho}f(x)$. At first, we introduce some definitions and notations. Let

$$\begin{aligned}&F:=[f_1,f_2,\cdots,f_N]^{\mathrm{T}}:=[f(x_1),f(x_2),\cdots,f(x_N)]^{\mathrm{T}},\\&C:=[c_1,c_2,\cdots,c_N]^{\mathrm{T}}.\end{aligned}$$

Define the norms of vector F and matrix $(a_{ij})_{m\times n}$ as follows:

$$\|F\|_\infty:=\max_j|f_j|,\quad \|(a_{ij})_{m\times n}\|_\infty:=\max_{1\le i\le m}\sum_{j=1}^{n}|a_{ij}|, \tag{5.2}$$

respectively. Let E be an $N\times N$ identity matrix, δE be the difference of G_{ϕ_ρ} and E, i.e., $\delta E=G_{\phi_\rho}-E$, and δF be the difference of C and F, i.e., $\delta F=C-F$. Then

$$EF=F,\quad (E+\delta E)(F+\delta F)=F.$$

From (5.2), we have

$$\begin{aligned}\|\delta E\|_\infty=\|G_{\phi_\rho}-E\|_\infty&=\max_i\frac{2\sum\limits_{j\ne i}\mathrm{e}^{2\rho(x_ix_j)}}{\mathrm{e}^{2\rho}+\sum\limits_{j\ne i}\mathrm{e}^{2\rho(x_ix_j)}}\\&=2\max_i\frac{\sum\limits_{j\ne i}\mathrm{e}^{-2\rho[1-(x_ix_j)]}}{1+\sum\limits_{j\ne i}\mathrm{e}^{-2\rho[1-(x_ix_j)]}}\\&\le 2\max_i\sum_{j\ne i}\mathrm{e}^{-4\rho\sin^2\frac{d(x_i,x_j)}{2}}\\&\le 2\max_i\sum_{j\ne i}\mathrm{e}^{-4\rho\left(\frac{2}{\pi}\cdot\frac{d(x_i,x_j)}{2}\right)^2}\\&\le 2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}.\end{aligned}$$

When $\|\delta E\|_\infty<1$, by Theorem 5.3 of [10], we obtain

$$\|\delta F\|_\infty\le\frac{\|\delta E\|_\infty}{1-\|\delta E\|_\infty}\|F\|_\infty\le\frac{2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}{1-2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}\|f\|.$$

Therefore

$$|I_{N,\rho}f(x)-G_{N,\rho}f(x)|=\Big|\sum_{j=1}^{N}c_j\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}}-\sum_{j=1}^{N}f_j\frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^{N}\mathrm{e}^{2\rho(xx_i)}}\Big|$$

$$\begin{aligned}
&\le \max_j |c_j - f_j| \sum_{j=1}^N \frac{\mathrm{e}^{2\rho(xx_j)}}{\sum\limits_{i=1}^N \mathrm{e}^{2\rho(xx_i)}} \\
&= \|C-F\|_\infty = \|\delta F\|_\infty \\
&\le \frac{2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}{1-2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}\|f\|.
\end{aligned}$$

From

$$|f(x) - I_{N,\rho}f(x)| \le |f(x) - G_{N,\rho}f(x)| + |G_{N,\rho}f(x) - I_{N,\rho}f(x)|,$$

we get

$$\begin{aligned}
|f(x) - I_{N,\rho}f(x)| &\le \omega(f, 2h_X) + 2N\|f\|\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2} + \frac{2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}{1-2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}\|f\| \\
&\le \omega(f, 2h_X) + 2N\|f\|\Big(\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2} + \frac{\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}{1-2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}\Big).
\end{aligned}$$

Hence, we have proved the following result.

Theorem 5.1 *Let $X = \{x_1, x_2, \cdots, x_N\} \subset \mathbb{S}^2$, and h_X and q_X be the mesh norm and the separation radius of X, respectively. If $f \in C(\mathbb{S}^2)$ and $h_X < \frac{3}{4}\pi$, then there holds*

$$\|f - I_{N,\rho}f\| \le \omega(f, 2h_X) + 2N\|f\|\Big(\mathrm{e}^{-\frac{8}{\pi^2}\rho h_X^2} + \frac{\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}{1-2N\mathrm{e}^{-\frac{16}{\pi^2}\rho q_X^2}}\Big).$$

When X is quasi-uniform, that is, (2.5) holds, we can choose $\rho = \frac{\pi^2}{8h_X^2}N$, and the result of Theorem 5.1 becomes

$$\|f - I_{N,\rho}f\| \le \omega(f, 2h_X) + 2N\|f\|\Big(\mathrm{e}^{-N} + \frac{\mathrm{e}^{-\frac{2N}{c_q^2}}}{1-2N\mathrm{e}^{-\frac{2N}{c_q^2}}}\Big).$$

Particularly, when $X = \{x_1, x_2, \cdots, x_N\}$ satisfies $c_q = \sqrt{2},\ 4N \le \mathrm{e}^N$, we have

$$\|f - I_{N,\rho}f\| \le \omega(f, 2h_X) + 6N\mathrm{e}^{-N}\|f\|.$$

Remark 5.1 Using the methods in Section 4 and Section 5, we can extend the Gaussian kernel to general cases.

6 Approximation by Quasi-interpolation Operators with Local Compact Support

We have discussed the approximation of $f \in C(\mathbb{S}^2)$ by the combinations of

$$\Phi_{\rho,j}(x) := \frac{\mathrm{e}^{-2\rho(1-xx_j)}}{\sum\limits_{i=1}^N \mathrm{e}^{-2\rho(1-xx_i)}}, \quad j = 1, 2, \cdots, N.$$

Clearly, each function $\Phi_{\rho,j}(x)$ $(j=1,2,\cdots,N)$ is continuous on $\mathbb{S}^2$. Now we slightly relax the condition of continuity, and the error estimates will be improved for compensation.

For a given mesh norm h_X of a finite set $X \subset \mathbb{S}^2$, we define

$$E_{\rho,h_X}(t) = \begin{cases} \mathrm{e}^{-2\rho(1-t)}, & \cos h_X \le t \le 1, \\ 0, & -1 \le t < \cos h_X, \end{cases}$$

and introduce the quasi-interpolation operators

$$\widehat{G}_{N,\rho} f(x) = \sum_{j=1}^{N} f(x_j) \frac{E_{\rho,h_X}(xx_j)}{\sum\limits_{i=1}^{N} E_{\rho,h_X}(xx_i)}. \tag{6.1}$$

Thus

$$\begin{aligned} |\widehat{G}_{N,\rho} f(x) - f(x)| &\le \sum_{j=1}^{N} |f(x_j) - f(x)| \frac{E_{\rho,h_X}(xx_j)}{\sum\limits_{i=1}^{N} E_{\rho,h_X}(xx_i)} \\ &= \sum_{j:d(x,x_j)\le h_X} |f(x_j) - f(x)| \frac{E_{\rho,h_X}(xx_j)}{\sum\limits_{i:d(x,x_i)\le h_X} E_{\rho,h_X}(xx_i)} \\ &\le \omega(f, h_X). \end{aligned}$$

Although the function $E_{\rho,h_X}(t)$ is discontinuous at h_X, the oscillation $\mathrm{e}^{-2\rho(1-\cos h_X)}$ becomes sufficiently small with ρ being large enough.

In fact, we can give the estimate of the number of point x_j which satisfies

$$d(x, x_j) \le h_X$$

as follows:

$$\begin{aligned} |C(x, h_X)| &\le \frac{\displaystyle\int_0^{h_X+q_X} \sin\theta \mathrm{d}\theta}{\displaystyle\int_0^{q_X} \sin\theta \mathrm{d}\theta} \\ &= \frac{1 - \cos(h_X + q_X)}{1 - \cos q_X} \\ &= \frac{\sin^2 \frac{h_X + q_X}{2}}{\sin^2 \frac{q_X}{2}} \\ &\le \frac{\frac{(h_X + q_X)^2}{4}}{\frac{q_X^2}{\pi^2}} \\ &= \frac{\pi^2}{4}\Big[1 + \frac{2h_X}{q_X} + \Big(\frac{h_X}{q_X}\Big)^2\Big]. \end{aligned}$$

When X is quasi-uniform, then

$$|C(x, h_X)| \le \frac{\pi^2}{4}(1 + 2c_q + c_q^2).$$

Remark 6.1 Naturally, we can use the technique to construct quasi-interpolation operators by means of the kernels with continuous compact support. We also refer the reader to [3, 5, 26, 28].

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