Chin. Ann. Math.
36B(3), 2015, 413–426
DOI: 10.1007/s11401-015-0911-y

Chinese Annals of Mathematics, Series B

A Relation in the Stable Homotopy Groups of Spheres*

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Abstract Let $p \geqslant 7$ be an odd prime. Based on the Toda bracket $\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle$, the authors show that the relation $\alpha_1\beta_1^{p-1}h_{2,0}\gamma_s = \beta_{p/p-1}\gamma_s$ holds. As a result, they can obtain $\alpha_1\beta_1^p h_{2,0}\gamma_s = 0 \in \pi_*(S^0)$ for $2 \leqslant s \leqslant p-2$, even though $\alpha_1 h_{2,0}\gamma_s$ and $\beta_1\alpha_1 h_{2,0}\gamma_s$ are not trivial. They also prove that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ is nontrivial in $\pi_*(S^0)$ and conjecture that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_s$ is nontrivial in $\pi_*(S^0)$ for $3 \leqslant s \leqslant p-2$. Moreover, it is known that $\beta_{p/p-1}\gamma_3 = 0 \in \mathrm{Ext}_{BP_*BP}^{5,*}(BP_*, BP_*)$, but $\beta_{p/p-1}\gamma_3$ is nontrivial in $\pi_*(S^0)$ and represents the element $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$.

Keywords Toda bracket, Stable homotopy groups of spheres, Adams-Novikov spectral sequence, Method of infinite descent
2000 MR Subject Classification 55Q10, 55Q45, 55T99

1 Introduction

Let p be an odd prime and $q = 2p - 2$. It is well-known that the Adams-Novikov spectral sequence (ANSS) based on the Brown-Peterson spectrum is one of the most powerful tools to compute the p-component of stable homotopy groups of spheres S^0, and the E_2-term of the ANSS is $\mathrm{Ext}_{BP_*BP}^{s,t}(BP_*, BP_*)$ (see [1, 7, 10–11, 15]). Moreover, we have the Adams spectral sequence (ASS) (see [1–2]) based on the Eilenberg-MacLane spectrum $K\mathbb{Z}/p$.

From [10–11], $\mathrm{Ext}_{BP_*BP}^{1}(BP_*, BP_*) = H^1BP_*$ is generated by $\alpha_{sp^n/n+1}$ for $n \geqslant 0$, $p \nmid s \geqslant 1$, where $\alpha_{sp^n/n+1}$ has order p^{n+1}. $\mathrm{Ext}_{BP_*BP}^{2}(BP_*, BP_*) = H^2BP_*$ is a direct sum of cyclic groups generated by $\beta_{sp^n/j,i+1}$ for $n \geqslant 0$, $p \nmid s \geqslant 1$, $j \geqslant 1$, $i \geqslant 0$ and is subject to

(1) $j \leqslant p^n$ if $s = 1$,

(2) $p^i | j \leqslant a_{n-i}$, and

(3) $a_{n-i-1} < j$ if $p^{i+1} \mid j$,

where $a_0 = 1$, $a_k = p^k + p^{k-1} - 1$ for $k \geqslant 1$. $\beta_{sp^n/j,i+1}$ has order p^{i+1} (see [10, 14–15]). There is only partial information for $\mathrm{Ext}_{BP_*BP}^{3}(BP_*, BP_*) = H^3BP_*$ which contains the p order generators γ_s, $s \geqslant 1$.

In 1985, D. C. Ravenel [13] first introduced the method of infinite descent and later used it to compute the first thousand stems of the stable homotopy groups of spheres at the prime 5.

Manuscript received September 25, 2013. Revised April 24, 2014.
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*This work was supported by the National Natural Science Foundation of China (Nos. 11071125, 11261062, 11471167) and the Specialized Research Fund for the Doctoral Program of Higher Education (No. 20120031110025).

This method is devoted to computing the Adams-Novikov E_2-term for a spheral spectrum S^0 by the following spectral sequence referred to as the small descent spectral sequence (SDSS)

$$E_1^{s,t,*} = \mathrm{Ext}^{t,*}_{BP_*BP}(BP_*, BP_*Y) \otimes E(\alpha_1) \otimes P(\beta_1) \Longrightarrow \mathrm{Ext}^{s+t,*}_{BP_*BP}(BP_*, BP_*)$$

and $d_r : E_r^{s,t,*} \longrightarrow E_r^{s+r,t-r+1,*}$.

The following relation about the Toda bracket is showed by Ravenel in the topological SDSS (see [14, Proposition 7.5.11] or [15, Proposition 7.6.11]). If x is an element in stable homotopy groups of spheres and satisfies $px = 0$, $\langle \alpha_1\beta_1, p, x\rangle = 0$ and $\alpha_1 x \neq 0$, then

$$\alpha_1\beta_1^{p-1}h_{2,0}x = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, x\rangle = \beta_{p/p-1}x.$$

In this paper, we show that the condition $\langle \alpha_1\beta_1, p, x\rangle = 0$ holds for the element $x = \gamma_s$ with $p \geqslant 7$ by using the cobar complex of BP-homology of the Smith-Toda spectrum $V(2)$. Therefore, it follows that the relation

$$\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_s = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle = \beta_{p/p-1}\gamma_s \tag{1.1}$$

holds for $p \geqslant 7$. Applying this relation, we can prove that $\beta_1^p\alpha_1 h_{2,0}\gamma_s$ is trivial in $\pi_*(S^0)$ for $2 \leqslant s \leqslant p-2$, $p \geqslant 7$, but $\alpha_1 h_{2,0}\gamma_s$ and $\beta_1\alpha_1 h_{2,0}\gamma_s$ are not trivial in $\pi_*(S^0)$ for $3 \leqslant s \leqslant p-2$, $p \geqslant 7$.

It is also proved that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ is nontrivial in $\pi_*(S^0)$, and we can further conjecture that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_s$ is nontrivial in $\pi_*(S^0)$ for $p \geqslant 7$, $2 \leqslant s \leqslant p-2$.

Let x and y be two elements in $\mathrm{Ext}^{*,*}_{BP_*BP}(BP_*, BP_*)$ and be permanent cycles. It is known that if $xy = 0$, then the homotopy product could still be nontrivial and represents an element in a higher Ext group. $\beta_{p/p-1}$ and γ_3 are two such elements. We know that $\beta_{p/p-1}$ and γ_3 are permanent cycles and $\beta_{p/p-1}\gamma_3 = 0 \in \mathrm{Ext}^{5,*}_{BP_*BP}(BP_*, BP_*)$, but $\beta_{p/p-1}\gamma_3$ is nontrivial in $\pi_*(S^0)$ and represents the element $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ from the relation (1.1).

The rest of this paper is organized as follows. In Section 2, the (topological) small descent spectral sequence will be introduced. In Section 3, we prove that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ is nontrivial in $\pi_*(S^0)$ by applying the May spectral sequence (MSS) and the small descent spectral sequence (SDSS). In Section 4, we recall the cobar complex and use it to calculate the E_2-term of the Adams-Novikov spectral sequence for $V(2)$. We show that the Toda bracket $\langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle$ is well defined. As a result, $\alpha_1\beta_1^{p-1}h_{2,0}\gamma_s = \beta_{p/p-1}\gamma_s$ holds for $p \geqslant 7$ and $\alpha_1\beta_1^p h_{2,0}\gamma_s = 0 \in \pi_*(S^0)$ because $\beta_1\beta_{p/p-1} = 0$ in $\pi_*(S^0)$.

2 The Small Descent Spectral Sequence

In this section, we recall the construction of the small descent spectral sequence. Ravenel computed the E_1-term of this spectral sequence and used it to determine the stable homotopy groups of spheres in a certain range, see [14–16] for more details.

Let $T(n)$ be the Ranevel spectrum (see [15]) characterized by

$$BP_*T(n) = BP_*[t_1, t_2, \cdots, t_n].$$

Then we have the following diagram:

$$S^0 = T(0) \longrightarrow T(1) \longrightarrow T(2) \longrightarrow \cdots \longrightarrow T(n) \longrightarrow \cdots \longrightarrow BP,$$

where S^0 denotes the sphere spectrum localized at an odd prime p. Let $T(0)_{p-1}$ and $T(0)_{p-2}$ denote the $q(p-1)$ and $q(p-2)$ skeletons of $T(1)$ respectively. They are denoted by Y and $\overline{Y}$ for simplicity. Then

$$Y = S^0 \bigcup_{\alpha_1} e^q \bigcup_{\alpha_1} \cdots \bigcup_{\alpha_1} e^{(p-2)q} \bigcup_{\alpha_1} e^{(p-1)q} \quad \text{and} \quad \overline{Y} = S^0 \bigcup_{\alpha_1} e^q \bigcup_{\alpha_1} \cdots \bigcup_{\alpha_1} e^{(p-2)q}.$$

The BP-homologies of them are

$$BP_*(Y) = BP_*[t_1]/\langle t_1^p \rangle \quad \text{and} \quad BP_*(\overline{Y}) = BP_*[t_1]/\langle t_1^{p-1} \rangle.$$

From the definition above, we get the following cofibre sequences:

$$S^0 \xrightarrow{i'} Y \xrightarrow{j'} \Sigma^q \overline{Y} \xrightarrow{k'} S^1, \tag{2.1}$$

$$\overline{Y} \xrightarrow{i''} Y \xrightarrow{j''} S^{(p-1)q} \xrightarrow{k''} \Sigma \overline{Y}, \tag{2.2}$$

and the short exact sequences of BP_* homology

$$0 \longrightarrow BP_* S^0 \xrightarrow{i'_*} BP_* Y \xrightarrow{j'_*} BP_* \Sigma^q \overline{Y} \longrightarrow 0, \tag{2.3}$$

$$0 \longrightarrow BP_* \overline{Y} \xrightarrow{i''_*} BP_* Y \xrightarrow{j''_*} BP_* S^{(p-1)q} \longrightarrow 0. \tag{2.4}$$

Putting (2.3) and (2.4) together, one has the following long exact sequence:

$$0 \longrightarrow BP_* S^0 \longrightarrow BP_*(Y) \longrightarrow BP_*(\Sigma^q Y) \longrightarrow BP_*(\Sigma^{pq} Y) \longrightarrow \cdots . \tag{2.5}$$

Putting (2.1) and (2.2) together, one has the following Adams diagram of cofibres:

$$\begin{array}{ccccccccc} S^0 & \longleftarrow & \Sigma^{q-1}\overline{Y} & \longleftarrow & S^{pq-2} & \longleftarrow & \Sigma^{(p+1)q-3}\overline{Y} & \longleftarrow & \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ Y & & \Sigma^{q-1}Y & & \Sigma^{pq-2}Y & & \Sigma^{(p+1)q-3}Y & & \end{array} \tag{2.6}$$

Thus one has the following proposition.

Proposition 2.1 (see [14, Proposition 7.4.2] and [15, Theorems 7.1.13 and 7.1.16]) *Let Y be as above.*

(a) *There is a spectral sequence converging to $\mathrm{Ext}_{BP_*BP}^{s+t,*}(BP_*, BP_*)$ with the E_1-term*

$$E_1^{s,t,*} = \mathrm{Ext}_{BP_*BP}^{t,*}(BP_*, BP_*Y) \otimes E(\alpha_1) \otimes P(\beta_1), \quad \alpha_1 \in E_1^{1,0,q},\ \beta_1 \in E_1^{2,0,pq}$$

and $d_r : E_r^{s,t,*} \longrightarrow E_r^{s+r,t-r+1,*}$, where $E(-)$ denotes the exterior algebra and $P(-)$ denotes the polynomial algebra on the indicated generators.

This spectral sequence is referred to as the small descent spectral sequence ($SDSS$).

(b) *There is a spectral sequence converging to* $\pi_*(S^0)$ *with the* E_1*-term*

$$E_1^{s,t} = \pi_*(Y) \otimes E(\alpha_1) \otimes P(\beta_1), \quad \alpha_1 \in E_1^{1,q},\ \beta_1 \in E_1^{2,pq}$$

and $d_r : E_r^{s,t} \longrightarrow E_r^{s+r,t-r+1}$.

This spectral sequence is referred to as the topological small descent spectral sequence ($TSDSS$).

The above two spectral sequences produce the $\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*)$ and $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*)$ or the corresponding elements in $\pi_*(S^0)$ by $\mathrm{Ext}^{0,*}_{BP_*BP}(BP_*, BP_*Y)$ and $\mathrm{Ext}^{1,*}_{BP_*BP}(BP_*, BP_*Y)$. $\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*(S^0))$ ($s \geqslant 2$) or the corresponding elements in $\pi_*(S^0)$ are produced by $\mathrm{Ext}^{s,*}_{BP_*BP}(BP_*, BP_*Y)$ ($s \geqslant 2$) as described in the following ABC Theorem.

Note that in the range $t - s < q(p^3 + p - 1) - 3$, there is no element with filtration $> 2p$, and the Adams-Novikov spectral sequence for the spectrum Y collapses from the E_2-term. So the E_2-term is actually $\pi_{t-s}(Y)$ for this range.

ABC Theorem (see [14–15]) *For* $p > 2$ *and* $t - s < q(p^3 + p - 1) - 3$, $s \geqslant 2$,

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*Y) = A \oplus B \oplus C,$$

where A *is the* $\mathbb{Z}/p$*-vector space spanned by*

$$A = \{\beta_{ip},\ \beta_{ip+1} \mid i \leqslant p-1\} \cup \{\beta_{p^2/p^2-j} \mid 0 \leqslant j \leqslant p-1\},$$
$$B = R \otimes \{\gamma_k \mid k \geqslant 2\},$$

where $R = P(b_{20}^p) \otimes E(h_{20}) \otimes \mathbb{Z}/p\{\{b_{11}^i \mid 0 \leqslant i \leqslant p-1\} \cup \{h_{11}b_{20}^i \mid 0 \leqslant i \leqslant p-2\}\}$, *and*

$$C^{s,t} = \bigoplus_{i \geqslant 0} R^{s+2i, t+i(p^2-1)q}.$$

From the generators of R, we can obtain precise generators of C as follows.

Let $i = jp + m$. Then $R^{s+2i,t+i(p^2-1)} \subset C^{s,t}$, so we have

(1) $b_{2,0}^{(j+1)p} \in R^{2(p-m)+2(\underline{jp+m});t+(\underline{jp+m})(p^2-1)q} \subset C^{2(p-m);t}$ is represented by

$$b_{2,0}^{p-m-1} u_{jp+m}$$

for $p - 1 \geqslant m \geqslant 1$, from which we have

$$b_{2,0}^{p-m-1} u_{jp+m} \otimes E(h_{2,0}) \otimes \{b_{11}^i \mid 0 \leqslant i \leqslant p-1\} \cup \{h_{11}b_{20}^i \mid 0 \leqslant i \leqslant p-2\},$$

where $u_k = v_2^k\big(\frac{v_3}{pv_1} - \frac{v_2^{1+p}}{c_k p v_1^{1+p}}\big)$, $c_k = \binom{1+p+k}{p}$ and $u_{jp+m} \in C^{2,q[(j+1)p^2+(j+m+1)p+m]}$.

(2) $b_{1,1}^k b_{2,0}^{jp} \in R^{2(k-m)+2(\underline{jp+m});t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m);t}$ is represented by

$$b_{1,1}^{k-m-1} \beta_{(j+1)p/p-m}$$

for $p-1 \geqslant k \geqslant m+1 \geqslant 1$, from which we have

$$b_{1,1}^{k-m-1}\beta_{(j+1)p/p-m} \otimes E(h_{2,0}),$$

where $\beta_{(j+1)p/p-m} \in C^{2,q[(j+1)p^2+jp+m]}$.

Especially $h_{2,0}b_{1,1}^{p-1}b_{2,0}^{jp} \in R^{3+2(\underline{jp+p-2});t+(\underline{jp+p-2})(p^2-1)q} \subset C^{3;t}$ is represented by

$$h_{1,1}\beta_{(j+1)p/1;2},$$

which is an element of order p^2.

(3) $h_{1,1}b_{2,0}^{k}b_{2,0}^{jp} \in R^{2(k-m)+1+2(\underline{jp+m});t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m)+1;t}$ is represented by

$$b_{2,0}^{k-m-1}\eta_{jp+m+1}$$

for $p-2 \geqslant k \geqslant m+1 \geqslant 1$, where $\eta_{jp+m+1} = h_{1,1}u_{jp+m} \in C^{3,q[(j+1)p^2+(j+m+2)p+m]}$.

(4) $h_{2,0}h_{1,1}b_{2,0}^{k}b_{2,0}^{jp} \in R^{2(k-m+1)+2(\underline{jp+m});t+(\underline{jp+m})(p^2-1)q} \subset C^{2(k-m+1);t}$ is represented by

$$b_{2,0}^{k-m}\beta_{jp+m+2}$$

for $p-2 \geqslant k \geqslant m \geqslant 0$, where $\beta_{jp+m+2} \in C^{2,q[jp^2+(j+m+2)p+m+1]}$.

Especially $h_{2,0}h_{1,1}b_{2,0}^{p-2}b_{2,0}^{jp} \in R^{2+2(\underline{jp+p-2});t+(\underline{jp+p-2})(p^2-1)q} \subset C^{2;t}$ is represented by

$$\beta_{(j+1)p/1;2},$$

which is an element of order p^2.

3 The Non-triviality of $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ in $\pi_*(S^0)$

It is known that $\alpha_1 h_{2,0}\gamma_s$ and $\beta_1\alpha_1 h_{2,0}\gamma_s$ are not trivial in $\pi_*(S^0)$ for $3 \leqslant s \leqslant p-2$, $p \geqslant 7$. Further, we conjecture that so is $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_s$. The proof includes complicated calculation. Here we only prove that this conjecture is right for $s=3$.

Let $\phi : BP \to KZ/p$ be the Thom map which induces the Thom reduction map between the Adams-Novikov Spectral Sequence and the Adams Spectral Sequence

$$\Phi : \mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*) \longrightarrow \mathrm{Ext}^*_{A_*}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Then it is known that

$$\Phi(\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3) = b_0^{p-1}g_0\widetilde{\gamma}_3 \in \mathrm{Ext}_{A_*}^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p, \mathbb{Z}/p),$$

where $\widetilde{\gamma}_3 \in \mathrm{Ext}_{A_*}^{3,q(3p^2+2p+1)}(\mathbb{Z}/p, \mathbb{Z}/p)$ is constructed by X. Wang and Q. Zheng in [18].

Next we prove that $b_0^{p-1}g_0\widetilde{\gamma}_3$ is not trivial in $\mathrm{Ext}_{A_*}^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p, \mathbb{Z}/p)$ by the May spectral sequence.

Let A_* denote the dual mod p Steenrod algebra which is isomorphic to

$$A_* = P[\xi_1, \xi_2, \cdots] \otimes E[\tau_0, \tau_1, \tau_2, \cdots],$$

where the inner degree of ξ_i is $q(1+p+\cdots+p^{i-1})$ and that of τ_i is $q(1+\cdots+p^{i-1})+1$. Set the May filtration on A_* by $M(\xi_i^{p^j}) = M(\tau_{i-1}) = 2i-1$. Applying the May filtration to the cobar construction $C^{s,t}(\mathbb{Z}/p)$, we get an increasingly filtered module $F^{s,t,M} = F^M(C^{s,t}(\mathbb{Z}/p))$, and then there is the May spectral sequence (MSS) $\{E_r^{s,t,M}, d_r\}$ which converges to $\mathrm{Ext}_{A_*}^{s,t}(Z/p, Z/p)$ with the E_1-term

$$E_1^{*,*,*} = E(h_{i,j} \mid i>0,\ j\geq 0) \otimes P(b_{i,j} \mid i>0,\ j\geq 0) \otimes P(a_i \mid i \geq 0),$$

where

$$h_{i,j} \in E_1^{1,2(p^i-1)p^j,2i-1}, \quad b_{i,j} \in E_1^{2,2(p^i-1)p^{j+1},p(2i-1)}, \quad a_i \in E_1^{1,2p^i-1,2i+1};$$

$h_{i,j}$, a_i and $b_{i,j}$ correspond respectively to $\xi_i^{p^j}$, τ_i and $\sum\limits_{0<k<p} \binom{p}{k}/p\ \xi_i^{kp^j} \otimes \xi_i^{(p-k)p^j}$ (see [14, Theorem 3.2.5] and [8–9]).

One has

$$d_r : E_r^{s,t,u} \to E_r^{s+1,t,u-r}. \tag{3.1}$$

If $x \in E_r^{s,t,*}$ and $y \in E_r^{s',t',*}$, then

$$d_r(x\cdot y) = d_r(x)\cdot y + (-1)^s x \cdot d_r(y).$$

The known May differentials d are given by

$$\begin{cases} d_1(h_{i,j}) = \sum\limits_{0<k<i} h_{i-k,k+j}h_{k,j}, \\ d_1(a_i) = \sum\limits_{0\leq k<i} h_{i-k,k}a_k, \\ d_1(b_{i,j}) = 0, \\ d_r(b_{1,j}) = 0 \quad \text{for all } r, \\ d_r(b_{2,0}) = 0 \quad \text{for } r < 2p-1, \\ d_{2p-1}(b_{2,0}) = h_{1,2}b_{1,0} - h_{1,1}b_{1,1}. \end{cases} \tag{3.2}$$

From the Thom map, we know that $b_0^{p-1}g_0\widetilde{\gamma}_3$ is represented by $b_{1,0}^{p-1}h_{2,0}h_{1,0}h_{3,0}h_{2,1}h_{1,2}$ up to a nonzero coefficient in the E_1-term of MSS. In order to prove that $b_0^{p-1}g_0\widetilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_{A_*}^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p,\mathbb{Z}/p)$, it is necessary to guarantee that there is no element $x \in E_r^{2p+2,q(4p^2+2p+3),*}$ in the MSS such that $d_r(x) = b_{1,0}^{p-1}h_{2,0}h_{1,0}h_{3,0}h_{2,1}h_{1,2}$. That is to say, we need to compute $E_r^{2p+2,q(4p^2+2p+3),*}$. Since $h_{2,0}h_{1,0}h_{3,0}h_{2,1}h_{1,2}$ converges non-trivially to $\mathrm{Ext}_{A_*}^{5,*}(\mathbb{Z}/p,\mathbb{Z}/p)$, it is easy to show that $b_{1,0}^{p-1}h_{2,0}h_{1,0}h_{3,0}h_{2,1}h_{1,2}$ should not be killed by some first May differential from (3.2).

Lemma 3.1 *In the May spectral sequence, $E_2^{2p+2,q(4p^2+2p+3),*}$ is generated by the following elements:*

$$G_1 = h_{1,1}l_1b_{1,0}^{p-3}b_{2,0}^2 = h_{3,0}h_{2,0}h_{1,0}h_{1,1}b_{1,0}^{p-3}b_{2,0}^2,$$
$$G_2 = h_{1,0}k_0\gamma_3b_{1,0}^{p-2} = h_{3,0}h_{2,1}h_{1,2}h_{1,0}h_{2,0}h_{1,1}b_{1,0}^{p-2},$$

and $E_2^{2p+3,q(4p^2+2p+3),*}$ *is generated by the following elements:*

$$G'_1 = g_0\gamma_3 b_{1,0}^{p-1} \in E_2^{2p+3,q(4p^2+2p+3),p^2-p+13},$$

$$G'_2 = h_{1,0}m_1 b_{1,0}^{p-2} b_{1,1} \in E_2^{2p+3,q(4p^2+2p+3),p^2-p+13},$$

$$G'_3 = l_1 b_{1,0}^{p-2} b_{2,0}^2 \in E_2^{2p+3,q(4p^2+2p+3),p^2+4p+9}.$$

Proof In our range, we only need to consider

$$H^*(E(h_{i,j} : i+j \leqslant 3)) \otimes P(b_{1,0}, b_{1,1}, b_{2,0}) \otimes P(a_0, a_1, a_2, a_3).$$

Note that the degree of a_i is of the form $t = q(p^{i-1} + \cdots + p + 1) + 1$. If there is a factor a_i in the generator $g \in E_2^{2p+2,q(4p^2+2p+3),*}$, then g should contain q a's, where $a \in \{a_0, a_1, a_2, a_3\}$. It is easy to verify that the generators in $E_2^{2p+2,q(4p^2+2p+3),*}$ do not contain a.

Therefore

$$E_2^{2p+2,q(4p^2+2p+3),*} \subset H^*(E(h_{i,j} : i+j \leqslant 3)) \otimes P(b_{1,0}, b_{1,1}, b_{2,0}),$$

and the generators of $E_2^{2p+2,q(4p^2+2p+3),*}$ are of the form

$$g = x b_{1,0}^{k_0} b_{1,1}^{k_1} b_{2,0}^{k_2}, \quad x \in H^*(E(h_{i,j} : i+j \leqslant 3)),$$

where $0 \leqslant k_0 \leqslant 2p+2$, $0 \leqslant k_1 \leqslant 4$ and $0 \leqslant k_2 \leqslant 3$.

The cohomology of $E(h_{i,j} : i+j \leqslant 3)$ was already computed by Toda in [17]. We list these elements in the Table 1 below.

Table 1 $H^*(E(h_{i,j} : i+j \leqslant 3))$

Generators	$(s,\ t/q)$
1	$(0,\ 0)$
$h_{1,0}$	$(1, 1)$
$h_{1,1}$	$(1, p)$
$h_{1,2}$	$(1, p^2)$
g_0	$(2, p+2)$
k_0	$(2, 2p+1)$
$h_{1,0}h_{1,2}$	$(2, p^2+1)$
g_1	$(2, p^2+2p)$
$k_1 = h_{1,2}h_{2,1}$	$(2, 2p^2+p)$
$h_{1,0}k_0$	$(3, 2p+2)$
$l_1 = h_{1,0}h_{2,0}h_{3,0}$	$(3, p^2+2p+3)$
$l_2 = h_{1,1}h_{2,0}h_{2,1}$	$(3, p^2+3p+1)$
$l_3 = h_{1,0}h_{1,2}h_{3,0}$	$(3, 2p^2+p+2)$
$h_{1,1}k_1$	$(3, 2p^2+2p)$
$\gamma_3 = h_{1,2}h_{2,1}h_{3,0}$	$(3, 3p^2+2p+1)$
$h_{1,1}l_1$	$(4, p^2+3p+3)$
$h_{1,2}l_1$	$(4, 2p^2+2p+3)$
$m_1 = h_{1,1}h_{2,0}h_{2,1}h_{3,0}$	$(4, 2p^2+4p+2)$
$h_{1,0}\gamma_3$	$(4, 3p^2+2p+2)$
$h_{1,1}\gamma_3$	$(4, 3p^2+3p+1)$
$h_{1,0}m_1$	$(5, 2p^2+4p+3)$
$g_0\gamma_3$	$(5, 3p^2+3p+3)$
$k_0\gamma_3$	$(5, 3p^2+4p+2)$
$h_{1,0}k_0\gamma_3$	$(6, 3p^2+4p+3)$

On the one hand, consider the inner degree of $b_{i,j}$. Since t/q is the multiple of the prime p, the inner degree of x is of the form $q(np+3)$ because

$$\mathrm{degree}(x)+\mathrm{degree}(b_{1,0}^{k_0}b_{1,1}^{k_1}b_{2,0}^{k_2})=q(4p^2+2p+3).$$

On the other hand, since $b_{i,j}$ and $xb_{1,0}^{k_0}b_{1,1}^{k_1}b_{2,0}^{k_2}$ have an even dimension, so is x.

Above all, the inspection of Table 1 shows that x must be

$$h_{1,1}l_1,\quad h_{1,2}l_1\quad \text{and}\quad h_{1,0}k_0\gamma_3.$$

Noting that $g=xb_{1,0}^{k_0}b_{1,1}^{k_1}b_{2,0}^{k_2}$ has the dimension $2p+2$ and degree $q(4p^2+2p+3)$, it is easy to get that

$$E_2^{2p+2,q(4p^2+2p+3),*}=\mathbb{Z}/p\{h_{1,1}l_1b_{1,0}^{p-3}b_{2,0}^2,h_{1,0}k_0\gamma_3b_{1,0}^{p-2}\}.$$

In the same way, we can determine the generators of $E_2^{2p+3,q(4p^2+2p+3),*}$.

There are the following higher May differentials in the MSS.

Lemma 3.2 *In the May spectral sequence,*
(i) $d_{4p-3}(h_{1,1}l_1b_{1,0}^{p-3}b_{2,0}^2)=2h_{1,0}m_1b_{1,1}b_{1,0}^{p-2}+2g_0\gamma_3b_{1,0}^{p-1}$,
(ii) $d_{2p-1}(h_{1,0}m_1b_{1,0}^{p-3}b_{2,0})=h_{1,0}k_0\gamma_3b_{1,0}^{p-2}$.

Proof We only prove (i), and another statement can be verified easily in the similar way.

To calculate these higher May differentials, we are required to work back in the cobar complex $C^{s,t}(\mathbb{Z}/p)$ whose tensor product is not commutative, and hence permuting the tensor product will give rise to higher May differentials.

Since $h_{1,1}l_1$ is a permanent cycle in the May spectral sequence, it can be represented by some element in the cobar complex $C^{4,q(p^2+3p+3)}(\mathbb{Z}/p)$, and we let $\widetilde{h_{1,1}l_1}$ denote this element. From the formula

$$d_{2p-1}(\widetilde{b}_{2,0})=-[\widetilde{b}_{1,1}|\xi_1^p]+[\xi_1^{p^2}|\widetilde{b}_{1,0}],$$

we obtain that in the filtered cobar complex $C^*(\mathbb{Z}/p)$,

$$\begin{aligned}d[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}^2]=&[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\xi_1^{p^2}|\widetilde{b}_{1,0}|\widetilde{b}_{2,0}]_{A_1}+[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{1,1}|\xi_1^p|\widetilde{b}_{2,0}]_{A_2}\\&+[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}|\xi_1^{p^2}|\widetilde{b}_{1,0}]_{B_1}+[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}|\widetilde{b}_{1,1}|\xi_1^p]_{B_2}.\end{aligned}$$

Applying the formula

$$d([\widetilde{b}_{2,0}\cdot\Delta\xi_1^{p^j}])=[\widetilde{b}_{2,0}|\xi_1^{p^j}]-[\xi_1^{p^j}|\widetilde{b}_{2,0}]+d(\widetilde{b}_{2,0})\cdot\Delta^2\xi_1^{p^j},$$

we achieve permutation between $\widetilde{b}_{2,0}$ and $\xi_1^{p^j}$ in the cobar complex. Moreover, we can also achieve permutation among $\widetilde{b}_{2,0}$, $\widetilde{b}_{1,i}$ and $\xi_1^{p^j}$ (see [6] for more details).

In conclusion, permutation among $\widetilde{b}_{2,0}$, $\widetilde{b}_{1,i}$ and $\xi_1^{p^j}$ can be achieved in the sense mod $F^{*,*,p^2-p+12}$, and thus, there is a chain $u_1\in C^{2p+2}(\mathbb{Z}/p)$ such that mod $F^{*,*,p^2-p+12}$,

$$d[\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}^2+u_1]=2[\widetilde{h_{1,1}l_1}|\xi_1^{p^2}|\widetilde{b}_{1,0}^{p-2}|\widetilde{b}_{2,0}]_A+2[\widetilde{h_{1,1}l_1}|\xi_1^p|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}]_B.\tag{3.3}$$

Applying the following relations in the E_1-term of MSS by formula (3.2)

$$h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{1,1} = 0,$$
$$d_1(h_{3,0}h_{2,0}h_{1,0}h_{2,1}) = h_{3,0}h_{2,0}h_{1,0}h_{1,1}h_{1,2},$$

one has a chain $u_2 = [\widetilde{l_1}|\xi_1^{2p}] \in C^{4,p^2+4p+3,11}$ such that

$$d[u_2|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}] \equiv -2[\widetilde{h_{1,1}l_1}|\xi_1^p|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}]_B \quad \bmod F^{*,*,p^2-p+12}, \tag{3.4}$$

and mod $F^{*,*,p^2-p+12}$,

$$d[2\widetilde{l_1}|\xi_2^p|\widetilde{b}_{1,0}^{p-2}|\widetilde{b}_{2,0}] \equiv -2[\widetilde{h_{1,1}l_1}|\xi_1^{p^2}|\widetilde{b}_{1,0}^{p-2}|\widetilde{b}_{2,0}]_A + 2[\widetilde{h_{1,0}m_1}|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-2}]_C + 2[\widetilde{g_0\gamma_3}|\widetilde{b}_{1,0}^{p-1}]_D. \tag{3.5}$$

Above all, there is a chain $u \in C^{2p+2}(\mathbb{Z}/p)$ such that

$$\begin{aligned} & d([\widetilde{h_{1,1}l_1}|\widetilde{b}_{1,0}^{p-3}|\widetilde{b}_{2,0}^2] + u]) \\ & = 2[\widetilde{h_{1,0}m_1}|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-2}]_C + 2[\widetilde{g_0\gamma_3}|\widetilde{b}_{1,0}^{p-1}]_D \quad \bmod F^{*,*,p^2-11p+11}. \end{aligned} \tag{3.6}$$

Notice that $[\widetilde{h_{1,0}m_1}|\widetilde{b}_{1,1}|\widetilde{b}_{1,0}^{p-2}]$ and $[\widetilde{g_0\gamma_3}|\widetilde{b}_{1,0}^{p-1}]$ are sent to $h_{1,0}m_1b_{1,1}b_{1,0}^{p-2}$ and $g_0\gamma_3b_{1,0}^{p-1}$ in the E_2-term of the May spectral sequence respectively. From Lemma 3.1, we know that

$$E_2^{2p+3,q(4p^2+2p+3),M} = 0 \quad \text{for } M \leqslant p^2 - p + 12.$$

Therefore, the following higher May differential follows:

$$d_{4p-3}(h_{1,1}l_1b_{1,0}^{p-3}b_{2,0}^2) = 2h_{1,0}m_1b_{1,1}b_{1,0}^{p-2} + 2g_0\gamma_3b_{1,0}^{p-1}.$$

Theorem 3.1 *In the Adams spectral sequence, for $p \geqslant 7$,*

$$b_0^{p-1}g_0\widetilde{\gamma}_3 \neq 0 \in \mathrm{Ext}_A^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p, \mathbb{Z}/p).$$

Therefore, in the Adams-Novikov spectral sequence, for $p \geqslant 7$,

$$\beta_1^{p-1}\alpha_1h_{2,0}\gamma_3 \neq 0 \in \mathrm{Ext}_{BP_*BP}^{2p+3,q(4p^2+2p+3)}(BP_*, BP_*).$$

Proof According to Lemma 3.1, $g_0\gamma_3b_{1,0}^{p-1} \in E_2^{2p+3,q(4p^2+2p+2),*}$ can only be killed by $G_1 = h_{1,1}l_1b_{1,0}^{p-3}b_{2,0}^2$ and $G_2 = h_{1,0}k_0\gamma_3b_{1,0}^{p-2}$. However, from Lemma 3.2, G_1 and G_2 do not kill $g_0\gamma_3b_{1,0}^{p-1}$, so $g_0\gamma_3b_{1,0}^{p-1}$ converges nontrivially to $b_0^{p-1}g_0\widetilde{\gamma}_3 \in \mathrm{Ext}_A^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p, \mathbb{Z}/p)$.

By the Thom reduction map

$$\Phi(\beta_1^{p-1}\alpha_1h_{2,0}\gamma_3) = b_0^{p-1}g_0\widetilde{\gamma}_3 \in \mathrm{Ext}_A^{2p+3,q(4p^2+2p+3)}(\mathbb{Z}/p, \mathbb{Z}/p),$$

it is obtained that

$$\beta_1^{p-1}\alpha_1h_{2,0}\gamma_3 \neq 0 \in \mathrm{Ext}_{BP_*BP}^{2p+3,q(4p^2+2p+3)}(BP_*, BP_*).$$

Theorem 3.2 *In the Adams-Novikov spectral sequence,*

$$\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3 \in \mathrm{Ext}_{BP_*BP}^{2p+3,q(4p^2+2p+3)}(BP_*, BP_*)$$

converges nontrivially to $\pi_{q(4p^2+2p+2)-5}(S^0)$.

Proof From Theorem 3.1, it is known that

$$\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3 \neq 0 \in \mathrm{Ext}_{BP_*BP}^{2p+3,q(4p^2+2p+3)}(BP_*, BP_*).$$

Meanwhile, β_1^{p-1} and $\alpha_1 h_{2,0}\gamma_3$ converge nontrivially to $\pi_*(S^0)$. Therefore, we need to prove that $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ is not killed by any Adams differential. Using the sparseness of the ANSS, it is sufficient to consider elements in $\mathrm{Ext}_{BP_*BP}^{4,q(4p^2+2p+2)}(BP_*, BP_*)$.

Let us see the small descent spectral sequence

$$E_1^{s,t,*} = \mathrm{Ext}_{BP_*BP}^{t,*}(BP_*, BP_*Y) \otimes E(\alpha_1) \otimes P(\beta_1) \Longrightarrow \mathrm{Ext}_{BP_*BP}^{s+t,*}(BP_*, BP_*)$$

and the ABC Theorem which describes the generators of $\mathrm{Ext}_{BP_*BP}^{t,*}(BP_*, BP_*Y)$ $(t \geqslant 2)$.

It is easy to show that only the element $b_{1,1}\beta_{3p/p-2}$ can survive to $\mathrm{Ext}_{BP_*BP}^{4,q(4p^2+2p+2)}(BP_*,BP_*)$. However, since $\beta_1\beta_{3p/p-2} = 0$ (see [12]) in the E_2-term of the ANSS, the relation

$$d_{2p-1}(b_{1,1}\beta_{3p/p-2}) = \alpha_1\beta_1^p\beta_{3p/p-2} = 0$$

holds. Thus $\beta_1^{p-1}\alpha_1 h_{2,0}\gamma_3$ is not killed by $b_{1,1}\beta_{3p/p-2}$. The theorem is obtained.

4 A Toda Bracket and Relative Results

Let p be an odd prime number and let BP denote the Brown-Peterson ring spectrum at p (see [3–4]). We have

$$BP_* = Z_{(p)}[v_1, v_2, \cdots] \quad \text{and} \quad BP_*BP = BP_*[t_1, t_2, \cdots],$$

where the homological degrees of v_i and t_i are given by $|v_i| = |t_i| = 2(p^i - 1)$.

Let (BP_*, Γ) be a Hopf algebroid. For any $BP_*(BP)$-comodule M, we write

$$\mathrm{Ext}^*(M) = \mathrm{Ext}_\Gamma^*(BP_*, M).$$

One method of calculating this Ext group is to use the cobar complex. Given any Γ-comodule M with coaction $\psi : M \to M \otimes \Gamma$, one has $\mathrm{Ext}^*(M) = H^*(C_\Gamma^* M, d)$, where the cobar complex $C_\Gamma^* M$ is the differential graded $Z_{(p)}$-module with

$$C_\Gamma^s M = M \otimes_{BP_*} \Gamma \otimes_{BP_*} \cdots \otimes_{BP_*} \Gamma$$

(s factors of Γ) and the differential d of degree $+1$ given by

$$\begin{aligned} d(m \otimes x_1 \otimes \cdots \otimes x_s) = &\sum m' \otimes m'' \otimes x_1 \otimes \cdots \otimes x_s \\ &+ \sum_{i=1}^{s} (-1)^i m \otimes x_1 \otimes \cdots \otimes x_i' \otimes x_i'' \otimes \cdots \otimes x_s \\ &- (-1)^s m \otimes x_1 \otimes \cdots \otimes x_s \otimes 1, \end{aligned} \tag{4.1}$$

where the coproduct $\Delta(x_i) = \sum x_i' \otimes x_i''$ and $\psi(m) = \sum m' \otimes m''$.

The element $m \otimes x_1 \otimes x_2 \otimes \cdots \otimes x_s$ is sometimes denoted by $m[x_1|x_2|\cdots|x_s]$ for simplicity.

Let $I_n = (p, v_1, \cdots, v_{n-1})$ be the ideal of BP_*. Then

$$BP_*V(2) = BP_*/I_3 = Z_{(p)}[v_1, v_2, \cdots]/(p, v_1, v_2).$$

Let $\Gamma = BP_*/I_3[t_1, t_2, t_3, \cdots]$. Then (BP_*, Γ) is a Hopf algebroid. Thus, there is a natural isomorphism

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*V(2)) \cong \mathrm{Ext}^*_{\Gamma}(BP_*, BP_*). \tag{4.2}$$

Theorem 4.1 *The $q(p^2 + 2p + 2) - 2$ dimension stable homology group of $V(2)$ is trivial, i.e.,*

$$\pi_{q(p^2+2p+2)-2}V(2) = 0.$$

Proof For complex $V(2)$, there is the Adams-Novikov spectral sequence converging to the stable homotopy groups of $V(2)$ at the prime p,

$$\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*(V(2))) \Longrightarrow \pi_{t-s}(V(2)).$$

It is known that the inner degree t of the E_2-term $\mathrm{Ext}^{s,t}_{BP_*BP}(BP_*, BP_*V(2))$ is the multiple of $q = 2p - 2$. In order to consider all possible elements converging to $\pi_{q(p^2+2p+2)-2}V(2)$, it is sufficient to consider only those of the form $s = 2 + nq$ and $t = q(p^2 + 2p + 2 + n)$ for $n \geqslant 0$.

For computing $\mathrm{Ext}^{2+nq, q(p^2+2p+2+n)}_{BP_*BP}(BP_*, BP_*(V(2)))$ $(n \geqslant 0)$, consider the isomorphism

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*V(2)) \cong \mathrm{Ext}^*_{\Gamma}(BP_*, BP_*).$$

Note that we only need to consider elements which have the homotopy degree $t - s \leqslant q(p^2 + 2p + 2) - 2$. Since $|v_i| > q(p^2 + 2p + 2) - 2,\ |t_i| > q(p^2 + 2p + 2) - 2$ for $i > 3$, we have the following isomorphism:

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*V(2)) \cong \mathrm{Ext}^*_{\Gamma'}(BP_*, BP_*),$$

where $\Gamma' = \mathbb{Z}/p[v_3][t_1, t_2, t_3] = P(3)[v_3]$, and $P(3)$ is the Hopf algebra $\mathbb{Z}/p[t_1, t_2, t_3]$.

Hopf algebroid $\Gamma' = P(3)[v_3]$ has the coproduct and the right unit as follows:

$$\begin{aligned} \Delta(t_1) &= t_1 \otimes 1 + 1 \otimes t_1, \\ \Delta(t_2) &= t_2 \otimes 1 + t_1 \otimes t_1^p + 1 \otimes t_2, \\ \Delta(t_3) &= t_3 \otimes 1 + t_2 \otimes t_1^{p^2} + t_1 \otimes t_2^p + 1 \otimes t_3, \\ \eta_R(v_3) &= v_3. \end{aligned} \tag{4.3}$$

Since the right unit $\eta_R(v_3) = v_3$ in the cobar complex $C_{\Gamma'}(BP_*)$, there is a natural isomorphism

$$\mathrm{Ext}^*_{BP_*BP}(BP_*, BP_*V(2)) \cong \mathrm{Ext}^*_{\Gamma'}(BP_*, BP_*) \cong \mathrm{Ext}^*_{P(3)}(\mathbb{Z}/p, \mathbb{Z}/p) \otimes P(v_3). \tag{4.4}$$

For computing $\mathrm{Ext}^*_{P(3)}(\mathbb{Z}/p,\mathbb{Z}/p)$, i.e., the cohomology of the Hopf algebra $P(3)$, we can use the modified form of the May spectral sequence introduced in [8–9, 15]. Let $P_*=P(t_1,t_2,\cdots)$ be the dual of Steenrod's reduced powers. Then there is the spectral sequence $\{E_r^{s,t,*},d_r\}$ which converges to $\mathrm{Ext}^{s,t}_{P_*}(\mathbb{Z}/p,\mathbb{Z}/p)$ with the E_1-term

$$E_1^{*,*,*}=E(h_{i,j}\mid i>0,\ j\geq 0)\otimes P(b_{i,j}\mid i>0,\ j\geq 0). \tag{4.5}$$

We only need to consider the elements $h_{i,j}$ with $i+j\leqslant 3$ and $b_{i,j}$ with $i+j\leqslant 2$, so, the modified May E_2-term is $P(b_{1,0},b_{1,1},b_{2,0})$ tensored with the cohomology of the complex

$$(E(h_{i,j}:i+j\leqslant 3),d_1)$$

described by Toda in [17]. We list its generators in the Table 1 in Section 3.

In the range $t-s\leqslant q(p^2+2p+2)-2$, the E_2-term of the modified May spectral sequence equals

$$\mathcal{G}=P(b_{1,0})\otimes E(b_{1,1})\otimes E(b_{2,0})\otimes\{1,h_{1,0},h_{1,1},h_{1,2},h_{1,2}h_{1,0},g_0,g_1,k_0,k_0h_{1,0}\}. \tag{4.6}$$

In our range, the Adams-Novikov E_2-term for $V(2)$ is isomorphic to $\mathrm{Ext}^*_{P(3)}(\mathbb{Z}/p,\mathbb{Z}/p)\otimes P(v_3)$ which is a subquotient of $\mathcal{G}\otimes P(v_3)$. It is easy to verify that

$$\mathrm{Ext}^{2+nq,q(p^2+2p+2+n)}_{BP_*BP}(BP_*,BP_*V(2))=0$$

for $n\geqslant 0$ because no element can have both the dimensions $2+nq$ and the inner degree $q(p^2+2p+2+n)$ in $\mathcal{G}\otimes P(v_3)$.

It now follows that the theorem holds from the Adams-Novikov spectral sequence for $V(2)$.

It is easily showed that the following theorem holds from the above theorem.

Theorem 4.2 *For $p\geqslant 7$, $s\geqslant 1$, the Toda bracket $\langle\alpha_1\beta_1,p,\gamma_s\rangle=0$.*

Proof Let $\widetilde{v}_3$ be the composite of the following maps:

$$S^{q(p^2+p+1)}\xrightarrow{\widetilde{i}}\Sigma^{q(p^2+p+1)}V(2)\xrightarrow{v_3}V(2),$$

where the first map is the inclusion of the bottom cell.

It is well-known that $\widetilde{v}_3$ is a p order element in $\pi_{q(p^2+p+1)}V(2)$, and then the Toda bracket $\langle\alpha_1\beta_1,p,\widetilde{v}_3\rangle$ is well defined and $\langle\alpha_1\beta_1,p,\widetilde{v}_3\rangle\in\pi_{q(p^2+2p+2)-2}V(2)$.

Let us use $\widetilde{j}$ to denote the projection from $V(2)$ to S^0. Then $\gamma_s=\widetilde{i}\cdot v_3^s\cdot\widetilde{j}$.

As a result,

$$\langle\alpha_1\beta_1,p,\gamma_s\rangle=\langle\alpha_1\beta_1,p,\widetilde{v}_3\cdot v_3^{s-1}\cdot\widetilde{j}\rangle=\langle\alpha_1\beta_1,p,\widetilde{v}_3\rangle\cdot v_3^{s-1}\cdot\widetilde{j}=0$$

because $\langle\alpha_1\beta_1,p,\widetilde{v}_3\rangle=0\in\pi_{q(p^2+2p+2)-2}V(2)=0$.

D. C. Ravenel proved the following proposition (see [14, Proposition 7.5.11] and [15, Proposition 7.6.11]).

Proposition 4.1 *If x is an element in the stable homotopy groups of spheres and satisfies $px = 0$, $\langle \alpha_1\beta_1, p, x\rangle = 0$ and $\alpha_1 x \neq 0$, then the following relation*

$$\alpha_1\beta_1^{p-1}h_{2,0}x = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, x\rangle = \beta_{p/p-1}x$$

holds.

Proof From the relation between Toda brackets and Massey products, we have the following Toda brackets:

$$\beta_{p/p-1} = \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\rangle \quad \text{and} \quad \alpha_1 h_{2,0}x = \langle \alpha_1, \alpha_1\beta_1, p, x\rangle.$$

On the other hand,

$$\begin{aligned}\alpha_1\beta_1^{p-1}h_{2,0}x &= \beta_1^{p-1}\langle \alpha_1, \alpha_1\beta_1, p, x\rangle \\ &= \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, x\rangle \\ &= \alpha_1\langle \beta_1^{p-1}, \alpha_1\beta_1, p, x\rangle \\ &= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1 x\rangle \\ &= \langle \beta_1^{p-1}, \alpha_1\beta_1, p, \alpha_1\rangle \cdot x \\ &= \beta_{p/p-1}x.\end{aligned}$$

Therefore, the proposition holds.

It is known that $p\gamma_s = 0$ since γ_s has order p. The condition $\alpha_1\gamma_s \neq 0$ holds as a result of R. Kato and K. Shimomura [5] who got that the elements $\alpha_1\gamma_t \neq 0$ for $p \geqslant 7$ and the positive integer t with $p \nmid t(t^2-1)$ using the cohomology of the third Morava stabilizer algebra. Thus we get the following result.

Theorem 4.3 *For $s \geqslant 2$, $p \geqslant 7$ and $p \nmid s(s^2-1)$, the following relation holds:*

$$\alpha_1\beta_1^{p-1}h_{2,0}\gamma_s = \langle \alpha_1\beta_1^{p-1}, \alpha_1\beta_1, p, \gamma_s\rangle = \beta_{p/p-1}\gamma_s.$$

Corollary 4.1 *In the stable homotopy groups of spheres $\pi_*(S^0)$, $\beta_{p/p-1}\gamma_3$ is nontrivial and represents the element $\alpha_1\beta_1^{p-1}h_{2,0}\gamma_3$.*

Proof In Section 3, we have already got that $\alpha_1\beta_1^{p-1}h_{2,0}\gamma_3$ is nontrivial in $\pi_*(S^0)$, so is $\beta_{p/p-1}\gamma_3$. Thus the corollary holds.

It is known that $\alpha_1 h_{2,0}\gamma_s$, $\beta_1\alpha_1 h_{2,0}\gamma_s$ are not trivial in $\pi_*(S^0)$ for $3 \leqslant s \leqslant p-2$, $p \geqslant 7$. However, we can prove that $\beta_1^p\alpha_1 h_{2,0}\gamma_s$ is trivial in $\pi_*(S^0)$ for $2 \leqslant s \leqslant p-2$, $p \geqslant 7$.

Corollary 4.2 *For $s \geqslant 2$, $p \geqslant 7$ and $p \nmid s(s^2-1)$,*

$$\beta_1^p\alpha_1 h_{2,0}\gamma_s = 0 \in \pi_*(S^0).$$

Proof The result can be easily got since $\beta_1^p\alpha_1 h_{2,0}\gamma_s = \beta_1\beta_{p/p-1}\gamma_s$ and $\beta_1\beta_{p/p-1} = 0$ in $\pi_*(S^0)$.

Acknowledgement The authors would like to express their gratitude to Prof. X. Wang for his suggestions.

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