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Vertex Representations of Toroidal Special Linear Lie Superalgebras*

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Abstract Based on the loop-algebraic presentation of 2-toroidal Lie superalgebras, a free field representation of toroidal Lie superalgebras of type $A(m, n)$ is constructed using both vertex operators and bosonic fields.

Keywords Toroidal Lie superalgebras, Vertex operators, Free fields
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1 Introduction

Let $\mathfrak{g}$ be a finite-dimensional simple Lie (super)algebra of type X and R be the ring of Laurent polynomials in commuting variables, and then the toroidal Lie (super)algebra $T(\mathfrak{g})$ is by definition the perfect central extension of the loop algebra $\mathfrak{g} \otimes R$. When $R = \mathbb{C}[t, t^{-1}]$, the toroidal Lie algebra is the affine Kac-Moody Lie algebra. The larger class of Lie (super)algebras $T(\mathfrak{g})$ shares many properties with the untwisted affine Lie (super)algebras.

In the case of untwisted toroidal Lie algebras, Moody, Rao and Yokonuma [18] gave a loop algebra presentation for the 2-toroidal Lie algebras similar to the affine Kac-Moody Lie algebras, which has set the stage for later developments such as free field realizations and vertex operator representations. Notably in [20] the toroidal Lie algebras of type B_n were constructed by using fermionic operators (see [8]). On the other hand, level-one representations of toroidal Lie algebras of the simply laced types were realized via McKay correspondence and wreath products of Kleinian subgroups of $SL_2(\mathbb{C})$ (see [5]). Bosonic realizations of higher-level toroidal Lie algebras $T(A_1)$ were also given in [10]. More recently, a unified realization (see [9, 11]) of all 2-toroidal Lie algebras of classical types was constructed by using bosonic or ferminoic fields, which has generalized the Feingold-Frenkel construction (see [4]) for affine Lie algebras.

Affine Lie superalgebras have been studied as early as their non-super counterparts. In fact, Feingold and Frenkel construction works for Lie superalgebras as well (see [4]). Vertex superalgebras and their representations were also given in [15]. Later in [14] integrable highest-weight modules were constructed for affine superalgebras of the orthosymplectic seises using fermionic and bosonic fields. All these constructions were based on the loop algebra realizations of affine Lie superalgebras.

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Irreducible highest-weight modules of classical toroidal Lie superalgebras can be constructed abstractly as in the affine cases (see [19]). Various other constructions of toroidal Lie superalgebras and their generalizations were known (see [1, 3, 9, 16–17]). In particular, [2] has constructed a certain vertex operator representation for the general toroidal cases. Recently we have given a loop algebra realization for 2-toroidal classical superalgebras (see [12]), which is a super analog of the MRY construction (see [7] for earlier development).

The aim of this work is to generalize Kac and Wakimoto's work on affine superalgebras of unitary series to the 2-toroidal setting using both vertex operators and Weyl bosonic fields, and the construction has utilized our recent MRY presentation exclusively. We remark that our work is different from [2] in that we use more bosonic fields while the latter used more vertex operators. This suggests that there could be a super boson-fermion correspondence for the 2-toroidal cases.

This paper is organized as follows. In Section 2 we recall the notions of 2-toroidal Lie superalgebras and the loop-algebra presentation. In Section 3 we construct certain vertex operators and Weyl bosonic fields to give a level-one representation of the 2-toroidal Lie special linear superalgebra.

2 The Toroidal Lie Superalgebra $\mathfrak{T}(A(m,n))$

Let $V=\mathbb{C}^{m|n+1}$ be the $\mathbb{Z}_2$-graded vector space of dimension $(m,n+1)$, where $m\neq n$. Let $\mathfrak{gl}(m\mid n+1)$ be the Lie superalgebra of the super-endomorphisms of V under the superbracket. Let $\mathfrak{g}$ be the traceless subalgebra, i.e., the simple Lie superalgebra of type $A(m,n)$. Let $R=\mathbb{C}[s^{\pm1},t^{\pm1}]$ be the complex commutative ring of Laurant polynomials in s,t. The loop Lie superalgebra $L(\mathfrak{g}):=\mathfrak{g}\otimes R$ is defined under the Lie superbracket $[x\otimes a,y\otimes b]=[x,y]\otimes ab$.

Let Ω_R be the R-module of Kähler differentials $\{bda\mid a,b\in R\}$, and let $d\Omega_R$ be the space of exact forms. The quotient space $\Omega_R/d\Omega_R$ has a basis consisting of $\overline{s^{m-1}t^nds}$, $\overline{s^nt^{-1}dt}$, $\overline{s^{-1}ds}$, where $m,n\in\mathbb{Z}$. Here $\overline{a}$ denotes the coset $a+d\Omega_R$.

The toroidal special linear superalgebra $T(\mathfrak{g})$ is defined to be the Lie superalgebra on the following vector space:

$$T(\mathfrak{g})=\mathfrak{g}\otimes R\oplus\Omega_R/d\Omega_R$$

with the Lie superbracket ($x,y\in\mathfrak{g},\ a,b\in R$):

$$[x\otimes a,y\otimes b]=[x,y]\otimes ab+(x\mid y)\overline{(da)b},\quad [T(\mathfrak{g}),\Omega_R/d\Omega_R]=0$$

and the parities are specified by:

$$p(x\otimes a)=p(x),\quad p(\Omega_R/d\Omega_R)=\overline{0}.$$

Let $A=(a_{ij})$ be the extended distinguished Cartan matrix of the affine Lie superalgebra of type $A(m,n)^{(1)}$, i.e.,

$$\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & \cdots & 0 & 1\\ -1 & 2 & -1 & \ddots & 0 & \cdots & 0 & 0\\ \vdots & \ddots & \ddots & \ddots & \ddots & \cdots & \cdots & \vdots\\ 0 & \cdots & -1 & 0 & 1 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots\\ 0 & \cdots & 0 & & & 2 & -1 & 0\\ 0 & \cdots & 0 & & & -1 & 2 & -1\\ -1 & \cdots & 0 & 0 & \cdots & 0 & -1 & 2 \end{pmatrix}$$

and let $Q = \mathbb{Z}\alpha_0 \oplus \cdots \oplus \mathbb{Z}\alpha_{m+n+1}$ be its root lattice. Here α_0, α_{m+1} are the odd simple roots. The standard invariant form is given by $(\alpha_i, \alpha_j) = d_i a_{ij}$, where $(d_0, d_1, \cdots, d_{m+n+1}) = (1, \underbrace{1, \cdots, 1}_{m}, \underbrace{-1, \cdots, -1}_{n+1})$.

We recall the loop algebra presentation of the 2-toroidal Lie superalgebras.

Theorem 2.1 (see [12]) *The toroidal special linear superalgebra $T(\mathfrak{g})$ is isomorphic to the Lie superalgebra $\mathfrak{T}(A(m,n))$ generated by*

$$\{\mathcal{K}, \alpha_i(k), x_i^{\pm}(k) \mid 0 \leqslant i \leqslant m+n+1, k \in \mathbb{Z}\}$$

with parities given as

$$p(\mathcal{K}) = p(\alpha_i(k)) = \overline{0}, \quad p(x_i^{\pm}(k)) = p(\alpha_i) \quad (0 \leqslant i \leqslant m+n+1, k \in \mathbb{Z}).$$

The defining relations of superbrackets are given by:

(1) $[\mathcal{K}, \alpha_i(k)] = [\mathcal{K}, x_i^{\pm}(k)] = 0$;

(2) $[\alpha_i(k), \alpha_j(l)] = k(\alpha_i \mid \alpha_j)\delta_{k,-l}\mathcal{K}$;

(3) $[\alpha_i(k), x_j^{\pm}(l)] = \pm(\alpha_i \mid \alpha_j)x_j^{\pm}(k+l)$;

(4) $[x_i^{+}(k), x_j^{-}(l)] = 0,$ *if* $i \neq j$;
$[x_i^{+}(k), x_i^{-}(l)] = -\{\alpha_i(k+l) + k\delta_{k,-l}\mathcal{K}\},$ *if* $(\alpha_i \mid \alpha_i) = 0$;
$[x_i^{+}(k), x_i^{-}(l)] = -\frac{2}{(\alpha_i|\alpha_i)}\{\alpha_i(k+l) + k\delta_{k,-l}\mathcal{K}\},$ *if* $(\alpha_i \mid \alpha_i) \neq 0$;

(5) $[x_i^{\pm}(k), x_i^{\pm}(l)] = 0$;
$[x_i^{\pm}(k), x_j^{\pm}(l)] = 0,$ *if* $a_{ii} = a_{ij} = 0, i \neq j$;
$[x_i^{\pm}(k), [x_i^{\pm}(k), (x_j^{\pm}(l)]] = 0,$ *if* $a_{ii} = 0, a_{ij} \neq 0, i \neq j$;
$\underbrace{[x_i^{\pm}(k), \cdots, [}_{1-a_{ij}} x_i^{\pm}(k), x_j^{\pm}(l)] \cdots] = 0,$ *if* $a_{ii} \neq 0, i \neq j$.

We define the formal power series with coefficients from $\mathfrak{T}(A(m,n))$:

$$\alpha_i(z) = \sum_{k\in\mathbb{Z}} \alpha_i(k) z^{-k-1}, \quad x_i^{\pm}(z) = \sum_{k\in\mathbb{Z}} x_i^{\pm}(k) z^{-k-1},$$

and then the defining relations of $\mathfrak{T}(A(m,n))$ can be rewritten in terms of the formal series as follows.

Proposition 2.1 *The relations of $\mathfrak{T}(A(m,n))$ can be written as follows*:

(1′) $[\mathcal{K}, \alpha_i(z)] = [\mathcal{K}, x_i^{\pm}(z)] = 0$;

(2′) $[\alpha_i(z), \alpha_j(w)] = (\alpha_i \mid \alpha_j)\partial_w\delta(z-w)\mathcal{K}$;

(3′) $[\alpha_i(z), x_j^{\pm}(w)] = \pm(\alpha_i \mid \alpha_j)x_j^{\pm}(w)\delta(z-w)$;

(4′) $[x_i^{+}(z), x_j^{-}(w)] = 0,$ *if* $i \neq j$;
$[x_i^{+}(z), x_i^{-}(w)] = -\{(\alpha_i(w)\delta(z-w) + \partial_w\delta(z-w)\mathcal{K}\},$ *if* $(\alpha_i \mid \alpha_i) = 0$;
$[x_i^{+}(z), x_i^{-}(w)] = -\frac{2}{(\alpha_i|\alpha_i)}\{(\alpha_i(w)\delta(z-w) + \partial_w\delta(z-w)\mathcal{K}\},$ *if* $(\alpha_i \mid \alpha_i) \neq 0$;

(5′) $[x_i^{\pm}(z), x_i^{\pm}(w)] = 0$;
$[x_i^{\pm}(z), x_j^{\pm}(w)] = 0,$ *if* $a_{ii} = a_{ij} = 0, i \neq j$;
$[x_i^{\pm}(z_1), [x_i^{\pm}(z_2), x_j^{\pm}(w)]] = 0,$ *if* $a_{ii} = 0, a_{ij} \neq 0, i \neq j$;
$[x_i^{\pm}(z_1), \cdots, [x_i^{\pm}(z_{1-a_{ij}}), x_j^{\pm}(w)] \cdots] = 0,$ *if* $a_{ii} \neq 0, i \neq j$.

Here we have used the formal delta function

$$\delta(z-w) = \sum_{n\in\mathbb{Z}} z^{-n-1}w^n.$$

Its derivatives are given by the power series expansions (see [13]):

$$\partial_w^{(j)}\delta(z-w)=i_{z,w}\frac{1}{(z-w)^{j+1}}-i_{w,z}\frac{1}{(-w+z)^{j+1}},$$

where $\partial_w^{(j)}=\frac{1}{j!}\partial_w^j$ and $i_{z,w}$ means power series expansion in the domain $|z|>|w|$. By convention, if we write a rational function in the variable $z-w$, it is usually assumed that the power series is expanded in the region $|z|>|w|$. Finally, the equation

$$f(z,w)\delta(z-w)=f(z,z)\delta(z-w)$$

holds when both sides are meaningful.

3 The Vertex Representation of $\mathfrak{T}(A(m,n))$

In this section, we will give a representation of the Lie superalgebra $\mathfrak{T}(A(m,n))$ using both vertex operators and bosonic fields.

Let ε_i $(0\leqslant i\leqslant n+m+3)$ be an orthomormal basis of the vector space $\mathbb{C}^{n+m+4}$ and denote $\delta_i=\sqrt{-1}\varepsilon_{m+1+i}$ $(1\leqslant i\leqslant n+2)$. Then the distinguished simple root systems, the positive root systems and the longest distinguished root of the Lie superalgebra of type $A(m,n)$ can be represented in terms of vectors ε_i's and δ_i's as follows:

$$\begin{aligned}\Pi=\{&\alpha_1=\varepsilon_1-\varepsilon_2,\cdots,\alpha_m=\varepsilon_m-\varepsilon_{m+1},\alpha_{m+1}=\varepsilon_{m+1}-\delta_1,\\&\alpha_{m+2}=\delta_1-\delta_2,\cdots,\alpha_{n+m+1}=\delta_n-\delta_{n+1}\};\\\triangle_+=\{&\varepsilon_i-\varepsilon_j,\delta_k-\delta_l\mid 1\leqslant i<j\leqslant n+1,1\leqslant k<l\leqslant m+1\}\\&\cup\big\{\delta_k-\varepsilon_i\mid 1\leqslant i\leqslant n+1,1\leqslant k\leqslant m+1\big\};\\\theta=\alpha_1&+\cdots+\alpha_{m+n+1}=\varepsilon_1-\delta_{n+1}.\end{aligned}$$

3.1 Vertex operators

Let $\Gamma=\mathbb{Z}\varepsilon_1\oplus\cdots\oplus\mathbb{Z}\varepsilon_{m+1}$ and $\mathfrak{h}=\Gamma\otimes_{\mathbb{Z}}\mathbb{C}$. We view $\mathfrak{h}$ as an abelian Lie algebra and consider the central extension of its affinization $\widehat{\mathfrak{h}}$, i.e.,

$$\widehat{\mathfrak{h}}=\bigoplus_{n\neq 0}\mathbb{C}\mathfrak{h}\otimes t^n\oplus\mathbb{C}K$$

with the following communication relations:

$$[\alpha(k),\beta(l)]=k(\alpha,\beta)\delta_{k,-l}K,\quad[\widehat{\mathfrak{h}},K]=0,$$

where $\alpha(k)=\alpha\otimes t^k$ and $\alpha,\beta\in\Gamma;k,l\in\mathbb{Z}$. This is an infinite-dimensional Heisenberg algebra.

For $i=0,1$, we let $\Gamma_{\bar{i}}=\{\alpha\in\Gamma\mid(\alpha,\alpha)\in 2\mathbb{Z}+i\}$, so then $\Gamma=\Gamma_{\bar{0}}\oplus\Gamma_{\bar{1}}$. Let $F:\Gamma\times\Gamma\to\{\pm1\}$ is the bimultiplicative map determined by

$$F(\varepsilon_i,\varepsilon_j)=\begin{cases}1, & \text{if } i\leqslant j;\\-1, & \text{if } i>j.\end{cases}$$

Then the map satisfies the following properties:

(1) $F(0,\alpha)=F(\alpha,0)=1,\ \forall\alpha\in\Gamma;$

(2) $F(\alpha,\beta)F(\alpha,\beta+\gamma)=F(\beta,\gamma)F(\alpha,\beta+\gamma),\ \forall\alpha,\beta,\gamma\in\Gamma;$

(3) $F(\alpha,\beta)F(\beta,\alpha)^{-1}=(-1)^{(\alpha,\beta)+ij},\ \forall\alpha\in\Gamma_{\bar{i}},\beta\in\Gamma_{\bar{j}}.$

Let $\mathbb{C}[\Gamma]$ be the vector space spanned by the basis $\{\mathrm{e}^\gamma|\gamma\in\Gamma\}$ over $\mathbb{C}$. We define a twisted group algebra structure on $\mathbb{C}[\Gamma]$ as follows:

$$\mathrm{e}^\alpha\mathrm{e}^\beta=F(\alpha,\beta)\mathrm{e}^{\alpha+\beta}.$$

We form the tensor space

$$V[\Gamma]=\mathbb{C}[\Gamma]\otimes S\Big(\bigoplus_{j<0}(\mathfrak{h}\otimes t^j)\Big),$$

and define the action of $\widehat{\mathfrak{h}}$ as follows: K acts as the identity operator, $\alpha(-k)$ $(k>0)$ acts as multiplication by $\alpha\otimes t^k$ for $\alpha\in\Gamma$, and $\alpha(k)$ $(k>0)$ acts as the derivation of $V[\Gamma]$ defined by the formula

$$\alpha(k)(v\otimes\mathrm{e}^\beta)=k(\alpha,\beta)(v\otimes\mathrm{e}^\beta), \tag{3.1a}$$

$$\alpha(k)(\mathrm{e}^\beta\otimes\gamma\otimes t^{-l})=\delta_{k,0}(\alpha,\beta)(\mathrm{e}^\beta\otimes\gamma\otimes t^{-l})+k\delta_{k,l}(\alpha,\gamma)\mathrm{e}^\beta\otimes\gamma\otimes t^{-k}. \tag{3.1b}$$

The space $V[\Gamma]$ has a natural $\mathbb{Z}_2$-gradation: $V[\Gamma]=V[\Gamma]_{\overline{0}}\oplus V[\Gamma]_{\overline{1}}$, where $V[\Gamma]_{\overline{0}}$ (resp. $V[\Gamma]_{\overline{1}}$) is the vector space spanned by $\mathrm{e}^\alpha\otimes\beta\otimes t^{-j}$ with $\alpha,\beta\in\Gamma; j\in\mathbb{Z}_+$ such that $(\alpha,\alpha)\in2\mathbb{Z}$ (resp. $(\alpha,\alpha)\in2\mathbb{Z}+1$).

For $\alpha\in\Gamma$, we define the vertex operator $Y(\alpha,z)$ as follows:

$$Y(\alpha,z)=\mathrm{e}^\alpha z^{\alpha(0)}\exp\Big(-\sum_{j<0}\frac{\alpha(j)}{j}z^{-j}\Big),\quad \exp\Big(-\sum_{j>0}\frac{\alpha(j)}{j}z^{-j}\Big),$$

where the operator $z^{\alpha(0)}$ is given by

$$z^{\alpha(0)}(\mathrm{e}^\beta\otimes\gamma\otimes t^{-j})=z^{(\alpha,\beta)}(\mathrm{e}^\beta\otimes\gamma\otimes t^{-j})$$

for $\beta,\gamma\in\Gamma; j\in\mathbb{Z}_+$ and we denote

$$X(\alpha,z)=\begin{cases}z^{\frac{(\alpha,\alpha)}{2}}Y(\alpha,z), & \text{if } \alpha\in\Gamma_{\overline{0}};\\ Y(\alpha,z), & \text{if } \alpha\in\Gamma_{\overline{1}}.\end{cases}$$

We expand $X(\alpha,z)$ in z

$$X(\alpha,z)=\sum_{j\in\mathbb{Z}}X(\alpha,j)z^{-j-1},$$

where the components $X(\alpha,j)$ are well-defined local operators. Similarly, for $\alpha\in\Gamma$, we define

$$\alpha(z)=\sum_{k\in\mathbb{Z}}\alpha(k)z^{-k-1}.$$

Lemma 3.1 *For $\alpha\in\Gamma_{\overline{i}},\beta\in\Gamma_{\overline{j}}$, one has that*

(1) $[Y(\alpha,z),Y(\beta,w)]=0$, *if* $(\alpha,\beta)\geqslant0$;

(2) $[Y(\alpha,z),Y(\beta,w)]=F(\alpha,\beta)Y(\alpha+\beta,z)\delta(z-w)$, *if* $(\alpha,\beta)=-1$;

(3) $[\alpha(z),Y(\beta,w)]=(\alpha,\beta)Y(\beta,z)\delta(z-w)$.

Proof The first and second parts have been proved in [1]. For the third part, we refer to [21].

Corollary 3.1 (1) $[X(\varepsilon_i,z),X(\varepsilon_j-\varepsilon_k,w)]=\delta_{ik}F(\varepsilon_i,\varepsilon_j-\varepsilon_k)X(\varepsilon_j,w)\delta(z-w),\ j\neq k$;

(2) $[X(\varepsilon_i,z),X(-\varepsilon_j,w)]=\delta_{ij}F(\varepsilon_i,-\varepsilon_j)\partial_w\delta(z-w)$;

(3) $[\alpha(z),X(\beta,w)]=(\alpha,\beta)X(\beta,z)\delta(z-w),\ \alpha,\beta\in\Gamma$.

Proof The corollary is a direct consequence of Lemma 3.1.

3.2 Bosonic fields

We introduce $\overline{c} = \varepsilon_0 + \delta_{n+2}$ and define $\beta = \delta_{n+1} + \overline{c}$, so then $\alpha_0 = \beta - \varepsilon_1$. Note that $(\beta \mid \beta) = -1, (\beta \mid \delta_i) = -\delta_{n+1,i}$. Let $\mathcal{P}$ be the vector spaces spanned by the set $\{\overline{c}, \delta_i \mid 1 \leqslant i \leqslant n+1\}$ and $\mathcal{P}^*$ be its dual space. Let $\mathcal{C} = \mathcal{P} \oplus \mathcal{P}^*$ and define the bilinear form on it as follows: For $a, b \in \mathcal{P}$,

$$\langle b^*, a\rangle = -\langle a, b^*\rangle = (a, b), \quad \langle b, a\rangle = \langle a^*, b^*\rangle = 0.$$

Let $\mathcal{A}(\mathbb{Z}^{2n+2})$ be the Weyl algebra generated by $\{u(k) \mid u \in \mathcal{C}, k \in \mathbb{Z}\}$ with the defining relations

$$u(k)v(l) - u(k)v(l) = \langle u, v\rangle \delta_{k,-l}$$

for $u, v \in \mathcal{C}$ and $k, l \in \mathbb{Z}$.

The representation space of the algebras $\mathcal{A}(\mathbb{Z}^{n+1})$ is defined to be the following vector space:

$$\mathfrak{F} = \bigotimes_{a_i} \Big(\bigotimes_{k\in\mathbb{Z}_+} \mathbb{C}[a_i(-k)] \bigotimes_{k\in\mathbb{Z}_+} \mathbb{C}[a_i^*(-k)] \Big),$$

where a_i runs through any basis in $\mathcal{P}$, consisting of $\overline{c}$ and δ_k's. The algebra $\mathcal{A}(\mathbb{Z}^{2n+2})$ acts on the space by the usual action: $a(-k)$ acts as creation operators and $a(k)$ as annihilation operators.

For $u \in \mathcal{C}$, we define the formal power series with coefficients from the associative algebra $\mathcal{A}(\mathbb{Z}^{2n+2})$:

$$u(z) = \sum_{k\in\mathbb{Z}} u(k) z^{-k-1}.$$

It is a bosonic field acting on the Fock space $\mathfrak{F}$.

In the following, we will give a representation of $\mathfrak{T}(A(m, n))$ on a quotient $\mathfrak{V}$ of the tensor space $V[\Gamma] \otimes \mathfrak{F}$:

$$\mathfrak{V} = \frac{V[\Gamma] \otimes \mathfrak{F}}{\Big(\sum_k : X(\pm\varepsilon_1, -n+k)\overline{c}(n) :\Big)}.$$

Therefore, the relation $: X(\pm\varepsilon_1, z)\overline{c}(z) := 0$ holds on $\mathfrak{V}$. Note that there is a natural homomorphism from $\mathfrak{V}$ onto $V[\overline{\Gamma}] \otimes \mathfrak{F}$, where $\overline{\Gamma} = \Gamma/(\varepsilon_0 + \delta_{n+2})$. For simplicity, we will use the same symbol to denote the coset elements in $\mathfrak{V}$. Observe that there is a $\mathbb{Z}_2$-gradation on this space with the parity given by $p(\mathrm{e}^\alpha \otimes x \otimes y) = p(\alpha)$ for $\alpha \in \Gamma, x \in S\big(\bigoplus_{j<0} (\mathfrak{h} \otimes t^j)\big), y \in \mathfrak{F}$. The vertex operators $X(\alpha, z), \alpha(z)$ act on the first component and the bosonic field $u(z)$ acts on the second component. It follows that

$$p(X(\alpha, z)) = p(\alpha), \quad p(\alpha(z)) = p(u(z)) = \overline{0}.$$

For any two fields $a(z), b(w)$ with fixed parity, we define the normal ordered product by

$$\begin{aligned} : a(z)b(w) : &= a(z)_+ b(w) - (-1)^{p(a)p(b)} b(w)a(z)_- \\ &= (-1)^{p(a)p(b)} : b(w)a(z) :, \end{aligned}$$

where $a_\pm(z)$ is defined as usual. Based on the normal ordering of two fields, one can define inductively the normal ordering of more than two fields "from right to left".

The following facts are well known in literature (see [6, 11]).

Proposition 3.1 *One has that*
(1) $[\alpha(z), \beta(w)] = (\alpha, \beta)\partial_w \delta(z - w),\ \alpha, \beta \in \Gamma;$
(2) $[X(\varepsilon_i - \varepsilon_j, z), X(\varepsilon_j - \varepsilon_i, w)] = F(\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_i)((\varepsilon_i - \varepsilon_j)(z)\delta(z - w) + \partial_w \delta(z - w));$
(3) $: X(\varepsilon_i, z)X(-\varepsilon_j, z) := F(\varepsilon_i, -\varepsilon_j)X(\varepsilon_i - \varepsilon_j, z),\ i \neq j;$
(4) $: X(-\varepsilon_j, z)X(\varepsilon_i, z) := F(-\varepsilon_j, -\varepsilon_i)X(\varepsilon_i - \varepsilon_j, z),\ i \neq j;$
(5) $: X(\varepsilon_i, z)X(-\varepsilon_i, z) := \varepsilon_i(z).$

Furthermore, we define the contraction of two fields $a(z), b(w)$ by

$$\underbrace{a(z)b(w)} = a(z)b(w) - : a(z)b(w) : .$$

Proposition 3.2 (see [13]) *Suppose that fields $a(z), b(w)$ satisfy the following equality*:

$$[a(z), b(w)] = \sum_{j=0}^{N-1} c^j(w)\partial_w^{(j)}\delta(z - w),$$

where N is a positive integer and $c^j(w)$ are formal distributions in the indeterminate z, and then we have that

$$\underbrace{a(z)b(w)} = \sum_{j=0}^{N-1} c^j(w)\frac{1}{(z - w)^{j+1}}.$$

The following well-known Wick's theorem is useful for calculating the operator product expansions (OPE) of normally ordered products of free fields.

Theorem 3.1 (see [13]) *Let $A^1, A^2, \cdots, A^M$ and $B^1, B^2, \cdots, B^N$ be two collections of fields with definite parity. Suppose that these fields satisfy the following properties*:
(1) $[\underbrace{A^i B^j}, Z^k] = 0,$ *for all i, j, k and $Z = A$ or B;*
(2) $[A^i_\pm, B^j_\pm] = 0,$ *for all i, j.*
Then we have that

$$\begin{aligned} & : A^1 \cdots A^M :: B^1 \cdots B^N : \\ = & \sum_{s=0}^{m} \sum_{\substack{i_1 < \cdots < i_s \\ j_1 \neq \cdots \neq j_s}} \pm(\underbrace{A^{i_1} B^{j_1}} \cdots \underbrace{A^{i_s} B^{j_s}} : A^1 \cdots A^M B^1 \cdots B^N :_{(i_1, \cdots, i_s, j_1, \cdots, j_s)}), \end{aligned}$$

where $m = \min\{M, N\}$ and the subscript $(i_1, \cdots, i_s, j_1, \cdots, j_s)$ means that the fields $A^{i_1}, \ldots,$ A^{i_s}, B^{j_1}, $\ldots$, B^{j_s} are removed and the sign $\pm$ is obtained by the rule: Each permutation of the adjacent odd fields changes the sign.

Now we state the main result of this paper.

Theorem 3.2 *The following map defines a level-one representation on the space $\mathfrak{V}$:*

$$x_i^+(z) \mapsto \begin{cases} \sqrt{-1} : X(-\varepsilon_1, z)\beta(z) :, & i = 0; \\ X(\varepsilon_i - \varepsilon_{i+1}, z), & 1 \leqslant i \leqslant m; \\ : X(\varepsilon_{m+1}, z)\delta_1^*(z) :, & i = m + 1; \\ \sqrt{-1} : \delta_{i-m-1}(z)\delta_{i-m}^*(z) :, & m + 2 \leqslant i \leqslant m + n + 1, \end{cases}$$

$$x_i^-(z) \mapsto \begin{cases} \sqrt{-1} : X(\varepsilon_1, z)\beta^*(z) :, & i = 0; \\ X(\varepsilon_{i+1} - \varepsilon_i, z), & 1 \leqslant i \leqslant m; \\ : X(-\varepsilon_{m+1}, z)\delta_1(z) :, & i = m + 1; \\ \sqrt{-1} : \delta_{i-m-1}(z)\delta_{i-m}^*(z) :, & m + 2 \leqslant i \leqslant m + n + 1, \end{cases}$$

$$\alpha_i(z) \mapsto \begin{cases} :\beta(z)\beta^*(z): -\varepsilon_1(z), & i=0; \\ (\varepsilon_i-\varepsilon_{i+1})(z), & 1\leqslant i\leqslant m; \\ \varepsilon_{m+1}(z)- :\delta_1(z)\delta_1^*(z):, & i=m+1; \\ :\delta_{i-m-1}(z)\delta_{i-m-1}^*(z): - :\delta_{i-m}(z)\delta_{i-m}^*(z):, & m+2\leqslant i\leqslant m+n+1. \end{cases}$$

Proof To prove the theorem, one needs to check that all the field operators on the right side of above map satisfy relations (1′)–(5′) listed in Proposition 2.1.

First of all, we check (4′)–(3′) with the help of Wick's theorem.

$$\begin{aligned} [x_0^+(z), x_0^-(w)] &= -(:\beta(z)\beta^*(z): + :X(-\varepsilon_1,z)X(\varepsilon_1,z):)\delta(z-w) - \partial_w\delta(z-w) \\ &= -(\alpha_0(z)\delta(z-w) + \partial_w\delta(z-w)\cdot 1), \end{aligned}$$

where we have used the fact $:X(-\varepsilon_1,z)X(\varepsilon_1,z): = -\varepsilon_1(z)$ and

$$[\alpha_0(z), x_0^\pm(w)] = 0 = \pm(\alpha_0,\alpha_0)x_0^\pm(w)\delta(z-w).$$

For $1\leqslant i\leqslant m$, we have by Proposition 3.1 that

$$\begin{aligned} [x_i^+(z), x_i^-(w)] &= -((\varepsilon_i-\varepsilon_{i+1})(z)\delta(z-w) + \partial_w\delta(z-w)) \\ &= -\frac{2}{(\alpha_i,\alpha_i)}(\alpha_i(z)\delta(z-w) + \partial_w\delta(z-w)\cdot 1). \end{aligned}$$

It follows from Corollary 3.1 that

$$\begin{aligned} [\alpha_i(z), x_i^\pm(w)] &= \pm(\alpha_i,\alpha_i)x_i^\pm(w)\delta(z-w), \\ [x_{m+1}^+(z), x_{m+1}^-(w)] &= (:\delta_1(z)\delta_1^*(z): - :X(\varepsilon_{m+1},z)X(-\varepsilon_{m+1},z):)\delta(z-w) - \partial_w\delta(z-w) \\ &= -(\alpha_{m+1}(z)\delta(z-w) + \partial_w\delta(z-w)\cdot 1) \end{aligned}$$

and

$$[\alpha_{m+1}(z), x_{m+1}^\pm(w)] = 0 = \pm(\alpha_{m+1},\alpha_{m+1})x_{m+1}^\pm(w)\delta(z-w).$$

For $m+2\leqslant i\leqslant m+n+1$, we have that

$$\begin{aligned} [x_i^+(z), x_i^-(w)] &= (:\delta_{i-m-1}(z)\delta_{i-m-1}^*(z): - :\delta_{i-m}(z)\delta_{i-m}^*(z):)\delta(z-w) + \partial_w\delta(z-w) \\ &= -\frac{2}{(\alpha_i,\alpha_i)}(\alpha_i(z)\delta(z-w) + \partial_w\delta(z-w)\cdot 1) \end{aligned}$$

and

$$\begin{aligned} [\alpha_i(z), x_i^+(w)] &= -2\sqrt{-1} :\delta_{i-m-1}(z)\delta_{i-m}^*(z): \delta(z-w) \\ &= (\alpha_i,\alpha_i)x_i^+(w)\delta(z-w), \\ [\alpha_i(z), x_i^-(w)] &= -(\alpha_i,\alpha_i)x_i^-(w)\delta(z-w). \end{aligned}$$

For all $i\neq j$, we have $[x_i^+(z), x_j^-(w)] = 0$ and for any disconnected vertices

$$[\alpha_i(z), x_j^\pm(w)] = 0 = \pm(\alpha_i,\alpha_j)x_j^\pm(w)\delta(z-w).$$

All the rest can be checked by straightforward calculation, for example,

$$\begin{aligned} [\alpha_0(z), x_1^+(w)] &= -X(\varepsilon_1-\varepsilon_2,z)\delta(z-w) \\ &= (\alpha_0,\alpha_1)x_1^+(w)\delta(z-w), \end{aligned}$$

$$\begin{aligned}[\alpha_{m+1}(z), x^+_{m+2}(w)] &=: \delta_1(w)\delta^*_2(w) : \delta(z-w)\\ &= (\alpha_{m+1}, \alpha_{m+2})x^+_{m+2}(w)\delta(z-w),\\ [\alpha_{m+n+1}(z), x^+_{m+n}(w)] &= \sqrt{-1} : \delta_{n-1}(w)\delta^*_n(w)\delta(z-w)\\ &= (\alpha_{m+n+1}, \alpha_{m+n})x^+_{m+n}(w)\delta(z-w).\end{aligned}$$

For the extremal vertices, one also has that

$$\begin{aligned}[\alpha_{m+n+1}(z), x^+_0(w)] &= \sqrt{-1} : X(-\varepsilon_1, w)\delta_{n+1}(w) : \delta(z-w)\\ &= \sqrt{-1} : X(-\varepsilon_1, w)\beta(w) : \delta(z-w)\\ &= (\alpha_{m+n+1}, \alpha_0)x^+_0(w)\delta(z-w),\end{aligned}$$

where we have used the fact that

$$X(-\varepsilon_1, w)\overline{c}(w) := 0$$

and others can be proved similarly.

Secondly, we can check (2′) case by case by using Proposition 3.1(1) and we include the following examples:

$$\begin{aligned}&[\alpha_0(z), \alpha_0(w)] = 0 = (\alpha_0, \alpha_0)\partial_w\delta(z-w)\cdot 1,\\ &[\alpha_0(z), \alpha_1(w)] = -\partial_w\delta(z-w) = (\alpha_0, \alpha_1)\partial_w\delta(z-w)\cdot 1,\\ &[\alpha_0(z), \alpha_{m+n+1}(w)] = \partial_w\delta(z-w) = (\alpha_0, \alpha_{m+1})\partial_w\delta(z-w)\cdot 1.\end{aligned}$$

Finally, we proceed to check the Serre relations. It is easy to verify that $[x^\pm_i(z), x^\pm_i(w)] = 0$ for $0 \leqslant i \leqslant m+n+1$ and $[x^\pm_i(z), x^\pm_j(w)] = 0$ for $i \neq j,\ a_{ij} = 0$. The rest can be checked directly:

$$\begin{aligned}&\quad [x^+_0(z_1), [x^+_0(z_2), x^+_1(w)]]\\ &= -[: X(-\varepsilon_1, z_1)\beta(z_1) :, [: X(-\varepsilon_1, z_2)\beta(z_2) :, X(\varepsilon_1-\varepsilon_2, w)]]\\ &= -[: X(-\varepsilon_1, z_1)\beta(z_1) :, X(-\varepsilon_2, w)]\delta(z_2-w)\\ &= 0,\\ &\quad [x^+_0(z_1), [x^+_0(z_2), x^+_{m+n+1}(w)]]\\ &= -\sqrt{-1}[: X(-\varepsilon_1, z_1)\beta(z_1) :, [: X(-\varepsilon_1, z_2)\beta(z_2) :, : \delta_n(w)\delta^*_{n+1}(w) :]]\\ &= -\sqrt{-1}[: X(-\varepsilon_1, z_1)\beta(z_1) :, : X(-\varepsilon_1, w)\delta_n(w) :]\delta(z_2-w)\\ &= 0,\\ &\quad [x^+_{m+1}(z_1), [x^+_{m+1}(z_2), x^+_m(w)]]\\ &= [: X(\varepsilon_{m+1}, z_1)\delta^*_1(z_1) :, [: X(\varepsilon_{m+1}, z_2)\delta^*_1(z_2) :, X(\varepsilon_m-\varepsilon_{m+1}, w)]]\\ &= [: X(\varepsilon_{m+1}, z_1)\delta^*_1(z_1) :, ; X(\varepsilon_m, w)\delta^*_1(w) :]\delta(z_2-w)\\ &= 0,\\ &\quad [x^+_{m+1}(z_1), [x^+_{m+1}(z_2), x^+_{m+2}(w)]]\\ &= \sqrt{-1}[: X(\varepsilon_{m+1}, z_1)\delta^*_1(z_1) :, [: X(\varepsilon_{m+1}, z_2)\delta^*_1(z_2) :, : \delta_1(w)\delta^*_2(w) :]]\\ &= -\sqrt{-1}[: X(\varepsilon_{m+1}, z_1)\delta^*_1(z_1) :, ; X(\varepsilon_{m+1}, w)\delta^*_2(w) :]\delta(z_2-w)\\ &= 0.\end{aligned}$$

The remaining relations follow similarly by Wick's theorem or Corollary 3.1. This completes the proof of the theorem.

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