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Superderivation Algebras of Modular Lie Superalgebras of $\mathscr{O}$-Type*

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Abstract The authors consider a family of finite-dimensional Lie superalgebras of $\mathscr{O}$-type over an algebraically closed field of characteristic $p > 3$. It is proved that the Lie superalgebras of $\mathscr{O}$-type are simple and the spanning sets are determined. Then the spanning sets are employed to characterize the superderivation algebras of these Lie superalgebras. Finally, the associative forms are discussed and a comparison is made between these Lie superalgebras and other simple Lie superalgebras of Cartan type.

Keywords Modular Lie superalgebras, $\mathbb{Z}$-graded Lie superalgebras, Superderivation algebras
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1 Introduction

During the last fifty years, the theory of Lie superalgebras over fields of characteristic zero has experienced a rather vigorous development in mathematics. For example, Kac [8–9] classified the finite-dimensional simple Lie superalgebras and infinite-dimensional simple linearly compact Lie superalgebras over algebraically closed fields of characteristic zero. The research on modular Lie superalgebras, i.e., Lie superalgebras over a field of prime characteristic, just began in recent years. The complete classification of the finite-dimensional simple modular Lie superalgebras remains an open problem. However, Many important results on modular Lie superalgebras were obtained (see, e.g., [1, 3–7, 10–15, 17–22]).

As is well-known, the derivation algebras are very useful subjects in the research of both Lie algebras and Lie superalgebras. In [2, 16], the derivation algebras of modular Lie algebras of Cartan type were discussed. Eight families of finite-dimensional simple modular Lie superalgebras of Cartan type W, S, H, K, HO, KO, SHO and SKO were constructed and their superderivation algebras were studied in [6, 11–13, 19, 23].

In this paper, we study a class of Lie superalgebras of $\mathscr{O}$-type over a field of prime characteristic. The article is organized as follows. In Section 2, we give the definition of Lie superalgebras of $\mathscr{O}$-type and prove that they are simple. In Section 3, the generator sets of these Lie superalgebras are investigated. In Section 4, we first establish some technical lemmas which will be used to determine the homogeneous derivations of Lie superalgebras of $\mathscr{O}$-type. Then an explicit description of the superderivation algebras is given.

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2 Definition and Simplicity

Let $\mathbb{F}$ be an algebraically closed field of characteristic $p > 3$ and assume that $\mathbb{F}$ is not equal to its prime field Π. For $m > 0$, let $\mathbb{E} = \{z_1, \cdots, z_m\} \in \mathbb{F}$ be linearly independent over the prime field Π, and H be the additive subgroup generated by $\mathbb{E}$ which dose not contain 1. If $\lambda \in H$, then let $\lambda = \sum\limits_{i=1}^{m} \lambda_i z_i$, where $0 \le \lambda_i < p$. We define $y^\lambda = y_1^{\lambda_1} \cdots y_m^{\lambda_m}$. Let $\mathbb{N}$ be the set of positive integers, and $\mathbb{N}_0$ be the set of nonnegative integers. Let $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$ denote the ring of integers modulo 2.

Given $n \in \mathbb{N}$, let $n > 2$ and $\underline{s} := (s_1, \cdots, s_n) \in \mathbb{N}_0^n$. Set $M = \{1, \cdots, n\}$. For $k_i \in \mathbb{N}_0$, k_i can be uniquely expressed in the p-adic form $k_i = \sum\limits_{v=0}^{s_i} \varepsilon_v(k_i) p^v$, where $0 \le \varepsilon_v(k_i) < p$. We define the truncated polynomial algebras

$$\mathcal{A}(n, \underline{s}) = \mathbb{F}[x_{10}, x_{11}, \cdots, x_{1s_1}, \cdots, x_{n0}, x_{n1}, \cdots, x_{ns_n}, y_1, \cdots, y_m],$$

such that $x_{ij}^p = 0$ for all $i \in M$ and $j = 0, 1, \cdots, s_i$; $y_i^p = 1$ for $i = 1, \cdots, m$. Let

$$Q = \{(k_1, \cdots, k_n) \mid 0 \le k_i \le \pi_i,\ \pi_i = p^{s_i+1} - 1,\ i \in M\}.$$

If $k = (k_1, \cdots, k_n) \in Q$, we let $x^k = x_1^{k_1} \cdots x_n^{k_n}$, where ${x_i}^{k_i} = \prod\limits_{v=0}^{s_i} x_{iv}^{\varepsilon_v(k_i)}$. For $0 \le k_i,\ k_i' \le \pi_i$, it is easy to see that

$$ {x_i}^{k_i} {x_i}^{k_i'} = {x_i}^{k_i + k_i'} \neq 0 \Leftrightarrow \varepsilon_v(k_i) + \varepsilon_v(k_i') < p, \quad v = 0, 1, \cdots, s_i,\ i \in M. \tag{2.1}$$

Let $\Lambda(n+1)$ be the Grassmann superalgebras over $\mathbb{F}$ in $n+1$ variables $\xi_{n+1}, \cdots, \xi_{2n+1}$. Denote the tensor product by $\mathcal{A} := \mathcal{A}(n, n+1, \underline{s}) = \mathcal{A}(n, \underline{s}) \otimes \Lambda(n+1)$. Obviously, $\mathcal{A}$ are associative superalgebras with a $\mathbb{Z}_2$-gradation induced by the trivial $\mathbb{Z}_2$-gradation of $\mathcal{A}(n, \underline{s})$ and the natural $\mathbb{Z}_2$-gradation of $\Lambda(n+1)$: $\mathcal{A}_{\overline{0}} = \mathcal{A}(n, \underline{s}) \otimes \Lambda(n+1)_{\overline{0}}$, $\mathcal{A}_{\overline{1}} = \mathcal{A}(n, \underline{s}) \otimes \Lambda(n+1)_{\overline{1}}$. For $f \in \mathcal{A}(n, \underline{s})$ and $g \in \Lambda(n+1)$, we abbreviate $f \otimes g$ to fg. For $k \in \{1, \cdots, n+1\}$, we let

$$\mathbb{B}_k := \{\langle i_1, i_2, \cdots, i_k \rangle \mid n+1 \le i_1 < i_2 < \cdots < i_k \le 2n+1\}$$

and $\mathbb{B}(n+1) = \bigcup\limits_{k=0}^{n+1} \mathbb{B}_k$, where $\mathbb{B}_0 := \emptyset$. Given $u = \langle i_1, \cdots, i_k \rangle \in \mathbb{B}_k$, we set $\{u\} = \{i_1, \cdots, i_k\}$, $|u| = k$,

$$[u] = \begin{cases} k-1, & 2n+1 \in \mathbb{B}_k, \\ k, & 2n+1 \notin \mathbb{B}_k, \end{cases} \qquad \|u\| = \begin{cases} k+1, & 2n+1 \in \mathbb{B}_k, \\ k, & 2n+1 \notin \mathbb{B}_k, \end{cases}$$

and $\xi^u = \xi_{i_1} \cdots \xi_{i_k}$. Put $|\emptyset| = 0$ and $\xi^\emptyset = 1$. Then $\{x^k y^\lambda \xi^u \mid k \in Q,\ \lambda \in H,\ u \in \mathbb{B}(n+1)\}$ is an $\mathbb{F}$-basis of $\mathcal{A}$.

If L is a superalgebra, then $h(L)$ denotes the set of all $\mathbb{Z}_2$-homogeneous elements of L, i.e., $h(L) = L_{\overline{0}} \cup L_{\overline{1}}$. If $|x|$ occurs in some expression in this paper, then we always regard x as a $\mathbb{Z}_2$-homogeneous element and $|x|$ as the $\mathbb{Z}_2$-degree of x.

Let $\mathcal{B}$ be a given $\mathbb{Z}_2$-graded vector space over $\mathbb{F}$, and σ be a given homogeneous linear mapping of degree $\overline{1}$,

$$\sigma : \mathcal{A} \to \mathcal{B},$$

such that $\sigma(f) \neq 0$ for all $0 \neq f \in \mathcal{A}$. It is easy to see that $\sigma(\mathcal{A}) \subseteq \mathcal{B}$ is a $\mathbb{Z}_2$-graded subspaces. For $\sigma(f) \in \sigma(\mathcal{A})$, one may easily verify that $\sigma(f)$ is a $\mathbb{Z}_2$-homogeneous element if and only if f is a $\mathbb{Z}_2$-homogeneous element of $\mathcal{A}$, and if $f \in \mathcal{A}_\alpha$, then $\sigma(f) \in \sigma(\mathcal{A})_{\alpha+\overline{1}}$, where $\alpha \in \mathbb{Z}_2$.

Set $e_i := (\delta_{i1}, \cdots, \delta_{in})$ for $i \in M$. Put $T = \{n+1, \cdots, 2n+1\}$ and $R = M \cup T$. Define $\widetilde{i} = \overline{0}$ if $i \in M$, and $\widetilde{i} = \overline{1}$ if $i \in T$. Put $T_1 = \{n+1, \cdots, 2n\}$. Let

$$i' = \begin{cases} i+n, & \text{if } \ i \in M, \\ i-n, & \text{if } \ i \in T_1. \end{cases}$$

Let D_i $(i \in R)$ be the linear transformations of $\sigma(\mathcal{A})$, such that

$$D_i(\sigma(f)) = \sigma(D_i(f)),$$

where D_i are the linear transformations of $\mathcal{A}$, such that

$$D_i(x^k y^\lambda \xi^u) = \begin{cases} k_i^* x^{k-e_i} y^\lambda \xi^u, & \text{if } \ i \in M, \\ x^k y^\lambda \cdot \dfrac{\partial \xi^u}{\partial \xi_i}, & \text{if } \ i \in T, \end{cases}$$

where k_i^* is the first nonzero number of $\varepsilon_0(k_i), \varepsilon_1(k_i), \cdots, \varepsilon_{s_i}(k_i)$. Then D_i is an even derivation of $\mathcal{A}$ for any $i \in M$, and D_i is an odd derivation of $\mathcal{A}$ for any $i \in T$.

Set

$$\overline{\partial} = I - 2^{-1} \sum_{j \in M} x_{j\,0} \frac{\partial}{\partial x_{j\,0}} - \sum_{j=1}^{m} z_j y_j \frac{\partial}{\partial y_j} - 2^{-1} \sum_{j \in T_1} \xi_j \frac{\partial}{\partial \xi_j}\,,$$

where I is the identity mapping of $\mathcal{A}$. It is easy to see that

$$\overline{\partial}(x^k y^\lambda \xi^u) = \Big(1 - 2^{-1} \sum_{j \in M} k_j - \lambda - 2^{-1}[u]\Big) x^k y^\lambda \xi^u.$$

We denote $\sigma(\mathcal{A})$ by $\overline{\mathscr{O}}$. For $\sigma(f)$, $\sigma(g) \in h(\overline{\mathscr{O}})$, we define a bilinear operation in $\overline{\mathscr{O}}$, such that

$$\begin{aligned}[] [\sigma(f), \sigma(g)] &= \sigma(\langle f, g \rangle) \\ &= \sigma(\overline{\partial}(f) D_{2n+1}(g) + (-1)^{|f|} D_{2n+1}(f) \overline{\partial}(g) \\ &\quad + \sum_{i \in M \cup T_1} (-1)^{\widetilde{i}|f|} D_i(f) D_{i'}(g)). \end{aligned} \tag{2.2}$$

Theorem 2.1 *$\overline{\mathscr{O}}$ become Lie superalgebras for the operation* [,] *defined above.*

Proof Clearly $\overline{\mathscr{O}}$ are superalgebras by (2.2). Let $\sigma(f) \in \overline{\mathscr{O}}_\alpha$, $\sigma(g) \in \overline{\mathscr{O}}_\beta$ and $\sigma(h) \in \overline{\mathscr{O}}_\gamma$, where α, β, $\gamma \in \mathbb{Z}_2$. Note that $\widetilde{i} + \widetilde{i'} = \overline{1}$, $\widetilde{i}\,\widetilde{i'} = \overline{0}$ for $i \in M \cup T_1$. By (2.2), one may easily verify that $[\sigma(f), \sigma(g)] = -(-1)^{\alpha\beta}[\sigma(g), \sigma(f)]$.

Put $\partial(f) f := \overline{\partial}(f)$. Then we have

$$\begin{aligned} \overline{\partial}(f D_{2n+1}(g)) &= \partial(fg) f D_{2n+1}(g) = (\partial(f) + \partial(g) - 1) f D_{2n+1}(g), \\ \overline{\partial}(D_i(f) D_{i'}(g)) &= (\partial(f) + \partial(g)) D_i(f) D_{i'}(g), \quad i = 1, \cdots, n+1. \end{aligned}$$

We will prove that the operation [,] satisfies the graded Jacobi identity.

According to (2.2), we have

$$(-1)^{\alpha\gamma}[\sigma(f), [\sigma(g), \sigma(h)]]$$

$$
\begin{aligned}
&= (-1)^{\alpha\gamma}\sigma\Big(\langle f, \partial(g) g D_{2n+1}(h)\rangle + \langle f, (-1)^{\beta+\overline{1}}\partial(h) D_{2n+1}(g) h\rangle \\
&\quad + \Big\langle f, \sum_{i\in M\cup T_1} (-1)^{\widetilde{i}(\beta+\overline{1})} D_i(g) D_{i'}(h)\Big\rangle\Big) \\
&= a + b + c,
\end{aligned}
$$

where

$$
\begin{aligned}
a &= (-1)^{\alpha\gamma}\sigma(\langle f, \partial(g) g D_{2n+1}(h)\rangle) \\
&= \sigma\Big((-1)^{\beta\gamma+\gamma}\partial(f)\partial(g) D_{2n+1}(h) f D_{2n+1}(g) \\
&\quad + (-1)^{\beta\gamma+\gamma+\alpha+\overline{1}}(\partial(g)^2 - \partial(g)) D_{2n+1}(h) D_{2n+1}(f) g \\
&\quad + (-1)^{\alpha\gamma+\alpha+\overline{1}}\partial(g)\partial(h) D_{2n+1}(f) g D_{2n+1}(h) \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\beta\gamma+\gamma+\widetilde{i}\alpha+\widetilde{i}}\partial(g) D_{2n+1}(h) D_i(f) D_{i'}(g) \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\alpha\beta+\widetilde{i}\gamma+\widetilde{i}'}\partial(g) g D_{2n+1} D_i(h) D_{i'}(f)\Big), \\
b &= (-1)^{\alpha\gamma}\sigma(\langle f, (-1)^{\beta+\overline{1}}\partial(h) D_{2n+1}(g) h\rangle) \\
&= \sigma\Big((-1)^{\alpha\beta+\beta+\gamma+\overline{1}}\partial(f)\partial(h) D_{2n+1}(g) D_{2n+1}(h) f \\
&\quad + (-1)^{\alpha\gamma+\alpha+\beta}(\partial(h)^2 - \partial(h)) D_{2n+1}(f) D_{2n+1}(g) h \\
&\quad + (-1)^{\alpha\gamma+\alpha+\beta}\partial(g)\partial(h) D_{2n+1}(f) D_{2n+1}(g) h \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\beta\gamma+\widetilde{i}'\alpha}\partial(h) h D_i(f) D_{2n+1} D_{i'}(g) + \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\alpha\beta+\widetilde{i}\gamma+\beta+i}\partial(h) D_{2n+1}(g) D_i(h) D_{i'}(f)\Big), \\
c &= (-1)^{\alpha\gamma}\sigma\Big(\Big\langle f, \sum_{i\in M\cup T_1} (-1)^{\widetilde{i}(\beta+\overline{1})} D_i(g) D_{i'}(h)\Big\rangle\Big) \\
&= \sigma\Big(\sum_{i\in M\cup T_1} (-1)^{\alpha\gamma+\widetilde{i}\beta+\widetilde{i}}\partial(f) f D_{2n+1} D_i(g) D_{i'}(h) \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\alpha\gamma+\widetilde{i}'\beta+\overline{1}}\partial(f) f D_i(g) D_{2n+1} D_{i'}(h) \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\alpha\gamma+\widetilde{i}\beta+\alpha+\widetilde{i}'}\partial(g) D_{2n+1}(f) D_i(g) D_{i'}(h) \\
&\quad + \sum_{i\in M\cup T_1} (-1)^{\alpha\gamma+\widetilde{i}\beta+\alpha+\widetilde{i}'}\partial(h) D_{2n+1}(f) D_i(g) D_{i'}(h) \\
&\quad + \sum_{i,j\in M\cup T_1} (-1)^{\alpha\gamma+\widetilde{j}\alpha+\widetilde{i}\beta+\widetilde{i}+\widetilde{j}} D_j(f) D_{j'}(D_i(g) D_{i'}(h))\Big).
\end{aligned}
$$

Similarly,

$$
(-1)^{\beta\alpha}[\sigma(g), [\sigma(h), \sigma(f)]] = a' + b' + c',
$$

where

$$
a' = \sigma\Big((-1)^{\alpha\gamma+\alpha}\partial(g)\partial(h) D_{2n+1}(f) g D_{2n+1}(h)
$$

$$
\begin{aligned}
&+ (-1)^{\alpha\gamma+\alpha+\beta+\overline{1}}(\partial(h)^2-\partial(h))D_{2n+1}(f)D_{2n+1}(g)h\\
&+ (-1)^{\alpha\beta+\beta+\overline{1}}\partial(h)\partial(f)D_{2n+1}(g)hD_{2n+1}(f)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\alpha\gamma+\widetilde{i}\beta+\alpha+\widetilde{i}}\partial(h)D_{2n+1}(f)D_i(g)D_{i'}(h)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i}\alpha+\widetilde{i}'}\partial(h)hD_{2n+1}D_i(f)D_{i'}(g)\Big),
\end{aligned}
$$

$$
\begin{aligned}
b' = \sigma\Big(&(-1)^{\beta\gamma+\gamma+\alpha+\overline{1}}\partial(g)\partial(f)D_{2n+1}(h)D_{2n+1}(f)g\\
&+ (-1)^{\beta\alpha+\beta+\gamma}(\partial(f)^2-\partial(f))D_{2n+1}(g)D_{2n+1}(h)f\\
&+ (-1)^{\beta\alpha+\beta+\gamma}\partial(h)\partial(f)D_{2n+1}(g)D_{2n+1}(h)f\\
&+ \sum_{i\in M\cup T_1}(-1)^{\alpha\gamma+\widetilde{i}'\beta}\partial(f)fD_i(g)D_{2n+1}D_{i'}(h)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i}\alpha+\gamma+\widetilde{i}}\partial(f)D_{2n+1}(h)D_i(f)D_{i'}(g)\Big),
\end{aligned}
$$

$$
\begin{aligned}
c' = \sigma\Big(&\sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}\gamma+\widetilde{i}}\partial(g)gD_{2n+1}D_i(h)D_{i'}(f)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}'\gamma+\overline{1}}\partial(g)gD_i(h)D_{2n+1}D_{i'}(f)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}\gamma+\beta+\widetilde{i}'}\partial(h)D_{2n+1}(g)D_i(h)D_{i'}(f)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}\gamma+\beta+\widetilde{i}'}\partial(f)D_{2n+1}(g)D_i(h)D_{i'}(f)\\
&+ \sum_{i,j\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{j}\beta+\widetilde{i}\gamma+\widetilde{i}+\widetilde{j}}D_j(g)D_{j'}(D_i(h)D_{i'}(f))\Big)
\end{aligned}
$$

and

$$(-1)^{\beta\gamma}[\sigma(h),[\sigma(f),\sigma(g)]] = a''+b''+c'',$$

where

$$
\begin{aligned}
a'' = \sigma\Big(&(-1)^{\beta\alpha+\beta}\partial(h)\partial(f)D_{2n+1}(g)hD_{2n+1}(f)\\
&+ (-1)^{\beta\alpha+\beta+\gamma+\overline{1}}(\partial(f)^2-\partial(f))D_{2n+1}(g)D_{2n+1}(h)f\\
&+ (-1)^{\beta\gamma+\gamma+\overline{1}}\partial(f)\partial(g)D_{2n+1}(h)fD_{2n+1}(g)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}\gamma+\beta+\widetilde{i}}\partial(f)D_{2n+1}(g)D_i(h)D_{i'}(f)\\
&+ \sum_{i\in M\cup T_1}(-1)^{\alpha\gamma+\widetilde{i}\beta+\widetilde{i}'}\partial(f)fD_{2n+1}D_i(g)D_{i'}(h)\Big),
\end{aligned}
$$

$$
\begin{aligned}
b'' = \sigma\Big(&(-1)^{\alpha\gamma+\alpha+\beta+\overline{1}}\partial(h)\partial(g)D_{2n+1}(f)D_{2n+1}(g)h\\
&+ (-1)^{\beta\gamma+\gamma+\alpha}(\partial(g)^2-\partial(g))D_{2n+1}(h)D_{2n+1}(f)g\\
&+ (-1)^{\beta\gamma+\gamma+\alpha}\partial(f)\partial(g)D_{2n+1}(h)D_{2n+1}(f)g\\
&+ \sum_{i\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{i}'\gamma}\partial(g)gD_i(h)D_{2n+1}D_{i'}(f)
\end{aligned}
$$

$$+\sum_{i\in M\cup T_1}(-1)^{\alpha\gamma+\widetilde{i}\beta+\alpha+\widetilde{i}}\partial(g)D_{2n+1}(f)D_i(g)D_{i'}(h)\Big),$$

$$\begin{aligned}c''=\sigma\Big(&\sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i}\alpha+\widetilde{i}}\partial(h)hD_{2n+1}D_i(f)D_{i'}(g)\\&+\sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i'}\alpha+\overline{1}}\partial(h)hD_i(f)D_{2n+1}D_{i'}(g)\\&+\sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i}\alpha+\gamma+\widetilde{i'}}\partial(f)D_{2n+1}(h)D_i(f)D_{i'}(g)\\&+\sum_{i\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{i}\alpha+\gamma+\widetilde{i'}}\partial(g)D_{2n+1}(h)D_i(f)D_{i'}(g)\\&+\sum_{i,j\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{j}\gamma+\widetilde{i}\alpha+\widetilde{i}+\widetilde{j}}D_j(h)D_{j'}(D_i(f)D_{i'}(g))\Big).\end{aligned}$$

Moreover, by a straightforward computation, we can obtain the following equation:

$$\begin{aligned}\sigma\Big(&\sum_{i,j\in M\cup T_1}(-1)^{\alpha\gamma+\widetilde{j}\alpha+\widetilde{i}\beta+\widetilde{i}+\widetilde{j}}D_j(f)D_{j'}(D_i(g)D_{i'}(h))\\&+\sum_{i,j\in M\cup T_1}(-1)^{\beta\alpha+\widetilde{j}\beta+\widetilde{i}\gamma+\widetilde{i}+\widetilde{j}}D_j(g)D_{j'}(D_i(h)D_{i'}(f))\\&+\sum_{i,j\in M\cup T_1}(-1)^{\beta\gamma+\widetilde{j}\gamma+\widetilde{i}\alpha+\widetilde{i}+\widetilde{j}}D_j(h)D_{j'}(D_i(f)D_{i'}(g))\Big)=0.\end{aligned}$$

By a careful comparison, we find that the elements on the right-hand side of a, b, c, a', b', c' and a'', b'', c'' can cancel each other out to be zero.

Therefore

$$(-1)^{\alpha\gamma}[\sigma(f),[\sigma(g),\sigma(h)]]+(-1)^{\beta\alpha}[\sigma(g),[\sigma(h),\sigma(f)]]+(-1)^{\beta\gamma}[\sigma(h),[\sigma(f),\sigma(g)]]=0.$$

Thus $\overline{\mathscr{O}}$ are Lie superalgebras.

Let $x_i=x_i^1=x_{i0}$ for all $i\in M$. Set $\pi=(\pi_1,\cdots,\pi_n)\in Q$ and $\omega=\langle n+1,\cdots,2n+1\rangle\in\mathbb{B}(n+1)$. Put $\omega-\langle n+i\rangle=\langle n+1,\cdots,n+i-1,n+i+1,\cdots,2n+1\rangle\in\mathbb{B}(n+1)$.

Lemma 2.1 *Let $f\in\mathcal{A}$. If $D_i(f)=0$ for all $i\in R$, then $f=\sum\limits_{\lambda\in H}a_\lambda y^\lambda$, where $a_\lambda\in\mathbb{F}$.*

Proof If $D_i(x^ky^\lambda\xi^u)=0$ for all $i\in R$, then we have $k=(0,\cdots,0)$ and $u=\emptyset$. Hence $x^ky^\lambda\xi^u=y^\lambda$, as desired.

Theorem 2.2 *Lie superalgebras $\overline{\mathscr{O}}$ are simple.*

Proof Let I be a nonzero ideal of $\overline{\mathscr{O}}$. Assume that $\sigma(f)$ is a $\mathbb{Z}_2$-homogeneous nonzero element of I. Suppose that $f=f_0\xi_{2n+1}+f_1$, where $f_0\neq 0$ and $D_{2n+1}(f_j)=0$ for $j=0,1$. By (2.2), we have

$$[\sigma(f),\sigma(1)]=(-1)^{|f|}\sigma(D_{2n+1}(f))=(-1)^{|f|}\sigma(f_0)\in I.$$

So we can assume $D_{2n+1}(f)=0$. Suppose that $f=f_0\xi_{i'}+f_1$, where $f_0\neq 0$ and $D_{i'}(f_j)=0$ for $i\in M$, $j=0,1$. Also by (2.2), we get

$$[\sigma(f),\sigma(x_i)]=(-1)^{\widetilde{i'}|f|}\sigma(D_{i'}(f))=(-1)^{|f|}\sigma(f_0)\in I.$$

Thus we can assume that $D_{i'}(f)=0$ for all $i\in M$. Now suppose that $f=x_i^tf_0+x_i^{t-1}f_1+\cdots+f_t$, where $f_0\neq 0$ and $D_i(f_j)=0$ for $i\in M$, $j=0,1,\cdots,t$. As

$$\begin{aligned}(\operatorname{ad}\sigma(\xi_{i'}))^t(\sigma(f))&=(\operatorname{ad}\sigma(\xi_{i'}))^{t-1}[\sigma(\xi_{i'}),\sigma(f)]=-(\operatorname{ad}\sigma(\xi_{i'}))^{t-1}\sigma(D_i(f))\\&=(-1)^t\sigma(D_i^t(f))=(-1)^t\Big(\prod_{j=1}^t j^*\Big)\sigma(f_0)\in I,\end{aligned}$$

we can assume that $D_i(f)=0$ for all $i\in M$. According to Lemma 2.1, $f=\sum\limits_{\lambda\in H}a_\lambda y^\lambda$. If f contains at least two nonzero terms, we can suppose that

$$f=a_\eta y^\eta+a_\mu y^\mu+\sum_{\lambda\in H\setminus\{\eta,\mu\}}a_\lambda y^\lambda,$$

where $a_\eta\neq 0$, $a_\mu\neq 0$. Let

$$\sigma(g):=[\sigma(\xi_{2n+1}),\sigma(f)]+(1-\eta)\sigma(f)=\sigma\Big((\mu-\eta)a_\mu y^\mu+\sum_{\lambda\in H\setminus\{\eta,\mu\}}(\lambda-\eta)a_\lambda y^\lambda\Big).$$

Obviously, $\sigma(g)$ is an element of I and $g\in\mathcal{A}$ with one term less than f. Thus we may assume that $\sigma(y^\lambda)\in I$. Since $1-\lambda\neq 0$, $(1-\lambda)^{-1}[\sigma(y^\lambda),\sigma(\xi_{2n+1}y^{-\lambda})]=\sigma(1)\in I$. In particular, $-[\sigma(x_i\xi_{2n+1}),\sigma(1)]=\sigma(x_i)\in I$ and $[\sigma(\xi_{i'}\xi_{2n+1}),\sigma(1)]=\sigma(\xi_{i'})\in I$ for all $i\in M$. Then

$$\begin{aligned}&[\sigma(x_1),\sigma(x^\pi y^\lambda\xi^\omega)]=\sigma(x^\pi y^\lambda\xi^{\omega-\langle n+1\rangle})\in I,\\&2(3-(-1)^n)^{-1}[\sigma(\xi_{n+1}\xi_{2n+1}),\sigma(x^\pi y^\lambda\xi^{\omega-\langle n+1\rangle})]=\sigma(x^\pi y^\lambda\xi^\omega)\in I.\end{aligned}$$

We will show that $\sigma(x^ky^\lambda\xi^u)\in I$ for all $k\in Q$, $\lambda\in H$, $u\in\mathbb{B}(n+1)$.

Case 1 ξ_{2n+1} is not contained in u. Due to (2.2), we have

$$[\sigma(1),\sigma(x^ky^\lambda\xi^u\xi_{2n+1})]=\sigma(x^ky^\lambda\xi^u)\in I.$$

Case 2 ξ_{2n+1} is contained in u. We let $x^ky^\lambda\xi^u=x^ky^\lambda\xi^v\xi_{2n+1}$, where $v\in\mathbb{B}(n+1)$. Suppose that $\langle n+1,\cdots,2n\rangle-v=\langle j_1,\cdots,j_s\rangle$. Then

$$[\sigma(x_{j_s'}),\cdots,[\sigma(x_{j_1'}),\sigma(x^\pi y^\lambda\xi^\omega)]\cdots]=\sigma(x^\pi y^\lambda\xi^u)\in I.$$

For $i\in M$, we have

$$[\sigma(\xi_{i'}),\sigma(x^\pi y^\lambda\xi^u)]=2^{-1}\sigma(x^\pi y^\lambda\xi_{i'}\xi^v)-\sigma(x^{\pi-e_i}y^\lambda\xi^u)\in I.$$

Case 1 implies that $\sigma(x^\pi y^\lambda\xi_{i'}\xi^v)\in I$. So $\sigma(x^{\pi-e_i}y^\lambda\xi^u)\in I$. Furthermore,

$$\begin{aligned}(\operatorname{ad}\sigma(\xi_{i'}))^2(\sigma(x^\pi y^\lambda\xi^u))&=[\sigma(\xi_{i'}),[\sigma(\xi_{i'}),\sigma(x^\pi y^\lambda\xi^u)]]\\&=2^{-1}[\sigma(\xi_{i'}),\sigma(x^\pi y^\lambda\xi_{i'}\xi^v)]-[\sigma(\xi_{i'}),\sigma(x^{\pi-e_i}y^\lambda\xi^u)]\\&=-\sigma(x^{\pi-e_i}y^\lambda\xi_{i'}\xi^v)+\sigma(x^{\pi-2e_i}y^\lambda\xi^u)\in I.\end{aligned}$$

Again by Case 1, we have $\sigma(x^{\pi-2e_i}y^\lambda\xi^u)\in I$. Similarly, by letting $(\operatorname{ad}\sigma(\xi_{i'}))^{\mu_i}$ act on $\sigma(x^\pi y^\lambda\xi^u)$, we can obtain $\sigma(x^{\pi-\mu_ie_i}y^\lambda\xi^u)\in I$. For $j\in M$ and $j\neq i$, we get

$$[\sigma(\xi_{j'}),\sigma(x^{\pi-\mu_ie_i}y^\lambda\xi^u)]=2^{-1}\sigma(x^{\pi-\mu_ie_i}y^\lambda\xi_{j'}\xi^v)-\sigma(x^{\pi-\mu_ie_i-e_j}y^\lambda\xi^u)\in I.$$

As $\sigma(x^{\pi-\mu_i e_i} y^\lambda \xi_{j'} \xi^v) \in I$, $\sigma(x^{\pi-\mu_i e_i - e_j} y^\lambda \xi^u) \in I$. Similarly, by letting $\prod_{i=1}^{n} (\mathrm{ad}\,\sigma(\xi_{i'}))^{\mu_i}$ act on $\sigma(x^\pi y^\lambda \xi^u)$, we can obtain $\sigma(x^k y^\lambda \xi^u) \in I$, where $k_i + \mu_i = \pi_i$ for all $i \in M$. Therefore $I = \overline{\mathscr{O}}$. The proof is completed.

In the sequel, we denote $[\overline{\mathscr{O}}, \overline{\mathscr{O}}]$ by $\mathscr{O}$ and write the element $\sigma(f)$ as f for simplicity.

Remark 2.1 Theorem 2.2 shows that $\mathscr{O} = \overline{\mathscr{O}}$ and $\mathscr{O}$ are finite-dimensional Lie superalgebras with $\dim \mathscr{O} = 2^{n+1} p^{\sum\limits_{i\in M}(s_i+1)+m}$. We call $\mathscr{O}$ the Lie superalgebras of $\mathscr{O}$-type.

3 Spanning Sets

Proposition 3.1 *Let $S = \{x_i^{k_i} \xi_{2n+1} \mid i \in M, 0 \le k_i \le \pi_i\} \cup \{y^\lambda \mid \lambda \in H\} \cup \{\xi_j \xi_{2n+1} \mid j \in T_1\}$. Then Lie superalgebras $\mathscr{O}$ are generated by S.*

Proof Let Y be the subalgebra generated by S. Firstly, we prove the following:
(i) $[1, x_i^{k_i} \xi_{2n+1}] = x_i^{k_i} \in Y$ for $0 \le k_i \le \pi_i$, $i \in M$.
(ii) $-[x_i \xi_{2n+1}, \xi_{i'} \xi_{2n+1}] = x_i \xi_{i'} \xi_{2n+1} \in Y$ for $i \in M$.
(iii) $\xi_j \in Y$ for $j \in T$. Clearly,

$$[1, \xi_j \xi_{2n+1}] = \xi_j \in Y, \quad j \in T_1.$$

According to (ii) and the equation above, we have

$$[x_i \xi_{i'} \xi_{2n+1}, 1] = x_i \xi_{i'} \in Y,$$
$$[\xi_{i'}, x_i \xi_{2n+1}] - 2^{-1} x_i \xi_{i'} = -\xi_{2n+1} \in Y.$$

(iv) $x_i^{k_i} x_j^{k_j} \in Y$ for $0 \le k_i \le \pi_i$, $0 \le k_j \le \pi_j$, $i, j \in M$. By virtue of (i), we get

$$[x_i^{k_i}, x_j^{k_j} \xi_{2n+1}] = (1 - 2^{-1} \varepsilon_0(k_i)) x_i^{k_i} x_j^{k_j}.$$

If $1 - 2^{-1}\varepsilon_0(k_i) \not\equiv 0 \,(\mathrm{mod}\, p)$, then $x_i^{k_i} x_j^{k_j} \in Y$. In particular,

$$[x_i^{\pi_i}, x_j^{k_j} \xi_{2n+1}] = (1 - 2^{-1}(p-1)) x_i^{\pi_i} x_j^{k_j} = \frac{3}{2} x_i^{\pi_i} x_j^{k_j} \in Y.$$

If $1 - 2^{-1}\varepsilon_0(k_i) \equiv 0 \,(\mathrm{mod}\, p)$, then $k_i \ne \pi_i$. As $0 \le \varepsilon_0(k_i) < p$, $\varepsilon_0(k_i) = 2$. Thus $(k_i+1)^* = 3$. By (i), we have

$$[x_i^{k_i+1}, x_j^{k_j} \xi_{2n+1}] = -2^{-1} x_i^{k_i} x_j^{k_j} \in Y.$$

(v) $x_i \xi_{i'} \xi_j \in Y$ for $i \in M$ and $j \in T_1$. (iii) implies that

$$[x_i^{k_i} \xi_{2n+1}, \xi_{i'}] - k_i^* x_i^{k_i - 1} \xi_{2n+1} = -2^{-1} x_i^{k_i} \xi_{i'} \in Y,$$
$$[\xi_{i'}, \xi_j \xi_{2n+1}] = 2^{-1} \xi_{i'} \xi_j \in Y.$$

It follows that $x_i^2 \xi_{i'} \in Y$. Hence

$$[x_i^2 \xi_{i'}, \xi_{i'} \xi_j] = 2 x_i \xi_{i'} \xi_j \in Y.$$

(vi) We will use induction on k to show that $\xi_{j_1}\xi_{j_2}\cdots\xi_{j_k}\in Y$, where $j_t\in T_1$, $1\leq t\leq k$. The conclusion is true for the case $k=1$ by (iii). Suppose that $\xi_{j_1}\xi_{j_2}\cdots\xi_{j_l}\in Y$ for $l\leq k-1$. According to (v), we get

$$[\xi_{j_1}\xi_{j_2}\cdots\xi_{j_{k-1}},x_{j_1'}\xi_{j_1}\xi_{j_k}]=(-1)^{k-2}\xi_{j_1}\xi_{j_2}\cdots\xi_{j_k}\in Y.$$

Let $\overline{\omega}=\omega-\langle 2n+1\rangle$. In particular, we have $\xi^{\overline{\omega}}\in Y$.

Now we verify that $\xi_{j_1}\xi_{j_2}\cdots\xi_{j_k}\xi_{2n+1}\in Y$ for $j_t\in T_1$, $1\leq t\leq k$. The conclusion above and (ii) yield that

$$[x_{j_1'}\xi_{j_1}\xi_{2n+1},\xi_{j_1}\xi_{j_2}\cdots\xi_{j_k}]=(-1)^{k-1}\xi_{j_1}\xi_{j_2}\cdots\xi_{j_k}\xi_{2n+1}\in Y.$$

In particular, we have $\xi^{\omega}\in Y$.

(vii) We propose to prove that $x^{\pi_1e_1+\cdots+\pi_ke_k}\xi_{2n+1}\in Y$ by induction on k.

Clearly the assertion is true for the case $k=1$. Suppose that $x^{\pi_1e_1+\cdots+\pi_{k-1}e_{k-1}}\xi_{2n+1}\in Y$. Thus

$$[x^{\pi_1e_1+\cdots+\pi_{k-1}e_{k-1}}\xi_{2n+1},x^{\pi_ke_k}\xi_{2n+1}]=(2^{-1}k-1)x^{\pi_1e_1+\cdots+\pi_ke_k}\xi_{2n+1}.$$

If $2^{-1}k-1\not\equiv 0\,(\operatorname{mod}p)$, then $x^{\pi_1e_1+\cdots+\pi_ke_k}\xi_{2n+1}\in Y$. If $2^{-1}k-1\equiv 0\,(\operatorname{mod}p)$, then $2^{-1}k+1\not\equiv 0\,(\operatorname{mod}p)$. The inductive hypothesis implies that

$$[x^{\pi_1e_1+\cdots+\pi_{k-1}e_{k-1}}\xi_{2n+1},\xi_{n+1}\xi_{2n+1}]=-(2^{-1}k+1)x^{\pi_1e_1+\cdots+\pi_{k-1}e_{k-1}}\xi_{n+1}\xi_{2n+1}\in Y.$$

By virtue of (iv), we see that

$$[x^{\pi_1e_1+\cdots+\pi_{k-1}e_{k-1}}\xi_{n+1}\xi_{2n+1},x_1x_k^{\pi_k}]=x^{\pi_1e_1+\cdots+\pi_ke_k}\xi_{2n+1}\in Y.$$

In particular, we have $x^{\pi}\xi_{2n+1}\in Y$. Since $1-\lambda\neq 0$,

$$(1-\lambda)^{-1}[y^{\lambda},x^{\pi}\xi_{2n+1}]=x^{\pi}y^{\lambda}\in Y.$$

(viii) $x^{\pi}y^{\lambda}\xi_{2n+1}\in Y$. By (ii) and (vii), we obtain

$$-[x^{\pi}y^{\lambda},x_i\xi_{i'}\xi_{2n+1}]=x^{\pi}y^{\lambda}\xi_{2n+1}\in Y.$$

(ix) $x^{\pi}y^{\lambda}\xi^{\omega}\in Y$ and $x^{\pi}y^{\lambda}\xi^{\overline{\omega}}\in Y$. (v) and (vii) yield that

$$[x^{\pi}y^{\lambda}\xi_{2n+1},x_2\xi_{n+2}\xi_{n+1}]=(p-1)x^{\pi-e_2}x_2y^{\lambda}\xi_{2n+1}\xi_{n+1}=x^{\pi}y^{\lambda}\xi_{n+1}\xi_{2n+1}\in Y.$$

Hence

$$\begin{aligned}&[x^{\pi}y^{\lambda}\xi_{n+1}\xi_{2n+1},x_3\xi_{n+3}\xi_{n+2}]=x^{\pi}y^{\lambda}\xi_{n+1}\xi_{n+2}\xi_{2n+1}\in Y,\\&[x^{\pi}y^{\lambda}\xi_{n+1}\xi_{n+2}\xi_{2n+1},x_4\xi_{4'}\xi_{n+3}]=x^{\pi}y^{\lambda}\xi_{n+1}\xi_{n+2}\xi_{n+3}\xi_{2n+1}\in Y,\ \ \cdots.\end{aligned}$$

Utilizing this procedure continuously, we can obtain $x^{\pi}y^{\lambda}\xi_{n+1}\cdots\xi_{2n-1}\xi_{2n+1}\in Y$. Thus

$$[x^{\pi}y^{\lambda}\xi_1\cdots\xi_{2n-1}\xi_{2n+1},\xi_{n+1}\xi_{2n}]=(p-1)x^{\pi-e_1}y^{\lambda}\xi_{n+1}\cdots\xi_{2n-1}\xi_{2n+1}\xi_{2n}=x^{\pi-e_1}y^{\lambda}\xi^{\omega}\in Y.$$

Since $4x_i\xi_{2n+1}-2[x_i^2\xi_{2n+1},\xi_{i'}]=x_i^2\xi_{i'}\in Y$, we have

$$[x^{\pi-e_1}y^{\lambda}\xi^{\omega},x_1^2\xi_{1'}]=(p-2)x^{\pi-2e_1}x_1^2y^{\lambda}\xi^{\omega}+(-1)^{n+1}2x_1x^{\pi-e_1}y^{\lambda}\xi_{n+2}\cdots\xi_{2n}\xi_{2n+1}\xi_{n+1}$$

$$= -4x^{\pi}y^{\lambda}\xi^{\omega} \in Y.$$

The proof of $x^{\pi}y^{\lambda}\xi^{\overline{\omega}} \in Y$ is completely similar to the proof above.

Then we prove the result of Proposition 3.1. Let $c := x^k y^{\lambda}\xi^u$ be any basis element of $\mathscr{O}$. We only need to show that $c \in Y$.

If $u = \overline{\omega}$, i.e., $c = x^k y^{\lambda}\xi^{\overline{\omega}}$, then we will prove $c \in Y$ by induction on $d_c := \sum\limits_{i\in M} \pi_i - \sum\limits_{i\in M} k_i$. If $d_c = 0$, it follows from (ix) that $c = x^{\pi}y^{\lambda}\xi^{\overline{\omega}} \in Y$. Let $d_c > 0$. Then there is an $i \in M$, such that $k_i < \pi_i$. By the induction hypothesis, we have $x^{k+e_i}y^{\lambda}\xi^{\overline{\omega}} \in Y$. Thus

$$[x^{k+e_i}y^{\lambda}\xi^{\overline{\omega}}, \xi_{i'}] = (k_i+1)^* x^k y^{\lambda}\xi^{\overline{\omega}} \in Y.$$

If $u \neq \overline{\omega}$, we let $u = \overline{\omega} - \langle j_1, \cdots, j_h\rangle$, and then

$$x^k y^{\lambda}\xi^u = \prod_{i=1}^{h}(\mathrm{ad}x_{j_i'})(x^k y^{\lambda}\xi^{\overline{\omega}}) \in Y.$$

If $u = \omega$, i.e., $c = x^k y^{\lambda}\xi^{\omega}$, then we still prove $c \in Y$ by induction on d_c. If $d_c = 0$, according to (ix), we have $c = x^{\pi}y^{\lambda}\xi^{\omega} \in Y$. Let $d_c > 0$. Then there is an $i \in M$, such that $k_i < \pi_i$. By the induction hypothesis, we see that $x^{k+e_i}y^{\lambda}\xi^{\omega} \in Y$. Hence

$$[x^{k+e_i}y^{\lambda}\xi^{\omega}, \xi_{i'}] = (k_i+1)^* x^k y^{\lambda}\xi^{\omega} \in Y.$$

If $u \neq \overline{\omega}$, we let $u = \overline{\omega} - \langle j_1, \cdots, j_h\rangle$. The conclusion above and (2.1) yield $x^k x_{j_i'} y^{\lambda}\xi^{\omega} = 0 \in Y$ or $x^k x_{j_i'} y^{\lambda}\xi^{\omega} = x^{k+e_{j_i'}} y^{\lambda}\xi^{\omega} \in Y$. Therefore,

$$(-1)^{n+1}[x^k y^{\lambda}\xi^{\omega}, x_{j_1'}] - 2^{-1}x^k x_{j_1'} y^{\lambda}\xi^{\omega} = x^k y^{\lambda}\xi^{\omega-\langle j_1\rangle} \in Y.$$

Clearly, the assertion above is true for all k. By (2.1), we get

$$x^k x_{j_2'} y^{\lambda}\xi^{\omega-\langle j_1\rangle} = 0 \quad \text{or} \quad x^k x_{j_2'} y^{\lambda}\xi^{\omega-\langle j_1\rangle} = x^{k+e_{j_2'}} y^{\lambda}\xi^{\omega-\langle j_1\rangle} \in Y.$$

Then

$$(-1)^{n}[x^k y^{\lambda}\xi^{\omega-\langle j_1\rangle}, x_{j_2'}] - 2^{-1}x^k x_{j_2'} y^{\lambda}\xi^{\omega-\langle j_1\rangle} = x^k y^{\lambda}\xi^{\omega-\langle j_1, j_2\rangle} \in Y.$$

Utilizing this procedure continuously, we have $x^k y^{\lambda}\xi^u \in Y$. Hence $\mathscr{O} \subseteq Y$. Consequently $Y = \mathscr{O}$.

4 Superderivations

We know that $\mathscr{O} = \bigoplus\limits_{\alpha\in\mathbb{Z}_2} \mathscr{O}_{\alpha}$, where

$$\mathscr{O}_{\alpha} = \mathrm{span}_{\mathbb{F}}\{x^k y^{\lambda}\xi^u \mid k \in Q,\, \lambda \in H,\, u \in \mathbb{B}(n+1),\, \alpha = \overline{|u|} + \overline{1}\}.$$

For $i \in \mathbb{Z}$, we let

$$\mathscr{O}_i = \mathrm{span}_{\mathbb{F}}\Big\{x^k y^{\lambda}\xi^u \,\Big|\, \sum_{j\in M} k_j + \|u\| - 2 = i\Big\}.$$

(2.2) shows that $[\mathscr{O}_i, \mathscr{O}_j] \subseteq \mathscr{O}_{i+j}$ for all $i, j \in \mathbb{Z}$. Hence $\mathscr{O} = \bigoplus\limits_{i=-2}^{\tau} \mathscr{O}_i$ are $\mathbb{Z}$-graded Lie superalgebras, where $\tau = \sum\limits_{i\in M} \pi_i + n$. Clearly, $\mathscr{O}_{-2} = \mathrm{span}_{\mathbb{F}}\{y^{\lambda} \mid \lambda \in H\}$. If $f \in \mathscr{O}_i$, then f is called a $\mathbb{Z}$-homogeneous element and i is the $\mathbb{Z}$-degree of f which is denoted by $\mathrm{zd}(f)$.

Let $\mathrm{Der}_\alpha\mathscr{O}$ denote the linear space of all derivations of degree α of $\mathscr{O}$, i.e.,

$$\mathrm{Der}_\alpha\mathscr{O} = \{\varphi \in \mathrm{Der}\mathscr{O} \mid \varphi(\mathscr{O}_\beta) \subseteq \mathscr{O}_{\alpha+\beta}, \forall\, \beta \in \mathbb{Z}_2\},$$

and let $\mathrm{Der}\mathscr{O} := \bigoplus_{\alpha\in\mathbb{Z}_2} \mathrm{Der}_\alpha\mathscr{O}$ be the superderivation algebras of $\mathscr{O}$. For $t \in \mathbb{Z}$, we let

$$\mathrm{Der}_t\mathscr{O} = \{\varphi \in \mathrm{Der}\mathscr{O} \mid \varphi(\mathscr{O}_i) \subseteq \mathscr{O}_{i+t}, \forall\, i \in \mathbb{Z}\}.$$

Then $\mathrm{Der}\mathscr{O} = \bigoplus_{t\in Y} \mathrm{Der}_t\mathscr{O}$ are $\mathbb{Z}$-graded Lie superalgebras, where $Y = \{-\zeta, -\zeta+1, \cdots, \zeta\}$ and $\zeta = \tau + 2$. Therefore, in order to determine the superderivation algebras $\mathrm{Der}\mathscr{O}$, we only need to determine $h(\mathrm{Der}_t\mathscr{O})$ for all $t \in Y$.

Lemma 4.1 *Let $\varphi \in h(\mathrm{Der}\mathscr{O})$ and $f \in \mathscr{O}$. Suppose $\varphi(x_i) = \varphi[f, x_i] = \varphi(\xi_j) = \varphi[f, \xi_j] = 0$ for all $i \in M$ and $j \in T_1$. Then $\varphi(f) \in \mathscr{O}_{-2}$.*

Proof Let $f = \sum_{\alpha\in\mathbb{Z}_2} f_\alpha$, where $f_\alpha \in \mathscr{O}_\alpha$. By $\varphi[f, x_i] = 0$ for all $i \in M$, we have

$$\sum_{\alpha\in\mathbb{Z}_2} \varphi[f_\alpha, x_i] = 0.$$

Since φ, f_α and x_i are all $\mathbb{Z}_2$-homogeneous elements, $\varphi[f_\alpha, x_i] \in h(\mathscr{O})$. Then $\sum_{\alpha\in\mathbb{Z}_2} \varphi[f_\alpha, x_i] = 0$ yields $\varphi[f_\alpha, x_i] = 0$ for all $\alpha \in \mathbb{Z}_2$, i.e., $[\varphi(f_\alpha), x_i] + (-1)^{\alpha|\varphi|}[f_\alpha, \varphi(x_i)] = 0$. As $\varphi(x_i) = 0$, $[\varphi(f_\alpha), x_i] = 0$ for all $i \in M$. Similarly, $[\varphi(f_\alpha), \xi_{i'}] = 0$ for all $i \in M$. Hence

$$[\varphi(f_\alpha), 1] = [\varphi(f_\alpha), [x_i, \xi_{i'}]] = [[\varphi(f_\alpha), x_i], \xi_{i'}] + (-1)^{\alpha+|\varphi|}[x_i, [\varphi(f_\alpha), \xi_{i'}]] = 0.$$

Let $h := \varphi(f_\alpha) \in \mathscr{O}$. (2.2) implies that $(-1)^{|h|}D_{2n+1}(h) = [\varphi(f_\alpha), 1] = 0$. Moreover, $[\varphi(f_\alpha), x_i] = (-1)^{\widetilde{i'}(|\varphi|+\alpha)}D_{i'}(h) = 0$ and $[\varphi(f_\alpha), \xi_{i'}] = D_i(h) = 0$ for all $i \in M$. Thus $D_i(h) = 0$ for all $i \in R$. By virtue of Lemma 2.1, we get $h \in \mathscr{O}_{-2}$, i.e., $\varphi(f_\alpha) \in \mathscr{O}_{-2}$. Hence $\varphi(f) \in \mathscr{O}_{-2}$, as desired.

Lemma 4.2 *Let $t \in \mathbb{Z}$ and $\varphi \in h(\mathrm{Der}_t\mathscr{O})$. If $\varphi(\mathscr{O}_j) = 0$ for $j = -2, -1, \cdots, s$, where $s \geq -1$ and $t + s \geq -2$, then $\varphi = 0$.*

Proof Let $j \geq s$. We will prove by induction on j that $\varphi(\mathscr{O}_j) = 0$. Let $j > s$ and $f \in \mathscr{O}_j$. It is easy to see that $[f, x_i]$, $[f, \xi_{i'}] \in \mathscr{O}_{j-1}$. Then the assumption $\varphi(\mathscr{O}_{j-1}) = 0$ implies that $\varphi(x_i) = \varphi[f, x_i] = \varphi(\xi_{i'}) = \varphi[f, \xi_{i'}] = 0$ for all $i \in M$. By Lemma 4.1, $\varphi(f) \in \mathscr{O}_{-2}$. Since $t + j > t + s \geq -2$, $\varphi(f) \in \mathscr{O}_{-2} \cap \mathscr{O}_{t+j} = 0$. So $\varphi(\mathscr{O}_j) = 0$, that is, $\varphi(\mathscr{O}) = 0$. Therefore $\varphi = 0$.

Proposition 4.1 $\mathrm{Der}_{-2}\mathscr{O} = \mathrm{ad}\, \mathscr{O}_{-2}$.

Proof Let $\varphi \in h(\mathrm{Der}_{-2}\mathscr{O})$. Clearly $\varphi(\mathscr{O}_{-1}) = \varphi(\mathscr{O}_{-2}) = 0$. Since $\varphi(\mathscr{O}_0) \subseteq \mathscr{O}_{-2}$, we may assume that $\varphi(\xi_{2n+1}y^\lambda) = \sum_{\eta\in H} a_\eta y^\eta$ with $a_\eta \in \mathbb{F}$. As $\eta - \lambda \in H$, $\eta - \lambda \neq 1$. Let $g = \sum_{\eta\in H} (\eta - \lambda - 1)^{-1} a_\eta y^{\eta-\lambda}$ and $\psi = \varphi - \mathrm{ad}\, g$. Then considering $\mathbb{Z}$-degree and by (2.2), we obtain

$$\psi(\mathscr{O}_{-1}) = \psi(\mathscr{O}_{-2}) = 0, \quad \psi(\xi_{2n+1}y^\lambda) = 0, \quad \forall\, \lambda \in H.$$

Clearly $\psi(x_ix_l), \psi(x_j\xi_{l'}y^\lambda) \in \mathscr{O}_{-2}$ for all $i, l, j \in M$ and $\lambda \in H$. Applying ψ to $x_ix_jy^\lambda = -[x_ix_l, x_j\xi_{l'}y^\lambda]$, we get $\psi(x_ix_jy^\lambda) = 0$. Similarly, we have

$$\psi(x_i\xi_jy^\lambda) = \psi(\xi_l\xi_\nu y^\lambda) = 0, \quad j \neq i', \ \forall\, i \in M, \ \forall\, l, j, \nu \in T_1, \ \forall\, \lambda \in H.$$

Suppose that $\psi(x_i\xi_{i'}y^\lambda)=\sum\limits_{\theta\in H}a_\theta y^\theta$, where $a_\theta\in\mathbb{F}$. Note that $\psi(\xi_{2n+1})=0$. By applying ψ to $\lambda x_i\xi_{i'}y^\lambda=[x_i\xi_{i'}y^\lambda,\xi_{2n+1}]$, we have $\sum\limits_{\theta\in H}(\theta-\lambda-1)a_\theta y^\theta=0$. Then $a_\theta=0$, i.e., $\psi(x_i\xi_{i'}y^\lambda)=0$. Thus $\psi(\mathscr{O}_0)=0$. By virtue of Lemma 4.2, $\psi=0$. Hence $\varphi=\operatorname{ad}g\in\operatorname{ad}\mathscr{O}_{-2}$, as desired.

If $i\in M$, then let $\tau(i)=\pi_i$. If $i\in T$, then let $\tau(i)=1$. An element f of $\mathscr{O}$ is called $\tau(i)$-truncated if $D_i^{\tau(i)}(f)=0$, where $i\in R$.

For $i\in R$, we define a linear transformation τ_i of $\mathscr{O}$, such that $\tau_i(\sigma(f))=\sigma(\tau_i(f))$ and

$$\tau_i(x^ky^\lambda\xi^u)=\begin{cases}(k_i+1)^*x^{k+e_i}y^\lambda\xi^u, & \text{if } i\in M,\\ x^ky^\lambda\xi_i\xi^u, & \text{if } i\in T,\end{cases}$$

where we set $x^{k+e_i}=0$ if $k+e_i\notin Q$.

By the convention before and the definition above, we still write $\tau_i(\sigma(f))$ as $\tau_i(f)$. Then we have the following lemma directly.

Lemma 4.3 (i) *If $f\in\mathscr{O}$ is $\mu(i)$-truncated, then $D_i\tau_i(f)=f$ for all $i\in R$.*

(ii) *$D_i\tau_j=(-1)^{\widetilde{i}\,\widetilde{j}}\tau_jD_i$, where $i,j\in R$ with $i\neq j$.*

Lemma 4.4 *Let $f_{t_1},\cdots,f_{t_k}\in\mathscr{O}$, where $t_1,\cdots,t_k\in R$. If f_i is $\mu(i)$-truncated for $i=t_1,\cdots,t_k$, and $D_i(f_j)=(-1)^{\widetilde{i}\,\widetilde{j}}D_j(f_i)$ for $i,j=t_1,\cdots,t_k$, then there is an $f\in L$, such that $D_i(f)=f_i$ for $i=t_1,\cdots,t_k$.*

Proof We will use induction on k. If $k=1$, then let $f=\tau_{t_1}(f_{t_1})$. By Lemma 4.3(i), we see that $D_{t_1}(f)=D_{t_1}\tau_{t_1}(f_{t_1})=f_{t_1}$. Assume that there is $g\in\mathscr{O}$, such that $D_i(g)=f_i$ for $i=t_1,\cdots,t_{k-1}$. Let $f=g+\tau_{t_k}(f_{t_k}-D_{t_k}(g))$. According to Lemma 4.3(ii), we obtain

$$\begin{aligned}D_i(f)&=f_i+D_i\tau_{t_k}(f_{t_k}-D_{t_k}(g))\\&=f_i+(-1)^{\widetilde{i}\widetilde{t_k}}\tau_{t_k}(D_i(f_{t_k})-D_iD_{t_k}(g))\\&=f_i+(-1)^{\widetilde{i}\widetilde{t_k}}\tau_{t_k}((-1)^{\widetilde{i}\widetilde{t_k}}D_{t_k}(f_i)-(-1)^{\widetilde{i}\widetilde{t_k}}D_{t_k}D_i(g))\\&=f_i.\end{aligned}$$

As $f_{t_k}-D_{t_k}(g)$ is $\mu(t_k)$-truncated, by virtue of Lemma 4.3(i), we have

$$\begin{aligned}D_{t_k}(f)&=D_{t_k}(g)+D_{t_k}\tau_{t_k}(f_{t_k}-D_{t_k}(g))\\&=D_{t_k}(g)+f_{t_k}-D_{t_k}(g)\\&=f_{t_k}.\end{aligned}$$

The result follows.

Lemma 4.5 *Assume that $\varphi\in h(\mathrm{Der}\mathscr{O})$. Let $f_{2n+1}=(-1)^{|\varphi|+\overline{1}}\varphi(1)$, $f_i=\varphi(\xi_{i'})+(-1)^{|\varphi|}2^{-1}f_{2n+1}\xi_{i'}$ and $f_{i'}=(-1)^{|\varphi|+\overline{1}}\varphi(x_i)-2^{-1}f_{2n+1}x_i$ for all $i\in M$. Then the following statements hold:*

(a) *$D_i(f_j)=(-1)^{\widetilde{i}\,\widetilde{j}}D_j(f_i)$ for all $i,j\in R$.*

(b) *f_i is $\mu(i)$-truncated for all $i\in R$.*

Proof (a) By the assumption, we have

$$\varphi(1)=(-1)^{|\varphi|+\overline{1}}f_{2n+1},\tag{4.1}$$

$$\varphi(x_i)=(-1)^{|\varphi|+\overline{1}}(f_{i'}+2^{-1}f_{2n+1}x_i),\quad\forall\, i\in M,\tag{4.2}$$

$$\varphi(\xi_{i'}) = f_i + (-1)^{|\varphi|+\overline{1}} 2^{-1} f_{2n+1}\xi_{i'}, \quad \forall i' \in T_1. \tag{4.3}$$

Note that $|f_i| = |\varphi| + \overline{1}$ and $|f_{i'}| = |f_{2n+1}| = |\varphi|$ for all $i \in M$. We will proceed in six steps.

(i) Applying φ to $[1, \xi_{i'}] = 0$ for all $i \in M$, we obtain

$$[\varphi(1), \xi_{i'}] + (-1)^{|\varphi|(|1|+\overline{1})}[1, \varphi(\xi_{i'})] = 0.$$

Utilizing (4.1) and (4.3), we get

$$[(-1)^{|\varphi|+\overline{1}} f_{2n+1}, \xi_{i'}] + (-1)^{|\varphi|}[1, f_i + (-1)^{|\varphi|+\overline{1}} 2^{-1} f_{2n+1}\xi_{i'}] = 0.$$

By (2.2), a direct computation shows that

$$D_i(f_{2n+1}) = (-1)^{|\varphi|+\overline{1}} D_{2n+1}(f_{2n+1})\xi_{i'} + D_{2n+1}(f_i).$$

So f_{2n+1} is $\mu(2n+1)$-truncated, i.e., $D_{2n+1}(f_{2n+1}) = 0$. Thus $D_i(f_{2n+1}) = D_{2n+1}(f_i)$ for all $i \in M$.

(ii) Similarly, applying φ to $[1, x_i] = 0$ for all $i \in M$, we have

$$[\varphi(1), x_i] + (-1)^{|\varphi|(|1|+\overline{1})}[1, \varphi(x_i)] = 0.$$

From (4.1)–(4.2), we get

$$[(-1)^{|\varphi|+\overline{1}} f_{2n+1}, x_i] + (-1)^{|\varphi|}[1, (-1)^{|\varphi|+\overline{1}}(f_{i'} + 2^{-1} f_{2n+1}x_i)] = 0.$$

Then

$$D_{2n+1}(f_{2n+1})x_i + D_{i'}(f_{2n+1}) + D_{2n+1}(f_{i'}) = 0.$$

Hence $D_{2n+1}(f_{i'}) = -D_{i'}(f_{2n+1})$ for all $i' \in T_1$.

(iii) Applying φ to $[\xi_{i'}, \xi_{j'}] = 0$ for all $i', j' \in T_1$, together with (4.3), yields

$$\begin{aligned}
&[\varphi(\xi_{i'}), \xi_{j'}] + [\xi_{i'}, \varphi(\xi_{j'})] = 0,\\
&[f_i + (-1)^{|\varphi|+\overline{1}} 2^{-1} f_{2n+1}\xi_{i'}, \xi_{j'}] + [\xi_{i'}, f_j + (-1)^{|\varphi|+\overline{1}} 2^{-1} f_{2n+1}\xi_{j'}] = 0.
\end{aligned}$$

A direct computation yields

$$\begin{aligned}
&(-1)^{|f_i|} 2^{-1} D_{2n+1}(f_i)\xi_{j'} + D_j(f_i) + 4^{-1} D_{2n+1}(f_{2n+1})\xi_{i'}\xi_{j'}\\
&+ (-1)^{|\varphi|+\overline{1}} D_j(f_{2n+1})\xi_{i'} + 2^{-1}\xi_{i'} D_{2n+1}(f_j) - D_i(f_j)\\
&- (-1)^{|\varphi|+\overline{1}} 4^{-1}\xi_{i'} D_{2n+1}(f_{2n+1})\xi_{j'} + (-1)^{|\varphi|} D_i(f_{2n+1})\xi_{j'} = 0.
\end{aligned}$$

As $D_i(f_{2n+1}) = D_{2n+1}(f_i)$ for all $i \in M$, $D_i(f_j) = D_j(f_i)$ for all $i, j \in M$.

(iv) Applying φ to $[x_i, x_j] = 0$ for all $i, j \in M$, and by (4.2), we have

$$\begin{aligned}
&[\varphi(x_i), x_j] + (-1)^{|\varphi|(|x_i|+\overline{1})}[x_i, \varphi(x_j)] = 0,\\
&[(-1)^{|\varphi|+\overline{1}}(f_{i'} + 2^{-1} f_{2n+1}x_i), x_j] + (-1)^{|\varphi|}[x_i, (-1)^{|\varphi|+\overline{1}}(f_{j'} + 2^{-1} f_{2n+1}x_j)] = 0.
\end{aligned}$$

A direct computation shows that

$$\begin{aligned}
&2^{-1} D_{2n+1}(f_{i'})x_j + D_{j'}(f_{i'}) + 4^{-1} D_{2n+1}(f_{2n+1})x_i x_j\\
&+ 2^{-1} x_i D_{j'}(f_{2n+1}) + 2^{-1} x_i D_{2n+1}(f_{j'}) + D_{i'}(f_{j'})
\end{aligned}$$

$$+ 4^{-1}x_ix_jD_{2n+1}(f_{2n+1}) + 2^{-1}x_jD_{i'}(f_{2n+1}) = 0.$$

By the claim above, we see that $D_{i'}(f_{j'}) = -D_{j'}(f_{i'})$ for all $i', j' \in T_1$.

(v) Applying φ to $[x_i, \xi_{i'}] = 1$ for all $i \in M$, and by (4.2)–(4.3), we get

$$[\varphi(x_i), \xi_{i'}] + (-1)^{|\varphi|(|x_i|+\overline{1})}[x_i, \varphi(\xi_{i'})] = \varphi(1),$$
$$[(-1)^{|\varphi|+\overline{1}}(f_{i'} + 2^{-1}f_{2n+1}x_i), \xi_{i'}] + (-1)^{|\varphi|}[x_i, f_i + (-1)^{|\varphi|+\overline{1}}2^{-1}f_{2n+1}\xi_{i'}] = (-1)^{|\varphi|+\overline{1}}f_{2n+1}.$$

A direct computation ensures that

$$\begin{aligned}
&(-1)^{|\varphi|}2^{-1}D_{2n+1}(f_{i'})\xi_{i'} + D_i(f_{i'}) + (-1)^{|\varphi|}4^{-1}D_{2n+1}(f_{2n+1})x_i\xi_{i'}\\
&+ 2^{-1}f_{2n+1} + 2^{-1}x_iD_i(f_{2n+1}) - 2^{-1}x_iD_{2n+1}(f_i) - D_{i'}(f_i)\\
&+ (-1)^{|\varphi|}4^{-1}D_{2n+1}(f_{2n+1})x_i\xi_{i'} + (-1)^{|\varphi|}2^{-1}D_{i'}(f_{2n+1})\xi_{i'} + 2^{-1}f_{2n+1} = f_{2n+1}.
\end{aligned}$$

Thus $D_i(f_{i'}) = D_{i'}(f_i)$ for all $i \in M$.

(vi) Applying φ to $[x_i, \xi_{j'}] = 0$ for all $i \in M$, $j' \in T_1$, $j \neq i$, and by (4.2)–(4.3), we obtain

$$[\varphi(x_i), \xi_{j'}] + (-1)^{|\varphi|(|x_i|+\overline{1})}[x_i, \varphi(\xi_{j'})] = 0,$$
$$[(-1)^{|\varphi|+\overline{1}}(f_{i'} + 2^{-1}f_{2n+1}x_i), \xi_{i'}] + (-1)^{|\varphi|}[x_i, f_j + (-1)^{|\varphi|+\overline{1}}2^{-1}f_{2n+1}\xi_{j'}] = 0.$$

By computation, it follows that

$$\begin{aligned}
&(-1)^{|\varphi|}2^{-1}D_{2n+1}(f_{i'})\xi_{j'} + D_j(f_{i'}) + (-1)^{|\varphi|}4^{-1}D_{2n+1}(f_{2n+1})x_i\xi_{j'}\\
&+ 2^{-1}x_iD_j(f_{2n+1}) - 2^{-1}x_iD_{2n+1}(f_j) - D_{i'}(f_j)\\
&+ (-1)^{|\varphi|}4^{-1}D_{2n+1}(f_{2n+1})x_i\xi_{j'} + (-1)^{|\varphi|}2^{-1}D_{i'}(f_{2n+1})\xi_{j'} = 0.
\end{aligned}$$

Therefore $D_j(f_{i'}) = D_{i'}(f_j)$ for all $i \in M$ and $j' \in T_1$ with $j \neq i$.

Now we conclude that $D_i(f_j) = (-1)^{\tilde{i}\tilde{j}}D_j(f_i)$ for all $i, j \in R$.

(b) By the first part, we obtain $2D_i(f_i) = 0$, that is, f_i is $\mu(i)$-truncated for all $i \in T$.

For $i \in M$, we let $f_i = ex_i^{\pi_i} + h_i$, where e does not contain x_i and h_i does not contain $x_i^{\pi_i}$. By the assumption of this lemma, we have

$$D_i(f_j) = D_j(f_i) = D_j(e)x_i^{\pi_i} + D_j(h_i), \quad j \neq i, \ \forall\, j \in R.$$

As $D_i(f_j)$ and $D_j(h_i)$ are $\mu(i)$-truncated, $D_j(e) = 0$ for all $j \in R$ with $j \neq i$. Noticing that $D_i(e) = 0$, it follows that $D_j(e) = 0$ for all $j \in R$. Lemma 2.1 yields $e \in \mathscr{O}_{-2}$.

Applying φ to $[1, \xi_{2n+1}] = 1$, we get

$$[\varphi(1), \xi_{2n+1}] + (-1)^{|\varphi|(|1|+\overline{1})}[1, \varphi(\xi_{2n+1})] = \varphi(1).$$

Put $\varphi(\xi_{2n+1}) = g$. Then by (4.1), we obtain

$$[(-1)^{|\varphi|+\overline{1}}f_{2n+1}, \xi_{2n+1}] + (-1)^{|\varphi|}[1, g] = (-1)^{|\varphi|+\overline{1}}f_{2n+1}.$$

Thus

$$(-1)^{|\varphi|}D_{2n+1}(f_{2n+1})\xi_{2n+1} + \overline{\partial}(f_{2n+1}) - D_{2n+1}(g) = f_{2n+1}.$$

By the convention before, we see that

$$D_{2n+1}(g) = \overline{\partial}(f_{2n+1}) - f_{2n+1} = (\partial(f_{2n+1}) - 1)f_{2n+1}.$$

Applying φ to $[\xi_{i'}, \xi_{2n+1}] = 2^{-1}\xi_{i'}$ for all $i \in M$, we have

$$[\varphi(\xi_{i'}), \xi_{2n+1}] + [\xi_{i'}, \varphi(\xi_{2n+1})] = 2^{-1}\varphi(\xi_{i'}).$$

Utilizing (4.3), we get

$$[f_i + (-1)^{|\varphi|+\overline{1}}2^{-1}f_{2n+1}\xi_{i'}, \xi_{2n+1}] + [\xi_{i'}, g] = 2^{-1}f_i + (-1)^{|\varphi|+\overline{1}}2^{-1}f_{2n+1}\xi_{i'}.$$

A direct computation shows that

$$\begin{aligned}
&\overline{\partial}(f_i) + (-1)^{|\varphi|+\overline{1}}D_{2n+1}(f_i)\xi_{2n+1} + (-1)^{|\varphi|+\overline{1}}2^{-1}\overline{\partial}(f_{2n+1}\xi_{i'})\\
&+ 2^{-1}D_{2n+1}(f_{2n+1})\xi_{i'}\xi_{2n+1} + 2^{-1}\xi_{i'}D_{2n+1}(g) - D_i(g)\\
=\ & 2^{-1}f_i + (-1)^{|\varphi|+\overline{1}}4^{-1}f_{2n+1}\xi_{i'}.
\end{aligned}$$

Since $D_{2n+1}(f_{2n+1}) = 0$, $D_{2n+1}(f_i) = D_i f_{2n+1}$ and

$$\begin{aligned}
\overline{\partial}(f_{2n+1}\xi_{i'}) &= (\partial(f_{2n+1}) - 2^{-1})f_{2n+1}\xi_{i'},\\
\xi_{i'}D_{2n+1}(g) &= (\partial(f_{2n+1}) - 1)\xi_{i'}f_{2n+1},\\
\overline{\partial}(f_i) &= \overline{\partial}(ex_i^{\pi_i}) + \overline{\partial}(h_i) = (\partial(e) + 2^{-1})ex_i^{\pi_i} + \overline{\partial}(h_i),
\end{aligned}$$

we have

$$2^{-1}f_i = 2^{-1}ex_i^{\pi_i} + 2^{-1}h_i = (\partial(e) + 2^{-1})ex_i^{\pi_i} + \overline{\partial}(h_i) + (-1)^{|\varphi|+\overline{1}}D_i(f_{2n+1}) - D_i(g).$$

It follows that

$$\overline{\partial}(e)x_i^{\pi_i} = (2^{-1} - \partial(h_i))h_i + (-1)^{|\varphi|}D_i(f_{2n+1}) + D_i(g).$$

Because every term on the right-hand side of the equation above is $\mu(i)$-truncated, $\overline{\partial}(e) = 0$. Since $e \in \mathscr{O}_{-2}$, $e = 0$. Thus f_i is $\mu(i)$-truncated for all $i \in M$. Hence the result holds.

Put $\Delta = \{\theta : H \to \mathbb{F} \mid \theta(\lambda + \eta) = \theta(\lambda) + \theta(\eta), \forall\, \lambda, \eta \in H\}$. For $\theta \in \Delta$, we define a linear transformation D_θ of $\mathscr{O}$, such that $D_\theta(\sigma(x^k y^\lambda \xi^u)) = \theta(\lambda)\sigma(x^k y^\lambda \xi^u)$. Clearly $D_\theta \in \mathrm{Der}_{\overline{0}}\mathscr{O}$.

Lemma 4.6 *Let $\varphi \in h(\mathrm{Der}\mathscr{O})$. If $\varphi(x_i) = \varphi(\xi_j) = \varphi(\xi_{2n+1}) = 0$ for all $i \in M$ and $j \in T_1$, then there is a $\theta \in \Delta$, such that $\varphi(y^\lambda) = \theta(\lambda)y^\lambda$ for all $\lambda \in H$.*

Proof Clearly, $\varphi(x_i) = \varphi[y^\lambda, x_i] = \varphi(\xi_j) = \varphi[y^\lambda, \xi_j] = 0$ for all $i \in M$ and $j \in T_1$. By Lemma 4.1, we may assume $\varphi(y^\lambda) = \sum\limits_{\eta \in H} a_\eta y^\eta \in \mathscr{O}_{-2}$ with $a_\eta \in \mathbb{F}$. Applying φ to $[y^\lambda, \xi_{2n+1}] = (1-\lambda)y^\lambda$, we have

$$\begin{aligned}
\Big[\sum_{\eta \in H} a_\eta y^\eta, \xi_{2n+1}\Big] &= (1-\lambda)\sum_{\eta \in H} a_\eta y^\eta,\\
\sum_{\eta \in H} a_\eta (1-\eta)y^\eta &= (1-\lambda)\sum_{\eta \in H} a_\eta y^\eta.
\end{aligned}$$

It follows that $a_\eta = 0$ for all $\eta \in H\backslash\{\lambda\}$. Thus $\varphi(y^\lambda) = \theta(\lambda)y^\lambda$, where $\theta(\lambda) = a_\lambda$. Note that $\varphi(1) = \varphi[x_i, \xi_{i'}] = 0$. Applying φ to $[1, y^\eta\xi_{2n+1}] = y^\eta$, we get $[1, \varphi(y^\eta\xi_{2n+1})] = \varphi(y^\eta)$. Let

$\varphi(y^\eta\xi_{2n+1}) = z$. By computation, we can conclude that $D_{2n+1}(z) = \theta(\eta)y^\eta$. Now applying φ to $[y^\lambda, y^\eta\xi_{2n+1}] = (1-\lambda)y^{\lambda+\eta}$, we obtain

$$[\theta(\lambda)y^\lambda, y^\eta\xi_{2n+1}] + [y^\lambda, z] = (1-\lambda)\theta(\lambda+\eta)y^{\lambda+\eta}.$$

Furthermore,

$$(1-\lambda)\theta(\lambda)y^{\lambda+\eta} + (1-\lambda)\theta(\eta)y^{\lambda+\eta} = (1-\lambda)\theta(\eta+\lambda)y^{\lambda+\eta}.$$

As $1-\lambda \neq 0$, $\theta(\lambda+\eta) = \theta(\lambda)+\theta(\eta)$, i.e., $\theta \in \Delta$.

Proposition 4.2 *Let $\varphi \in h(\mathrm{Der}_t\mathscr{O})$ with $t \geq -1$. Then there exist $g \in \mathscr{O}$ and $\theta \in \Delta$, such that $\varphi = \mathrm{ad}\, g + D_\theta$.*

Proof We first prove that there exist $g \in \mathscr{O}$ and $\theta \in \Delta$, such that $(\varphi - \mathrm{ad}g - D_\theta)(\mathscr{O}_j) = 0$ for $j = -2, -1$.

In fact, we can suppose that f_i is defined as in Lemma 4.5. Then $D_i(f_j) = (-1)^{\widetilde{i}\widetilde{j}}D_j(f_i)$ and f_i is $\mu(i)$-truncated for all $i, j \in R$. According to Lemma 4.4, there is an $f \in \mathscr{O}$, such that $D_i(f) = f_i$ for all $i \in R$.

Let $\varphi_1 = \varphi - \mathrm{ad}\, f$. Note that $|f| + \widetilde{i} = |f_i| = |\varphi| + \widetilde{i'}$. Due to Lemma 4.5, we know that

$$\varphi_1(x_i) = \varphi(x_i) - [f, x_i] = \varphi(x_i) - ((-1)^{|f|}2^{-1}D_{2n+1}(f)x_i + (-1)^{|f|}D_i(f)) = 0, \quad \forall\, i \in M.$$

Similarly, $\varphi_1(\xi_j) = 0$ for all $j \in T_1$. Moreover,

$$\varphi_1[\xi_{2n+1}, x_i] = -2^{-1}\varphi_1(x_i) = 0, \quad \varphi_1[\xi_{2n+1}, \xi_j] = -2^{-1}\varphi_1(\xi_j) = 0.$$

By Lemma 4.1, we can suppose $\varphi_1(\xi_{2n+1}) = \sum\limits_{\lambda\in H}\alpha_\lambda y^\lambda$ with $\alpha_\lambda \in \mathbb{F}$. Put $z := \sum\limits_{\lambda\in H}(\lambda-1)^{-1}\alpha_\lambda y^\lambda$ and $\varphi_2 = \varphi_1 - \mathrm{ad}z$. Then

$$\varphi_2(\xi_{2n+1}) = \varphi_1(\xi_{2n+1}) - \Big[\sum_{\lambda\in H}(\lambda-1)^{-1}\alpha_\lambda y^\lambda, \xi_{2n+1}\Big] = 0,$$
$$\varphi_2(x_i) = \varphi_1(x_i) - \Big[\sum_{\lambda\in H}(\lambda-1)^{-1}\alpha_\lambda y^\lambda, x_i\Big] = 0, \quad \forall\, i \in M,$$
$$\varphi_2(\xi_j) = \varphi_1(\xi_j) - \Big[\sum_{\lambda\in H}(\lambda-1)^{-1}\alpha_\lambda y^\lambda, \xi_j\Big] = 0, \quad \forall\, j \in T_1.$$

By virtue of Lemma 4.6, there is $\theta \in \Delta$, such that $\varphi_2(y^\lambda) = \theta(\lambda)y^\lambda$. Let $\varphi_3 = \varphi_2 - D_\theta$. Then $\varphi_3(y^\lambda) = 0$ for all $\lambda \in H$, that is, $\varphi_3(\mathscr{O}_{-2}) = 0$. Moreover, $\varphi_3(x_i) = \varphi_3(\xi_j) = \varphi_3(\xi_{2n+1}) = 0$ for all $i \in M$ and $j \in T_1$. Since $\varphi_3[x_iy^\lambda, x_i] = \varphi_3[x_iy^\lambda, \xi_j] = 0$ for all $i \in M$ and $j \in T_1$, Lemma 4.1 yields $\varphi_3(x_iy^\lambda) \in \mathscr{O}_{-2}$ for all $i \in M$. Similarly, $\varphi_3(\xi_jy^\lambda) \in \mathscr{O}_{-2}$, $\varphi_3(x_i\xi_{i'}) \in \mathscr{O}_{-2}$, $\forall i \in M$, $\forall j \in T_1$. Applying φ_3 to $x_iy^\lambda = -[x_i\xi_{i'}, x_iy^\lambda]$ and by $[\mathscr{O}_{-2}, x_iy^\lambda] = [x_i\xi_{i'}, \mathscr{O}_{-2}] = 0$, we have

$$\varphi_3(x_iy^\lambda) = -[\varphi_3(x_i\xi_{i'}), x_iy^\lambda] - [x_i\xi_{i'}, \varphi_3(x_iy^\lambda)] = 0, \quad \forall\, i \in M.$$

Similarly, $\varphi_3(\xi_jy^\lambda) = 0$ for all $j \in T_1$. Thus $\varphi_3(\mathscr{O}_{-1}) = 0$. Lemma 4.2 implies $\varphi_3 = 0$. Set $g := f + z$. Then $\varphi = \mathrm{ad}(g) + D_\theta$.

Lemma 4.7 *Let $t > 2$ and $\varphi \in h(\mathrm{Der}_{-t}\mathscr{O})$. Then $\varphi(x_i^{t-1}x_l) = \varphi(x_i^{t-1}\xi_j) = 0$ for all $j \in T_1$, $i, l \in M$ with $l \neq i$.*

Proof For $i \in M$, if $j = i'$, let $\varphi(x_i^{t-1}\xi_{i'}) = \sum_{\eta\in H} a_{i\eta}y^\eta \in \mathscr{O}_{-2}$ with $a_{i\eta} \in \mathbb{F}$. Applying φ to

$$[x_i^{t-1}\xi_{i'}, x_i\xi_{i'}] = (t-1)^* x_i^{t-2}x_i\xi_{i'} - x_i^{t-1}\xi_{i'} = \begin{cases} -x_i^{t-1}\xi_{i'}, & \varepsilon_0(t-1) = 0, \\ ((t-1)^* - 1)x_i^{t-1}\xi_{i'}, & \varepsilon_0(t-1) \neq 0, \end{cases}$$

we have

$$[\varphi(x_i^{t-1}\xi_{i'}), x_i\xi_{i'}] + [x_i^{t-1}\xi_{i'}, \varphi(x_i\xi_{i'})] = \begin{cases} -\varphi(x_i^{t-1}\xi_{i'}), & \varepsilon_0(t-1) = 0, \\ ((t-1)^* - 1)\varphi(x_i^{t-1}\xi_{i'}), & \varepsilon_0(t-1) \neq 0, \end{cases}$$

which combined with $[\mathscr{O}_{-2}, x_i\xi_{i'}] = 0$ and $\varphi(x_i\xi_{i'}) \in \mathscr{O}_{-t} = 0$ for $t > 2$ yields the following:

If $\varepsilon_0(t-1) = 0$, then it is obvious that $\varphi(x_i^{t-1}\xi_{i'}) = 0$.

If $\varepsilon_0(t-1) \neq 0$ and $(t-1)^* - 1 \neq 0$, then $\varphi(x_i^{t-1}\xi_{i'}) = 0$. When $(t-1)^* - 1 = 0$, we have $\varepsilon_0(t-1) = 1$, since $(t-1)^* = \varepsilon_0(t-1)$ and $0 < \varepsilon_0(t-1) < p$. Letting φ act on

$$[x_i^{t-1}\xi_{i'}, \xi_{2n+1}] = (1 - 2^{-1}\varepsilon_0(t-1) - 2^{-1})x_i^{t-1}\xi_{i'} = 0,$$

from the assumption above and $\varphi(\xi_{2n+1}) \in \mathscr{O}_{-t} = 0$ for $t > 2$, we get

$$[\varphi(x_i^{t-1}\xi_{i'}), \xi_{2n+1}] + [x_i^{t-1}\xi_{i'}, \varphi(\xi_{2n+1})] = \Big[\sum_{\eta\in H} a_{i\eta}y^\eta, \xi_{2n+1}\Big] = \sum_{\eta\in H} a_{i\eta}(1-\eta)y^\eta = 0.$$

Hence $a_{i\eta} = 0$ for all $\eta \in H$, that is, $\varphi(x_i^{t-1}\xi_{i'}) = 0$.

Now let $j \neq i'$ and $l \neq i$. By applying φ to

$$[x_i^{t-1}\xi_{i'}, x_ix_l] = -x_i^{t-1}x_l \quad \text{and} \quad [x_i^{t-1}\xi_{i'}, x_i\xi_j] = -x_i^{t-1}\xi_j,$$

we see that $\varphi(x_i^{t-1}x_l) = \varphi(x_i^{t-1}\xi_j) = 0$. Thus for every $i \in M$, we have $\varphi(x_i^{t-1}x_l) = \varphi(x_i^{t-1}\xi_j) = 0$ for all $l \in M$ and $j \in T_1$ with $l \neq i$.

Lemma 4.8 *Let $t > 2$ and $\varphi \in h(\mathrm{Der}_{-t}\mathscr{O})$. If $\varphi(x_i^t) = 0$, then $\varphi(x_i^k\xi_{2n+1}) = \varphi(\xi_j\xi_{2n+1}) = 0$ for all $i \in M$, $j \in T_1$ and $0 \leq k \leq \pi_i$.*

Proof We proceed in two steps.

(i) We propose to prove that

$$\varphi(x_i^k x_l) = \varphi(x_i^k\xi_j) = 0, \quad \forall\, i \in M, \forall\, j \in T_1,\, 0 \leq k \leq \pi_i + 1.$$

We first show that $\varphi(x_i^k) = 0$ by induction on k. If $0 \leq k < t$, then $\varphi(x_i^k) \in \mathscr{O}_{-t+k-2} = 0$. Moreover, $\varphi(x_i^t) = 0$. Suppose that $k > t$ and $\varphi(x_i^{k-1}) = 0$. Clearly, $\varphi(x_i) = \varphi(\xi_j) = 0$. Applying φ to

$$[x_i^k, \xi_j] = k^* x_i^{k-1}\delta_{i'j} \quad \text{and} \quad [x_i^k, x_\nu] = 0,$$

we obtain $\varphi[x_i^k, \xi_j] = \varphi[x_i^k, x_\nu] = 0$ for all $\nu \in M$ and $j \in T_1$. Lemma 4.1 ensures that $\varphi(x_i^k) \in \mathscr{O}_{-2} \cap \mathscr{O}_{-t+k-2} = 0$. It is easy to see that the claim $\varphi(x_i^k) = 0$ is true for all $k \geq 0$.

Then we prove that $\varphi(x_i^k x_l) = 0$. If $l = i$, then by the argument above, we have

$$\varphi(x_i^k x_i) = \begin{cases} 0, & \varepsilon_0(k+1) = 0, \\ \varphi(x_i^{k+1}) = 0, & \varepsilon_0(k+1) \neq 0. \end{cases}$$

Let $l \neq i$. We use induction on k. If $k < t-1$, then $\varphi(x_i^k x_l) \in \mathscr{O}_{-t+k-1} = 0$. Lemma 4.7 implies $\varphi(x_i^{t-1}x_l) = 0$. Assume that $k > t-1$ and $\varphi(x_i^{k-1}x_l) = 0$. By the induction hypothesis and $\varphi(x_i^k) = 0$, we see that

$$\varphi([x_i^k x_l, x_\nu]) = 0 \quad \text{and} \quad \varphi([x_i^k x_l, \xi_j]) = \varphi(\delta_{ij} k^* x_i^{k-1} x_l + \delta_{lj} x_i^k) = 0, \quad \forall\, \nu \in M, \forall\, j \in T_1.$$

It follows from Lemma 4.1 that $\varphi(x_i^k x_l) \in \mathscr{O}_{-2} \cap \mathscr{O}_{-t+k-1} = \{0\}$.

Finally, we prove $\varphi(x_i^k \xi_j) = 0$ also by induction on k. If $k < t-1$, then $\varphi(x_i^k \xi_j) \in \mathscr{O}_{-t+k-1} = 0$. If $k = t-1$, by Lemma 4.7, we get $\varphi(x_i^{t-1}\xi_j) = 0$. Now let $k > t-1$ and $\varphi(x_i^{k-1}\xi_j) = 0$. Similarly, by letting φ act on the equalities below,

$$[x_i^k \xi_j, x_\nu] = -\delta_{j\nu} x_i^k \quad \text{and} \quad [x_i^k \xi_j, \xi_\iota] = \delta_{i\iota} k^* x_i^{k-1} \xi_j, \quad \forall\, \nu \in M,\ \forall\, \iota \in T_1,$$

we see that the result is zero. Again Lemma 4.1 yields $\varphi(x_i^k \xi_j) \in \mathscr{O}_{-2} \cap \mathscr{O}_{-t+k-1} = \{0\}$.

(ii) Now we return to the proof of this lemma. If $t > 3$, then $\varphi(\xi_j \xi_{2n+1}) \in \mathscr{O}_{-t+1} = 0$. Put $t = 3$. Then let $\varphi(\xi_j \xi_{2n+1}) = \sum\limits_{\eta \in H} a_{j\eta} y^\eta \in \mathscr{O}_{-2}$ with $a_{j\eta} \in \mathbb{F}$. Applying φ to $[x_{j'}\xi_j, \xi_j \xi_{2n+1}] = \xi_j \xi_{2n+1}$, we have

$$[\varphi(x_{j'}\xi_j), \xi_j \xi_{2n+1}] + [x_{j'}\xi_j, \varphi(\xi_j \xi_{2n+1})] = \varphi(\xi_j \xi_{2n+1}).$$

Since $\varphi(x_{j'}\xi_j) = 0$ and $[x_{j'}\xi_j, \mathscr{O}_{-2}] = 0$, $\varphi(\xi_j \xi_{2n+1}) = 0$.

For $0 \leq k \leq \pi_i$, by applying φ to

$$(k+1)^* x_i^k \xi_{2n+1} = [x_i^{k+1}, \xi_{i'} \xi_{2n+1}] - (1 - 2^{-1}\varepsilon_0(k+1)) x_i^{k+1} \xi_{i'}$$

and by the known results $\varphi(x_i^{k+1}) = 0$, $\varphi(x_i^{k+1}\xi_{i'}) = 0$ and $\varphi(\xi_{i'}\xi_{2n+1}) = 0$ above, we obtain $\varphi(x_i^k \xi_{2n+1}) = 0$.

Proposition 4.3 *Let $t > 2$ and $t \neq p^v$ for all $v \in \mathbb{N}$. Then $h(\mathrm{Der}_{-t}\mathscr{O}) = \{0\}$.*

Proof Let $\varphi \in h(\mathrm{Der}_{-t}\mathscr{O})$. Considering the $\mathbb{Z}$-degree, we have $\varphi(x_i^t) \in \mathscr{O}_{-2}$. Suppose $\varphi(x_i^t) = \sum\limits_{\eta \in H} a_{i\eta} y^\eta$ with $a_{i\eta} \in \mathbb{F}$. By applying φ to

$$[x_i^t, x_i \xi_{i'}] = t^* x_i^{t-1} x_i,$$

if $\varepsilon_0(t) \neq 0$, then $x_i^{t-1} x_i = x_i^t$. It follows from $[\mathscr{D}_{-2}, x_i \xi_{i'}] = 0$ and $\varphi(x_i \xi_{i'}) = 0$ that

$$[\varphi(x_i^t), x_i \xi_{i'}] + [x_i^t, \varphi(x_i \xi_{i'})] = t^* \varphi(x_i^t) = 0.$$

If $\varepsilon_0(t) = 0$, assume that $t = \sum\limits_{s=1}^{l} \varepsilon_s(t) p^s$ for some $0 \leq \varepsilon_s(t) < p$ and $\varepsilon_l(t) \neq 0$. Since $t \neq p^l$, considering $\mathbb{Z}$-degree, we have $\varphi(x_i^{t-p^l+1}) = \varphi(x_i^{p^l}\xi_{i'}) = 0$. Applying φ to

$$-[x_i^{p^l}\xi_{i'}, x_i^{t-p^l+1}] = x_i^{p^l} x_i^{t-p^l} = x_i^t,$$

we get $\varphi(x_i^t) = 0$. Lemma 4.8 implies $\varphi(x_i^k \xi_{2n+1}) = \varphi(\xi_j \xi_{2n+1}) = 0$ for all $i \in M$ and $j \in T_1$. Moreover, $\varphi(y^\lambda) \in \mathscr{O}_{-t-2} = 0$. According to Proposition 3.1, we see that $\varphi = 0$. Hence $h(\mathrm{Der}_{-t}\mathscr{O}) = \{0\}$.

Proposition 4.4 *If $t = p^v$ for some $v \in \mathbb{N}$, then $\mathrm{Der}_{-t}\mathscr{O} = \langle\, D_i^{p^v} \mid i \in M \rangle$.*

Proof Clearly $D_i^{p^v} \in \mathrm{Der}_{-t}\mathscr{O}$ for all $i \in M$. Let $t = p^v$ and $\varphi \in h(\mathrm{Der}_{-t}\mathscr{O})$. It is easy to see that $\varphi(x_i^t) = \sum\limits_{\eta \in H} a_{i\eta} y^\eta \in \mathscr{D}_{-2}$ with $a_{i\eta} \in \mathbb{F}$. As $t = p^v$, $\varepsilon_0(t) = 0$. Applying φ to

$$[x_i^t, \xi_{2n+1}] = (1 - 2^{-1}\varepsilon_0(t))x_i^t = x_i^t,$$

together with $\varphi(\xi_{2n+1}) = 0$, yields

$$\Big[\sum_{\eta \in H} a_{i\eta} y^\eta, \xi_{2n+1}\Big] = \sum_{\eta \in H} a_{i\eta} y^\eta.$$

A direct computation implies that

$$\sum_{\eta \in H} a_{i\eta}(1-\eta) y^\eta = \sum_{\eta \in H} a_{i\eta} y^\eta.$$

Thus $a_{i\eta} = 0$ for all $\eta \neq 0$. Hence $\varphi(x_i^t) = a_{i0}1$. Put $\psi := \varphi - \sum\limits_{j \in M} c_j D_j^t$, where $c_j = a_{j0}\big(\prod\limits_{h=1}^{t} h^*\big)^{-1}$. Then

$$\psi(x_i^t) = \varphi(x_i^t) - \sum_{j \in M} c_j D_j^t(x_i^t) = a_{i0}1 - c_i D_i^t(x_i^t) = a_{i0}1 - c_i \prod_{h=1}^{t} h^* = 0.$$

By virtue of Lemma 4.8, we have $\psi(x_i^k \xi_{2n+1}) = \psi(\xi_j \xi_{2n+1}) = 0$ for all $i \in M$ and $j \in T_1$. Moreover, $\psi(y^\lambda) = 0$. Proposition 3.1 shows that $\psi = 0$. Consequently, $\varphi \in \langle D_i^{p^v} \mid i \in M\rangle$.

If $v_i > s_i$, then $D_i^{p^{v_i}} = 0$ for all $i \in M$. By Propositions 4.1–4.4, we obtain the following theorem.

Theorem 4.1 $\mathrm{Der}\mathscr{O} = \mathrm{ad}\,\mathscr{O} \oplus \{D_\theta \mid \theta \in \Delta\} \oplus \langle D_i^{p^{v_i}} \mid \forall\, i \in M,\, 0 < v_i \leq s_i\,\rangle$.

Theorem 4.2 *For each algebra in the family, $\mathscr{O}$ has no nondegenerate associative form.*

Proof Assume that λ is a nondegenerate associative form on $\mathscr{O}$. [23, Proposition 2.3] implies that $\lambda\big|_{\mathscr{O}_\tau \times \mathscr{O}_{-2}}$ is nonsingular. It follows that $\lambda(1, x^\pi \xi^\omega) \neq 0$. Since λ is associative,

$$\begin{aligned}
\lambda(1, x^\pi \xi^\omega) &= \lambda([1, \xi_{2n+1}], x^\pi \xi^\omega) = \lambda(1, [\xi_{2n+1}, x^\pi \xi^\omega]) \\
&= \lambda\Big(1, (-1)^n x^\pi \xi^\omega - \Big(1 - \frac{1}{2}\sum_{i=1}^{n} \pi_i - \frac{1}{2}n\Big) x^\pi \xi^\omega\Big) \\
&= ((-1)^n - 1)\lambda(1, x^\pi \xi^\omega).
\end{aligned}$$

Hence $(2 - (-1)^n)\lambda(1, x^\pi \xi^\omega) = 0$. As $2 - (-1)^n \neq 0 \pmod p$, $\lambda(1, x^\pi \xi^\omega) = 0$, a contradiction. As a result, $\mathscr{O}$ has no nonsingular associative form.

Theorem 4.3 *For each algebra in the family, $\mathscr{O}$ is not isomorphic to the simple Lie superalgebras of Cartan type W, S, H, HO, SHO, K, KO, SKO.*

Proof Recall that $\dim\mathscr{O} = 2^{n+1} p^{\sum_{i \in M}(s_i+1)+m}$. By means of [13, 19], we see that the dimension of modular Lie superalgebras HO is odd and the dimension of modular Lie superalgebras H can not be divided by p. So $\mathscr{O}$ is not isomorphic to modular Lie superalgebras H and HO, respectively. The outer derivations of W, S, K and KO are all ad-nilpotent in

[6, 23], but $\mathscr{O}$ possesses outer derivations D_θ which are not ad-nilpotent. It follows that $\mathscr{O}$ is not isomorphic to modular Lie superalgebras W, S, K and KO, respectively. Using Theorem 4.2, we can also prove that $\mathscr{O}$ is not isomorphic to modular Lie superalgebras SHO and SKO, which possess nondegenerate associative forms on them (see [12]).

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