Chin. Ann. Math.
36B(3), 2015, 467–484
DOI: 10.1007/s11401-015-0931-7

Chinese Annals of Mathematics, Series B

Convolutions, Tensor Products and Multipliers of the Orlicz-Lorentz Spaces*

Hongliang LI[1] Jiecheng CHEN[2]

Abstract In this paper, the authors first give the properties of the convolutions of Orlicz-Lorentz spaces $\Lambda_{\varphi_1,w}$ and $\Lambda_{\varphi_2,w}$ on the locally compact abelian group. Secondly, the authors obtain the concrete representation as function spaces for the tensor products of Orlicz-Lorentz spaces $\Lambda_{\varphi_1,w}$ and $\Lambda_{\varphi_2,w}$, and get the space of multipliers from the space $\Lambda_{\varphi_1,w}$ to the space $M_{\varphi_{2*},w}$. Finally, the authors discuss the homogeneous properties for the Orlicz-Lorentz space $\Lambda^{p,q}_{\varphi,w}$.

Keywords Orlicz-Lorentz spaces, Convolution, Tensor products, Multipliers, Hardy operator
2010 MR Subject Classification 43A15, 43A22

1 Introduction

The convolution operator has been studied for many years. The classical type, called Young inequality, $L^p * L^q \hookrightarrow L^r$ $(1 < p, q, r < \infty)$ is well known. Subsequently, the convolutions of Lorentz spaces $L^{p,q}$ was studied in [1] by O'Neil and [2] by Yap. In [3], Kamińska and Musielak got the sufficient and necessary conditions for embedding $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow L_{\varphi_3}$, $E_{\varphi_1} * E_{\varphi_2} \hookrightarrow L_{\varphi_3}$, $E_{\varphi_1} * E_{\varphi_2} \hookrightarrow L_{\varphi_3}$, and $L_{\varphi_1} * L_{\varphi_2} \hookrightarrow E_{\varphi_3}$, where L_{φ_i} are Orlicz spaces and E_{φ_i} are their subspaces consisting of all order continuous elements, some parts of which generalize the results of Hewitt and Ross [4], Hudzik et al [5], O'Neil [6] and Zelazko [7]. In the present paper, we will obtain the corresponding results on Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and use the conclusion of the embedding on Orlicz-Lorentz spaces to get the representation of the tensor products and multipliers on Orlicz-Lorentz spaces.

Let A be a Banach algebra. By a left (right) Banach A-module we mean (see [8]) a Banach space V, which is a left (right) A-module in the algebraic sense, and for which

$$\|av\| \leqslant k\|a\|\|v\|, \quad \forall a \in A,\ v \in V,$$

where k is a constant independent of $a,\ v$.

If V and W are left (right) Banach A-modules, then $\mathrm{Hom}_A(V, W)$ will denote the Banach space of all continuous A-module homomorphisms from V to W with the operator norm. $\mathrm{Hom}_A(V, W)$, as a rule, is called the space of multipliers from V to W.

Manuscript received October 27, 2012. Revised October 12, 2014.
[1]Department of Mathematics, Zhejiang International Studies University, Hangzhou 310012, China.
E-mail: hongli@126.com
[2]Department of Mathematics, Zhejiang Normal University, Jinhua 321004, Zhejiang, China.
E-mail: jcchen@zjnu.edu.cn
*This work was supported by the National Natural Science Foundation of China (Nos. 11401530, 11461033, 11271330) and the Natural Science Foundation of Zhejiang Province (No. LQ13A010018).

The definition of the tensor product of Banach modules can be found in [8–9]. Let A be a Banach algebra, and V and W be left and right Banach A-modules respectively. Suppose that $V \otimes_\gamma W$ denotes the projective tensor product (see [10]) of V and W as Banach spaces (γ is the greatest crossnorm in [11], [12, p. 36]). Let K be the closed linear subspace of $V \otimes_\gamma W$ which is spanned by all the elements of the form

$$av \otimes w - v \otimes aw, \quad a \in A, \ v \in V, \ w \in W.$$

Now the quotient Banach space $V \otimes_\gamma W/K$ is called the A-module tensor product $V \otimes_A W$.

The following isomorphism

$$\mathrm{Hom}_A(V, W^*) \cong (V \otimes_A W)^* \tag{1.1}$$

was proved by Rieffel [8], where the notation W^* is the dual of W. The linear functional on $\mathrm{Hom}_A(V, W^*)$, which corresponds to $t = \sum_{i=1}^{\infty} v_i \otimes w_i \in V \otimes_A W$, has value

$$\langle t, T \rangle = \sum_{i=1}^{\infty} \langle w_i, T v_i \rangle$$

at $T \in \mathrm{Hom}_A(V, W^*)$. The topology on $\mathrm{Hom}_A(V, W^*)$ defined by the linear functional of this form corresponds to the weak $*$-topology $(V \otimes_A W)^*$, which is called ultraweak $*$-operators topology (see [9, 13]).

In this paper, we get the concrete representation of the tensor products of the Orlicz-Lorentz spaces $\Lambda_{\varphi,w}$ and obtain the multipliers of the Orlicz-Lorentz spaces by (1.1). For more details about tensor products and multipliers, one can also refer to [14–18] and so on.

2 Preliminaries for Orlicz-Lorentz Spaces

Let $\mathcal{M}(G, \mu)$ be the class of all measurable and almost everywhere finite functions on (G, μ). For $f \in \mathcal{M}(G, \mu)$, a non-increasing rearrangement of f, is a non-increasing function f^* on $\mathbf{R}_+ \equiv (0, +\infty)$ which is equimeasurable with $|f|$. The rearrangement f^* is defined by the equality (see [19])

$$f^*(t) = \inf\{s : \lambda_f(s) \le t\}, \quad 0 < t < \infty,$$

where

$$\lambda_f(s) = \mu\{x \in X : |f(x)| > s\}, \quad s \ge 0.$$

We say $\varphi : [0, \infty) \to [0, \infty)$ is a Young function if φ is non-decreasing and convex with $\varphi(0) = 0$, and $\lim_{x \to \infty} \varphi(x) = \infty$. The Young conjugate φ_* of the Young function φ is defined by

$$\varphi_*(x) = \sup_{y \ge 0} \{xy - \varphi(y)\}, \quad x \ge 0.$$

The Orlicz-Lorentz spaces $\Lambda_{\varphi,w}(G)$ (see [20–21]) associated to the Young function φ and a weight w on $\mathbf{R}_+$ (nonnegative locally integrable functions in $\mathbf{R}_+$), are the set of $f \in \mathcal{M}(G, \mu)$ such that for some $\lambda > 0$, we have $I_{\varphi,w}(\lambda f) < \infty$, where

$$I_{\varphi,w}(f) = \int_0^{\infty} \varphi(f^*(t)) w(t) \mathrm{d}t$$

(we assume that the weight w vanishes in $[\mu(G),\infty)$). Let

$$\|f\|_{\Lambda_{\varphi,w}(G)} = \inf\Big\{\epsilon > 0 :\ I_{\varphi,w}\Big(\frac{f}{\epsilon}\Big) \le 1\Big\}.$$

If there is no ambiguity, we indicate $\Lambda_{\varphi,w}(G) = \Lambda_{\varphi,w}$. If $w(t) = 1$, then $\Lambda_{\varphi,w} = L^{\varphi}$ is an Orlicz space (see [22–23]); if $\varphi(t) = t^p$ $(1 \le p < \infty)$, then $\Lambda_{\varphi,w} = \Lambda^p(w)$ is a Lorentz space (see [24–25]). Additionally, let

$$E_{\varphi,w} = \{f : I_{\varphi,w}(\lambda f) < \infty \text{ for all } \lambda > 0\},$$

called the subspace of finite elements of $\Lambda_{\varphi,w}$. If group G is discrete, the notations $l_{\varphi,w}$ and $h_{\varphi,w}$ are used instead of $\Lambda_{\varphi,w}$ and $E_{\varphi,w}$.

Given an arbitrary function $D : [0,\infty) \to [0,\infty)$, we say that D satisfies condition Δ_2 in symbol $G \in \Delta_2$ when

$$\sup_{t>0} \frac{D(2t)}{D(t)} < \infty.$$

A Young function F is said to satisfy Δ' condition in symbol $F \in \Delta'$ if there exists $C > 0$ such that

$$F(xy) \le CF(x)F(y), \quad \forall x, y \ge 0.$$

Clearly if $F \in \Delta'$, then $F \in \Delta_2$. By [26, Thm. 3.1], we know that if $\mu(G) = \infty$, $W \in \Delta_2$, $W(\infty) = \infty$ or $\mu(G) < \infty$, then

$$(\Lambda_{\varphi,w})^* = M_{\varphi_*,w}. \tag{2.1}$$

There are many papers devoted to researching Hardy-type inequalities on monotone functions. Let f be a nonnegative function on $\mathbf{R}^+$, the Hardy operator be

$$Sf(x) = \frac{1}{x}\int_0^x f(t)\mathrm{d}t, \quad x \in [0,\infty),$$

and $f\downarrow$ indicate that f is a nonnegative nonincreasing function in $\mathbf{R}_+$. In [27, Thm. 2.3], the author got that if $\varphi \in \Delta'$, then

$$\int_0^\infty \varphi(Sf(x))w(x)\mathrm{d}x \le C\int_0^\infty \varphi(f(x))w(x)\mathrm{d}x, \quad \forall f\downarrow \tag{2.2}$$

if and only if there is a constant $H > 0$ such that

$$\int_r^\infty \varphi\Big(\frac{br}{t}\Big)w(t)\mathrm{d}t \le H\varphi(b)\int_0^r w(t)\mathrm{d}t, \quad \forall r > 0,\ b > 0. \tag{2.3}$$

Obviously, (2.2) implies that $\Lambda_{\varphi,w}$ can be normable if $\varphi \in \Delta'$. If $\varphi = t^p$, then (2.3) implies that $w \in B_p$ (see [28–29]). If w, φ satisfy the inequality (2.2), we say $w \in B_\varphi$, and let

$$B_\varphi(w) = \sup_{f\downarrow} \frac{\displaystyle\int_0^\infty \varphi(Sf(x))w(x)\mathrm{d}x}{\displaystyle\int_0^\infty \varphi(f(x))w(x)\mathrm{d}x}.$$

As usual, $f \approx g$ indicates the existence of a universal constant $B > 0$ (independent of all parameters involved) so that $\big(\frac{1}{B}\big)f \le g \le Bf$. In the sequel, C denotes a positive constant which need not be the same at different occurrences. If w is a weight on $\mathbf{R}^+$, we denote $W(t) = \int_0^t w(s)\mathrm{d}s$.

3 Convolutions of Orlicz-Lorentz Spaces

In the rest of this paper, G will be a unimodular locally compact abelian group, with Haar measure μ. Let φ be a Young function. A generalized inverse function $\varphi^{-1}:[0,\infty]\to[0,\infty]$ is defined as

$$\varphi^{-1}(y)=\inf\{x\geq 0,\varphi(x)>y\},\quad \text{where } \inf\emptyset=\infty.$$

It is said in [3] that φ_i $(i=1,2,3)$ satisfy condition $(+)$ for l.a. (s.a.) [a.a] if there exist $k>0$, $\delta>0$, such that

$$kuv\leq\varphi_1(u)\varphi_3^{-1}(\varphi_2(v))+\varphi_2(v)\varphi_3^{-1}(\varphi_1(u)),$$

when $\varphi_1(u)\geq\delta$, $\varphi_2(v)\geq\delta$ ($\varphi_1(u)\leq\delta$ and $\varphi_2(v)\leq\delta$) $[u,v\geq 0]$. It is said that φ_i $(i=1,2,3)$ satisfy condition $(++)$ for l.a. (s.a.) [a.a], if for every $\alpha>0$, there exist $k>0$, $\delta>0$, such that

$$\alpha uv\leq\varphi_1(u)\varphi_3^{-1}(k\varphi_2(v))+\varphi_2(v)\varphi_3^{-1}(k\varphi_1(u)),$$

when $\varphi_1(u)\geq\delta$, $\varphi_2(v)\geq\delta$ ($\varphi_1(u)\leq\delta$ and $\varphi_2(v)\leq\delta$) $[u,v\geq 0]$. [3, Prop. 2] showed that condition $(+)$ for l.a. (s.a.) [a.a] is equivalent to the following one: There exist $l,\delta>0$, such that

$$\varphi_1^{-1}(u)\varphi_2^{-1}(u)\leq lu\varphi_3^{-1}(u),$$

if $u\geq\delta$ ($u\leq\delta$) $[u\geq 0]$.

Since G is a unimodular locally compact group, by virtue of the definition of convolution (see [1]), Hewitt and Ross [4, Ch. 5, Sec. 20] indicated that the operator

$$T(f,g)=f*g$$

satisfies that

$$\|T(f,g)\|_1\leq\|f\|_1\|g\|_1,$$
$$\|T(f,g)\|_\infty\leq\|f\|_1\|g\|_\infty,$$
$$\|T(f,g)\|_\infty\leq\|f\|_\infty\|g\|_1.$$

So such T is a convolution operator. Thus by [1],

$$(f*g)^{**}(t)\leq\int_t^\infty f^{**}(u)g^{**}(u)\mathrm{d}u,\quad t>0. \tag{3.1}$$

Lemma 3.1 *Let $w\in B_{\varphi_i}$ $(i=1,2)$ and $w\geq 1$. Suppose that φ_i $(i=1,2,3)$ satisfy condition $(+)$ for a.a. and $I_{\varphi_1,w}\big(\frac{2\lambda}{k}f\big)\leq\frac{1}{B_{\varphi_1}(w)}$, $I_{\varphi_2,w}(g)\leq\frac{1}{B_{\varphi_2}(w)}$ (or $I_{\varphi_1,w}(f)\leq\frac{1}{B_{\varphi_1}(w)}$, $I_{\varphi_2,w}\big(\frac{2\lambda}{k}g\big)\leq\frac{1}{B_{\varphi_2}(w)}$), where k is the constant from $(+)$, so then $I_{\varphi_3,w}(\lambda f*g)\leq 1$.*

Proof By (3.1) we have

$$\begin{aligned}I_{\varphi_3,w}(\lambda f*g)&=\int_0^\infty\varphi_3(\lambda(f*g)^*(t))w(t)\mathrm{d}t\\&\leq\int_0^\infty\varphi_3\Big(\lambda\int_t^\infty f^{**}(s)g^{**}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t\\&\leq\frac{1}{2}\int_0^\infty\varphi_3\Big(\int_0^\infty k\cdot\frac{2\lambda}{k}f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t.\end{aligned}$$

Since φ_i $(i=1,2,3)$ satisfy condition $(+)$ for a.a. and $w(s)\geq 1,\ s>0$, the right hand of the last inequality

$$\begin{aligned}
&\frac{1}{2}\int_0^\infty \varphi_3\Big(\int_0^\infty k\cdot\frac{2\lambda}{k}f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t\\
\leq{}&\frac{1}{2}\int_0^\infty \varphi_3\Big(\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)w(s)\varphi_3^{-1}(\varphi_2(g^{**}(s)\chi_{(t,\infty)}(s)))\mathrm{d}s\Big)w(t)\mathrm{d}t\\
&+\frac{1}{2}\int_0^\infty \varphi_3\Big(\int_0^\infty \varphi_2(g^{**}(s)\chi_{(t,\infty)}(s))w(s)\varphi_3^{-1}\Big(\varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)\Big)\mathrm{d}s\Big)w(t)\mathrm{d}t.
\end{aligned}\tag{3.2}$$

Since

$$\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)w(s)\mathrm{d}t\leq\frac{1}{B_{\varphi_1}(w)}\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{*}(s)\Big)w(s)\mathrm{d}t\leq 1,$$

$$\int_0^\infty \varphi_2(g^{**}(s)\chi_{(t,\infty)}(s))w(s)\mathrm{d}s\leq\frac{1}{B_{\varphi_2}(w)}\int_0^\infty \varphi_2(g^{*}(s))w(s)\mathrm{d}s\leq 1,$$

by Jensen's inequality and $\varphi_3(\varphi_3^{-1}(t))\leq t,\ t>0$, we get that the right part of (3.2)

$$\begin{aligned}
&\leq\frac{1}{2}\int_0^\infty\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)w(s)\varphi_2(g^{**}(s)\chi_{(t,\infty)}(s))w(t)\mathrm{d}s\mathrm{d}t\\
&=\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)w(s)\mathrm{d}s\int_0^s \varphi_2(g^{**}(s))w(t)\mathrm{d}t\\
&=\int_0^\infty \varphi_1\Big(\frac{2\lambda}{k}f^{**}(s)\Big)w(s)\mathrm{d}s\int_0^\infty \varphi_2(g^{**}(t))w(t)\mathrm{d}t\leq 1.
\end{aligned}$$

Remark 3.1 If $w\geq 1$ is replaced by $w\geq c$ $(c>0$ is a nonnegative constant) in the preceding lemma, then the result also holds.

The next theorems give sufficient conditions for embedding of the spaces $\Lambda_{\varphi_1}(w)*\Lambda_{\varphi_2}(w)$ $(l_{\varphi_1}(w)*l_{\varphi_2}(w))$ into $\Lambda_{\varphi_3}(w)$ $(l_{\varphi_3}(w))$.

Theorem 3.1 (I) *Let G be nondiscrete, $w\in B_{\varphi_i}$ $(i=1,2)$, $w\geq c$ $(c>0$ be a nonnegative constant) and φ_i $(i=1,2,3)$ satisfy condition $(+)$ for l.a. if G is compact and $(+)$ for a.a. if G is noncompact. Then $\Lambda_{\varphi_1,w}*\Lambda_{\varphi_2,w}\hookrightarrow\Lambda_{\varphi_3,w}$. If additionally φ_3 is finite, then $E_{\varphi_1,w}*E_{\varphi_2,w}\hookrightarrow E_{\varphi_3,w}$.*

(II) *Let G be discrete, $w\in B_{\varphi_i}$ $(i=1,2)$, and φ_i satisfy $(+)$ for s.a. Then $l_{\varphi_1,w}*l_{\varphi_2,w}\hookrightarrow l_{\varphi_3,w}$ and $h_{\varphi_1,w}*h_{\varphi_2,w}\hookrightarrow h_{\varphi_3,w}$.*

Proof (I) By [3] it is sufficient to prove only inclusion. Let first G be noncompact and $(+)$ for a.a. Take $f\in\Lambda_{\varphi_1,w}$ and $g\in\Lambda_{\varphi_2,w}$ satisfying $\max(I_{\varphi_1,w}(f),I_{\varphi_2,w}(g))\leq\min\limits_{i=1,2}\big(\frac{1}{B_{\varphi_i}(w)}\big)$. Then applying Lemma 3.1 with $\lambda=\frac{k}{2}$, we obtain $I_{\varphi_3,w}\big(\frac{k}{2}f*g\big)\leq 1$, which means by [3, Thm. 1.2] that $f*g\in\Lambda_{\varphi_3,w}$ and $\Lambda_{\varphi_1,w}*\Lambda_{\varphi_2,w}\hookrightarrow\Lambda_{\varphi_3,w}$. If G is compact and φ_i $(i=1,2,3)$ satisfy condition $(+)$ for l.a., then by [3, Lem. 5] there exist functions $\overline{\varphi}_i$ $(i=1,2,3)$ satisfying $(+)$ for a.a. and equivalent to φ_i for l.a., which implies that $\Lambda_{\varphi_i,w}=\Lambda_{\overline{\varphi}_i,w}$. Thus, the embedding follows in the same way as the above.

To prove the inclusion $E_{\varphi_1,w}*E_{\varphi_2,w}\subset E_{\varphi_3,w}$, take $f\in E_{\varphi_1,w},\ g\in E_{\varphi_2,w}$. Let $\max(I_{\varphi_1,w}(f),$ $I_{\varphi_2,w}(g))\leq\min\big(1,\frac{1}{B_{\varphi_1}(w)},\frac{1}{B_{\varphi_2}(w)}\big)$. For any $\beta>0$, suppose $\lambda=2\beta$. Then by (3.1) it follows that

$$\begin{aligned} I_{\varphi_3,w}(\beta f*g) &= I_{\varphi_3,w}\Big(\frac{\lambda}{2}f*g\Big) \\ &= \int_0^\infty \varphi_3\Big(\frac{\lambda}{2}(f*g)^*(t)\Big)w(t)\mathrm{d}t \\ &\le \int_0^\infty \varphi_3\Big(\int_0^\infty \frac{\lambda}{2}f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t. \end{aligned} \tag{3.3}$$

Since $w \in B_{\varphi_1}$ and $f \in E_{\varphi_1,w}$,

$$\int_0^\infty \varphi_1(\lambda f^{**}(t))w(t)\mathrm{d}t < \infty.$$

Thus we can choose $t_1 > 0$ such that $\int_{t_1}^\infty \varphi_1\big(\frac{2\lambda}{k}f^{**}(t)\big)w(t)\mathrm{d}t \le 1$. Then the right side of (3.3)

$$\begin{aligned} &\le \frac{1}{2}\int_0^\infty \varphi_3\Big(\int_0^{t_1} \lambda f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\ &\quad + \frac{1}{2}\int_0^\infty \varphi_3\Big(\int_{t_1}^\infty \lambda f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\ &\le \frac{1}{2}\int_0^\infty \varphi_3\Big(\int_0^{t_1} \lambda f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t + \frac{1}{2}. \end{aligned} \tag{3.4}$$

Since $f \in E_{\varphi_1,w}$, there exists a constant $u_0 > 0$, such that

$$\int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^*(s)\Big)w(s)\mathrm{d}s \le \frac{1}{B_{\varphi_1}(w)}, \quad \text{where } T_1 = \mu\{x : |f(x)| \ge u_0\}.$$

Thus

$$\begin{aligned} &\int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)w(s)\mathrm{d}s \\ &= \int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}(f^*(\cdot)\chi_{(0,T_1)}(\cdot))^{**}(s)\Big)\chi_{(0,T_1)}(s)w(s)\mathrm{d}s \\ &\le B_{\varphi_1}(w)\int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^*(s)\Big)w(s)\mathrm{d}s \le 1. \end{aligned} \tag{3.5}$$

Now we get

$$\begin{aligned} &\int_0^\infty \varphi_3\Big(\int_0^{t_1} \lambda f^{**}(s)g^{**}(s)\chi_{(t,\infty)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\ &= \int_0^{t_1} \varphi_3\Big(\int_t^{t_1} \lambda f^{**}(s)g^{**}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\ &\le \frac{1}{2}\int_0^{t_1} \varphi_3\Big(\int_t^{T_1} 2\lambda f^{**}(s)g^{**}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t + \frac{1}{2}\int_0^{t_1} \varphi_3\Big(\int_{T_1}^{t_1} 2\lambda f^{**}(s)g^{**}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\ &= \frac{1}{2}\mathrm{I}_1 + \frac{1}{2}\mathrm{I}_2. \end{aligned}$$

But since each function in $\Lambda_\varphi(w)$ is locally integrable, we get $f^{**}(T_1) < \infty$, $g^{**}(T_1) < \infty$, and thus

$$\mathrm{I}_2 \le \int_0^{t_1} \varphi_3\Big(2\lambda u_0\int_{T_1}^{t_1} g^{**}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t = \varphi_3\Big(2\lambda u_0\int_{T_1}^{t_1} g^{**}(s)\mathrm{d}s\Big)W(t_1) < \infty.$$

On the other hand, in view of the condition (+), (3.5), $I_{\varphi_2,w}(g) \le \min\big(1, \frac{1}{B_{\varphi_2}(w)}\big)$ and Jensen's inequality, we also know

$$\begin{aligned}
\mathrm{I}_1 &= \int_0^{t_1} \varphi_3\Big(\int_0^{T_1} 2\lambda f^{**}(s)g^{**}(s)\chi_{(t,T_1)}(s)\mathrm{d}s\Big)w(t)\mathrm{d}t \\
&\le \frac{1}{2}\int_0^{t_1} \varphi_3\Big(\int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)\varphi_3^{-1}(\varphi_2(g^{**}(s)\chi_{(t,T_1)}(s)))\mathrm{d}s\Big)w(t)\mathrm{d}t \\
&\quad + \frac{1}{2}\int_0^{t_1} \varphi_3\Big(\int_0^{T_1} \varphi_2(g^{**}(s)\chi_{(t,T_1)}(s))\varphi_3^{-1}\Big(\varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)\Big)\mathrm{d}s\Big)w(t)\mathrm{d}t \\
&\le \int_0^{t_1} \Big(\int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)\varphi_2(g^{**}(s)\chi_{(t,T_1)}(s))\mathrm{d}s\Big)w(t)\mathrm{d}t \\
&= \int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)\Big(\int_0^{s} \varphi_2(g^{**}(s))w(t)\mathrm{d}t\Big)\mathrm{d}s \\
&\le \int_0^{T_1} \varphi_1\Big(\frac{4\lambda}{k}f^{**}(s)\Big)\mathrm{d}s\Big(\int_0^{\infty} \varphi_2(g^{**}(t))w(t)\mathrm{d}t\Big) \\
&\le 1.
\end{aligned}$$

Now, we see that the right hand of (3.4) is less than infinity, which completes the proof.

(II) For this case, using [3, Lem. 5], we can assume condition (+) for a.a., and get the corresponding embedding by the same arguments as in (I).

Theorem 3.2 *Let G be nondiscrete and φ_i $(i = 1, 2, 3)$ satisfy condition (++) for l.a. if G is compact and (++) for a.a. if G is noncompact. Let $w \in B_{\varphi_i},\ i = 1, 2$ and $W(t) \ge C_1 t,\ \forall 0 < t < \mu(G)$, where C_1 is a positive constant. If φ_3 is finite, then $\Lambda_{\varphi_1,w} * \Lambda_{\varphi_2,w} \hookrightarrow E_{\varphi_3,w}$.*

Proof Suppose that G is compact. Take $f \in \Lambda_{\varphi_1,w},\ g \in \Lambda_{\varphi_2,w}$ such that $I_{\varphi_1,w}(f) \le \frac{C_1}{B_{\varphi_1}(w)}$, $I_{\varphi_2,w}(g) \le \frac{C_1}{B_{\varphi_2}(w)}$. Let $G_1 = \{x : \varphi_1(|f(x)|) \ge \delta\},\ G_2 = \{x : \varphi_2(|g(x)|) \ge \delta\}$, where δ is from the condition (++) and $a = \min(\mu(G_1), \mu(G_2))$. Let $\lambda > 0$. Then

$$\begin{aligned}
I_{\varphi_3,w}(\lambda f * g) &= \int_0^{\mu(G)} \varphi_3(\lambda(f * g)^*(t))w(t)\mathrm{d}t \\
&= \int_0^{\mu(G)} \varphi_3(\lambda(f\chi_{G_1} * g\chi_{G_2} + f\chi_{G\setminus G_1} * g + f\chi_{G_1} * g\chi_{G\setminus G_2})^*(t))w(t)\mathrm{d}t \\
&\le \int_0^{\mu(G)} \varphi_3\Big(\lambda(f\chi_{G_1} * g\chi_{G_2})^*\Big(\frac{t}{3}\Big) \\
&\quad + \lambda(f\chi_{G\setminus G_1} * g)^*\Big(\frac{t}{3}\Big) + \lambda(f\chi_{G_1} * g\chi_{G\setminus G_2})^*\Big(\frac{t}{3}\Big)\Big)w(t)\mathrm{d}t \\
&\le \frac{1}{3}\int_0^{\mu(G)} \varphi_3\Big(3\lambda(f\chi_{G_1} * g\chi_{G_2})^*\Big(\frac{t}{3}\Big)\Big)w(t)\mathrm{d}t \\
&\quad + \frac{1}{3}\int_0^{\mu(G)} \varphi_3\Big(3\lambda(f\chi_{G\setminus G_1} * g)^*\Big(\frac{t}{3}\Big)\Big)w(t)\mathrm{d}t \\
&\quad + \frac{1}{3}\int_0^{\mu(G)} \varphi_3\Big(3\lambda(f\chi_{G_1} * g\chi_{G\setminus G_2})^*\Big(\frac{t}{3}\Big)\Big)w(t)\mathrm{d}t \\
&= \frac{1}{3}\mathrm{I}_1 + \frac{1}{3}\mathrm{I}_2 + \frac{1}{3}\mathrm{I}_3.
\end{aligned}$$

First check I_2 and I_3.

$$\begin{aligned}
\mathrm{I}_2 &= 3\int_0^{\mu(G)} \varphi_3(3\lambda(f\chi_{G/G_1} * g\chi_{G_2})^*(s))w(3s)\chi_{(0,\frac{\mu(G)}{3})}(s)\mathrm{d}s \\
&\le 3\int_0^{\mu(G)} \varphi_3\Big(3\lambda\varphi_1^{-1}(\delta)\int_G |g|\Big)w(3s)\chi_{(0,\frac{\mu(G)}{3})}(s)\mathrm{d}s \\
&\le \varphi_3\Big(3\lambda\varphi_1^{-1}(\delta)\int_G |g|\Big)\int_0^{\mu(G)} w(s)\mathrm{d}s < \infty.
\end{aligned}$$

Analogously, $\mathrm{I}_3 < \infty$. On the other hand, if $a = 0$, then $\mathrm{I}_1 = 0$; if $a \neq 0$, then

$$\begin{aligned}
\mathrm{I}_1 &= 3\int_0^{\frac{\mu(G)}{3}} \varphi_3(3\lambda(f\chi_{G_1} * g\chi_{G_2})^*(s))w(3s)\mathrm{d}s \\
&\le 3\int_0^{\frac{\mu(G)}{3}} \varphi_3(3\lambda(f\chi_{G_1} * g\chi_{G_2})^{**}(s))w(3s)\mathrm{d}s \\
&\le 3\int_0^{\frac{\mu(G)}{3}} \varphi_3\Big(3\lambda\int_s^\infty (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&\le \frac{3}{2}\int_0^a \varphi_3\Big(6\lambda\int_s^a (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&\quad + \frac{3}{2}\int_0^a \varphi_3\Big(6\lambda\int_a^\infty (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&\quad + 3\int_a^{\frac{\mu(G)}{3}} \varphi_3\Big(3\lambda\int_s^\infty (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&= \frac{3}{2}\mathrm{J}_1 + \frac{3}{2}\mathrm{J}_2 + 3\mathrm{J}_3.
\end{aligned}$$

But $(f\chi_{G_1})^{**}(t) \le \frac{\int_{G_1}|f|}{t}$, $(g\chi_{G_2})^{**}(t) \le \frac{\int_{G_2}|g|}{t}$, so

$$\int_a^\infty (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t \le \int_a^\infty \frac{1}{t^2}\mathrm{d}t\int_{G_1}|f|\int_{G_2}|g| < \infty.$$

Thus $\mathrm{J}_2 < \infty$, likewise $\mathrm{J}_3 < \infty$.

On the other hand, due to

$$\begin{aligned}
&\varphi_1((f\chi_{G_1})^{**}(t)) \ge \varphi_1((f\chi_{G_1})^*(t)) > \delta, \\
&\varphi_2((g\chi_{G_1})^{**}(t)) \ge \varphi_1((g\chi_{G_1})^*(t)) > \delta, \quad \forall t \in [0,a),
\end{aligned}$$

by the condition $(++)$ we obtain

$$\begin{aligned}
\mathrm{J}_1 &\le \int_0^a \varphi_3\Big(6\lambda\int_s^a (f\chi_{G_1})^{**}(t)(g\chi_{G_2})^{**}(t)\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&\le \frac{1}{2}\int_0^a \varphi_3\Big(\int_s^a \varphi_1((f\chi_{G_1})^{**}(t))\varphi_3^{-1}(K\varphi_2((g\chi_{G_2})^{**}(t)))\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&\quad + \frac{1}{2}\int_0^a \varphi_3\Big(\int_s^a \varphi_2((g\chi_{G_2})^{**}(t))\varphi_3^{-1}(K\varphi_1((f\chi_{G_1})^{**}(t)))\mathrm{d}t\Big)w(3s)\mathrm{d}s \\
&= \frac{1}{2}\mathrm{J}_4 + \frac{1}{2}\mathrm{J}_5,
\end{aligned}$$

where K is from the condition $(++)$ for $\alpha = 12\lambda$. Since $W(t) \geq C_1 t,\ t \in [0, \mu(G)]$, by Hardy lemma (see [19]) we get

$$\int_s^a \varphi_1((f\chi_{G_1})^{**}(t))\mathrm{d}t \leq \int_0^{\mu(G)} \varphi_1(f^{**}(t))\mathrm{d}t \leq \frac{1}{C_1}\int_0^{\mu(G)} \varphi_1(f^{**}(t))w(t)\mathrm{d}t$$
$$\leq \frac{B_{\varphi_1}(w)}{C_1}\int_0^{\mu(G)} \varphi_1(f^{*}(t))w(t)\mathrm{d}t \leq 1. \tag{3.6}$$

In the same way, $\int_s^a \varphi_2((g\chi_{G_2})^{**}(t))\mathrm{d}t \leq 1$. Thus, by Jensen's inequality, (3.6), Hardy lemma, and $W \in \Delta_2$,

$$\begin{aligned}\mathrm{J}_4 + \mathrm{J}_5 &\leq K\int_0^a \int_s^a \varphi_1(f^{**}(t))\varphi_2(g^{**}(t))\mathrm{d}t w(3s)\mathrm{d}s\\ &\leq K\int_0^a \varphi_2(g^{**}(s))w(3s)\Big(\int_s^a \varphi_1(f^{**}(t))\mathrm{d}t\Big)\mathrm{d}s\\ &\leq CK < \infty,\end{aligned}$$

which ends the proof.

Lemma 3.2 *Let $W \in \Delta_2$. Then*

(i) *if $w \in L^1(G)$, S is dense in $\Lambda_{\varphi,w}$, where S is the set of the simple functions in G;*

(ii) *if $w \notin L^1(G)$, S_0 is dense in $\Lambda_{\varphi,w}$, where S_0 is the subset of S with support in a set of finite measures.*

Proof Similar to Theorem 2.3.11 and Theorem 2.3.12 of [30].

From now on, let the weight w in $\Lambda_{\varphi,w}$ satisfy $W \in \Delta_2$.

Let $f_s(x) = f(x-s)$. Then we have the following result.

Proposition 3.1 *If φ is finite, then for every $f \in \Lambda_{\varphi,w}$, the mapping $s \to f_s$ of G into $\Lambda_{\varphi,w}$ is continuous.*

Proof By Lemma 3.2, it is sufficient to show that for any simple function f, $s \to f_s$ is continuous. Let $f = \sum\limits_{i=1}^n k_i\chi_{E_i}$, and then $f_s = \sum\limits_{i=1}^n k_i\chi_{E_i+s}$. Now

$$\begin{aligned}I_{\varphi,w}(f_s - f) &= \int_0^\infty \varphi\Big(\Big[\sum_{i=1}^n k_i(\chi_{E_i+s} - \chi_{E_i})\Big]^*(t)\Big)w(t)\mathrm{d}t\\ &\leq \varphi\Big(\max_{1\leq i\leq n} k_i\Big)\Big[W\Big(\sum_{j=1}^n \lambda((E_j+s) \triangle E_j)\Big)\Big] \to 0 \quad (s \to 0),\end{aligned}$$

where $\triangle$ denotes the symmetric difference of sets. Then the following relation

$$I_{\varphi,w}(f_s - f) \to 0 \Leftrightarrow \|f_s - f\|_{\Lambda_{\varphi,w}} \to 0 \quad (s \to 0)$$

derives the result.

Proposition 3.2 *Let φ, $\widetilde{\varphi}$ be two Young functions and $\lim\limits_{u\to 0}\frac{\widetilde{\varphi}(u)}{u} > 0$. Let w be a weight on $\mathbf{R}^+$ which satisfies $W(t) \geq Ct,\ C > 0,\ w \in B_\varphi,\ w \in B_{\widetilde{\varphi}}$. Then there is an approximate identity $\{a_\alpha\}$ in $\Lambda_{\widetilde{\varphi},w}$ such that $\|a_\alpha\|_{\Lambda_{\widetilde{\varphi},w}} = 1$ and $f * a_\alpha \to f$ for every $f \in \Lambda_{\varphi,w}$.*

Proof Let $\{U_\alpha\}$ be a decreasing neighborhood system at the origin in G. For each α, there exists a non-negative continuous function a_α with support in U_α such that $\|a_\alpha\|_{\Lambda_{\widetilde{\varphi},w}} = 1$. Thus by the Hardy lemma (see [19]), the conditions $W(t) \geq Ct$ and $\lim\limits_{u\to 0} \frac{\widetilde{\varphi}(u)}{u} > 0$, we get $a_\alpha \in L^1(G)$ and $\|a_\alpha\|_{L^1(G)} \leq C$. Let $f \in \Lambda_{\varphi,w}$. Then by the condition $\lim\limits_{u\to 0} \frac{\widetilde{\varphi}(u)}{u} > 0$ again and Theorem 3.1, we get $f * a_\alpha \in \Lambda_{\varphi,w}$ and

$$\begin{aligned}\|f * a_\alpha - f\|_{\Lambda_{\varphi,w}} &= \Big\| \int_G \big(f_y(\cdot) - f(\cdot)\big) a_\alpha(y) \mathrm{d}\lambda(y) \Big\|_{\Lambda_{\varphi,w}} \\ &\leq C \int_G \|f_y - f\|_{\Lambda_{\varphi,w}} a_\alpha(y) \mathrm{d}\lambda(y) \\ &\leq C \sup_{y\in U_\alpha} \|f_y - f\|_{\Lambda_{\varphi,w}}.\end{aligned}$$

This shows by Proposition 3.1 that

$$\|f * a_\alpha - f\|_{\Lambda_{\varphi,w}} \to 0,$$

where the limit is taken over the net of α.

4 Tensor Products and Multipliers on Orlicz-Lorentz Spaces

In this section, we let G be a locally compact unimodular group (unless otherwise indicated). Set $\widetilde{f}(x) = f(x^{-1})$. If for two Young functions φ_i $(i = 1, 2)$ and a weight w on $\mathbf{R}^+$ there exists a Young function φ_3 which makes φ_i $(i = 1, 2, 3)$ satisfy condition (+) for l.a., if G is compact and (+) for a.a., if G is noncompact, $w \in B_{\varphi_i}$ $(i = 1, 2)$, and $w \geq c$ ($c > 0$ is a nonnegative constant), then in view of Theorem 3.1 we may define the bounded bilinear map d as

$$d(f, g) = \widetilde{f} * g, \quad f \in \Lambda_{\varphi_1,w},\ g \in \Lambda_{\varphi_2,w},$$

which lifts to a linear map naturally, D, from $\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}$ into $\Lambda_{\varphi_3,w}$. In addition, letting $w \in B_{\widetilde{\varphi}}$ and $\lim\limits_{u\to 0} \frac{\widetilde{\varphi}(u)}{u} > 0$, by Theorem 3.1 again, we can get that $\Lambda_{\varphi_1,w},\ \Lambda_{\varphi_2,w}$ can be looked as right $\Lambda_{\widetilde{\varphi},w}$-modules.

Definition 4.1 *The range of D, with the quotient norm, will be denoted by $A_{\varphi_1}^{\varphi_2}(w)$.*

According to the definition of $V \otimes_\gamma W$, $A_{\varphi_1}^{\varphi_2}(w)$ consists of exactly those functions h, on G at least one expansion of the form $h = \sum\limits_{i=1}^{\infty} \widetilde{f}_i * g_i$, where $f \in \Lambda_{\varphi_1,w},\ g \in \Lambda_{\varphi_2,w}$, and

$$\sum_{i=1}^{\infty} \|f_i\|_{\Lambda_{\varphi_1,w}} \|g\|_{\Lambda_{\varphi_2,w}} < \infty,$$

and for any $h \in A_{\varphi_1}^{\varphi_2}(w)$, the norm of h is

$$\|h\| = \inf \Big\{ \sum_{i=1}^{\infty} \|f_i\|_{\Lambda_{\varphi_1,w}} \|g_i\|_{\Lambda_{\varphi_2,w}};\ h = \sum_{i=1}^{\infty} \widetilde{f}_i * g_i,\ f \in \Lambda_{\varphi_1,w},\ g \in \Lambda_{\varphi_2,w} \Big\}.$$

It can be seen that $A_{\varphi_1}^{\varphi_2}(w)$ is a Banach space of functions.

Let K be the closed linear subspace of $\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}$ spanned by all elements of the form $(h * f) \otimes g - f \otimes (\widetilde{h} * g)$, where $f \in \Lambda_{\varphi_1,w},\ g \in \Lambda_{\varphi_2,w}$ and $h \in \Lambda_{\widetilde{\varphi},w}$. Then the $\Lambda_{\widetilde{\varphi},w}$-module tensor product $\Lambda_{\varphi_1,w} \otimes_{\Lambda_{\widetilde{\varphi},w}} \Lambda_{\varphi_2,w}$ is the quotient space $\big(\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}\big)/K$.

In the next, we need the following conditions for weight w on $\mathbf{R}^+$ and Young functions $\varphi_1,\ \varphi_2, \widetilde{\varphi}$.

(i) $w \geq c$ ($c > 0$ is a nonnegative constant).

(ii) $w \in B_{\varphi_i}$ $(i = 1, 2)$.

(iii) $w \in B_{\widetilde{\varphi}}$ and $\lim\limits_{u\to 0} \frac{\widetilde{\varphi}(u)}{u} > 0$.

(iv) There exists a Young function φ_3 which makes φ_i $(i = 1, 2, 3)$ satisfy condition (+) for l.a., if G is compact and (+) for a.a., if G is noncompact.

(v) $w \in B_{\varphi_{2*}}$ and $\underline{\lim}_{u\to 0^+} \frac{(\varphi_{2*})^{-1}(u)}{\varphi_1^{-1}(u)} > 0$.

Remark 4.1 By [6], if

$$E_2 = \Big\{(x, y) : x \geq 0,\ 0 \leq y \leq \frac{\varphi_1^{-1}(x)\varphi_2^{-1}(x)}{x}\Big\}$$

and $\overline{E}_2$ = convex closure of E_2, then $\overline{E}_2 \neq$ first quadrant can deduce (iv).

Theorem 4.1 *Let G be compact, weight w be nonincreasing and Young functions $\varphi_1,\ \varphi_2$, $\widetilde{\varphi}$ satisfy* (i)–(iv). *Then $\Lambda_{\varphi_1,w} \otimes_{\Lambda_{\widetilde{\varphi},w}} \Lambda_{\varphi_2,w}$ is isomorphic to the space $A^{\varphi_2}_{\varphi_1}(w)$.*

Proof It suffices to show that the kernel of D is exactly K. Since

$$D((h * f) \otimes g) = (h * f)\widetilde{\ } * g = \widetilde{f} * (\widetilde{h} * g) = D(f \otimes (\widetilde{h} * g))$$

(d is $\Lambda_{\widetilde{\varphi},w}$-balanced), the kernel of D contains K.

On the contrary, suppose that t is an element of the kernel of D. Then

$$t = \sum_{i=1}^{\infty} f_i \otimes g_i \quad \text{with} \ \sum_{i=1}^{\infty} \|f_i\|_{\Lambda_{\varphi_1,w}} \|g\|_{\Lambda_{\varphi_2,w}} < \infty$$

and

$$\sum_{i=1}^{\infty} \widetilde{f}_i * g_i = 0,$$

where the summation converges in $\Lambda_{\varphi_3,w}$. Let $\{\psi_n\}$ be an approximate identity of $\Lambda_{\widetilde{\varphi},w}$ satisfying the condition in Proposition 3.2. For each n, define $t_n \in \Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}$ by

$$t_n = \sum_{i=1}^{\infty} (f_i * \psi_n) \otimes g_i.$$

Then, from Proposition 3.2, $f_i * \psi_n$ converges to f_i for each i, and by this one can prove that t_n converges to t in $\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}$. Now given n, s, and $\epsilon > 0$, choose m_0 such that

$$\Big\| \sum_{i=1}^{m} \widetilde{f}_i * g_i \Big\|_{\Lambda_{\varphi_3,w}} \leq \frac{\epsilon}{2\|\psi_n\|_{\Lambda_{\varphi_1,w}}}, \quad \forall m > m_0.$$

Choose $m_1 > m_0$ so that

$$\Big\| t_n - \sum_{i=1}^{m} (f_i * \psi_n) \otimes g_i \Big\| < \frac{\epsilon}{2}, \quad \forall m > m_1.$$

We observe that the second term on the right side of the following equality

$$\sum_{i=1}^{m}(f_i*\psi_n)\otimes g_i=\sum_{i=1}^{m}\psi_n\otimes(\widetilde{f}_i*g_i)+\sum_{i=1}^{m}[(f_i*\psi_n)\otimes g_i-\psi_n\otimes(\widetilde{f}_i*g_i)]$$

is in K and can be written

$$\Big\|\sum_{i=1}^{m}\psi_n\otimes(\widetilde{f}_i*g_i)\Big\|=\|\psi_n\|_{\Lambda_{\varphi_1,w}}\Big\|\sum_{i=1}^{m}\widetilde{f}_i*g_i\Big\|_{\Lambda_{\varphi_2,w}}$$

by the definition of the cross norm. Let $\varphi(y)=\sup\limits_{x\geq 0}[\varphi_2(xy)-\varphi_3(x)]$. Since $\mu(G)<\infty,\ 1\in\Lambda_{\varphi,w}$, similar to proving the theorem of Andô [31, Thm. 4, 6, Thm. 2.3], we easily get

$$\Lambda_{\varphi_3,w}\subset\Lambda_{\varphi_2,w}.$$

Thus

$$\begin{aligned}\Big\|\sum_{i=1}^{m}\psi_n\otimes(\widetilde{f}_i*g_i)\Big\|&\leq C\|\psi_n\|_{\Lambda_{\varphi_1,w}}\Big\|\sum_{i=1}^{m}\widetilde{f}_i*g_i\Big\|_{\Lambda_{\varphi_3,w}}\\&\leq C\|\psi_n\|_{\Lambda_{\varphi_1,w}}\frac{\epsilon}{2\|\psi_n\|_{\Lambda_{\varphi_1,w}}}=\frac{C\epsilon}{2}.\end{aligned}$$

Then the distance from t_n to K is less than $\frac{(1+C)\epsilon}{2}$ for every $\epsilon>0$, and so $t_n\in K$. For K is closed, $t\in K$.

Lemma 4.1 *Suppose that weight w is on $\mathbf{R}^+$ and Young functions φ_1, φ_2, $\widetilde{\varphi}$ satisfy* (i)–(iii) *and* (v). *Let $\varphi\in C_c(G)$ and define $T_\varphi f=f*\varphi$ for $f\in\Lambda_{\varphi_1,w}$. Then*

$$T_\varphi\in\mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w},(\Lambda_{\varphi_2,w})^*).$$

Proof It is obvious that $\varphi\in\Lambda_{\psi,w}$ with ψ a Young function. By [26], since $w\geq c$ ($c>0$ is a nonnegative constant), there holds

$$(\Lambda_{\varphi_2,w})^*=M_{\varphi_{2*},w},\tag{4.1}$$

where $\|f\|_{M_{\varphi_{2*},w}}=\|S(f^*)\|_{L_{\varphi_{2*},w}}$, $S(f)=\frac{\int_0^t f(s)\mathrm{d}s}{W(t)}$. In view of $\lim\limits_{u\to 0^+}\frac{(\varphi_{2*})^{-1}(u)}{\varphi_1^{-1}(u)}>0$ and a simple fact that the function $\frac{u(\varphi_{2*})^{-1}(u)}{\varphi_1^{-1}(u)}$ is increasing, there exists a Young function φ such that

$$\varphi_1^{-1}(u)\varphi^{-1}(u)\leq u(\varphi_2^*)^{-1}(u),\quad\forall u\geq 0,$$

i.e., $\varphi_1,\ \varphi,\ \varphi_2^*$ satisfy (+) a.a. So by Theorem 3.1 and (4.1), it follows that if $f\in\Lambda_{\varphi_1,w}$, then

$$f*\varphi\in\Lambda_{\varphi_{2*},w}\subset M_{\varphi_{2*},w}=(\Lambda_{\varphi_2,w})^*,$$

and T_φ is a bounded linear operator from $\Lambda_{\varphi_1,w}$ to $\big(\Lambda_{\varphi_2,w}\big)^*$. On the other hand,

$$T_\varphi(g*f)=(g*f)*\varphi=g*(f*\varphi)=g*T_\varphi f$$

for all $f\in\Lambda_{\varphi_1,w}$ and $g\in\Lambda_{\widetilde{\varphi},w}$, which ends the proof.

The above lemma induces the following concept.

Definition 4.2 *A locally compact unimodular group G is said to satisfy the property $P_{\widetilde{\varphi}}^{\varphi_1,\varphi_2}(w)$, if every element of* $\mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w},(\Lambda_{\varphi_2,w})^*)$ *can be approximated in the ultraweak $*$-operator topology by operators $T_\varphi,\ \varphi\in C_c(G)$.*

Theorem 4.2 *Let G be a locally compact unimodular group, w be a weight on $\mathbf{R}^+$ and Young functions $\varphi_1,\ \varphi_2,\ \widetilde{\varphi}$ satisfy* (i)–(v)*. Then the following statements are equivalent:*

(A) *G satisfies the property $P_{\widetilde{\varphi}}^{\varphi_1,\varphi_2}(w)$.*

(B) *The kernel of D is K such that*

$$\Lambda_{\varphi_1,w}\otimes_{\Lambda_{\widetilde{\varphi},w}}\Lambda_{\varphi_2,w}\cong A_{\varphi_1}^{\varphi_2}(w).$$

Proof Suppose that G satisfies the property $P_{\widetilde{\varphi}}^{\varphi_1,\varphi_2}(w)$. Since $K\subset \mathrm{Ker}\ D$, to show that $\mathrm{Ker}\,D=K$, it is enough to show that $\mathrm{Ker}\,D\subset K$. In other words, it suffices to show that by the Banach theorem, any bounded linear functional on $\Lambda_{\varphi_1,w}\otimes_{\Lambda_{\widetilde{\varphi},w}}\Lambda_{\varphi_2,w}$ which annihilates K also annihilates $\mathrm{Ker}\,D$. By (1.1), we know

$$\mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w},(\Lambda_{\varphi_2,w})^*)\cong(\Lambda_{\varphi_1,w}\otimes_{\Lambda_{\widetilde{\varphi},w}}\Lambda_{\varphi_2,w})^*. \tag{4.2}$$

It can be seen from this that if F is a linear functional that annihilates K, there is an operator $T\in \mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w},(\Lambda_{\varphi_2,w})^*)$ corresponding to F, such that

$$\langle t,F\rangle=\sum_{i=1}^{\infty}\langle g_i,Tf_i\rangle \tag{4.3}$$

for all $t\in\Lambda_{\varphi_1,w}\otimes_{\Lambda_{\widetilde{\varphi},w}}\Lambda_{\varphi_2,w}$ with expansion

$$t=\sum_{i=1}^{\infty}f_i\otimes g_i. \tag{4.4}$$

Suppose that $t\in\mathrm{Ker}\,D$ and has the form (4.4). Then

$$\sum_{i=1}^{\infty}\widetilde{f_i}*g_i=0,$$

the summation converging in the norm of $\Lambda_{\varphi_3,w}$. We will show that $\langle t,F\rangle=0$, or equivalently, by (4.3),

$$\sum_{i=1}^{\infty}\langle g_i,Tf_i\rangle=0. \tag{4.5}$$

Since G is assumed to satisfy the property $P_{\widetilde{\varphi}}^{\varphi_1,\varphi_2}(w)$, there is a net $\{\psi_j:j\in I\}$ of $C_c(G)$ such that the operators T_{ψ_j} converge to T in the ultraweak $*$-operator topology. Thus

$$\sum_{i=1}^{\infty}\langle g_i,Tf_i\rangle=\lim_{j\in I}\sum_{i=1}^{\infty}\langle g_i,T_{\psi_j}f_i\rangle=\lim_{j\in I}\sum_{i=1}^{\infty}\langle g_i,f_i*\psi_j\rangle.$$

So, to check (4.5), it is enough to show that

$$\sum_{i=1}^{\infty}\langle g_i,f_i*\psi_j\rangle=0,\quad \forall j\in I.$$

Since $\psi_j \in C_c(G) \subset (\Lambda_{\varphi_3,w})^*$, we deduce that

$$\sum_{i=1}^{\infty} \langle g_i, f_i * \psi_j \rangle = \sum_{i=1}^{\infty} \langle \widetilde{f}_i * g_i, \psi_j \rangle = \Big\langle \sum_{i=1}^{\infty} \widetilde{f}_i * g_i, \psi_j \Big\rangle = 0.$$

This implies that

$$\Lambda_{\varphi_1,w} \otimes_{\Lambda_{\widetilde{\varphi},w}} \Lambda_{\varphi_2,w} \cong A_{\varphi_1}^{\varphi_2}(G,w).$$

Suppose conversely that $\operatorname{Ker} D = K$. To show that the operators of the form T_φ of $\varphi \in C_c(G)$ are dense in

$$\mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w}, (\Lambda_{\varphi_2,w})^*)$$

in the ultraweak $*$-operator topology, we only need to show that the corresponding functionals are dense in $(\Lambda_{\varphi_1,w} \otimes_{\Lambda_{\widetilde{\varphi},w}} \Lambda_{\varphi_2,w})^*$ in the weak $*$-topology. Furthermore, it is sufficient to show that if these functionals, say N, are viewed as functionals on $\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w}$, then their annihilators $N^\perp = K$. Since

$$\Lambda_{\varphi_1,w} \otimes_{\Lambda_{\widetilde{\varphi},w}} \Lambda_{\varphi_2,w} = (\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w})/K$$

and

$$((\Lambda_{\varphi_1,w} \otimes_\gamma \Lambda_{\varphi_2,w})/K)^* \cong K^\perp$$

(see [32]), we have $N \subset K^\perp$. So $K \subset N^\perp$. Due to the assumption that $\operatorname{Ker} D = K$, we only need to prove that $N^\perp \subset \operatorname{Ker} D$.

Now, let $t \in N^\perp$. Then $\langle t, F \rangle = 0$ for all $F \in N$ and there exist $f_i \in \Lambda_{\varphi_1,w},\ g_i \in \Lambda_{\varphi_2,w}$, so that

$$t = \sum_{i=1}^{\infty} f_i \otimes g_i, \quad \sum_{i=1}^{\infty} \|f_i\|_{\Lambda_{\varphi_1,w}} \|g\|_{\Lambda_{\varphi_2,w}} < \infty.$$

For any $F \in N$, there is an operator $T_\varphi \in \mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w}, (\Lambda_{\varphi_2,w})^*)$ corresponding to F such that

$$\begin{aligned} \langle t, F \rangle = \langle t, T_\varphi \rangle &= \sum_{i=1}^{\infty} \langle g_i, T_\varphi f_i \rangle \\ &= \sum_{i=1}^{\infty} \langle \widetilde{f}_i * g_i, \varphi \rangle = \Big\langle \sum_{i=1}^{\infty} \widetilde{f}_i * g_i, \varphi \Big\rangle = 0. \end{aligned}$$

It follows that

$$D(t) = \sum_{i=1}^{\infty} \widetilde{f}_i * g_i = 0,$$

that is, $N^\perp \subset \operatorname{Ker} D$. This proves the theorem.

Then by (4.1)–(4.2), we have the following result.

Corollary 4.1 *Let G be a locally compact unimodular group, w be a weight on $\mathbf{R}^+$, Young functions φ_1, φ_2, $\widetilde{\varphi}$ satisfy* (i)–(v)*, and G satisfy the property $P_{\widetilde{\varphi}}^{\varphi_1,\varphi_2}(w)$. Then*

$$\mathrm{Hom}_{\Lambda_{\widetilde{\varphi},w}}(\Lambda_{\varphi_1,w}, M_{\varphi_{2*},w}) \cong (A_{\varphi_1}^{\varphi_2}(G,w))^*. \tag{4.6}$$

In the following, we illustrate the convolutions, tensor products, multipliers of Orlicz-Lorentz spaces $\Lambda^{p,q}_{\varphi,w}$. Let $0 < p, q < \infty$, w be a weight on $\mathbf{R}^+$ and φ be a Young function, and define the Orlicz-Lorentz space as $\Lambda^{p,q}_{\varphi,w} = \Lambda_{\varphi_0,w_0}$, where $\varphi_0 = \varphi^q,\ w_0 = W^{\frac{q}{p}-1}w$. Simultaneously, we introduce a modular of $\mathcal{M}(G,\mu)$ as

$$\Theta_{\varphi,w}(f) = \int_0^\infty \varphi(f^{**}(x))w(x)\mathrm{d}x,$$

and the modular space as

$$\Gamma_{\varphi,w} = \{f \in \mathcal{M}(G,\mu) :\ \exists\ \lambda > 0,\ \text{such that}\ \Theta_{\varphi,w}(\lambda f) < \infty\},$$

which induces

$$\|f\|_{\Gamma_{\varphi,w}} = \inf\Big\{\lambda > 0 : \Theta_{\varphi,w}\Big(\frac{f}{\lambda}\Big) < 1\Big\}.$$

Additionally, let $\Theta^{p,q}_{\varphi,w}(f) = \Theta_{\varphi_0,w_0}(f),\ \Gamma^{p,q}_{\varphi,w} = \Gamma_{\varphi_0,w_0}$, where $\varphi_0 = \varphi^q,\ w_0 = W^{\frac{q}{p}-1}w$.

The next theorem needs a certain generalized Hardy-type inequality (see [33]). In [33, Thm. 1.7], Bloom and Kerman give the sufficient and necessary condition of establishing the weighted integral inequality:

$$\varphi_2^{-1}\Big(\int \varphi_2(w(x)|Tf(x)|)t(x)\mathrm{d}x\Big) \le \varphi_1^{-1}\Big(\int \varphi_1(Cu(x)|f(x)|)v(x)\mathrm{d}x\Big),$$

where φ_1, φ_2 are N-functions (an N-function φ is a continuous Young function such that $\varphi(x) = 0$ if and only if $x = 0$ and $\lim\limits_{x\to 0}\frac{\varphi(x)}{x} = 0$, $\lim\limits_{x\to 0}\frac{\varphi(x)}{x} = +\infty$) and T is a generalized Hardy operator. We need the special form of the above inequality:

$$\int_0^\infty \varphi^s\Big(\Big|\int_0^t f(x)\mathrm{d}x\Big|\Big)\frac{1}{t^{\frac{s}{r}+1}}\mathrm{d}t \le \int_0^\infty \varphi^s(C|f(x)|)\frac{\varphi^s(t)}{t^{\frac{s}{r}+1}}\mathrm{d}t, \tag{4.7}$$

which holds if and only if there exists a positive constant D such that

$$\int_0^x (\varphi^s)_*\Big(\frac{\varphi^s(\lambda)y^{\frac{s}{r}+1}}{D\lambda\varphi^s(y)x^{\frac{s}{r}}}\Big)\frac{\varphi^s(y)}{y^{\frac{s}{r}+1}}\mathrm{d}y \le \varphi^s(\lambda)x^{-\frac{s}{r}}, \quad \forall x, \lambda > 0. \tag{4.8}$$

When $\varphi(t) = t$, (4.7) is a classical Hardy inequality (see [19]) and it is obvious that (4.8) always holds.

Now we have the following theorem.

Theorem 4.3 *Let G be a unimodular locally compact group, T be a convolution operator $k = T(f,g) = f * g$. Suppose that the following conditions hold:* (a$_1$) $\frac{1}{p_1} + \frac{1}{p_2} > 1$; (a$_2$) $w \ge c_1 > 0$, *where c_1 is a constant;* (a$_3$) $\varphi(t) \le c_2 t$, *where c_2 is a constant, $\varphi \in \Delta'$;* (a$_4$) $\varphi^{q_1},\ \varphi^{q_2}$ *are N-functions. Then there exists a constant C such that*

$$\Theta^{r,s}_{\varphi,w}(k) \le C[\Theta^{p_1,mq_1}_{\varphi,w}(f)]^{\frac{s}{mq_1}}[\Theta^{p_2,mq_2}_{\varphi,w}(g)]^{\frac{s}{mq_2}}, \quad \forall f \in \Gamma^{p_1,q_1}_{\varphi,w},\ g \in \Gamma^{p_2,q_2}_{\varphi,w},$$

where $\frac{1}{p_1} + \frac{1}{p_2} = 1 + \frac{1}{r}$, $\frac{s}{q_1} + \frac{s}{q_2} = m$ and $s \ge 1$ is a number such that $\frac{1}{q_1} + \frac{1}{q_2} \ge \frac{1}{s}$ and the inequality (4.8) *holds. In particular, if $f \in \Gamma^{p_1,q_1}_{\varphi,w}$, $g \in \Gamma^{p_2,q_2}_{\varphi,w}$, then $k \in \Gamma^{r,s}_{\varphi,w}$.*

Proof Since (see [1–2])

$$(f * g)^{**}(t) \le \int_t^\infty f^{**}(u)g^{**}(u)\mathrm{d}u,$$

we get

$$\Theta^{r,s}_{\varphi,w}(k) \le \int_0^\infty \varphi^s\Big(\int_x^\infty f^{**}(t)g^{**}(t)\mathrm{d}t\Big)W(x)^{\frac{s}{r}-1}w(x)\mathrm{d}x.$$

For convenience, let $f^{**}g^{**} = h$. Then

$$\begin{aligned}\Theta^{r,s}_{\varphi,w}(k) &\le \int_0^\infty \varphi^s\Big(\int_x^\infty h(t)\mathrm{d}t\Big)W(x)^{\frac{s}{r}-1}w(x)\mathrm{d}x\\ &= \int_0^\infty \varphi^s\Big[\int_{W^{-1}(\frac{1}{y})}^\infty h(t)\mathrm{d}t\Big]\frac{\mathrm{d}y}{y^{\frac{s}{r}+1}}\\ &= \int_0^\infty \varphi^s\Big[\int_0^y h\Big(W^{-1}\Big(\frac{1}{u}\Big)\Big)\frac{1}{u^2 w\Big(W^{-1}\Big(\frac{1}{u}\Big)\Big)}\mathrm{d}u\Big]\frac{\mathrm{d}y}{y^{\frac{s}{r}+1}}.\end{aligned}$$

And by (4.8) it follows that the right part of the last inequality

$$\begin{aligned}&\int_0^\infty \varphi^s\Big[\int_0^y h\Big(W^{-1}\Big(\frac{1}{u}\Big)\Big)\frac{1}{u^2 w\Big(W^{-1}\Big(\frac{1}{u}\Big)\Big)}\mathrm{d}u\Big]\frac{\mathrm{d}y}{y^{\frac{s}{r}+1}}\\ &\le C\int_0^\infty \varphi^s\Big[h\Big(W^{-1}\Big(\frac{1}{y}\Big)\Big)\frac{1}{y^2 w\Big(W^{-1}\Big(\frac{1}{y}\Big)\Big)}\Big]\frac{\varphi^s(y)}{y^{\frac{s}{r}+1}}\mathrm{d}y.\end{aligned}$$

Now letting $y = \frac{1}{W(x)}$ and noticing $w(t) \ge c > 0$, $\varphi \in \Delta'$ and $\varphi(t) \le Ct$, we have

$$\begin{aligned}\Theta^{r,s}_{\varphi,w}(k) &\le C\int_0^\infty [\varphi(W^2(x)h(x))]^s\frac{\varphi^s\Big(\frac{1}{W(x)}\Big)W(x)^{\frac{s}{r}}w(x)}{W(x)}\mathrm{d}x\\ &\le C\int_0^\infty [\varphi(f^{**}(x))\varphi(g^{**}(x))W(x)^{\frac{1}{r}+1}]^s\frac{w(x)}{W(x)}\mathrm{d}x\\ &= C\int_0^\infty \frac{[W(x)^{\frac{1}{p_1}}\varphi(f^{**}(x))]^s}{W(x)^{\frac{s}{mq_1}}}\frac{[W(x)^{\frac{1}{p_2}}\varphi(g^{**}(x))]^s}{W(x)^{\frac{s}{mq_2}}}w(x)\mathrm{d}x.\end{aligned}$$

By Hölder inequality, it is easy to get that

$$\begin{aligned}\Theta^{r,s}_{\varphi,w}(k) &\le C\Big[\int_0^\infty \frac{[W(x)^{\frac{1}{p_1}}\varphi(f^{**}(x))]^{mq_1}}{W(x)}w(x)\mathrm{d}x\Big]^{\frac{s}{mq_1}}\\ &\quad\cdot\Big[\int_0^\infty \frac{[W(x)^{\frac{1}{p_2}}\varphi(g^{**}(x))]^{mq_2}}{W(x)}w(x)\mathrm{d}x\Big]^{\frac{s}{mq_2}}\\ &= C[\Theta^{p_1,mq_1}_{\varphi,w}(f)]^{\frac{s}{mq_1}}[\Theta^{p_2,mq_2}_{\varphi,w}(g)]^{\frac{s}{mq_2}}.\end{aligned}$$

On the other hand, we can get that $\Gamma^{p_1,q_1}_{\varphi,w} \subset \Gamma^{p_1,mq_1}_{\varphi,w}$ and $\Gamma^{p_2,q_2}_{\varphi,w} \subset \Gamma^{p_2,mq_2}_{\varphi,w}$ (one can take the same method which is used for the proof of $\Lambda^{r,s_1}_G(w) \subset \Lambda^{r,s_2}_G(w)$ if $s_1 < s_2$ (see [30])). Now the lemma is proved.

Remark 4.2 In view of the above theorem, we obtain

$$\Gamma^{p_1,q_1}_{\varphi,w} * \Gamma^{p_2,q_2}_{\varphi,w} \hookrightarrow \Gamma^{r,s}_{\varphi,w},$$

if $\varphi,\ w,\ p_1,\ p_2,\ q_1,\ q_2,\ r,\ s$ satisfy the conditions in the above theorem. At the same time, if there exists a constant $C_0 > 0$ such that

$$\Theta^{p,q}_{\varphi,w}(f) \le C_0 I^{p,q}_{\varphi,w}(f)$$

for any $0 < p, q < \infty$, where $I^{p,q}_{\varphi,w}(f) = I_{\varphi_0,w_0}(f)$, $\varphi_0 = \varphi^q$, $w_0 = W^{\frac{q}{p}-1}w$ and $I_{\varphi,w}(f)$ is defined in Section 2, then we get

$$\Lambda^{p_1,q_1}_{\varphi,w} * \Lambda^{p_2,q_2}_{\varphi,w} \hookrightarrow \Lambda^{r,s}_{\varphi,w}, \tag{4.9}$$

and $\Lambda^{p,q}_{\varphi,w}$ is $\Lambda_{\varphi,w}$-module. Especially, if $\varphi(t) = t$, $w(t) = 1$, $1 \le p_1, p_2 < \infty$, $1 < q_1, q_2 < \infty$ (obviously they satisfy the preceding conditions), then (4.9) is

$$L^{p_1,q_1} * L^{p_2,q_2} \hookrightarrow L^{r,s}.$$

Additionally, using (4.9) and the same method in Theorem 4.1, Theorem 4.3 and Corollary 4.1, we can get the representations of the tensor products and multipliers on $\Lambda^{p,q}_{\varphi,w}$, which contain the result of [13].

References

[1] O'Neil, R., Convolution operators and $L(p, q)$ spaces, *Duke Math. J.*, **30**, 1963, 129–142.

[2] Yap, L. Y. H., Some remarks on convolution operators and $L(p, q)$ spaces, *Duke Math. J.*, **36**, 1969, 647–658.

[3] Kamińska, A. and Musielak, J., On convolution operator in Orlicz spaces. Congress on Functional Analysis, Madrid, 1988, *Rev. Mat. Univ. Complut. Madrid*, **2**, 1989, 157–178.

[4] Hewitt, E. and Ross, K. A., Abstract Harmonic Analysis, Vol. I, Springer-Verlag, Berlin, 1963.

[5] Hudzik, H., Kamińska, A. and Musielak, J., On Some Banach Algebras Given by a Modular, A. Haar memorial conference, Vol. I, II, Budapest, 1985, 445–463, Colloq. Math. Soc. Jnos Bolyai, **49**, North-Holland, Amsterdam, 1987.

[6] O'Neil, R., Fractional integration in Orlicz spaces, I., *Trans. Amer. Math. Soc.*, **115**, 1965, 300–328.

[7] Zelazko, W., On the algebras L_p of locally compact groups, *Colloq. Math.*, **8**, 1961, 115–120.

[8] Rieffel, M. A., Induced Banach representations of Banach algebras and locally compact groups, *J. Functional Analysis*, **1**, 1967, 443–491.

[9] Rieffel, M. A., Multipliers and tensor products of L^p-spaces of locally compact groups, *Studia Math.*, **33**, 1969, 71–82.

[10] Grothendieck, A., Produits tensoriels topologiques et espaces nuclaires, *Mem. Amer. Math. Soc.*, **16**(16), 1955, 140 pp.

[11] Bonsall, F. F. and Duncan, J., Complete Normed Algebras, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 80, Springer-Verlag, New York-Heidelberg, 1973.

[12] Schatten, R., A Theory of Cross-Spaces, Annals of Mathematics Studies, **26**, Princeton University Press, Princeton, N. J., 1950.

[13] Avci, H. and Gürkanli, A. T., Multipliers and tensor products of $L(p, q)$ Lorentz spaces, *Acta Math. Sci. Ser. B Engl. Ed.*, **27**, 2007, 107–116.

[14] Kerlin, J. E., Representations of generalized multipliers of L^p-spaces of locally compact groups, *Bull. Amer. Math. Soc.*, **79**, 1973, 1223–1228.

[15] Candeal Haro, J. C. and Lai, H. C., Multipliers in vector-valued function spaces under convolution, *Acta Math. Hungar.*, **67**(3), 1995, 175–192.

[16] Öztop, S., Multipliers of Banach valued weighted function spaces, *Int. J. Math. Math. Sci.*, **24**(8), 2000, 511–517.

[17] Özto, S. and Gürkanli, A. T., Multipliers and tensor products of weighted L^p-spaces, *Acta Math. Sci. Ser. B Engl. Ed.*, **21**(1), 2001, 41–49.

[18] Sağir, B., Multipliers and tensor products of vector valued $L^p(G, A)$ spaces, *Taiwanese J. Math.*, **7**(3), 2003, 493–501.

[19] Bennett, C. and Sharpley, R., Interpolation of operators, Pure and Applied Mathematics, **129**, Academic Press, Inc., Boston, MA, 1988, xiv+469 pp.

[20] Mastyło, M., Interpolation of linear operators in Calderón-Lozanovskiĭ spaces, *Comment. Math. Prace Mat.*, **26**(2), 1986, 247–256.

[21] Maligranda, L., Indices and interpolation, *Dissert. Math.*, **234**, 1984, 1–49.

[22] Orlicz, W., Über eine gewisse Klasse von Räumen vom Typus B, Bull, *Intern. Acad. Pol.*, **8**, 1932, 207–220.

[23] Luxemburg, W. A. J., Banach function spaces, Thesis, Delft Technical Univ., Delft, 1955.

[24] Lorentz, G. G., Some new functional spaces, *Ann. of Math.*, **51**, 1950, 37–55.

[25] Lorentz, G. G., On the theory of spaces Λ, *Pacific J. Math.*, **1**, 1951, 411–429.

[26] Kamińska, A. and Mastyło, M., Abstract duality Sawyer formula and its applications, *Monatsh. Math.*, **151**(3), 2007, 223–245.

[27] Li, H. L., Hardy-type inequalities on strong and weak Orlicz-Lorentz spaces, *Sci. China Math.*, **55**(12), 2012, 2493–2505.

[28] Ariño, M. A. and Muckenhoupt, B., Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for nonincreasing functions, *Trans. Amer. Math. Soc.*, **320**, 1990, 727–735.

[29] Sawyer, E., Boundedness of classical operators on classical Lorentz spaces, *Studia Math.*, **96**, 1990, 145–158.

[30] Carro, M. J., Raposo, J. A. and Soria, J., Recent developements in the theory of Lorentz spaces and weighted inequalities, Mem. Amer. Math. Soc., **187**, Amer. Math. Soc., Providence, RI, 2007.

[31] Andô, T., On products of Orlicz spaces, *Math. Ann.*, **140**, 1960, 174–186.

[32] Conway, J. B., A Course in Functional Analysis, Springer-Verlag, New York, 1985.

[33] Bloom, S. and Kerman, R., Weighted L_φ integral inequalities for operators of Hardy type, *Studia Math.*, **110**(1), 1994, 35–52.