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On CAP Representations for Even Orthogonal Groups I: A Correspondence of Unramified Representations*

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Abstract The authors prove the local unramified correspondence for a new type of construction of CAP representations of even orthogonal groups by a generalized automorphic descent method. This method is expected to work for all classical groups.

Keywords Automorphic representations, Fourier coefficients, Satake parameter transfer
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1 Introduction

Let π be an irreducible, automorphic, cuspidal representation of $\mathrm{SO}_{4n+2k}(\mathbb{A})$, where $\mathbb{A}$ is the Adele ring of a number field F, and SO_{4n+2k} is the split special orthogonal group in $4n+2k$ variables, regarded as an algebraic group over F. Langlands functoriality predicts that π lifts to an irreducible automorphic representation Π of $\mathrm{GL}_{4n+2k}(\mathbb{A})$. Moreover, Π is expected to lie inside a parabolic induction from a tensor product of Speh blocks (see [1]). In particular, there will be irreducible, automorphic, cuspidal representations τ of $\mathrm{GL}_{n_\tau}(\mathbb{A})$, such that $L^S(\pi\times\tau,s)$ (the partial L-function) has a pole at a positive half integer. For example, if $n_\tau = 4n+2k$, then this pole should be at $s=1$, and one expects that $\Pi=\tau$. Otherwise, when $n_\tau < 4n+2k$ and the largest (real) pole of $L^S(\pi\times\tau,s)$ is strictly larger than 1, one expects that π should be a CAP representation, with τ figuring in the cuspidal data of Π.

In this paper and its sequel (see [6]), we will consider π with the property that there exists an irreducible, automorphic, cuspidal representation τ of $\mathrm{GL}_{2n}(\mathbb{A})$, such that $L^S(\pi\times\tau,s)$ has a pole at $s=\frac{3}{2}$, and $L^S(\pi\times\tau,s)$ is holomorphic at $\Re(s)>\frac{3}{2}$, that is, $s=\frac{3}{2}$ is the right most pole of $L^S(\pi\times\tau,s)$. The conjecture is then that Π is nearly equivalent to an irreducible automorphic representation, which is parabolically induced from $\Delta(\tau,2)\otimes\Pi'$, where $\Delta(\tau,2)$ is a Speh block of length two (this is an irreducible, square integrable representation of $\mathrm{GL}_{4n}(\mathbb{A})$) and Π' is an irreducible, automorphic representation of $\mathrm{GL}_{2k}(\mathbb{A})$, lifted from an irreducible, automorphic, cuspidal representation σ of $\mathrm{SO}_{2k}(\mathbb{A})$. We will add one more assumption on π, namely,

$$\mathcal{O}(\pi) = [(2n+2k-1)(2n+1)],$$

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where $\mathcal{O}(\pi)$ is the set of all unipotent orbits $\mathcal{O}$ in SO_{4n+2k}, such that π has a non-trivial Fourier coefficient corresponding to $\mathcal{O}$, and for all $\mathcal{O}' > \mathcal{O}$, π has no non-trivial Fourier coefficients corresponding to $\mathcal{O}'$ (see [8] and [11, Section 4] for more detailed discussions of these notions). We remark that in the spirit of Conjecture 4.1 in [11], $[(2n+2k-1)(2n+1)]$ catches the maximal possible partition of $4n+2k$, whose corresponding unipotent orbits in SO_{4n+2k} support non-zero Fourier coefficients for the irreducible cuspidal automorphic representations π of $\mathrm{SO}_{4n+2k}(\mathbb{A})$ with the above L-function conditions. With this additional assumption, we will be able to prove that σ above can be taken to be (cuspidal and) generic. Moreover, we can construct such a representation σ by use of a generalized descent construction, as is called the endoscopy descent in [11], starting with π and τ as above. We remark that the work of Arthur on the endoscopy classification of the discrete spectrum in [1] provides each cuspidal automorphic representation π a global Arthur parameter, while the work presented here and in [11] in general is to give (to construct) explicitly the corresponding global Arthur parameter for π based on the conditions on L-functions and on Fourier coefficients (i.e., invariants) attached to π.

Let us outline the construction of σ. The full details will appear in a sequel to this paper (see [6]), where we will also show that τ above must be of symplectic type, namely, its exterior square L-function has a pole at $s = 1$. Then it follows that the Eisenstein series on (split) $\mathrm{SO}_{4n(k+1)}(\mathbb{A})$ corresponding to the parabolic induction from $\Delta(\tau, k+1)|\det\cdot|^s$ has a simple pole at $s = \frac{k+1}{2}$. Denote the residual representation by Θ_τ. The representation $\Delta(\tau, k+1)$ is the Speh block (corresponding to τ) of length $k+1$, which is the residual representation of $\mathrm{GL}_{2n(k+1)}(\mathbb{A})$ generated by the residues of the Eisenstein series on $\mathrm{GL}_{2n(k+1)}(\mathbb{A})$ corresponding to the parabolic induction from

$$\tau|\det\cdot|^{\frac{k}{2}} \otimes \tau|\det\cdot|^{\frac{k}{2}-1} \otimes \tau|\det\cdot|^{\frac{k}{2}-2} \otimes \cdots \otimes \tau|\det\cdot|^{-\frac{k}{2}}.$$

We apply to the elements of Θ_τ a Fourier coefficient corresponding to the orthogonal partition $[(2n-1)^{2k}1^{4n+2k}]$ of $4n(k+1)$ (see [8, 11]). The corresponding unipotent group turns out to be the unipotent radical $Z_{n,k}$ of the standard parabolic subgroup $Q_{(2k)^{n-1}}$ of $\mathrm{SO}_{4n(k+1)}$ whose Levi part is isomorphic to $\mathrm{GL}_{2k}^{\times(n-1)} \times \mathrm{SO}_{4(n+k)}$. It is clear that

$$Z_{n,k}/[Z_{n,k}, Z_{n,k}] \cong M_{2k\times 2k}^{\oplus(n-2)} \oplus M_{2k\times 2n+k} \oplus M_{2k\times 2k} \oplus M_{2k\times 2n+k}, \tag{1.1}$$

where $M_{m\times n}$ denotes the space of matrices of size $m \times n$. Denote a typical element of the right-hand side by

$$(X_1, \cdots, X_{n-2}; Y_1, Y_2, Y_3). \tag{1.2}$$

Let ψ be a non-trivial additive character of $F\backslash\mathbb{A}$, and define a character of $Z_{n,k}(\mathbb{A})$ by

$$\psi_{Z_{n,k}}(v) := \psi(\mathrm{tr}(X_1 + \cdots + X_{n-2} + Y_2)), \tag{1.3}$$

where $v \in Z_{n,k}(\mathbb{A})$ projects to (1.2). This character is left $Z_{n,k}(F)$-invariant, and it corresponds to the partition above. It is easy to check that the connected part of the stabilizer of $\psi_{Z_{n,k}}$ in the Levi subgroup $\mathrm{GL}_{2k}^{\times(n-1)} \times \mathrm{SO}_{4(n+k)}$ is isomorphic to $\mathrm{SO}_{2k} \times \mathrm{SO}_{4n+2k}$. The element

$$(g, h) = \left(g, \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix}\right) \in \mathrm{SO}_{2k} \times \mathrm{SO}_{4n+2k} \tag{1.4}$$

corresponds to

$$\Big(g^{\Delta(n-1)}, \begin{pmatrix} h_1 & & h_2 \\ & g & \\ h_3 & & h_4 \end{pmatrix}\Big) \in \mathrm{GL}_{2k}^{\times(n-1)} \times \mathrm{SO}_{4(n+k)}, \tag{1.5}$$

where

$$g^{\Delta(n-1)} = \mathrm{diag}(g, \cdots, g) \quad (n-1 \text{ times}).$$

We denote the group of elements (1.5) by $\mathrm{SO}_{2k}^{\Delta n} \times \mathrm{SO}_{4n+2k}$. The Fourier coefficient corresponding to the partition $[(2n-1)^{2k}1^{4n+2k}]$ of $\xi \in \Theta_\tau$ is defined as a function of $(g,h) \in \mathrm{SO}_{2k}(\mathbb{A}) \times \mathrm{SO}_{4n+2k}(\mathbb{A})$ by

$$\mathcal{F}_{\psi_{Z_{n,k}}}(\xi)(g,h) := \int_{Z_{n,k}(F)\backslash Z_{n,k}(\mathbb{A})} \xi(v(g,h))\psi^{-1}_{Z_{n,k}}(v)\mathrm{d}v. \tag{1.6}$$

This is an automorphic function on $\mathrm{SO}_{2k}(\mathbb{A}) \times \mathrm{SO}_{4n+2k}(\mathbb{A})$. We use it as a kernel function and integrate it against cusp forms in the space of π to get automorphic functions in $g \in \mathrm{SO}_{2k}(\mathbb{A})$. These functions span a space, which is invariant to right translations. We will prove in the sequel to this paper that this space is non-trivial; its elements are cuspidal, and there exists a Whittaker coefficient which is non-trivial on this space. This global construction is a special case of a more general construction which also applies to other classical groups (see [4, 11]). Thus, there exists an irreducible, automorphic, cuspidal and generic representation σ of $\mathrm{SO}_{2k}(\mathbb{A})$, such that the following integral is not identically zero:

$$I^{\psi}_{n,k}(\tau,\pi;\sigma) := \int_{[\mathrm{SO}_{2k}]}\int_{[\mathrm{SO}_{4n+2k}]} \mathcal{F}_{\psi_{Z_{n,k}}}(\xi)(g,h)\varphi_\pi(h)\varphi_{\sigma^\vee}(g)\mathrm{d}h\mathrm{d}g, \tag{1.7}$$

where $[\mathrm{SO}_m] := \mathrm{SO}_m(F)\backslash\mathrm{SO}_m(\mathbb{A})$. The goal of this paper is to prove that the non-triviality of (1.7) implies that π is a CAP representation, up to an outer conjugation, with respect to the parabolic induction from $\tau|\det\cdot|^{\frac{1}{2}} \otimes \sigma$. In detail, we will prove the following theorem.

Theorem 1.1 *Let τ, π and σ be irreducible, automorphic, cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A})$, $\mathrm{SO}_{4n+2k}(\mathbb{A})$ and $\mathrm{SO}_{2k}(\mathbb{A})$, respectively. Assume that τ is of symplectic type (i.e., $L^S(\tau,\Lambda^2,s)$ has a pole at $s=1$). Assume that either (1) π supports a Fourier coefficient corresponding to the partition $[(2n+2k-1)(2n+1)]$, or (2) σ is (globally) generic. Suppose that for some choice of data, the integral (1.7) is nonzero. Then at all finite places ν, where π_ν, τ_ν, σ_ν and ψ_v are unramified, π_ν is isomorphic to the unique unramified constituent of the (normalized) induced representation*

$$\mathrm{Ind}_{Q_{2n}(F_\nu)}^{\mathrm{SO}_{4n+2k}(F_\nu)}(\tau_\nu|\det|^{\frac{1}{2}} \otimes \sigma'_\nu),$$

where Q_{2n} is the standard maximal parabolic subgroup of SO_{4n+2k} with the Levi subgroup M_{2n} isomorphic to $\mathrm{GL}_{2n} \times \mathrm{SO}_{2k}$; the representation σ' is either σ, or an outer conjugation of σ, depending on the parities of n and k (this will be specified in Theorem 2.1). In particular, π is a CAP representation with respect to

$$\Big(M_{2n}, \tau \otimes \sigma', \frac{1}{2}\Big).$$

Note that in the proof, in case (1) of the theorem, we will only use the fact that π supports a Fourier coefficient corresponding to an orthogonal partition of the form $[(2n+2k-1)\cdots]$.

The work of this paper and its sequel to come fit as a special case of a general conjecture. For this, let τ be an irreducible, automorphic, cuspidal representation of $\mathrm{GL}_{2n}(\mathbb{A})$, which is self-dual and has a trivial central character. If τ is of symplectic type, let $c=2$, and if τ is of orthogonal type, let $c=1$. Consider the Eisenstein series on (split) $\mathrm{SO}_{2n(2k+c)}(\mathbb{A})$ induced from $\Delta(\tau,k+1)|\det|^s$, when $c=2$; and when $c=1$, it is induced from $\Delta(\tau,k)|\det|^s\otimes\epsilon$, where ϵ is an irreducible, automorphic, cuspidal, generic representation of $\mathrm{SO}_{2n}(\mathbb{A})$, which lifts to τ. It can be constructed by automorphic descent (see [7, 9]). See also [2]. The corresponding Eisenstein series has a pole at $s=\frac{k+1}{2}$. Denote the corresponding residual representation by $\Theta_{\tau,2k+c}$, which is square-integrable as proved in [12].

Consider the standard parabolic subgroup of $\mathrm{SO}_{2n(2k+c)}$, whose Levi part is isomorphic to $\mathrm{GL}_{2k}^{\times(n-1)}\times\mathrm{SO}_{2nc+4k}$. Denote its unipotent radical by $Z_{n,k,c}$. Then we have the analogue of (1.1) with $M_{2k\times 2n+k}$ replaced by $M_{2k\times nc+k}$. Thus, in (1.2), Y_1 and Y_3 lie in $M_{2k\times nc+k}$, and $Y_2\in M_{2k\times 2k}$. Let $\psi_{Z_{n,k,c}}$ be the character of $Z_{n,k,c}(\mathbb{A})$ given by (1.3). Then the elements of its stabilizer in the Levi subgroup $\mathrm{GL}_{2k}^{\times(n-1)}\times\mathrm{SO}_{2nc+4k}$ have the form (1.4)–(1.5) with SO_{4n+2k} replaced by SO_{2nc+2k}, and $\mathrm{SO}_{4(n+k)}$ replaced by SO_{2nc+4k} (respectively). Now we can define the Fourier coefficient $\mathcal{F}_{\psi_{Z_{n,k,c}}}(\xi)(g,h)$ by analogy to (1.6) and the integral $I^{\psi}_{n,k,c}(\tau,\pi;\sigma)$ by analogy to (1.7).

Conjecture 1.1 Let τ, σ and π be irreducible, automorphic, cuspidal representations of $\mathrm{GL}_{2n}(\mathbb{A})$, $\mathrm{SO}_{2k}(\mathbb{A})$ and $\mathrm{SO}_{2nc+2k}(\mathbb{A})$, respectively. Assume that τ is self-dual, with trivial central character. Let $c=2$, when τ is of symplectic type, and let $c=1$, when τ is of orthogonal type. Assume that the integral $I^{\psi}_{n,k,c}(\tau,\sigma;\pi)$ is not identically zero. Then σ has a global Arthur parameter $\psi_{\mathrm{SO}_{2k}}$ if and only if π has a global Arthur parameter $\psi_{\mathrm{SO}_{2k}}\oplus(\tau,c)$.

We refer to [1] for the notion of global Arthur parameters. The condition on τ, relative to the parity of c, makes (τ,c) a global Arthur parameter for SO_{2nc}. A more general conjecture of this type was discussed in [11].

The main results of this paper and its sequel ([6]) will be used to prove the case $c=2$ of the following theorem.

Theorem 1.2 *The above conjecture holds for tempered representations σ, and for either $c=1$, if τ is orthogonal, or $c=2$, if τ is symplectic.*

Note that the case $c=1$ was discussed in [3, 5].

The proof of Theorem 1.1 is quite complicated and technical, although the basic idea is simple. We view (1.7) as an equivariant trilinear form at one unramified place ν, replacing Θ_τ, π and σ by their local unramified components at ν, and replacing the Fourier coefficient $\mathcal{F}_{\psi_{Z_{n,k}}}$ by the analogous twisted Jacquet functor. This leads to the local formulation of Theorem 1.1, which is Theorem 2.1 in Section 2. The main ingredient of our proof of Theorem 2.1 is the Mackey theory. Since the proof and the calculations are quite involved and technical, we prefer, for clarity of exposition (hopefully), to break it to four steps. Each of the four steps consists of an application of the Mackey theory in order to obtain a semi-simplification of a given induced representation, when restricted to a given subgroup. In Section 3, we calculate the twisted Jacquet module of the unramified kernel representation on $\mathrm{SO}_{4n(k+1)}$ and reduce the problem to a calculation of a certain twisted Jacquet module on $\mathrm{GL}_{2n(k+1)}$, which we carry out in Section 4. The proof of Theorem 2.1 is completed in Section 5.

2 The Set-up and Formulation of the Main Theorem

From now on, let F be a p-adic local field of characteristic zero. Let SO_{2m} be the F-split special orthogonal group of rank m. We will realize it with respect to the standard anti-diagonal matrix of size $2m$. To simplify notation, for an F-algebraic group G, we will keep denoting by G its group of F-rational points. Let $m_1, \cdots, m_r$ be positive integers, such that $m_1 + \cdots + m_r \leq m$. We will denote by $Q_{m_1,\cdots,m_r}$ the standard parabolic subgroup of SO_{2m}, whose Levi part is isomorphic to

$$\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r} \times \mathrm{SO}_{2(m-(m_1+\cdots+m_r))}.$$

If necessary, we will add the superscript (m), $Q^{(m)}_{m_1,\cdots,m_r}$, to stress the fact that this is a parabolic subgroup of SO_{2m}. If $m_1 = \cdots = m_r = \ell$, we will denote $Q^{(m)}_{m_1,\cdots,m_r} = Q_{(\ell)^r}$. Similarly, when $m_1+\cdots+m_r = m$ we will denote by $P_{m_1,\cdots,m_r}$ the standard parabolic subgroup of GL_m, whose Levi part is isomorphic to

$$\mathrm{GL}_{m_1} \times \cdots \times \mathrm{GL}_{m_r}.$$

We will similarly use the notation $P^{(m)}_{m_1,\cdots,m_r}$. When $m_1 = \cdots = m_r = \ell$ (so that $r\ell = m$), we will denote $P_{m_1,\cdots,m_r} = P_{(\ell)^r}$.

Consider the standard parabolic subgroup $Q_{(2k)^{n-1}}$ of $\mathrm{SO}_{4n(k+1)}$. Its unipotent radical $Z_{n,k}$ consists of elements of the following type:

$$z = \begin{pmatrix} X & E & C \\ & I_{4(n+k)} & E' \\ & & X' \end{pmatrix} \in \mathrm{SO}_{4n(k+1)}, \tag{2.1}$$

where X is a $2k(n-1) \times 2k(n-1)$ upper triangular matrix of the form

$$X = \begin{pmatrix} I_{2k} & X_1 & * & \cdots & * & * \\ & I_{2k} & X_2 & \cdots & * & * \\ & & I_{2k} & \cdots & * & * \\ & & & \ddots & \vdots & \vdots \\ & & & & I_{2k} & X_{n-2} \\ & & & & & I_{2k} \end{pmatrix}.$$

Write the $2k(n-1)\times 4(n+k)$ matrix E in the form $E = \begin{pmatrix} * \\ Y \end{pmatrix}$, with Y having size $2k\times 4(n+k)$, and also write $Y = (Y_1, Y_2, Y_3)$, with Y_1 and Y_3 being of size $2k \times (2n+k)$ and Y_2 being of size $2k \times 2k$. Fix a non-trivial character ψ of F. The character $\psi_{Z_{n,k}}$ of $Z_{n,k}$ analogous to (1.3) is given by

$$\psi_{Z_{n,k}}(z) = \psi(\mathrm{tr}(X_1 + X_2 + \cdots + X_{n-2} + Y_2)), \tag{2.2}$$

where z is of the form (2.1). The connected component of the stabilizer of $\psi_{Z_{n,k}}$ in the Levi subgroup $\mathrm{GL}_{2k}^{\times(n-1)} \times \mathrm{SO}_{4(n+k)}$ is easily computed to be the subgroup $\mathrm{SO}_{2k}^{\Delta n} \times \mathrm{SO}_{4n+2k}$ of elements of the form (1.5).

Let τ be an irreducible, unitary, self-dual, generic, unramified representation of $\mathrm{GL}_{2n}(F)$, having a trivial central character. Thus, τ has the form

$$\tau = \mathrm{Ind}_{B_{\mathrm{GL}_{2n}}}^{\mathrm{GL}_{2n}}(\chi_1 \otimes \cdots \otimes \chi_n \otimes \chi_n^{-1} \otimes \cdots \otimes \chi_1^{-1}). \tag{2.3}$$

Here $B_{\mathrm{GL}_{2n}}$ is the standard Borel subgroup of GL_{2n}. We know, by a theorem of Jacquet-Shalika [10] that if we write, for $i = 1, \cdots, n$, $\chi_i = u_i|\cdot|^{\alpha_i}$, where u_i is a unitary character, and α_i is real, then $-\frac{1}{2} < \alpha_i < \frac{1}{2}$. Re-ordering the characters $\chi_1^{\pm 1}, \cdots, \chi_n^{\pm 1}$, if necessary, we may assume that

$$0 \le \alpha_i < \frac{1}{2}, \quad i = 1, \cdots, n. \tag{2.4}$$

Let $\Delta(\tau, k+1)$ be the representation of $\mathrm{GL}_{2n(k+1)}$, which is the unramified constituent of the parabolic induction from

$$\tau|\det\cdot|^{\frac{k}{2}} \otimes \tau|\det\cdot|^{\frac{k}{2}-1} \otimes \tau|\det\cdot|^{\frac{k}{2}-2} \otimes \cdots \otimes \tau|\det\cdot|^{-\frac{k}{2}}.$$

Consider the unramified constituent Θ_τ of the parabolic induction

$$\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}} \Delta(\tau, k+1)|\det\cdot|^{\frac{k+1}{2}}.$$

Using a conjugation by a suitable Weyl element within the Levi part of $Q_{2n(k+1)}$, it is clear that Θ_τ is the unramified constituent of the representation induced from the following character of the Borel subgroup

$$\bigotimes_{i=1}^n [(\chi_i|\cdot|^{\frac{2k+1}{2}} \otimes \chi_i|\cdot|^{\frac{2k-1}{2}} \otimes \cdots \otimes \chi_i|\cdot|^{\frac{1}{2}}) \otimes (\chi_i^{-1}|\cdot|^{\frac{1}{2}} \otimes \chi_i^{-1}|\cdot|^{\frac{3}{2}} \otimes \cdots \otimes \chi_i^{-1}|\cdot|^{\frac{2k+1}{2}})]. \tag{2.5}$$

Let $\alpha_{n,k}$ be the Weyl element in $\mathrm{O}_{4n(k+1)}$, which flips the character (2.5) to the character

$$\bigotimes_{i=1}^n [\chi_i|\cdot|^{\frac{2k+1}{2}} \otimes \chi_i|\cdot|^{\frac{2k-1}{2}} \otimes \cdots \otimes \chi_i|\cdot|^{\frac{1}{2}} \otimes \chi_i|\cdot|^{-\frac{1}{2}} \otimes \chi_i|\cdot|^{-\frac{3}{2}} \otimes \cdots \otimes \chi_i|\cdot|^{-\frac{2k+1}{2}})]. \tag{2.6}$$

Note that $\det(\alpha_{n,k}) = (-1)^{n(k+1)}$. The i-th factor in (2.6) is

$$(\chi_i \circ \det)\Big|_{B_{\mathrm{GL}_{2k+2}}} \cdot \delta_{B_{\mathrm{GL}_{2k+2}}}^{\frac{1}{2}}.$$

Therefore the unramified constituent of the representation of GL_{2k+2} induced from the i-th factor in (2.6) is $\chi_i \circ \det_{\mathrm{GL}_{2k+2}}$. Thus, it follows that the unramified constituent of the representation of $\mathrm{SO}_{4n(k+1)}$, parabolically induced from (2.6), is equal to the unramified constituent of

$$\mathrm{Ind}_{Q_{(2k+2)^n}}^{\mathrm{SO}_{4n(k+1)}}(\chi_1 \circ \det \otimes \cdots \otimes \chi_n \circ \det), \tag{2.7}$$

where det denotes the determinant of GL_{2k+2}. By induction in stages, we write this induced representation as

$$\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau')$$

with

$$\tau' := \mathrm{Ind}_{P_{(2k+2)^n}}^{\mathrm{GL}_{2n(k+1)}}(\chi_1 \circ \det \otimes \cdots \otimes \chi_n \circ \det).$$

Denote

$$\omega_0 = \omega_{0,m} = \begin{pmatrix} I_{m-1} & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & I_{m-1} \end{pmatrix} \in \mathrm{O}_{2m}. \tag{2.8}$$

We will drop the index m when m is clear from the context. Denote by $\Theta_\tau^{\omega_0}$ the outer conjugation of Θ_τ by ω_0. Then we conclude that $\Theta_\tau^{\omega_0^{n(k+1)}}$ is the unramified constituent of $\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau')$. The local counter part of the global Fourier coefficient (1.6) is the twisted Jacquet module $\mathcal{J}_{\psi_{Z_{n,k}}}(\Theta_\tau)$ of Θ_τ, with respect to $(Z_{n,k}, \psi_{Z_{n,k}})$.

Let σ and π be irreducible, unramified representations of SO_{2k} and SO_{4n+2k}, respectively. The local analogue of the family of global integrals (1.7) is the following Hom-space:

$$\mathrm{Hom}_{\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}}(\mathcal{J}_{\psi_{Z_{n,k}}}(\Theta_\tau)\otimes\sigma^\vee\otimes\pi, 1). \tag{2.9}$$

Note that for any element $\omega' \in \mathrm{O}_{4n(k+1)} - \mathrm{SO}_{4n(k+1)}$, and any (smooth) representation θ of $\mathrm{SO}_{4n(k+1)}$, $\theta^{\omega_0} \cong \theta^{\omega'}$. Consider the element ω_0' defined by the image of $(1_{2k}, \omega_{0,2n+k}) \in \mathrm{O}_{2k}\times\mathrm{O}_{4n+2k}$ inside $\mathrm{O}_{4n(k+1)}$, via (1.5). Then the conjugation by this element preserves $Z_{n,k}$ and $\psi_{Z_{n,k}}$. Thus, it is clear that, as modules over $\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}$,

$$\mathcal{J}_{\psi_{Z_{n,k}}}(\theta^{\omega_0'}) \cong (\mathcal{J}_{\psi_{Z_{n,k}}}(\theta))^{(1_{2k},\omega_{0,2n+k})}.$$

For the formulation of our main local theorem, we will use the notion of degenerate Whittaker models for π. These were defined and studied by Moeglin and Waldspurger in [14] and are the local analogs of Fourier coefficients corresponding to nilpotent orbits. Degenerate Whittaker models of π are obtained by considering twisted Jacquet modules, applied to π, with respect to certain characters of unipotent subgroups, which correspond to nilpotent orbits in the Lie algebra of SO_{4n+2k}, and hence correspond to orthogonal partitions. Therefore we will speak about orbits of degenerate Whittaker models of π, corresponding to orthogonal partitions. In particular, since there exists a partial order among partitions, we may speak of maximal degenerate Whittaker models of π, namely, their corresponding partitions are not majorized by other orthogonal partitions which support degenerate Whittaker models of π. Theorem 1.1 will follow from the following theorem.

Theorem 2.1 *Let τ be an irreducible, unitary, self-dual, unramified, generic representation of GL_{2n}, with a trivial central character. Let σ and π be irreducible, unramified representations of SO_{2k} and SO_{4n+2k}, respectively. Assume that either (1) π has a degenerate Whittaker model corresponding to an orthogonal partition of $4n+2k$ of the form $[(2n+2k-1)\cdots]$ (e.g., $[(2n+2k-1)(2n+1)]$), or (2) σ is unitary and generic. If*

$$\mathrm{Hom}_{\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}}(\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))\otimes\sigma^\vee\otimes\pi^{\omega_0^{n(k+1)}}, 1) \neq 0,$$

then π is isomorphic to the unique unramified constituent of

$$\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}}\otimes\sigma^{\omega_0^{(n+1)k}}).$$

The proof of Theorem 2.1 is quite technical and will be given in the following sections. The most technical part is a detailed analysis of the twisted Jacquet module $\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))$.

3 Analysis of the Twisted Jacquet Module $\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))$: A First Reduction

We study the twisted Jacquet module

$$\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))$$

by a series of applications of the Mackey theory. We start by considering the restriction of $\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau')$ to the standard maximal parabolic subgroup $Q_{2k(n-1)}$ of $\mathrm{SO}_{4n(k+1)}$. Note that its Levi part is isomorphic to $\mathrm{GL}_{2k(n-1)} \times \mathrm{SO}_{4(n+k)}$, and the restriction

$$\mathrm{Res}_{Q_{2k(n-1)}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau')) \tag{3.1}$$

is of finite length, with subquotients parameterized by the double cosets

$$Q_{2n(k+1)}\backslash \mathrm{SO}_{4n(k+1)}/Q_{2k(n-1)}. \tag{3.2}$$

Here it is a set of representatives for this set of double cosets. For $0 \le r \le m = 2k(n-1)$,

$$w_r := \omega_0^{m-r} \cdot \begin{pmatrix} I_r &&&& \\ & 0 & 0 & I_{m-r} & \\ & 0 & I_{4(n+k)} & 0 & \\ & I_{m-r} & 0 & 0 & \\ &&&& I_r \end{pmatrix}. \tag{3.3}$$

See [9, Chapter 4] for an elementary proof. The contribution of the double coset of w_r to the semi-simplification of (3.1) is

$$\mathrm{ind}_{Q^{(r)}}^{Q_m}(\sigma_{(r)}), \tag{3.4}$$

where the notation, ind, denotes non-normalized compact induction,

$$Q^{(r)} := Q_m \cap w_r^{-1} \cdot Q_{2n(k+1)} \cdot w_r,$$

and $\sigma_{(r)}$ is the representation of $Q^{(r)}$ given by

$$x \mapsto \delta_{Q_{2n(k+1)}}^{\frac{1}{2}} \cdot \tau'(w_r x w_r^{-1}), \quad x \in Q^{(r)}.$$

The elements of $Q^{(r)}$ are of the form

$$\begin{pmatrix} a_1 & a_2 & y_1 & y_2 & z_1 & z_2 \\ 0 & a_4 & 0 & y_4 & 0 & z_1' \\ & & d & v & y_4' & y_2' \\ & & 0 & d^* & 0 & y_1' \\ & & & & a_4^* & a_2' \\ & & & & 0 & a_1^* \end{pmatrix}^{\omega_0^r}, \tag{3.5}$$

where $a_1 \in \mathrm{GL}_r$, $a_4 \in \mathrm{GL}_{m-r}$, and $d \in \mathrm{GL}_{2(n+k)}$. The representation $\sigma_{(r)}$ assigns to an element of the form (3.5) the operator

$$\left|\frac{\det(d)\cdot\det(a_1)}{\det(a_4)}\right|^{\frac{2n(k+1)-1}{2}} \cdot \tau'\Big(\begin{pmatrix} a_1 & z_1 & y_1 \\ 0 & a_4^* & 0 \\ 0 & y_4' & d \end{pmatrix}\Big). \tag{3.6}$$

Next, we consider the restriction

$$\mathrm{Res}_{Q_{(2k)^{n-1}}}(\mathrm{ind}_{Q^{(r)}}^{Q_m}(\sigma_{(r)})), \tag{3.7}$$

and analyze it by the Mackey theory. Again, we need to consider the set of double cosets

$$Q^{(r)}\backslash Q_m/Q_{(2k)^{n-1}}.$$

It has the following set of representatives:

$$\widehat{\epsilon} := \mathrm{diag}(\epsilon, I_{4(n+k)}, \epsilon^*),$$

where ϵ varies in a set of Weyl elements, which form a set of representatives for

$$P_{r,m-r}\backslash \mathrm{GL}_{2k(n-1)}/P_{(2k)^{n-1}}.$$

Then, up to semi-simplification, we have

$$\mathrm{Res}_{Q_{(2k)^{n-1}}}(\mathrm{ind}_{Q^{(r)}}^{Q_m}(\sigma_{(r)})) \equiv \bigoplus_{\epsilon} \mathrm{ind}_{Q_{(2k)^{n-1}}\cap\widehat{\epsilon}^{-1}Q^{(r)}\widehat{\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma_{(r)}^{\epsilon}). \tag{3.8}$$

The representation $\sigma_{(r)}^{\epsilon}$ is obtained by composing $\sigma_{(r)}$ with conjugation by $\widehat{\epsilon}$ (on $Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1}Q^{(r)}\widehat{\epsilon}$). Let $L_{(2k)^{n-1}}$ be the subgroup of the Levi part of $Q_{(2k)^{n-1}}$, consisting of all matrices

$$\mathrm{diag}(g_1, \cdots, g_{n-1}, I_{4(n+k)}, g_{n-1}^*, \cdots, g_1^*),$$

where $g_i \in \mathrm{GL}_{2k}$, for $i \le n-1$. Denote

$$R_{n,k} = (\mathrm{SO}_{2k}^{\Delta n} \times \mathrm{SO}_{4n+2k})Z_{n,k},$$

where $\mathrm{SO}_{2k}^{\Delta n} \times \mathrm{SO}_{4n+2k}$ is embedded in $\mathrm{SO}_{4n(k+1)}$ via (1.5). For each ϵ, restrict the representation

$$\mathrm{ind}_{Q_{(2k)^{n-1}}\cap\widehat{\epsilon}^{-1}Q^{(r)}\widehat{\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma_{(r)}^{\epsilon}) \tag{3.9}$$

to $L_{(2k)^{n-1}}R_{n,k}$. For this, we need, once again, to consider the set

$$Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1}Q^{(r)}\widehat{\epsilon}\backslash Q_{(2k)^{n-1}}/L_{(2k)^{n-1}}R_{n,k}, \tag{3.10}$$

which is in one-to-one correspondence with the set

$$[Q_{2(n+k)}^{(2(n+k))}]^{\omega_0^r}\backslash \mathrm{SO}_{4(n+k)}/(\mathrm{SO}_{2k} \times \mathrm{SO}_{4n+2k}). \tag{3.11}$$

We choose the following set of representatives $g_l^{\omega_0^r}$ of (3.10), where

$$g_l := \begin{pmatrix} I_{2k(n-1)} & & \\ & g_l' & \\ & & I_{2k(n-1)} \end{pmatrix} \tag{3.12}$$

with the g'_l, $0 \le l \le k$, given by

$$g'_l := \begin{pmatrix} I_{k-l} & & & & & & & \\ & I_{2n+l} & & & & & & \\ & & I_l & & & & & \\ & & & I_{k-l} & & & & \\ -I_{k-l} & & & & I_{k-l} & & & \\ & 0 & & & & I_l & & \\ & & 0 & & & & I_{2n+l} & \\ & & & I_{k-l} & & & & I_{k-l} \end{pmatrix}, \tag{3.13}$$

and the elements $(g'_l)^{\omega_0^r}$ form a set of representatives for (3.11).

To prove this, it is enough to show that the elements g'_l form a set of representatives for $Q_{2(n+k)}^{(2(n+k))}\backslash \mathrm{SO}_{4(n+k)}/(\mathrm{SO}_{2k}\times \mathrm{SO}_{4n+2k})$, since $\mathrm{SO}_{2k}\times \mathrm{SO}_{4n+2k}$ is invariant to conjugation by ω_0^r. Denote by V the column space $F^{4(n+k)}$, and consider the left action of $\mathrm{SO}_{4(n+k)}$ on V, preserving the symmetric form $(u,v) = {}^t u J_{4(n+k)} v$ (J_ℓ is the standard anti-diagonal matrix of size $\ell\times\ell$). Let

$$e_1, \cdots, e_{2(n+k)}, e_{-2(n+k)}, \cdots, e_{-1}$$

be the standard basis of V. For $\ell \le 2(n+k)$, denote by $V_\ell^{\pm}$ the span of $e_1,\cdots,e_\ell$, and of $e_{-1},\cdots,e_{-\ell}$, respectively. These are dual isotropic subspaces of dimension ℓ. Denote by W_{2k} the ortho-complement of $V_{2n+k}^+ + V_{2n+k}^-$ inside V. Note that the subgroup of elements in $\mathrm{SO}_{4(n+k)}$ that preserve W_{2k} is given, as in (1.5), by

$$(g,h) = \begin{pmatrix} h_1 & & h_2 \\ & g & \\ h_3 & & h_4 \end{pmatrix} \in \mathrm{SO}_{4(n+k)}, \tag{3.14}$$

where $g \in \mathrm{O}_{2k}$, $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \in \mathrm{O}_{4n+2k}$ and $\det(g) = \det(h)$. Denote by H the connected component of the subgroup of elements (3.14). Then H is naturally identified with $\mathrm{SO}_{2k}\times \mathrm{SO}_{4n+2k}$. Let us realize $Q_{2(n+k)}\backslash \mathrm{SO}_{4(n+k)}$ as the variety $\mathcal{Y}$ of all (maximal) $2(n+k)$-dimensional isotropic subspaces of V, which are in the orientation class of $V_{2(n+k)}^+$ (i.e., in the $\mathrm{SO}_{4(n+k)}$-orbit of $V_{2(n+k)}^+$). Note that we dropped the superscript $(2(n+k))$ from the notation $Q_{2(n+k)}$. Thus, the coset $Q_{2(n+k)}g$ is identified with $X = g^{-1}V_{2(n+k)}^+$. We consider the action of H on $\mathcal{Y}$. Given $X, Y \in \mathcal{Y}$ in a given H-orbit $\mathcal{O}$, it is clear that

$$\dim(X\cap W_{2k}) = \dim(Y \cap W_{2k}).$$

Denote this number by $d_{\mathcal{O}}$.

Lemma 3.1 *The number $d_{\mathcal{O}}$ is the only invariant of the orbit $\mathcal{O}$.*

Proof This is a generalization of Lemma 2.1 of [15]. Let $X \in \mathcal{Y}$ be such that $\dim(X \cap W_{2k}) = l$. Denote $\dim(X \cap (V_{2n+k}^+ + V_{2n+k}^-)) = c$. Choose bases $B_1 = \{x_1, \cdots, x_l\}$ and $B_2 = \{x_{l+1}, \cdots, x_{l+c}\}$ to $X \cap W_{2k}$ and $X\cap(V_{2n+k}^+ + V_{2n+k}^-)$ respectively. Choose a set of linearly independent isotropic vectors in W_{2k}, $B_{-1} = \{x_{-1}, \cdots, x_{-l}\}$, dual to B_1 (that is $(x_i, x_{-j}) = \delta_{i,j}$, for $1 \le i, j \le l$). Similarly, choose a set of linearly independent isotropic vectors in $V_{2n+k}^+ + V_{2n+k}^-$, $B_{-2} = \{x_{-(l+1)}, \cdots, x_{-(l+c)}\}$, dual to B_2. Denote by W' the ortho-complement of the span of $B_1 \cup B_{-1}$ inside W_{2k}, and denote by V' the ortho-complement of the span of $B_2 \cup B_{-2}$ inside $V^+_{2n+k} + V^-_{2n+k}$. Let us choose vectors $z_i + w'_i + v'_i + f_i$, $c + l + 1 \le i \le 2(n+k)$, which complete $B_1 \cup B_2$ to a basis of X, such that z_i lies in the span of $B_1 \cup B_2$, $w'_i \in W'$, $v'_i \in V'$, and f_i lies in the span of $B_{-1} \cup B_{-2}$. Since X is isotropic, it is easy to see that we must have $f_i = 0$. Also, by construction, it is easy to see that the elements $B_3 = \{w'_{c+l+1}, \cdots, w'_{2(n+k)}\}$ are linearly independent, as well as the elements $B_4 = \{v'_{c+l+1}, \cdots, v'_{2(n+k)}\}$. Moreover, for all $c + l + 1 \le i, j \le 2(n+k)$, we have

$$(w'_i, w'_j) = -(v'_i, v'_j). \tag{3.15}$$

Note that the linearly independent set $B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_{-2} \cup B_{-1}$ contains $4(n+k)$ elements, and hence it is a basis for V. We conclude that $B_1 \cup B_3 \cup B_{-1}$ is a basis for W_{2k} and $B_2 \cup B_4 \cup B_{-2}$ is a basis for $V^+_{2n+k} + V^-_{2n+k}$. In particular, $c = 2n + l$ is determined by l. We can choose and re-denote

$$w'_{2n+2l+1} = w(1), \cdots, w'_{2n+l+k} = w(k-l)$$

and

$$w'_{2n+l+k+1} = w(-(k-l)), \cdots, w'_{2(n+k)} = w(-1)$$

with the corresponding Gram matrix $J_{2(k-l)}$. Note that all these make sense when $l = 0$ or $l = k$. Similarly we can choose and re-denote

$$v'_{2n+2l+1} = v(1), \cdots, v'_{2n+l+k} = v(k-l)$$

and

$$v'_{2n+l+k+1} = v(-(k-l)), \cdots, v'_{2(n+k)} = v(-1)$$

with the corresponding Gram matrix $-J_{2(k-l)}$. Clearly, (3.15) is satisfied. Now, let $Y \in \mathcal{Y}$ be such that $\dim(Y \cap W_{2k}) = l$. Let us construct for Y the basis of V as above. Denote the corresponding subsets by B'_1, B'_2, etc. For example B'_1 is a basis of $Y \cap W_{2k}$ and B'_{-1} is a subset of l linearly independent isotropic vectors in W_{2k}, dual to B'_1, and so on. Denote the elements of B'_i by the same letters and indices as for B_i, with the addition of primes. For example, we denote the elements of B'_3 by

$$w'(1), \cdots, w'(k-l), w'(-(k-l)), \cdots, w'(-1).$$

Let t be the linear transformation of V to itself, which sends the elements of each subset B_i to the corresponding elements in B'_i, $i = \pm 1, \pm 2, 3, 4$. Then, by construction, $t(X) = Y$ and $t(W_{2k}) = W_{2k}$ (and also $t(V^+_{2n+k} + V^-_{2n+k}) = V^+_{2n+k} + V^-_{2n+k}$). It is easy to correct t, if necessary, such that the determinant of the restriction of t to W_{2k} is one. We conclude that $t \in H$ takes X to Y, and hence X and Y lie in the same H-orbit.

Set $g_{l,r} = g_l^{\omega_0^r}$ and $g'_{l,r} = (g'_l)^{\omega_0^r}$. Recall again that we use the same notation ω_0 as an element of $\mathrm{O}_{2n'}$, for any n'. Up to semi-simplification, we get

$$\begin{aligned}
&\mathrm{Res}_{L_{(2k)^{n-1}} R_{n,k}}\left(\mathrm{ind}_{Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma^{\epsilon}_{(r)})\right) \\
\equiv &\bigoplus_{l=0}^{k} \mathrm{ind}_{L_{(2k)^{n-1}} R_{n,k} \cap g_{l,r}^{-1}(Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}) g_{l,r}}^{L_{(2k)^{n-1}} R_{n,k}}(\sigma^{\epsilon}_{(r),l}),
\end{aligned} \tag{3.16}$$

where for $x \in L_{(2k)^{n-1}} R_{n,k} \cap g_{l,r}^{-1}(Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}) g_{l,r}$,

$$\sigma^{\epsilon}_{(r),l}(x) = \sigma^{\epsilon}_{(r)}(g_{l,r} x g_{l,r}^{-1}).$$

We need to compute

$$\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{ind}^{L_{(2k)^{n-1}} R_{n,k}}_{L_{(2k)^{n-1}} R_{n,k} \cap g_{l,r}^{-1}(Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}) g_{l,r}}(\sigma^{\epsilon}_{(r),l})) \tag{3.17}$$

for all (r,ϵ,l). Let us write necessary conditions for (r,ϵ,l) to be relevant with respect to $\psi_{Z_{n,k}}$. More precisely, we say that (r,ϵ,l) is not relevant with respect to $\psi_{Z_{n,k}}$ if for each $h \in L_{(2k)^{n-1}}$, there exists $z \in Z_{n,k} \cap h^{-1} g_{l,r}^{-1}(Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}) g_{l,r} h$, such that $\psi_{Z_{n,k}}(z) \neq 1$ and $\sigma^{\epsilon}_{(r),l}(hzh^{-1}) = \mathrm{id}$. Clearly, in this case, the Jacquet module (3.17) is zero. Note that $R_{n,k}$ normalizes $Z_{n,k}$ and preserves $\psi_{Z_{n,k}}$.

Proposition 3.1 *Assume that (r,ϵ,l) is relevant with respect to $\psi_{Z_{n,k}}$. Then there exists a sequence of nonnegative integers $r_1, \cdots, r_{n-1}$ satisfying*

$$\begin{aligned} r_1 + \cdots + r_{n-1} &= r, \\ r_1 \leq \ldots \leq r_{n-1} &\leq l, \end{aligned} \tag{3.18}$$

such that ϵ can be chosen (modulo $P_{r,m-r}$ from the left, and modulo $P_{(2k)^{n-1}}$ from the right) as follows:

$$\epsilon = \begin{pmatrix} I_{r_1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_{r_2} & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & I_{r_{n-1}} & 0 \\ 0 & I_{t_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & I_{t_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & I_{t_{n-1}} \end{pmatrix}, \tag{3.19}$$

where for $1 \leq i \leq n-1$,

$$r_i + t_i = 2k.$$

Proof Modulo $P_{r,m-r}$ from the left, and modulo $P_{(2k)^{n-1}}$ from the right, we may assume that ϵ is of the form (3.19), where for $1 \leq i \leq n-1$, r_i, t_i are nonnegative integers, such that $r_i + t_i = 2k$ and $r_1 + \cdots + r_{n-1} = r$. We want to show that (3.18) holds (for relevant (r,ϵ,l)). Let $h = \mathrm{diag}(h_1, \cdots, h_{n-1})$, where $h_1, \cdots, h_{n-1} \in \mathrm{GL}_{2k}$, and denote $\widehat{h} = \mathrm{diag}(h, I_{4(n+k)}, h^*)$. Let also $y_2 \in M_{r \times 2(n+k)}$ and $z_2 \in M_r$, such that $(J_r z_2)$ is anti-symmetric. Consider

$$z = \begin{pmatrix} I_m & X & C \\ & I_{4(n+k)} & X' \\ & & I_m \end{pmatrix} \in Z_{n,k},$$

where

$$X = h^{-1}\epsilon^{-1} \begin{pmatrix} 0_{r \times 2(n+k)} & y_2 \\ 0_{(m-r) \times 2(n+k)} & 0_{(m-r) \times 2(n+k)} \end{pmatrix} \omega_0^r g'_{l,r},$$

$$C = h^{-1}\epsilon^{-1} \begin{pmatrix} 0_{r \times (m-r)} & z_2 \\ 0_{m-r} & 0_{(m-r) \times r} \end{pmatrix} \epsilon^* h^*.$$

By construction,

$$z \in Z_{n,k} \cap h^{-1} g_{l,r}^{-1} (Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}) g_{l,r} h,$$

and by (3.6),

$$\sigma^{\epsilon}_{(r),l}(hzh^{-1}) = \sigma_{(r)}(\widehat{\epsilon} g_{l,r} \widehat{h} z \widehat{h}^{-1} g_{l,r}^{-1} \widehat{\epsilon}^{-1}) = \mathrm{id}.$$

Thus, for (r, ϵ, l) to be relevant, we must have that $\psi_{Z_{n,k}}(z)$ is identically one, for all y_2, z_2 as above. We have

$$\psi_{Z_{n,k}}(z) = \psi\Big(\mathrm{tr}\Big(g'_l \omega_0^r \begin{pmatrix} 0_{(2n+k)\times 2k(n-2)} & 0_{(2n+k)\times 2k} \\ 0_{2k\times 2k(n-2)} & h_{n-1}^{-1} \\ 0_{(2n+k)\times 2k(n-2)} & 0_{(2n+k)\times 2k} \end{pmatrix} \epsilon^{-1} \begin{pmatrix} 0_{r\times 2(n+k)} & y_2 \\ 0_{(m-r)\times 2(n+k)} & 0_{(m-r)\times 2(n+k)} \end{pmatrix}\Big)\Big).$$

For this to be trivial for all y_2, we must have

$$\begin{pmatrix} 0_{2(n+k)} & \\ & I_{2(n+k)} \end{pmatrix} \omega_0^r g'_l \begin{pmatrix} 0_{(2n+k)\times 2k(n-2)} & 0_{(2n+k)\times 2k} \\ 0_{2k\times 2k(n-2)} & h_{n-1}^{-1} \\ 0_{(2n+k)\times 2k(n-2)} & 0_{(2n+k)\times 2k} \end{pmatrix} \epsilon^{-1} \begin{pmatrix} I_r & \\ & 0_{m-r} \end{pmatrix} = 0.$$

Now, by a simple verification, we see that this means that $\omega_0^r h_{n-1}^{-1}$ has the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \tag{3.20}$$

where $A \in M_{l\times r_{n-1}}$, and we conclude that

$$r_{n-1} \le l.$$

We continue in a similar fashion and obtain more conditions on ϵ by considering $z \in Z_{n,k}$ of the form $\widehat{\zeta_i} = \mathrm{diag}(\zeta_i, I_{4(n+k)}, \zeta_i^*)$, $i = 1, 2, \cdots, n-2$, where

$$\zeta_i = \mathrm{diag}\Big(I_{2k}, \cdots, I_{2k}, \begin{pmatrix} I_{2k} & X_i \\ & I_{2k} \end{pmatrix}, I_{2k}, \cdots, I_{2k}\Big) \in \mathrm{GL}_{2k(n-1)}.$$

Note that g_l and ω_0 commute with $\widehat{\zeta_i}$ and $\widehat{h}$. Let $e \in M_{r_i\times t_{i+1}}$ and

$$X_i = h_i^{-1} \begin{pmatrix} 0_{r_i\times r_{i+1}} & e \\ 0_{t_i\times r_{i+1}} & 0_{t_i\times t_{i+1}} \end{pmatrix} h_{i+1}.$$

Then

$$\widehat{\epsilon} g_l \widehat{h} \widehat{\zeta_i} \widehat{h}^{-1} g_l^{-1} \widehat{\epsilon}^{-1} = \widehat{\epsilon} \widehat{h} \widehat{\zeta_i} \widehat{h}^{-1} \widehat{\epsilon}^{-1} \in Q^{(r)},$$

and by (3.6), $\sigma_{(r)}$ acts as the identity on this last element. Thus, for (r, ϵ, l) to be relevant, we must have that $\psi(\mathrm{tr}(X_i))$ is identically trivial for all e, that is,

$$\psi\Big(\mathrm{tr}\Big(h_{i+1} h_i^{-1} \begin{pmatrix} 0_{r_i\times r_{i+1}} & e \\ 0_{t_i\times r_{i+1}} & 0_{t_i\times t_{i+1}} \end{pmatrix}\Big)\Big) \equiv 1. \tag{3.21}$$

This means that $h_{i+1} h_i^{-1}$ is of the form

$$h_{i+1} h_i^{-1} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \tag{3.22}$$

where $A \in M_{r_{i+1}\times r_i}$. This implies that

$$r_i \leq r_{i+1}, \quad 1 \leq i \leq n-2.$$

This proves the proposition.

Note that (3.18) implies that

$$r \leq l(n-1) \leq k(n-1). \tag{3.23}$$

Let us go back to the representation (3.9) for (r, ϵ) as in Proposition 3.1, satisfying (3.18). The elements of $Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}$ have the form

$$\begin{pmatrix} A & Y & C \\ & D & Y' \\ & & A^* \end{pmatrix}^{\omega_0^r}, \tag{3.24}$$

where $D \in \mathrm{SO}_{4(n+k)}$ has the form

$$D = \begin{pmatrix} d & v \\ & d^* \end{pmatrix}, \quad d \in \mathrm{GL}_{2(n+k)}. \tag{3.25}$$

The form of the rest of the elements is described as follows. The matrix $A \in \mathrm{GL}_m$ has the form

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & \cdots & A_{1,n-1} \\ & A_{2,2} & \cdots & A_{2,n-1} \\ & & \ddots & \vdots \\ & & & A_{n-1,n-1} \end{pmatrix}, \tag{3.26}$$

where for $1 \leq i \leq j \leq n-1$, $A_{i,j} \in M_{2k}$ and has the form

$$A_{i,j} = \begin{pmatrix} A_{i,j}^{(1)} & A_{i,j}^{(2)} \\ 0_{t_i \times r_j} & A_{i,j}^{(4)} \end{pmatrix}; \tag{3.27}$$

that is $A_{i,j}^{(1)} \in M_{r_i \times r_j}$, etc. The matrix $Y \in M_{m \times 4(n+k)}$ has the form

$$Y = \begin{pmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_{n-1} \end{pmatrix}, \tag{3.28}$$

where for $1 \leq i \leq n-1$, $Y_i \in M_{2k \times 4(n+k)}$ is of the form

$$Y_i = \begin{pmatrix} Y_i^{(1)} & Y_i^{(2)} \\ 0_{t_i \times 2(n+k)} & Y_i^{(4)} \end{pmatrix}. \tag{3.29}$$

Write

$$Y' = (Y'_{n-1}, Y'_{n-2}, \cdots, Y'_1). \tag{3.30}$$

Then, for $1 \le i \le n-1$, $Y_i' \in M_{4(n+k)\times 2k}$ is of the form

$$Y_i' = \begin{pmatrix} (Y_i')^{(4)} & Y_i^{(2)} \\ 0_{2(n+k)\times t_i} & (Y_i')^{(1)} \end{pmatrix}. \tag{3.31}$$

Finally, C has the form

$$C = \epsilon^{-1} \begin{pmatrix} C_1 & C_2 \\ 0_{(m-r)\times(m-r)} & C_1' \end{pmatrix} \epsilon^*, \tag{3.32}$$

where $C_1 \in M_{r\times m-r}$. The action of $\sigma^{\epsilon}_{(r)}$ on the element (3.24) described by (3.25)–(3.32) is given by

$$\left| \frac{\det(d) \prod_{i=1}^{n-1} \det(A_{i,i}^{(1)})}{\prod_{i=1}^{n-1} \det(A_{i,i}^{(4)})} \right|^{n(k+1)-\frac{1}{2}} \tau' \begin{pmatrix} A^{(1)} & C_1 & Y^{(1)} \\ 0 & (A^{(4)})^* & 0 \\ 0 & (Y')^{(4)} & d \end{pmatrix}, \tag{3.33}$$

where $A^{(1)} \in \mathrm{GL}_r$ is the matrix

$$A^{(1)} = \begin{pmatrix} A_{1,1}^{(1)} & A_{1,2}^{(1)} & \cdots & A_{1,n-1}^{(1)} \\ 0 & A_{2,2}^{(1)} & \cdots & A_{2,n-1}^{(1)} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & A_{n-1,n-1}^{(1)} \end{pmatrix}. \tag{3.34}$$

Similarly $(A^{(4)})^* = J_{m-r}({}^tA^{(4)})^{-1}J_{m-r}$, and $A^{(4)} \in \mathrm{GL}_{m-r}$ is the matrix obtained from A by replacing in (3.34) each super index (1) by (4). The matrix $Y^{(1)} \in M_{r\times 2(n+k)}$ is

$$Y^{(1)} = \begin{pmatrix} Y_1^{(1)} \\ Y_2^{(1)} \\ \vdots \\ Y_{n-1}^{(1)} \end{pmatrix}. \tag{3.35}$$

The matrix $(Y')^{(4)} \in M_{2(n+k)\times m-r}$ is

$$(Y')^{(4)} = ((Y_{n-1}')^{(4)}\ (Y_{n-2}')^{(4)}\ \cdots\ (Y_1')^{(4)}). \tag{3.36}$$

Let $h = \mathrm{diag}(h_1, \cdots, h_{n-1})$, where $h_i \in \mathrm{GL}_{2k}$, for $1 \le i \le n-1$, and let $g \in \mathrm{SO}_{4(n+k)}$. Set $m(h,g) = \mathrm{diag}(h, g, h^*)$, and $Q_{n,k,r:\epsilon} = Q_{(2k)^{n-1}} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon}$. The following map on $\mathrm{ind}_{Q_{n,k,r:\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma^{\epsilon}_{(r)})$ factors through its twisted Jacquet module with respect to $\psi_{Z_{n,k}}$ and induces an injective homomorphism on the twisted Jacquet module $\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{ind}_{Q_{n,k,r:\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma^{\epsilon}_{(r)}))$:

$$f \mapsto \Big((h,g) \mapsto \int_{Z'_{n,k;\epsilon}\backslash Z_{n,k}} \mathcal{J}_{\psi^{h,g}_{Z_{n,k;\epsilon}}}(f(zm(h,g)))\psi^{-1}_{Z_{n,k}}(m(h,g)^{-1}zm(h,g))\mathrm{d}z\Big), \tag{3.37}$$

where

$$Z'_{n,k;\epsilon} = Z_{n,k} \cap \widehat{\epsilon}^{-1} Q^{(r)} \widehat{\epsilon},$$

$Z_{n,k;\epsilon}$ is the projection to $\mathrm{GL}_{2n(k+1)}$ of the subgroup $w_r\widehat{\epsilon}Z'_{n,k;\epsilon}\widehat{\epsilon}^{-1}w_r^{-1}$, and the character $\psi^{h,g}_{Z_{n,k;\epsilon}}$ is obtained by pulling back $z \in Z_{n,k;\epsilon}$ to any element $z' \in w_r\widehat{\epsilon}Z'_{n,k;\epsilon}\widehat{\epsilon}^{-1}w_r^{-1}$ and then

$$\psi^{h,g}_{Z_{n,k;\epsilon}}(z) = \psi_{Z_{n,k}}(m(h,g)^{-1}\widehat{\epsilon}^{-1}w_r^{-1}z'w_r\widehat{\epsilon}m(h,g)).$$

$\mathcal{J}_{\psi^{h,g}_{Z_{n,k;\epsilon}}}(f(zm(h,g)))$ denotes the application of the twisted Jacquet functor with respect to $(Z_{n,k;\epsilon}, \psi^{h,g}_{Z_{n,k;\epsilon}})$ to the vector $f(zm(h,g))$ in the space of τ'. This statement (about the map (3.37)) is easily verified. Denote, for $0 \leq l \leq k$,

$$\gamma_{l,r} = \mathrm{diag}(\omega_0^r, \cdots, \omega_0^r, g'_{l,r}, \omega_0^r, \cdots, \omega_0^r),$$

where ω_0^r is repeated $n-1$ times to the left of $g'_{l,r}$ and to its right.

Proposition 3.2 *Let ϵ be as in* (3.19), *such that* (3.18) *is satisfied. Then the function on the r.h.s. of* (3.37) *is supported inside*

$$\bigcup_{l=0}^{k} Q_{n,k,r:\epsilon}\gamma_{l,r}(\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}).$$

Proof Let $m(h,g)$ be in the support of the function on the r.h.s. of (3.37). Then by (3.33) and the description (3.24)–(3.32) of $Q_{n,k,r:\epsilon}$, we must have

$$\psi_{Z_{n,k}}\left(m(h,g)^{-1}\begin{pmatrix} Z & Y\omega_0^r & C \\ & I_{4(n+k)} & \omega_0^r Y' \\ & & Z^* \end{pmatrix} m(h,g)\right) \equiv 1 \tag{3.38}$$

for all upper unipotent Z of the form (3.26)–(3.27), with $A_{i,i} = I_{2k}$ and for $i < j$, $A^{(1)}_{i,j} = 0, A^{(4)}_{i,j} = 0$, and similarly, all Y, as in (3.28)–(3.29) with $Y^{(1)}_i = 0, Y^{(4)}_i = 0$, and all C as in (3.32) with $C_1 = 0$. Now, we already did this calculation in the proof of Proposition 3.1. By (3.21)–(3.22), we must have, for all $1 \leq i \leq n-2$,

$$h_{i+1} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} h_i,$$

where $A \in M_{r_{i+1}\times r_i}$. Note that we have already assumed that $r_i \leq r_{i+1}$, $1 \leq i \leq n-2$. Assume that we already proved that, modulo $Q_{n,k,r:\epsilon}$ from the left, $h_1 = \cdots = h_i$, $1 \leq i \leq n-2$. Multiplying h_{i+1} from the left by an element in GL_{2k} of the form $\begin{pmatrix} A' & B' \\ 0 & D' \end{pmatrix}$, where $A' \in \mathrm{GL}_{r_{i+1}}$, we may assume that $A = \begin{pmatrix} I_{r_i} \\ 0_{r_{i+1}-r_i\times r_i} \end{pmatrix}$, so we have a relation

$$h_{i+1} = \begin{pmatrix} I_{r_i} & \beta \\ 0 & \delta \end{pmatrix} h_i.$$

Note that, for all $1 \leq j < i$,

$$\begin{pmatrix} I_{r_i} & \beta \\ 0 & \delta \end{pmatrix} = \begin{pmatrix} I_{r_j} & \beta' \\ 0 & \delta' \end{pmatrix}, \tag{3.39}$$

where $\delta' = \begin{pmatrix} I_{r_i-r_j} & \beta_2 \\ 0 & \delta \end{pmatrix}$, $\beta' = \begin{pmatrix} 0_{r_j\times r_i-r_j} & \beta_1 \end{pmatrix}$, and $\beta = \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$. This shows that multiplying from the left each one of $h_1 = \cdots = h_i$ by the matrix (3.39) is a result of a left multiplication by an element in $Q_{n,k,r:\epsilon}$. Therefore we may assume that $h_1 = \cdots = h_{i+1}$. This shows that, modulo $Q_{n,k,r:\epsilon}$ from the left, we may assume that $h_1 = \cdots = h_{n-1}$. As for g, we may assume that $g = g'_{l,r} \cdot (h_0, R)$, $0 \le l \le k$, where $(h_0, R) \in \mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}$. Then, as in (3.20), we must have

$$g_l \begin{pmatrix} 0_{(2n+k)\times 2k} \\ \omega_0^r h_0 \\ 0_{(2n+k)\times 2k} \end{pmatrix} = \begin{pmatrix} A & B \\ 0_{2(n+k)\times r_{n-1}} & D \end{pmatrix} h_{n-1}.$$

Recall again that we use the same notation ω_0 for the matrix obtained from an even-sized identity matrix by interchanging the order of its two middle rows. Thus,

$$\begin{pmatrix} 0 & 0 & 0 \\ I_l & 0 & 0 \\ 0 & I_{k-l} & 0 \\ 0 & 0 & I_k \\ 0 & 0 & 0 \\ 0 & I_{k-l} & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ 0_{2(n+k)\times r_{n-1}} & D \end{pmatrix} h_{n-1} h_0^{-1} \omega_0^r.$$

Here, in the first row, the zero blocks have $2n + k$ rows and, in the fifth row, the zero blocks have $2n + l$ rows. We deduce equalities of the form

$$\begin{aligned} I_{2k} &= \begin{pmatrix} \alpha & \beta \\ 0_{k\times r_{n-1}} & \gamma \end{pmatrix} h_{n-1} h_0^{-1} \omega_0^r, \\ (0_{(k-l)\times l}, I_{k-l}, 0_{(k-l)\times k}) &= (0_{(k-l)\times r_{n-1}}, \eta) h_{n-1} h_0^{-1} \omega_0^r, \end{aligned} \tag{3.40}$$

where $\alpha \in M_{k\times r_{n-1}}$ and $\gamma \in M_{k\times t_{n-1}}$, $\eta \in M_{k-l\times t_{n-1}}$. Recall that we have already known that $t_{n-1} \ge k$. Thus γ must be of rank k. Since we may multiply from the left all $h_1 = \cdots = h_{n-1}$ by $\begin{pmatrix} I_{r_{n-1}} & \\ & \mathrm{GL}_{t_{n-1}} \end{pmatrix}$, we may assume that $\gamma = (0_{k\times t_{n-1}-k}, I_k)$. Hence we may replace the first relation in (3.40) by a relation of the form

$$I_{2k} = \begin{pmatrix} \alpha' & \beta' \\ 0 & I_k \end{pmatrix} h_{n-1} h_0^{-1} \omega_0^r,$$

where $\alpha' \in \mathrm{GL}_{2k}$. Again, we may multiply from the left all $h_1 = \cdots = h_{n-1}$ by any upper unipotent $2k \times 2k$ matrix, and hence we may assume that $\beta' = 0$. We conclude that

$$h_{n-1} h_0^{-1} \omega_0^r = \begin{pmatrix} (\alpha')^{-1} & \\ & I_k \end{pmatrix},$$

and we have a relation of the form

$$(0_{(k-l)\times l}, I_{k-l}, 0_{(k-l)\times k}) \begin{pmatrix} \alpha' & \\ & I_{2k} \end{pmatrix} = (0_{(k-l)\times r_{n-1}}, \eta).$$

This implies that α' has the form

$$\alpha' = \begin{pmatrix} \alpha_1 & \alpha_2 \\ 0_{(k-l)\times r_{n-1}} & \alpha_4 \end{pmatrix},$$

where α_2 and α_4 have $k - r_{n-1} = t_{n-1} - k$ columns. We conclude that α_4 has rank $k - l$. Once again, since we may multiply from the left all $h_1 = \cdots = h_{n-1}$ by $\begin{pmatrix} I_{r_{n-1}} & \\ & \mathrm{GL}_{t_{n-1}} \end{pmatrix}$, we may assume that $\alpha_4 = (0_{k-l\times l-r_{n-1}}, I_{k-l})$, that is, α' has the form

$$\alpha' = \begin{pmatrix} \alpha'_1 & \alpha'_2 \\ 0 & I_{k-l} \end{pmatrix},$$

where $\alpha'_1 \in \mathrm{GL}_l$, and exactly as in the previous step, we may now assume that $\alpha'_2 = 0$. We get (after modification by $Q_{n,k,r;\epsilon}$ from the left) a relation of the form

$$I_{2k} = \begin{pmatrix} e & \\ & I_{2k-l} \end{pmatrix} h_{n-1} h_0^{-1} \omega_0^r,$$

where $e \in \mathrm{GL}_l$. Since $t_{n-1} \geq k \geq l$, we may multiply from the left all $h_1 = \cdots = h_{n-1}$ by $\mathrm{diag}(I_{2k-l}, e^*)$, and hence we may assume that

$$\omega_0^r \Big(\begin{pmatrix} e & & \\ & I_{2(k-l)} & \\ & & e^* \end{pmatrix}^{-1} \Big)^{\omega_0^r} h_0 = h_{n-1}.$$

Finally, since

$$Q_{2(n+k)}^{\omega_0^r} g'_{l,r}(h_0, R) = Q_{2(n+k)}^{\omega_0^r} g'_{l,r} \Big(\Big(\begin{pmatrix} e & & \\ & I_{2(k-l)} & \\ & & e^* \end{pmatrix}^{-1} \Big)^{\omega_0^r} h_0, R \Big),$$

we may assume that $\omega_0^r h_0 = h_{n-1}$. Recall that $(h_0, R) \in \mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}$. Therefore, $m(g,h)$ which lies in the support of the r.h.s. of (3.37) is equal, modulo $Q_{n,k,r:\epsilon}$ from the left, and modulo $\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}$ from the right, to an element of the form $\gamma_{l,r}$, for some $0 \leq l \leq k$. This completes the proof of the proposition.

Upon restriction of the r.h.s. of (3.37) to the support we found in Proposition 3.2, we see that, up to semi-simplification of $\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}$- modules,

$$\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{ind}_{Q_{n,k,r:\epsilon}}^{Q_{(2k)^{n-1}}}(\sigma_{(r)}^\epsilon)) \equiv \bigoplus_{l=0}^{k} \mathrm{ind}_{\gamma_{l,r}^{-1} Q_{n,k,r;\epsilon} \gamma_{l,r} \cap (\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k})}^{\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}} \sigma_{[r];l}^\epsilon, \tag{3.41}$$

where the representation $\sigma_{[r];l}^\epsilon$ is described as follows. First, $\gamma_{l,r}^{-1} Q_{n,k,r;\epsilon} \gamma_{l,r} \cap (\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k})$ is the subgroup consisting of all elements of the form:

$$\Big(\left[\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e'_3 \\ & -b_2 & a_1 & e'_2 \\ & & & e_1^* \end{pmatrix}^{\omega_0^r} \right]^{\Delta_n}, \begin{pmatrix} a_1 & 0 & b'_4 & b_2 \\ a_3 & a_4 & b_3 & b_4 \\ 0 & 0 & a_4^* & 0 \\ c_3 & 0 & a'_3 & d_4 \end{pmatrix} \Big) \in \mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}, \tag{3.42}$$

where $\begin{pmatrix} d_4 & -c_3 \\ -b_2 & a_1 \end{pmatrix} \in \mathrm{SO}_{2(k-l)}$, a_1 is of size $(k-l) \times (k-l)$, $a_4 \in \mathrm{GL}_{2n+l}$, and $e_1 \in \mathrm{GL}_l$ has the following form:

$$e_1 = \begin{pmatrix} h_1 & * & \cdots & * & * \\ & h_2 & \cdots & * & * \\ & & \ddots & \vdots & \vdots \\ & & & h_{n-1} & * \\ & & & & h_n \end{pmatrix}, \tag{3.43}$$

where $h_1 \in \mathrm{GL}_{r_1}, h_2 \in \mathrm{GL}_{r_2 - r_1}, \cdots, h_{n-1} \in \mathrm{GL}_{r_{n-1} - r_{n-2}}$, and $h_n \in \mathrm{GL}_{l - r_{n-1}}$ (see (3.24)–(3.27)). Next, we describe $(Z_{n,k;\epsilon}, \psi_{Z_{n,k;\epsilon}}^{h,g})$ which defines the inner Jacquet module in the integral

(3.37), for $h = \omega_0^r$ and $g = g'_{l,r}$. The group $Z_{n,k;\epsilon}$ consists of the matrices of the form

$$z = \begin{pmatrix} Z_1 & C & Y \\ 0 & Z_2 & 0 \\ 0 & S & I_{2(n+k)} \end{pmatrix}, \tag{3.44}$$

where $Z_1 \in \mathrm{GL}_r$ has the form

$$Z_1 = \begin{pmatrix} I_{r_1} & x_{1,2} & \cdots & x_{1,n-2} & x_{1,n-1} \\ & I_{r_2} & \cdots & x_{2,n-2} & x_{2,n-1} \\ & & \ddots & \vdots & \vdots \\ & & & I_{r_{n-2}} & x_{n-2,n-1} \\ & & & & I_{r_{n-1}} \end{pmatrix}; \tag{3.45}$$

$Z_2 \in \mathrm{GL}_{2k(n-1)-r}$ has the form

$$Z_2 = \begin{pmatrix} I_{t_{n-1}} & v_{n-2,n-1} & \cdots & v_{2,n-1} & v_{1,n-1} \\ & I_{t_{n-2}} & \cdots & v_{2,n-2} & v_{1,n-2} \\ & & \ddots & \vdots & \vdots \\ & & & I_{t_2} & v_{1,2} \\ & & & & I_{t_1} \end{pmatrix}. \tag{3.46}$$

This is directly deduced from (3.33)–(3.36). Write

$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n-1} \end{pmatrix}, \quad S = (s_{n-1}, s_{n-2}, \cdots, s_1), \tag{3.47}$$

where $y_i \in M_{r_i \times 2(n+k)}$ and $s_i \in M_{2(n+k)\times t_i}$, $1 \le i \le n-1$. Now, for z as in (3.44)–(3.46), let $z' = \begin{pmatrix} z & \\ & z^* \end{pmatrix}$, and write

$$z^* = \begin{pmatrix} I_{2(n+k)} & 0 & Y' \\ S' & Z_2^* & C' \\ 0 & 0 & Z_1^* \end{pmatrix} \tag{3.48}$$

with

$$Z_2^* = J_{m-r}({}^tZ_2^{-1})J_{m-r} = \begin{pmatrix} I_{t_1} & v'_{1,2} & \cdots & v'_{1,n-1} \\ & I_{t_2} & \cdots & v'_{2,n-1} \\ & & \ddots & \vdots \\ & & & v'_{n-2,n-1} \\ & & & I_{t_{n-1}} \end{pmatrix}, \tag{3.49}$$

$Y' = -J_{2(n+k)}({}^tY)J_r$, $C' = -J_{m-r}({}^tC)J_r$, and

$$S' = -J_{m-r}({}^tS)J_{2(n+k)} = \begin{pmatrix} s'_1 \\ s'_2 \\ \vdots \\ s'_{n-1} \end{pmatrix}, \tag{3.50}$$

where $s'_i \in M_{t_i\times 2(n+k)},\ 1\leq i\leq n-1$. We have

$$w_r^{-1}z'w_r = \begin{pmatrix} Z_1 & 0 & Y & 0 & C & 0\\ 0 & Z_2^* & 0 & S' & 0 & C'\\ 0 & 0 & I_{2(n+k)} & 0 & S & 0\\ 0 & 0 & 0 & I_{2(n+k)} & 0 & Y'\\ 0 & 0 & 0 & 0 & Z_2 & 0\\ 0 & 0 & 0 & 0 & 0 & Z_1^* \end{pmatrix}^{\omega_0^r}. \tag{3.51}$$

Now conjugate $w_r^{-1}z'w_r$ by $\widehat{\epsilon}^{-1}$, and note that

$$\epsilon^{-1}\begin{pmatrix} Z_1 & \\ & Z_2^* \end{pmatrix}\epsilon = \begin{pmatrix} I_{2k} & X_1 & * & \cdots & *\\ & I_{2k} & X_2 & \cdots & *\\ & & \ddots & \vdots & \vdots\\ & & & & X_{n-2}\\ & & & & I_{2k} \end{pmatrix}, \tag{3.52}$$

where for $1\leq i\leq n-2$,

$$X_i = \begin{pmatrix} x_{i,i+1} & 0\\ 0 & v'_{i,i+1} \end{pmatrix}. \tag{3.53}$$

Recall that $x_{i,i+1}\in M_{r_i\times r_{i+1}}$ and $v'_{i,i+1}\in M_{t_i\times t_{i+1}}$. Note also that

$$\epsilon^{-1}\begin{pmatrix} Y & 0\\ 0 & S' \end{pmatrix}\omega_0^r = \begin{pmatrix} y_1 & 0\\ 0 & s'_1\\ y_2 & 0\\ 0 & s'_2\\ \vdots & \vdots\\ y_{n-1} & 0\\ 0 & s'_{n-1} \end{pmatrix}\omega_0^r. \tag{3.54}$$

Now it is straight-forward to compute $\gamma_{l,r}^{-1}\widehat{\epsilon}^{-1}w_r^{-1}z'w_r\widehat{\epsilon}\gamma_{l,r}\in Z_{n,k}$, and get, using (3.44)–(3.54), that

$$\begin{aligned} &\psi_{Z_{n,k;\epsilon}}^{\omega_0^r,g'_{l,r}}(z')\\ &=\psi_{Z_{n,k}}(\gamma_{l,r}^{-1}\widehat{\epsilon}^{-1}w_r^{-1}z'w_r\widehat{\epsilon}\gamma_{l,r})\\ &=\psi\Big(\sum_{i=1}^{n-2}\mathrm{tr}\begin{pmatrix} x_{i,i+1} & 0\\ 0 & v'_{i,i+1}\end{pmatrix}^{\omega_0^r}\Big)\psi\Big(\mathrm{tr}\omega_0^r\begin{pmatrix} y_{n-1} & 0\\ 0 & s'_{n-1}\end{pmatrix}g'_l\omega_0^r\begin{pmatrix} 0_{(2n+k)\times 2k}\\ I_{2k}\\ 0_{(2n+k)\times 2k}\end{pmatrix}\Big). \end{aligned} \tag{3.55}$$

This product can be expressed as a product of the following four terms:

$$\prod_{i=1}^{n-2}\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} I_{r_i} \\ 0_{(r_{i+1}-r_i)\times r_i}\end{pmatrix} x_{i,i+1}\Big)\Big),$$

$$\prod_{i=1}^{n-2}\psi^{-1}\Big(\mathrm{tr}\Big(\begin{pmatrix} I_{t_{i+1}} \\ 0_{(t_i-t_{i+1})\times t_{i+1}}\end{pmatrix} v_{i,i+1}\Big)\Big),$$

$$\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} 0_{(2n+k)\times r_{n-1}} \\ I_{r_{n-1}} \\ 0_{(k-r_{n-1})\times r_{n-1}}\end{pmatrix} y_{n-1}\Big)\Big),$$

$$\psi^{-1}\Big(\mathrm{tr}\Big(\begin{pmatrix} 0_{k\times(k-l)} & 0_{k\times(2n+l)} & I_k \\ I_{k-l} & 0_{(k-l)\times(2n+l)} & 0_{(k-l)\times k} \\ 0_{(l-r_{n-1})\times(k-l)} & 0_{(l-r_{n-1})\times(2n+l)} & 0_{(l-r_{n-1})\times k}\end{pmatrix} s_{n-1}\Big)\Big).$$

Let us re-denote

$$\psi^{l,r}_{Z_{n,k;\epsilon}} = \psi^{\omega_0^r, g'_{l,r}}_{Z_{n,k;\epsilon}}.$$

The integrand in (3.37) depends on the twisted Jacquet module of τ' with respect to

$$(Z_{n,k;\epsilon}, \psi^{l,r}_{Z_{n,k;\epsilon}}), \quad \mathcal{J}_{\psi^{l,r}_{Z_{n,k;\epsilon}}}(\tau').$$

Now, the representation $\sigma^{\epsilon}_{[r];l}$ is obtained, up to a certain positive character, by applying $\sigma^{\epsilon}_{(r)}$ to $\gamma_{l,r}x\gamma_{l,r}^{-1}$, for

$$x \in \gamma_{l,r}^{-1}Q_{n,k,r;\epsilon}\gamma_{l,r} \cap (\mathrm{SO}_{2k}^{\Delta_n} \times \mathrm{SO}_{4n+2k}),$$

and applying $\mathcal{J}_{\psi^{l,r}_{Z_{n,k;\epsilon}}}(\tau')$. The precise form of the positive character can be determined from (3.33) and from an appropriate Jacobian resulting from conjugation on $Z'_{n,k;\epsilon}\backslash Z_{n,k}$ in (3.37). At the moment, this is not important to us. Thus we reduce the calculation of $\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))$ to that of the twisted Jacquet modules $\mathcal{J}_{\psi^{l,r}_{Z_{n,k;\epsilon}}}(\tau')$, for $0 \le l \le k$ and (r,ϵ) as in Proposition 3.1. This we will do in the next section. It will be more convenient to apply to the last Jacquet module a conjugation by

$$w = \begin{pmatrix} I_r & & & \\ & & & I_{k-l} \\ & & I_{2n+k+l} & \\ & I_{m-r} & & \end{pmatrix}. \tag{3.56}$$

Denote $Z^{\epsilon}_{n,k} = wZ_{n,k;\epsilon}w^{-1}$, and let $\psi^{l,r}_{Z^{\epsilon}_{n,k}}$ be the character of $Z^{\epsilon}_{n,k}$ given by

$$\psi^{l,r}_{Z^{\epsilon}_{n,k}}(z) = \psi^{l,r}_{Z_{n,k;\epsilon}}(w^{-1}zw).$$

The elements of $Z^{\epsilon}_{n,k}$ have the form

$$z = \begin{pmatrix} Z_1 & Y & C \\ & I_{2(n+k)} & S \\ & & Z_2\end{pmatrix}, \tag{3.57}$$

where Z_1, Z_2, Y, S, C are as in (3.44)–(3.47), and with the same notation, the character $\psi^{l,r}_{Z^{\epsilon}_{n,k}}(z)$ is equal to the product of the following two terms:

$$\prod_{i+1}^{n-2}\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} I_{r_i} \\ 0_{(r_{i+1}-r_i)\times r_i}\end{pmatrix} x_{i,i+1}\Big)\Big)\psi^{-1}\Big(\mathrm{tr}\Big(\begin{pmatrix} I_{t_{i+1}} \\ 0_{(t_i-t_{i+1})\times t_{i+1}}\end{pmatrix} v_{i,i+1}\Big)\Big) \tag{3.58}$$

and

$$\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} 0_{(2n+l)\times r_{n-1}} \\ I_{r_{n-1}} \\ 0_{(2k-l-r_{n-1})\times r_{n-1}} \end{pmatrix} y_{n-1}\Big)\Big)\psi^{-1}\Big(\mathrm{tr}\Big(\begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix} s_{n-1}\Big)\Big).$$

In the next section, we will study $\mathcal{J}_{\psi^{l,r}_{Z^{\epsilon}_{n,k}}}(\tau')$.

4 Analysis of $\mathcal{J}_{\psi^{l,r}_{Z^{\epsilon}_{n,k}}}(\tau')$

We apply the Mackey theory once again, and consider the set of double cosets

$$P_{(2k+2)^n}\backslash \mathrm{GL}_{2n(k+1)}/P_{\underline{r};2(n+k);\underline{t}}, \tag{4.1}$$

where we denote

$$\underline{r}=(r_1,\cdots,r_{n-1}),\quad \underline{t}=(t_{n-1},\cdots,t_1).$$

By [16, p. 170], we can choose a set of representatives for (4.1) as follows. Consider $n\times(2n-1)$ matrices $\underline{k}=(k_{i,j})$, with $k_{i,j}$ being non-negative integers, satisfying the following conditions:

$$\begin{aligned} &\sum_{i=1}^{n} k_{i,j}=r_j,\quad 1\le j\le n-1;\\ &\sum_{i=1}^{n} k_{i,n}=2(n+k);\\ &\sum_{i=1}^{n} k_{i,n+j}=t_{n-j},\quad 1\le j\le n-1;\\ &\sum_{j=1}^{2n-1} k_{i,j}=2(k+1),\quad 1\le i\le n. \end{aligned} \tag{4.2}$$

For such a matrix $\underline{k}=(k_{i,j})$, consider the Weyl element

$$w_{\underline{k}}=(w_{i,j})\in \mathrm{GL}_{2n(k+1)}, \tag{4.3}$$

where $1\le i\le n,\ 1\le j\le 2n-1$, and for $i=1,2,\cdots,n$, $w_{i,j}$ are matrices of sizes $2(k+1)\times r_j$ when $1\le j\le n-1$, and of size $2(k+1)\times 2(n+k)$ when $j=n$, and finally, $w_{i,n+j}$ is of size $2(k+1)\times t_{n-j}$, when $1\le j\le n-1$. Each matrix $w_{i,j}$ is divided into blocks: The rows in $w_{i,j}$ are divided into $2n-1$ blocks of sizes $k_{i,1},k_{i,2},\cdots,k_{i,2n-1}$, respectively; and the columns in $w_{i,j}$ are divided into n blocks of sizes $k_{1,j},k_{2,j},\cdots,k_{n,j}$, respectively. Thus, $w_{i,j}$ is a block matrix $(w^{f,s}_{i,j})$, where $1\le f\le 2n-1,\ 1\le s\le n$; the matrix $w^{f,s}_{i,j}$ is of size $k_{i,f}\times k_{s,j}$. Finally, all blocks $w^{f,s}_{i,j}$ of $w_{i,j}$ are zero, except the (j,i)-th block, $w^{j,i}_{i,j}$, where we have $w^{j,i}_{i,j}=I_{k_{i,j}}$. The semi-simplification of the restriction of τ' to $P_{\underline{r};2(n+k);\underline{t}}$ is the direct sum of the representations

$$\mathrm{ind}^{P_{\underline{r};2(n+k);\underline{t}}}_{P_{\underline{r};2(n+k);\underline{t}}\cap w_{\underline{k}}^{-1}P_{(2k+2)^n}w_{\underline{k}}}\chi^{w_{\underline{k}}}, \tag{4.4}$$

where for $x\in P_{\underline{r};2(n+k);\underline{t}}\cap w_{\underline{k}}^{-1}P_{(2k+2)^n}w_{\underline{k}}$,

$$\chi^{w_{\underline{k}}}(x)=\delta^{\frac12}_{P_{(2k+2)^n}}(\chi_1\circ\det\otimes\cdots\otimes\chi_n\circ\det)(w_{\underline{k}}xw_{\underline{k}}^{-1}),$$

as $w_{\underline{k}}$ varies over the Weyl elements (4.3) described above, with the matrix $\underline{k}$ as in (4.2). As in the previous section, only relevant $w_{\underline{k}}$ contributes to our Jacquet module. That is, for irrelevant $w_{\underline{k}}$, the Jacquet module of the representation (4.4) with respect to $\psi^{l,r}_{Z^{\epsilon}_{n,k}}$ is zero. Here $w_{\underline{k}}$ is called irrelevant (with respect to $\psi^{l,r}_{Z^{\epsilon}_{n,k}}$) if for each γ in the Levi part of $P_{\underline{r};2(n+k);\underline{t}}$, there exists $z \in Z^{\epsilon}_{n,k}$, such that $w_{\underline{k}} z w_{\underline{k}}^{-1} \in P_{(2k+2)^n}$, and $\psi^{l,r}_{Z^{\epsilon}_{n,k}}(\gamma^{-1} z \gamma) \neq 1$. Note that $Z^{\epsilon}_{n,k}$ is the unipotent radical of $P_{\underline{r};2(n+k);\underline{t}}$, and that for z as above, since $w_{\underline{k}} z w_{\underline{k}}^{-1}$ is a unipotent element of $P_{\underline{r};2(n+k);\underline{t}}$, one must have $\chi^{\underline{k}}(z) = 1$. Denote, for $1 \leq i \leq n$, $1 \leq j \leq n-1$,

$$k'_{i,j} = k_{i,2n-j}.$$

Proposition 4.1 *Assume that $w_{\underline{k}}$ is relevant. Then the matrix $\underline{k}$ satisfies the following properties*:

(1) *For all $1 \leq i \leq j \leq n-1$, $k'_{i,j} = 0$.*

(2) *For all $1 \leq i \leq n-2$ and $1 \leq j \leq n-i$,*

$$k'_{i+1,i+1} + k'_{i+2,i+1} + \cdots + k'_{i+j,i+1} \leq k'_{i,i} + k'_{i+1,i} + \cdots + k'_{i+j-1,i}.$$

In particular, $k'_{n-1,n-1} \leq k'_{n-2,n-2} \leq \cdots \leq k'_{2,2} \leq k'_{1,1}$.

(3) $k_{n,n}, k_{n-1,n-1} \geq 2k - l$.

(4) *For all $1 \leq i \leq n-1$ and $1 \leq j \leq n-i$, $k_{i,j} = 0$.*

(5) *For all $1 \leq i \leq n-1$ and $1 \leq j \leq n-i$,*

$$k_{i+1,n-i} + \cdots + k_{i+j,n-i} \leq k_{i,n-i+1} + \cdots + k_{i+j-1,n-i+1}.$$

In particular, $k_{n-1,2} \leq k_{n-2,3} \leq \cdots \leq k_{2,n-1} \leq k_{1,n}$.

Proof Let $1 \leq i \leq n-2$, and let $z \in Z^{\epsilon}_{n,k}$ be of the form (3.57), with $Z_1 = I_r$, $Y = 0$, $C = 0$, $S = 0$, and Z_2 be of the form (3.46), such that all blocks $v_{i,j}$ are zero, except the block $v_{i,i+1}$, which we now denote by X. Note that $X \in M_{t_{i+1} \times t_i}$. Write X as a block matrix $(X_{k'_{s,i+1},k'_{e,i}})_{1 \leq s,e \leq n}$, where $X_{k'_{s,i+1},k'_{e,i}} \in M_{k'_{s,i+1} \times k'_{e,i}}$. Then a simple calculation shows that $w_{\underline{k}} z w_{\underline{k}}^{-1} \in P_{(2k+2)^n}$, if and only if $X_{k'_{s,i+1},k'_{e,i}} = 0$ for all $1 \leq e < s \leq n-1$. If $w_{\underline{k}}$ is relevant, then we must have, for all X as above, and all $g \in \mathrm{GL}_{t_{i+1}}$ and $h \in \mathrm{GL}_{t_i}$,

$$\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} I_{t_{i+1}} \\ 0_{(t_i - t_{i+1}) \times t_{i+1}} \end{pmatrix} g^{-1}\Big) X h\Big) = 1.$$

This means that the matrix $h \begin{pmatrix} I_{t_{i+1}} \\ 0_{(t_i - t_{i+1}) \times t_{i+1}} \end{pmatrix} g^{-1}$ has the form of a block matrix

$$\begin{pmatrix} 0 & * & * & \cdots & * & * \\ 0 & 0 & * & \ldots & * & * \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & * \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

where the rows are of sizes, $k'_{1,i}, k'_{2,i}, \cdots, k'_{n,i}$, respectively, and the columns are of sizes $k'_{1,i+1}$, $k'_{2,i+1}, \cdots, k'_{n,i+1}$, respectively. Since the rank of this matrix is

$$t_{i+1} = k'_{1,i+1} + k'_{2,i+1} + \cdots + k'_{n,i+1},$$

we must have

$$\begin{aligned} &k'_{1,i+1} = 0, \\ &k'_{2,i+1} + k'_{3,i+1} + \cdots + k'_{j,i+1} \le k'_{1,i} + k'_{2,i} + \cdots + k'_{j-1,i}, \quad 1 \le j \le n. \end{aligned} \tag{4.5}$$

From this it is easy to conclude the first two parts of the proposition. Next, we repeat the argument with $z \in Z^{\epsilon}_{n,k}$ of the form (3.57), with $Z_1 = I_r$, $Y = 0$, $C = 0$, $Z_2 = I_{m-r}$, and S with $s_i = 0$, for $1 \le i \le n-2$, and $s_{n-1} = X \in M_{2(n+k)\times t_{n-1}}$. As before, If $w_{\underline{k}}$ is relevant, then we must have, for all $g \in \mathrm{GL}_{2(n+k)}$ and $h \in \mathrm{GL}_{t_{n-1}}$,

$$\psi\Big(\mathrm{tr}\Big(\begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix}\Big) g^{-1} X h\Big) = 1$$

for all X of the form $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$, where the size of the columns is $k'_{n-1,n-1}, k'_{n,n-1}$ and the size of the rows is $2(n+k) - k_{n,n}, k_{n,n}$. This means that

$$h\begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix} g^{-1}$$

has the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$, where $*$ is of size $k'_{n-1,n-1} \times k_{n,n}$. Comparing ranks, we get the third part of the proposition. The rest of the proposition is obtained similarly, by repeating the argument with z of the form (3.57), with $Z_1 = I_r$, $S = 0$, $C = 0$, $Z_2 = I_{m-r}$, and Y with $y_i = 0$, for $1 \le i \le n-2$, and $y_{n-1} = X \in M_{r_{n-1}\times 2(n+k)}$, and next by considering z of the form (3.57), with $Z_2 = I_{m-r}$, $Y = 0$, $C = 0$, $S = 0$, and S and Z_1 such that all blocks $x_{i,j}$ are zero except for the block $x_{i,i+1} \in M_{r_i \times r_{i+1}}$.

Proposition 4.1 implies that a relevant $w_{\underline{k}}$ has the form

$$w_{\underline{k}} = (w, w'), \tag{4.6}$$

where

$$w = \begin{pmatrix} 0 & 0 & \cdots & 0 & w_{1,n} \\ 0 & 0 & \cdots & w_{2,n-1} & w_{2,n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & w_{n-1,2} & \cdots & w_{n-1,n-1} & w_{n-1,n} \\ w_{n,1} & w_{n,2} & & w_{n,n-1} & w_{n,n} \end{pmatrix}$$

and

$$w' = \begin{pmatrix} 0 & 0 & \cdots & 0 & w_{1,2n-1} \\ 0 & 0 & \cdots & w_{2,2n-2} & w_{2,2n-1} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & w_{n-2,n+1} & \cdots & w_{n-2,2n-2} & w_{n-2,2n-1} \\ w_{n-1,n+1} & w_{n-1,n+2} & & w_{n-1,2n-2} & w_{n-1,2n-1} \\ w_{n,n+1} & w_{n,n+2} & & w_{n,2n-2} & w_{n,2n-1} \end{pmatrix}.$$

Note that in w the columns are of sizes $r_1, r_2, \cdots, r_{n-1}, 2(n+k)$ and in w', the columns are of sizes $t_{n-1}, t_{n-2}, \cdots, t_2, t_1$. The rows in both w and w' are all of size $2(k+1)$. Consider the stabilizer $S_{\underline{k}}$, of $w_{\underline{k}}$ inside the Levi part of $P_{\underline{r};2(n+k);\underline{t}}$, namely, the subgroup of all $\mathrm{diag}(g_1, \cdots, g_{n-1}, b, h_{n-1}, \cdots, h_1)$, with $g_i \in \mathrm{GL}_{r_i}$ and $h_i \in \mathrm{GL}_{t_i}$; $1 \le i \le n-1$ and $b \in \mathrm{GL}_{2(n+k)}$, whose conjugation by $w_{\underline{k}}$ lies in $P_{2(k+1)^n}$. This stabilizer can be found directly, using similar computations as in the last proof, so we omit the details.

Proposition 4.2 *For relevant $w_{\underline{k}}$,*

$$S_{\underline{k}} = \mathrm{GL}_{r_1} \times P^{r_2}_{k_{n-1,2},k_{n,2}} \times P^{r_3}_{k_{n-2,3},k_{n-1,3},k_{n,3}} \times \cdots \times P^{r_{n-1}}_{k_{2,n-1},k_{3,n-1},\cdots,k_{n,n-1}}$$
$$\times P^{2(n+k)}_{k_{1,n},k_{2,n},\cdots,k_{n,n}} \times P^{t_{n-1}}_{k'_{n-1,n-1},k'_{n,n-1}} \times P^{t_{n-2}}_{k'_{n-2,n-2},k'_{n-1,n-2},k'_{n,n-2}} \times \cdots \times P^{t_1}_{k'_{1,1},k'_{2,1},\cdots,k'_{n,1}}.$$

The proof of Proposition 4.1 shows more. It shows that, for relevant $w_{\underline{k}}$, if $w_{\underline{k}}\mathrm{diag}(g_1, \cdots, g_{n-1}, b, h_{n-1}, \cdots, h_1)$ supports the Jacquet module of the representation (4.4) with respect to $\psi^{l,r}_{Z^\epsilon_{n,k}}$, then we have, in particular, the following relations:

$$h_{n-1}\begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix} = \begin{pmatrix} 0_{k'_{n-1,n-1}\times(2(n+k)-k_{n,n})} & a \\ 0 & 0 \end{pmatrix} b, \tag{4.7}$$

where $a \in M_{k'_{n-1,n-1}\times k_{n,n}}$;

$$b\begin{pmatrix} 0_{(2n+l)\times r_{n-1}} \\ I_{r_{n-1}} \\ 0_{(2k-l-r_{n-1})\times r_{n-1}} \end{pmatrix} = \begin{pmatrix} c_{1,2} & c_{1,3} & \cdots & c_{1,n} \\ 0 & c_{2,3} & \cdots & c_{2,n} \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & c_{n-1,n} \\ 0 & 0 & \cdots & 0 \end{pmatrix} g_{n-1}, \tag{4.8}$$

where $c_{i,j} \in M_{k_{i,n}\times k_{j,n-1}}$ for $1 \le i \le n-1$, $2 \le j \le n$. By Proposition 4.2, we may multiply h_{n-1}, b and g_{n-1} from the left by $P^{t_{n-1}}_{k'_{n-1,n-1},k'_{n,n-1}}$, $P^{2(n+k)}_{k_{1,n},k_{2,n},\cdots,k_{n,n}}$ and $P^{r_{n-1}}_{k_{2,n-1},k_{3,n-1},\cdots,k_{n,n-1}}$, respectively. Since $\mathrm{rank}(a) = 2k-l$, we may multiply b and h_{n-1} from the left by matrices of the form $\mathrm{diag}(I_{2(n+k)}, \alpha)$ and $\mathrm{diag}(\beta, I_{k'_{n,n-1}})$, respectively, where $\alpha \in \mathrm{GL}_{k_{n,n}}$ and $\beta \in \mathrm{GL}_{k'_{n-1,n-1}}$, and replace (4.7) by

$$h_{n-1}\begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix} = \begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{(t_{n-1}-2k+l)\times(2n+l)} & 0 \end{pmatrix} b. \tag{4.9}$$

Recall that $k'_{n-1,n-1}, k_{n,n} \ge 2k-l$. Note also that the above modification of b from the left does not change the form of (4.8). The last equation implies that b has the form

$$b = \begin{pmatrix} b_1 & b_2 \\ 0 & b_4 \end{pmatrix}, \quad b_1 \in \mathrm{GL}_{2n+l},\ b_4 \in \mathrm{GL}_{2k-l}.$$

Using this in (4.8), and the fact that $k_{n,n} \ge 2k-l$, we conclude that the first r_{n-1} columns of b_4 must be zero. Since b_4 is invertible, this forces r_{n-1} to be zero! By Proposition 3.1, this means that $r = 0$ and $\epsilon = I_m$. This proves the following result.

Proposition 4.3 *Assume that (r, ϵ, l) is relevant with respect to $\psi_{Z_{n,k}}$. Then $r = 0$ and $\epsilon = I_m$.*

This simplifies the form of $w_{\underline{k}}$ in (4.6). Now $r_1 = \cdots = r_{n-1} = 0$, and hence $k_{i,j} = 0$ for all $1 \le i \le n$ and $1 \le j \le n-1$. Also, $t_1 = \cdots = t_{n-1} = 2k$. From Proposition 4.1, we have, for $1 \le i \le n-2$,

$$\begin{aligned} 2k &= t_{i+1} \\ &= k'_{i+1,i+1} + k'_{i+2,i+1} + \cdots + k'_{n,i+1} \\ &\le k'_{i,i} + k'_{i+1,i} + \cdots + k'_{n-1,i} \\ &= t_i - k'_{n,i} = 2k - k'_{n,i}. \end{aligned}$$

This implies that $k'_{n,i}=0$ for $1\le i\le n-2$. Repeating this argument, for $1\le i\le n-3$, we get

$$\begin{aligned}2k&=k'_{i+1,i+1}+k'_{i+2,i+1}+\cdots+k'_{n-1,i+1}\\&\le k'_{i,i}+k'_{i+1,i}+\cdots+k'_{n-2,i}\\&=2k-k'_{n-1,i},\end{aligned}$$

and hence $k'_{n-1,i}=0$ for $1\le i\le n-3$. We continue by induction (using Proposition 4.1) and get that

$$k'_{i,j}=0, \tag{4.10}$$

when $j\le i-2$ and $3\le i\le n$. Since

$$k'_{1,1}\ge k'_{2,2}\ge\cdots\ge k'_{n-1,n-1}\ge 2k-l\ge k,$$

let us write

$$k'_{i,i}=k+j_i,\quad 1\le i\le n-1, \tag{4.11}$$

where

$$k-l\le j_{n-1}\le\cdots\le j_2\le j_1\le k. \tag{4.12}$$

Denote $\underline{j}=(j_1,j_2,\cdots,j_{n-1})$. Sometimes, it will be convenient to denote $j_0=k$ and $j_n=k-l$. It follows that

$$\begin{aligned}k'_{i,i-1}&=k-j_{i-1},\quad 2\le i\le n,\\k_{i,n}&=j_{i-1}-j_i+2,\quad 1\le i\le n-1,\\k_{1,n}&=k+j_{n-1}+2.\end{aligned} \tag{4.13}$$

We will re-denote $w(\underline{j})=w_{\underline{k}}$, and also, for $1\le i,j\le n$,

$$w^{i,j}=w_{i,n-1+j}.$$

Thus $w(\underline{j})$ has the form

$$w(\underline{j})=\begin{pmatrix}w^{1,1}&0&0&0&0&\cdots&0&0&w^{1,n}\\w^{2,1}&0&0&0&0&\cdots&0&w^{2,n-1}&w^{2,n}\\w^{3,1}&0&0&0&0&\cdots&w^{3,n-2}&w^{3,n-1}&0\\\vdots&\vdots&\vdots&\vdots&\vdots&&\vdots&\vdots&\vdots\\w^{n-2,1}&0&w^{n-2,3}&w^{n-2,4}&0&\cdots&0&0&0\\w^{n-1,1}&w^{n-1,2}&w^{n-1,3}&0&0&\cdots&0&0&0\\w^{n,1}&w^{n,2}&0&0&0&\cdots&0&0&0\end{pmatrix}, \tag{4.14}$$

where the blocks $w^{i,j}$ are described through Proposition 4.1, with $r=0$ and (4.10)–(4.13).

Re-denote $\widetilde{Z}_{n,k}=Z^{I_m}_{n,k}$ ($\epsilon=I_m$) and $\psi^l_{\widetilde{Z}_{n,k}}=\psi^{l,0}_{Z^{I_m}_{n,k}}$ ($r=0$). Then $\widetilde{Z}_{n,k}$ is the subgroup of elements of the form

$$\widetilde{z}_{n,k}:=\begin{pmatrix}I_{2(n+k)}&S&*&\cdots&*&*\\&I_{2k}&X_1&\cdots&*&*\\&&I_{2k}&\cdots&*&*\\&&&\ddots&\vdots&\vdots\\&&&&I_{2k}&X_{n-2}\\&&&&&I_{2k}\end{pmatrix}, \tag{4.15}$$

and the character is given by

$$\psi^{l}_{\widetilde{Z}_{n,k}}(\widetilde{z}_{n,k}) = \psi^{-1}\Big(\mathrm{tr}(X_1 + \cdots + X_{n-2}) + \mathrm{tr}\Big(\begin{pmatrix} 0 & I_{2k-l} \\ 0_{l\times(2n+l)} & 0 \end{pmatrix} S\Big)\Big). \tag{4.16}$$

(see (3.57)–(3.58)). Denote the Levi subgroup of $P_{2(n+k),(2k)^{n-1}}$ by $M_{2(n+k),(2k)^{n-1}}$. We identify

$$M_{2(n+k),(2k)^{n-1}} = \mathrm{GL}_{2(n+k)} \times \mathrm{GL}_{2k}^{\times(n-1)}.$$

Note that the stabilizer of the character $\psi^{l}_{\widetilde{Z}_{n,k}}$ in $M_{2(n+k),(2k)^{n-1}}$ is the subgroup of $\mathrm{GL}_{2(n+k)} \times \mathrm{GL}_{2k}^{\times(n-1)}$ consisting of elements of the following type

$$\begin{pmatrix} \begin{pmatrix} h_1 & h_2 \\ 0 & a \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{\Delta(n-1)} \end{pmatrix} \in \mathrm{GL}_{2(n+k)} \times \mathrm{GL}_{2k}^{\times(n-1)}, \tag{4.17}$$

where $h_1 \in \mathrm{GL}_{2n+l}$, $a \in \mathrm{GL}_{2n-l}$, and $d \in \mathrm{GL}_l$. We denote this stabilizer by $\mathcal{S}_{n,k,l}$. Denote by $\mathcal{S}(w(\underline{j}))$ the stabilizer of $P_{(2k+2)^n}w(\underline{j})$ in $M_{2(n+k),(2k)^{n-1}}$. By Proposition 4.2, $\mathcal{S}(w(\underline{j}))$ consists of all elements of the form

$$\begin{pmatrix} h & & & \\ & g_2 & & \\ & & \ddots & \\ & & & g_n \end{pmatrix}, \tag{4.18}$$

where

$$h = \begin{pmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_n \end{pmatrix} \in \mathrm{GL}_{2(n+k)} \tag{4.19}$$

with $h_1 \in \mathrm{GL}_{k-j_1+2}$; $h_i \in \mathrm{GL}_{j_{i-1}-j_i+2}$, for $2 \le i \le n-1$; $h_n \in \mathrm{GL}_{k+j_{n-1}+2}$, and, for $2 \le s \le n$,

$$g_s = \begin{pmatrix} a_s & b_s \\ & d_s \end{pmatrix} \in \mathrm{GL}_{2k}, \tag{4.20}$$

where $a_s \in \mathrm{GL}_{k+j_{n+1-s}}$.

Proposition 4.4 *Let $\mathcal{A}_{\underline{j}}$ be the subset of elements $a \in M_{2(n+k),(2k)^{n-1}}$, such that $w(\underline{j})a$ supports $\mathcal{J}_{\psi^{l}_{\widetilde{Z}_{n,k}}}(\tau')$. Then*

$$\mathcal{A}_{\underline{j}} = \mathcal{S}(w(\underline{j}))\mathcal{S}_{n,k,l}.$$

Proof Let $a = \mathrm{diag}(h, g_2, \cdots, g_n)$ be such that $w(\underline{j})a$ supports $\mathcal{J}_{\psi^{l}_{\widetilde{Z}_{n,k}}}(\tau')$; $h \in \mathrm{GL}_{2(n+k)}$, $g_i \in \mathrm{GL}_{2k}$, $2 \le i \le n$. Then we see in the proof of Proposition 4.1 that we have the following relations:

$$g_{s+1}g_s^{-1} = \begin{pmatrix} \alpha_s & \beta_s \\ 0 & \delta_s \end{pmatrix}, \tag{4.21}$$

where $\alpha_s \in M_{k+j_{n-s}\times k+j_{n-s+1}}$, $2 \le s \le n-1$, and

$$g_2 \begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{l\times(2n+l)} & 0 \end{pmatrix} = \begin{pmatrix} 0_{(k+j_{n-1})\times(2n+k-j_{n-1}-2)} & \gamma \\ 0_{(k-j_{n-1})\times(2n+k-j_{n-1}-2)} & 0 \end{pmatrix} h, \tag{4.22}$$

where $\gamma \in M_{k+j_{n-1}\times k+j_{n-1}+2}$. Since $k+j_{n-s} \ge k+j_{n-s+1}$, we see that $\mathrm{rank}(\alpha_s) = k+j_{n-s+1}$. We may multiply g_{s+1} from the left by any matrix from $P^{2k}_{k+j_{n-s},k-j_{n-s}}$ by the description of (4.20). Thus, considering a, modulo $\mathcal{S}(w(\underline{j}))$ from the left, we may replace (4.21) by

$$g_{s+1} = \begin{pmatrix} I_{k+j_{n-s+1}} & \beta'_s \\ 0 & \delta'_s \end{pmatrix} g_s,$$

where $\delta'_s \in \mathrm{GL}_{k-j_{n-s+1}}$. Again, by (4.20), we conclude that modulo $\mathcal{S}(w(\underline{j}))$ from the left, we may assume that a is such that $g_2 = g_3 = \cdots = g_n = g$. Now in (4.22), $\mathrm{rank}(\gamma) = 2k-l$, and since $k+j_{n-1}+2 > 2k-l$, we may multiply h from the left by an element of the form $\mathrm{diag}(I_{2n+k-j_{n-1}-2}, \eta)$, $\eta \in \mathrm{GL}_{k+j_{n-1}+2}$, and replace (4.22) by

$$g \begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{l\times(2n+l)} & 0 \end{pmatrix} = \begin{pmatrix} 0_{(2k-l)\times(2n+l)} & I_{2k-l} \\ 0_{l\times(2n+l)} & 0 \end{pmatrix} h.$$

This implies that $g = \begin{pmatrix} \alpha & \beta \\ 0 & \delta \end{pmatrix}$, where $\alpha \in \mathrm{GL}_{2k-l}$, and $h = \begin{pmatrix} c & x \\ 0 & \alpha \end{pmatrix}$. By (4.17), this means that modulo $\mathcal{S}(w(\underline{j}))$ from the left, a lies in $\mathcal{S}_{n,k,l}$. This proves the proposition.

As an immediate corollary, we get the following result.

Proposition 4.5 *Up to semi-simplification,*

$$\mathcal{J}_{\psi^l_{\widetilde{Z}_{n,k}}}(\tau') \cong \bigoplus_{\underline{j}} \mathcal{J}_{\psi^l_{\widetilde{Z}_{n,k}}}(\mathrm{ind}^{\mathcal{S}_{n,k,l}\cdot\widetilde{Z}_{n,k}}_{\mathcal{S}_{n,k,l}\cdot\widetilde{Z}_{n,k}\cap w(\underline{k})^{-1}P_{(2k+2)^n}w(\underline{j})}(\delta^{\frac{1}{2}}_{P_{(2k+2)^n}}\cdot\chi)^{w(\underline{j})}), \tag{4.23}$$

where $w(\underline{j})$ varies over all Weyl elements (4.14).

Each of the summands in (4.23) is isomorphic to

$$\mathrm{ind}^{\mathcal{S}_{n,k,l}}_{\mathcal{S}^{(w(\underline{j}))}_{n,k,l}}(\delta_{\widetilde{Z}^{w(\underline{j})}_{n,k}\backslash\widetilde{Z}_{n,k}}\cdot(\delta^{\frac{1}{2}}_{P_{(2k+2)^n}}\cdot\chi)^{w(\underline{j})}), \tag{4.24}$$

where

$$\widetilde{Z}^{w(\underline{j})}_{n,k} = \widetilde{Z}_{n,k}\cap w(\underline{j})^{-1}P_{(2k+2)^n}w(\underline{j})$$

and

$$\mathcal{S}^{(w(\underline{j}))}_{n,k,l} := \mathcal{S}_{n,k,l}\cap w(\underline{j})^{-1}P_{(2k+2)^n}w(\underline{j}).$$

The isomorphism is induced by the map which sends a function f in the space of $\mathrm{ind}^{\mathcal{S}_{n,k,l}\cdot\widetilde{Z}_{n,k}}_{\mathcal{S}_{n,k,l}\cdot\widetilde{Z}_{n,k}\cap w(\underline{k})^{-1}P_{(2k+2)^n}w(\underline{j})}(\delta^{\frac{1}{2}}_{P_{(2k+2)^n}}\cdot\chi)^{w(\underline{j})}$ to the function on $\mathcal{S}_{n,k,l}$ given by

$$e \mapsto \int_{\widetilde{Z}^{w(\underline{j})}_{n,k}\backslash\widetilde{Z}_{n,k}} f(ze)\psi^l_{\widetilde{Z}_{n,k}}(z^{-1})\mathrm{d}z.$$

Since

$$\mathcal{S}^{(w(\underline{j}))}_{n,k,l} = \mathcal{S}_{n,k,l}\cap\mathcal{S}(w(\underline{j})),$$

we see from (4.18)–(4.20) and (4.17) that an element in $\mathcal{S}_{n,k,\underline{l}}^{(w(\underline{j}))}$ must be of the form

$$\left(\begin{pmatrix} c & x \\ 0 & a \end{pmatrix}, \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}^{\Delta(n-1)}\right) \in \mathrm{GL}_{2n+2k} \times \mathrm{GL}_{2k}^{\Delta(n-1)}$$

and

$$\mathrm{diag}(h; g_2, \cdots, g_n)$$

with h as in (4.19) and g_s as in (4.20) for $s = 2, 3, \cdots, n$. Hence $\mathcal{S}_{n,k,\underline{l}}^{(w(\underline{j}))}$ is the subgroup of the elements of the form

$$\left(\begin{pmatrix} h_1 & * & \cdots & * & * \\ & h_2 & \cdots & * & * \\ & & \ddots & \vdots & \vdots \\ & & & h_n & * \\ & & & & \epsilon_{n+1} \end{pmatrix}, \begin{pmatrix} \epsilon_{n+1} & * & \cdots & * & * \\ & \epsilon_n & \cdots & * & * \\ & & \ddots & \vdots & \vdots \\ & & & \epsilon_2 & * \\ & & & & \epsilon_1 \end{pmatrix}^{\Delta(n-1)}\right) \tag{4.25}$$

in $\mathrm{GL}_{2(n+k)} \times \mathrm{GL}_{2k}^{\times(n-1)}$, where $h_1 \in \mathrm{GL}_{k-j_1+2}$, $\epsilon_1 \in \mathrm{GL}_{k-j_1}$; for $i = 2, 3, \cdots, n-1$, $h_i \in \mathrm{GL}_{j_{i-1}-j_i+2}$ and $\epsilon_i \in \mathrm{GL}_{j_{i-1}-j_i}$; $h_n \in \mathrm{GL}_{j_{n-1}-k+l+2}$, $\epsilon_n \in \mathrm{GL}_{j_{n-1}-k+l}$, and $\epsilon_{n+1} \in \mathrm{GL}_{2k-l}$.

Denote in (4.24)

$$\delta_{n,k,\underline{l}}^{(w(\underline{j}))} = \delta_{\widetilde{Z}_{n,k}^{w(\underline{j})} \backslash \widetilde{Z}_{n,k}} (\delta_{P_{(2k+2)^n}}^{\frac{1}{2}})^{w(\underline{j})}, \quad \chi_{n,k,\underline{l}}^{(w(\underline{j}))} = \chi^{w(\underline{j})}. \tag{4.26}$$

Proposition 4.6 *The character $\delta_{n,k,\underline{l}}^{(w(\underline{j}))}$ in (4.26) takes an element in $\mathcal{S}_{n,k,\underline{l}}^{(w(\underline{j}))}$ of the form (4.25) to*

$$\prod_{i=1}^{n} |\det h_i|^{m_i} \cdot \prod_{i=1}^{n+1} |\det \epsilon_i|^{s_i} \tag{4.27}$$

with $m_i = n(k+1) - 2(i+k) + 1 + j_{i-1}$, $s_i = (n+k)(2-n) - 1 - j_{i-1}$ for $i = 1, 2, \cdots, n$, and $s_{n+1} = -n(n-1)$; and the character $\chi_{n,k,\underline{l}}^{(w(\underline{j}))}$ in (4.26) takes an element in $\mathcal{S}_{n,k,\underline{l}}^{(w(\underline{j}))}$ of the form (4.25) to

$$\prod_{i=1}^{n} \chi_i(\det h_i \cdot \det \epsilon_1 \cdots \widehat{\det \epsilon_i} \cdots \det \epsilon_{n+1}), \tag{4.28}$$

which can also be written as

$$\prod_{i=1}^{n} \chi_i(\det h_i) \cdot \prod_{i=1}^{n} (\chi_1 \cdots \widehat{\chi}_i \cdots \chi_n)(\det \epsilon_i) \cdot \Big(\prod_{i=1}^{n} \chi_i\Big)(\det \epsilon_{n+1}). \tag{4.29}$$

Proof Let $a = \mathrm{diag}(h, g, \cdots, g)$ be an element in $\mathcal{S}_{n,k,\underline{l}}^{(w(\underline{j}))}$ of the form (4.25) ($h \in \mathrm{GL}_{2(n+k)}$, and $g \in \mathrm{GL}_{2k}$ is repeated $n-1$ times). Then $w(\underline{j}) a w(\underline{j})^{-1} \in P_{(2k+2)^n}$. A straightforward multiplication shows that the block diagonal of $w(\underline{j}) a w(\underline{j})^{-1}$ consists of the following matrices in GL_{2k+2}: The first block is

$$w^{1,1} h^t w^{1,1} + w_{1,n} g^t w^{1,n}; \tag{4.30}$$

the next $n-2$ blocks are

$$w^{s,1}h^t w^{s,1} + w^{s,n-s+1}g^t w^{s,n-s+1} + w^{s,n-s+2}g^t w^{s,n-s+2}, \quad s = 2, \cdots, n-1; \tag{4.31}$$

and the last block is

$$w^{n,1}h^t w^{n,1} + w^{n,2}g^t w^{n,2}. \tag{4.32}$$

We used the form (4.14). Each matrix (4.30), (4.31), or (4.32) is a block upper triangular, and its diagonal blocks are, in the notation of (4.25), for $1 \le i \le n$,

$$\mathrm{diag}(h_i, \epsilon_{n+1}, \epsilon_n, \cdots, \widehat{\epsilon_i}, \cdots, \epsilon_1).$$

This contributes to the character $(\delta^{\frac{1}{2}}_{P_{(2k+2)^n}} \cdot \chi)^{w(\underline{j})}$ the product of the following two terms:

$$|\det(h_i)\det(\epsilon_1)\cdots\widehat{\det(\epsilon_i)}\cdots\det(\epsilon_{n+1})|^{(n-2i+1)(k+1)}$$

and

$$\chi_i(\det(h_i)\det(\epsilon_1)\cdot\cdots\cdot\widehat{\det(\epsilon_i)}\cdot\cdots\cdot\det(\epsilon_{n+1})).$$

Altogether, we get that the character $(\delta^{\frac{1}{2}}_{P_{(2k+2)^n}} \cdot \chi)^{w(\underline{j})}$ evaluated at an element in $\mathcal{S}^{(w(\underline{j}))}_{n,k,\underline{l}}$ of the form (4.25) is given by the product of

$$\Big(\prod_{i=1}^n |\det h_i|^{n-2i+1}\Big)^{k+1} \cdot \prod_{i=1}^n \chi_i(\det h_i)$$

and

$$\Big(\prod_{i=1}^n |\det \epsilon_i|^{2i-1-n}\Big)^{k+1} \cdot \prod_{i=1}^n (\chi_1\cdots\widehat{\chi_i}\cdots\chi_n)(\det \epsilon_i) \cdot \Big(\prod_{i=1}^n \chi_i\Big)(\det \epsilon_{n+1}).$$

Let us describe the subgroup $\widetilde{Z}^{w(\underline{j})}_{n,k}$. Write an element of $\widetilde{Z}^{w(\underline{j})}_{n,k}$ in the form (4.15)

$$\begin{pmatrix} I_{2(n+k)} & Z_{1,2} & Z_{1,3} & \cdots & Z_{1,n-1} & Z_{1,n} \\ & I_{2k} & Z_{2,3} & \cdots & Z_{2,n-1} & Z_{2,n} \\ & & I_{2k} & \cdots & Z_{3,n-1} & Z_{3,n} \\ & & & \ddots & \vdots & \vdots \\ & & & & I_{2k} & Z_{n-1,n} \\ & & & & & I_{2k} \end{pmatrix}. \tag{4.33}$$

For $2 \le r \le n$, write $Z_{1,r}$ as a block matrix $(Z^{(\ell,t)}_{1,r})_{\ell,t}$ of size $n \times 2$ ($1 \le \ell \le n$, $t = 1,2$) where $Z^{(1,1)}_{1,r} \in M_{(k-j_1+2)\times(k+j_{n-r+1})}$ and $Z^{(1,2)}_{1,r} \in M_{(k-j_1+2)\times(k-j_{n-r+1})}$; for $2 \le \ell \le n-1$, $Z^{(\ell,1)}_{1,r} \in M_{(j_{\ell-1}-j_\ell+2)\times(k+j_{n-r+1})}$ and $Z^{(\ell,2)}_{1,r} \in M_{(j_{\ell-1}-j_\ell+2)\times(k-j_{n-r+1})}$; finally, $Z^{(n,1)}_{1,r} \in M_{(k+j_{n-1}+2)\times(k+j_{n-r+1})}$ and $Z^{(n,2)}_{1,r} \in M_{(k+j_{n-1}+2)\times(k-j_{n-r+1})}$. Then these blocks must satisfy the conditions

$$Z^{(n+2-r,1)}_{1,r} = Z^{(n+3-r,1)}_{1,r} = \cdots = Z^{(n,1)}_{1,r} = 0, \quad 2 \le r \le n; \tag{4.34}$$

$$Z^{(n+3-r,2)}_{1,r} = \cdots = Z^{(n,2)}_{1,r} = 0, \quad 3 \le r \le n. \tag{4.35}$$

The rest of the $2k \times 2k$ matrices $Z_{i,j}$ in (4.33) has the following form. Let $2 \le i \le n-1$. Then

$$Z_{i,i+2} = \cdots = Z_{i,n} = 0. \tag{4.36}$$

Write

$$Z_{i,i+1} = \begin{pmatrix} Z_{i,i+1}^{(1,1)} & Z_{i,i+1}^{(1,2)} \\ Z_{i,i+1}^{(2,1)} & Z_{i,i+1}^{(2,2)} \end{pmatrix},$$

where $Z_{i,i+1}^{(1,1)} \in M_{(k+j_{n-i+1})\times(k+j_{n-i})}$.

Now we can compute the value of the character $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}}$ on elements $\mathrm{diag}(h, g, \cdots, g)$ in $\mathcal{S}_{n,k,l}^{(w(\underline{j}))}$ of the form (4.25). The contribution of h to this character is

$$\Big(\prod_{i=1}^{n} |\det h_i|^{2k(n-i)+(k-j_{i-1})}\Big)|\det \epsilon_{n+1}|^{k-j_{n-1}},$$

$j_0 = k$. The contribution of g to $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}}$ is

$$\Big(\prod_{i=1}^{n} |\det \epsilon_i|^{-(n+k)(n-1)-(n-i)(k+2)+(i-2)k+j_{i-1}}\Big)|\det \epsilon_{n+1}|^{-(n+k)(n-1)+(n-2)k+j_{n-1}}.$$

Altogether, the character $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}}$ evaluated at elements in $\mathcal{S}_{n,k,l}^{(w(\underline{k}))}$ of the form (4.25) is

$$|\det \epsilon_{n+1}|^{n(1-n)} \prod_{i=1}^{n} |\det h_i|^{2k(n-i)+(k-j_{i-1})}|\det \epsilon_i|^{(n+k)(1-n)+(1-n)(k+2)+(i-2)k+j_{i-1}}.$$

Now, the character $\delta_{\widetilde{Z}_{n,k}}$ evaluated at the elements in $\mathcal{S}_{n,k,l}^{(w(\underline{k}))}$ of the above form is

$$|\det h_1 \cdots \det h_n \cdot \det \epsilon_{n+1}|^{2k(n-1)} \cdot |\det \epsilon_1 \cdots \det \epsilon_{n+1}|^{2(n+k)(1-n)}.$$

Since the character $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}\backslash \widetilde{Z}_{n,k}}$ (of $\mathcal{S}_{n,k,l}^{(w(\underline{j}))}$) is the quotient of $\delta_{\widetilde{Z}_{n,k}}$ by $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}}$, we get that $\delta_{\widetilde{Z}_{n,k}^{w(\underline{j})}\backslash \widetilde{Z}_{n,k}}$ evaluated at the elements of $\mathcal{S}_{n,k}^{(w(\underline{j}))}$ of the above form is

$$|\det \epsilon_{n+1}|^{n(1-n)} \prod_{i=1}^{n} |\det h_i|^{2k(i-1)-(k-j_{i-1})}|\det \epsilon_i|^{(n+k)(1-n)+(n-i)(k+2)-(i-2)k-j_{i-1}}.$$

Putting together the above calculations, we obtain the desired formula.

Summarizing the last three propositions, we obtain the following result.

Proposition 4.7 *Up to semi-simplification of $\mathcal{S}_{n,k,l}$-modules*

$$\mathcal{J}_{\psi^l_{\widetilde{Z}_{n,k}}}(\tau') \cong \bigoplus_{w(\underline{j})} \mathrm{ind}_{\mathcal{S}_{n,k,l}^{(w(\underline{j}))}}^{\mathcal{S}_{n,k,l}} (\delta_{n,k,l}^{(w(\underline{j}))} \cdot \chi_{n,k,l}^{(w(\underline{j}))}),$$

where $w(\underline{j})$ runs over all relevant representatives given in (4.14) *and* (4.10)–(4.13), *and the inducing characters are given in Proposition* 4.6.

5 Proof of Theorem 2.1

By Proposition 4.3 and (3.41), we have, up to semi-simplification,

$$\begin{aligned}\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau')) &\cong \mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{ind}_{Q_{n,k,r=0:\epsilon=I_m}}^{Q_{(2k)^{n-1}}}(\sigma_{(0)}^{I_m}))\\ &\equiv \bigoplus_{l=0}^{k} \mathrm{ind}_{\gamma_{l,0}^{-1}Q_{n,k,0;I_m}\gamma_{l,0}\cap(\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k})}^{\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}}\sigma_{[0];l}^{I_m}.\end{aligned} \tag{5.1}$$

The subgroup $\gamma_{l,0}^{-1}Q_{n,k,0;I_m}\gamma_{l,0}\cap(\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k})$ is described in (3.42). Applying to the second factor in (3.42) conjugation by

$$\begin{pmatrix} & I_{2n+l} & & \\ I_{k-l} & 0 & & \\ & & 0 & I_{k-l} \\ & & I_{2n+l} & 0 \end{pmatrix},$$

the typical summand in (5.1) becomes

$$\mathrm{ind}_{R_{n,k}^l}^{\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}}\sigma_l', \tag{5.2}$$

where $R_{n,k}^l$ is the subgroup of elements

$$\left(\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta n}, \begin{pmatrix} a_4 & a_3 & b_4 & b_3 \\ 0 & a_1 & b_2 & b_4' \\ 0 & c_3 & d_4 & a_3' \\ 0 & 0 & 0 & a_4^* \end{pmatrix}\right) \in \mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}, \tag{5.3}$$

where $e_1\in\mathrm{GL}_l$, $\begin{pmatrix} d_4 & -c_3 \\ -b_2 & a_1\end{pmatrix}\in\mathrm{SO}_{2(k-l)}$, a_1 is of size $(k-l)\times(k-l)$, and $a_4\in\mathrm{GL}_{2n+l}$, as before. The representation σ_l' is expressed in terms of $\mathcal{J}_{\psi^{l,0}_{Z_{n,k;I_m}}}(\tau')$. Taking into account the conjugation by the element (3.56), we get that the representation (5.2) is isomorphic to

$$\mathrm{ind}_{R_{n,k}^l}^{\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}}\sigma_l, \tag{5.4}$$

where

$$\begin{aligned}&\sigma_l\left(\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta n}, \begin{pmatrix} a_4 & a_3 & b_4 & b_3 \\ 0 & a_1 & b_2 & b_4' \\ 0 & c_3 & d_4 & a_3' \\ 0 & 0 & 0 & a_4^* \end{pmatrix}\right)\\ &= |\det a_4\cdot\det e_1|^{\frac{2n+4k-2nk-1}{2}}\mathcal{J}_{\psi^l_{\tilde{Z}_{n,k}}}(\tau')\\ &\quad\cdot\left(\begin{pmatrix} \begin{pmatrix} a_4 & 0 & -b_4 & a_3 \\ 0 & e_1 & e_2 & e_3 \\ 0 & 0 & d_4 & -c_3 \\ 0 & 0 & -b_2 & a_1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta(n-1)} \end{pmatrix}\right).\end{aligned} \tag{5.5}$$

Let φ be the isomorphism which takes the element

$$\left(\begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta n}, \begin{pmatrix} a_4 & a_3 & b_4 & b_3 \\ 0 & a_1 & b_2 & b_4' \\ 0 & c_3 & d_4 & a_3' \\ 0 & 0 & 0 & a_4^* \end{pmatrix} \right) \in R^l_{n,k}$$

to

$$\begin{pmatrix} \begin{pmatrix} a_4 & 0 & -b_4 & a_3 \\ 0 & e_1 & e_2 & e_3 \\ 0 & 0 & d_4 & -c_3 \\ 0 & 0 & -b_2 & a_1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta(n-1)} \end{pmatrix}.$$

Denote by $\widehat{R}^l_{n,k} \subset \mathrm{GL}_{2n(k+1)}$ the image of φ. By (5.5) and Proposition 4.7, σ_l is expressed through the representations

$$\mathrm{Res}_{\widehat{R}^l_{n,k}}(\mathrm{ind}^{\mathcal{S}_{n,k,l}}_{\mathcal{S}^{(w(\underline{j}))}_{n,k,l}}(\delta^{(w(\underline{j}))}_{n,k,l} \cdot \chi^{(w(\underline{j}))}_{n,k,l})).$$

It is clear that $\mathcal{S}_{n,k,l} = \mathcal{S}^{(w(\underline{j}))}_{n,k,l} \cdot \widehat{R}^l_{n,k}$. Hence

$$\mathrm{Res}_{\widehat{R}^l_{n,k}}(\mathrm{ind}^{\mathcal{S}_{n,k,l}}_{\mathcal{S}^{(w(\underline{j}))}_{n,k,l}}(\delta^{(w(\underline{j}))}_{n,k,l} \cdot \chi^{(w(\underline{j}))}_{n,k,l})) \cong \mathrm{ind}^{\widehat{R}^l_{n,k}}_{\widehat{R}^l_{n,k} \cap \mathcal{S}^{(w(\underline{j}))}_{n,k,l}}(\widehat{\delta}^{(w(\underline{j}))}_{n,k,l} \cdot \widehat{\chi}^{(w(\underline{j}))}_{n,k,l}), \tag{5.6}$$

where $\widehat{\delta}^{(w(\underline{j}))}_{n,k,l}$ and $\widehat{\chi}^{(w(\underline{j}))}_{n,k,l}$ are the restriction of the characters $\delta^{(w(\underline{j}))}_{n,k,l}$ and $\chi^{(w(\underline{j}))}_{n,k,l}$, respectively, to the subgroup $\widehat{\mathcal{S}}^{(w(\underline{j}))}_{n,k,l} := \widehat{R}^l_{n,k} \cap \mathcal{S}^{(w(\underline{j}))}_{n,k,l}$. The subgroup $\widehat{\mathcal{S}}^{(w(\underline{j}))}_{n,k,l}$ consists of elements of the type

$$\begin{pmatrix} \begin{pmatrix} a_4 & 0 & -b_4 & a_3 \\ 0 & e_1 & e_2 & e_3 \\ 0 & 0 & d_4 & -c_3 \\ 0 & 0 & -b_2 & a_1 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} e_1 & e_2 & e_3 & e_4 \\ & d_4 & -c_3 & e_3' \\ & -b_2 & a_1 & e_2' \\ & & & e_1^* \end{pmatrix}^{\Delta(n-1)} \end{pmatrix} \tag{5.7}$$

with

$$a_4 = \begin{pmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_n \end{pmatrix} \in \mathrm{GL}_{2n+l}, \quad e_1 = \begin{pmatrix} \epsilon_1^* & * & \cdots & * \\ & \epsilon_2^* & \cdots & * \\ & & \ddots & \vdots \\ & & & \epsilon_n^* \end{pmatrix} \in \mathrm{GL}_l,$$

where $h_i \in \mathrm{GL}_{j_{i-1}-j_i+2}$, $\epsilon_i \in \mathrm{GL}_{j_{i-1}-j_i}$ for $i = 1, \cdots, n$ (letting $j_0 = k$, $j_n = k - l$). From (4.27), the character $\widehat{\delta}^{(w(\underline{j}))}_{n,k,l}$ evaluated at elements in $\widehat{\mathcal{S}}^{(w(\underline{j}))}_{n,k,l}$ of the form (5.7) is

$$\prod_{i=1}^{n} |\det h_i|^{n(k+1)-2(i+k)+1+j_{i-1}} \cdot \prod_{i=1}^{n} |\det \epsilon_i|^{(k-1)(1-n)+k-j_{i-1}}, \tag{5.8}$$

and from (4.29), the character $\widehat{\chi}_{n,k,l}^{(w(\underline{j}))}$ is

$$\prod_{i=1}^{n}\chi_i(\det h_i)\cdot\prod_{i=1}^{n}\chi_i^{-1}(\det\epsilon_i). \tag{5.9}$$

Define

$$\mathcal{S}_{n,k}^{l}(w(\underline{j})) := \varphi^{-1}(\widehat{\mathcal{S}}_{n,k,l}^{(w(\underline{j}))}). \tag{5.10}$$

This is a subgroup of $R_{n,k}^{l}$. Define also

$$\delta_{n,k}^{l}(w(\underline{j})) := \widehat{\delta}_{n,k,l}^{(w(\underline{j}))}\circ\varphi \quad\text{and}\quad \chi_{n,k}^{l}(w(\underline{j})) := \widehat{\chi}_{n,k,l}^{(w(\underline{j}))}\circ\varphi. \tag{5.11}$$

Then the induced representation (5.6) is pulled back to

$$\mathrm{ind}_{\mathcal{S}_{n,k}^{l}(w(\underline{j}))}^{R_{n,k}^{l}}(\delta_{n,k}^{l}(w(\underline{j}))\cdot\chi_{n,k}^{l}(w(\underline{j}))). \tag{5.12}$$

Hence, by (5.1) and Proposition 4.7, we obtain the following proposition.

Proposition 5.1 *Up to semi-simplification of* $\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}$*-modules,*

$$\begin{aligned}&\mathcal{J}_{\psi_{Z_{n,k}}}(\mathrm{Ind}_{Q_{2n(k+1)}}^{\mathrm{SO}_{4n(k+1)}}(\tau'))\\ \equiv&\bigoplus_{l=0}^{k}\Big[\bigoplus_{w(\underline{j})}\mathrm{ind}_{R_{n,k}^{l}}^{\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}}(\delta\cdot\mathrm{ind}_{\mathcal{S}_{n,k}^{l}(w(\underline{j}))}^{R_{n,k}^{l}}(\delta_{n,k}^{l}(w(\underline{j}))\cdot\chi_{n,k}^{l}(w(\underline{j}))))\Big],\end{aligned}$$

where δ *is given by*

$$\delta\left(\begin{pmatrix}e_1&e_2&e_3&e_4\\&d_4&-c_3&e_3'\\&-b_2&a_1&e_2'\\&&&e_1^*\end{pmatrix}^{\Delta n},\begin{pmatrix}a_4&a_3&b_4&b_3\\0&a_1&b_2&b_4'\\0&c_3&d_4&a_3'\\0&0&0&a_4^*\end{pmatrix}\right)=|\det a_4\cdot\det e_1|^{\frac{2n+4k-2nk-1}{2}},$$

and $w(\underline{j})$ *runs over all relevant representatives given in* (4.14) *and* (4.10)–(4.13).

From now on, in order to simplify our notation, we identify

$$\mathrm{SO}_{2k}^{\Delta n}\times\mathrm{SO}_{4n+2k}=\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}.$$

However, we still use the same notation for other data. Hence we consider the following induced representations:

$$\mathrm{ind}_{R_{n,k}^{l}}^{\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}}(\delta\cdot\mathrm{ind}_{\mathcal{S}_{n,k}^{l}(w(\underline{j}))}^{R_{n,k}^{l}}(\delta_{n,k}^{l}(w(\underline{j}))\cdot\chi_{n,k}^{l}(w(\underline{j})))), \tag{5.13}$$

which is the same as

$$\mathrm{ind}_{\mathcal{S}_{n,k}^{l}(w(\underline{j}))}^{\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}}(\delta\cdot\delta_{n,k}^{l}(w(\underline{j}))\cdot\chi_{n,k}^{l}(w(\underline{j}))). \tag{5.14}$$

Let us specify the data in (5.14). By (5.7) and the definition of φ, the subgroup $\mathcal{S}_{n,k}^{l}(w(\underline{j}))$ consists of the elements of the form

$$\left(\begin{pmatrix}e_1&&*&*\\&d_4&-c_3&\\&-b_2&a_1&*\\&&&e_1^*\end{pmatrix},\begin{pmatrix}a_4&&*&*\\&a_1&b_2&\\&c_3&d_4&*\\&&&a_4^*\end{pmatrix}\right)\in\mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k},$$

where $e_1 \in \mathrm{GL}_l$ and $a_4 \in \mathrm{GL}_{2n+l}$ are given respectively by

$$e_1 = \begin{pmatrix} \epsilon_1 & * & \cdots & * \\ & \epsilon_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & \epsilon_n \end{pmatrix}, \quad a_4 = \begin{pmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_n \end{pmatrix},$$

as in (5.7). By (5.8)–(5.9) and (5.11), and by the definition of δ in the last proposition, we get that the character $\delta \cdot \delta^l_{n,k}(w_{\underline{k}}) \cdot \chi^l_{n,k}(w_{\underline{k}})$ evaluated at the elements above of the subgroup $\mathcal{S}^l_{n,k}(w(\underline{j}))$ is given by

$$\prod_{i=1}^n \chi_i(\det h_i) \cdot |\det h_i|^{\frac{4(n-i)+2j_{i-1}+1}{2}} \cdot \prod_{i=1}^n \chi_i(\det \epsilon_i) \cdot |\det \epsilon_i|^{\frac{2j_{i-1}+1}{2}}. \tag{5.15}$$

In order to prove Theorem 2.1, it is enough to consider, by Proposition 5.1 and (5.13)–(5.14), for $0 \leq l \leq k$ and $w(\underline{j})$, as above,

$$\mathrm{Hom}_{\mathrm{SO}_{2k}\times \mathrm{SO}_{4n+2k}}([\mathrm{ind}^{\mathrm{SO}_{2k}\times \mathrm{SO}_{4n+2k}}_{\mathcal{S}^l_{n,k}(w(\underline{j}))}(\delta \cdot \delta^l_{n,k}(w(\underline{j})) \cdot \chi^l_{n,k}(w(\underline{j})))] \otimes \sigma^\vee \otimes \pi^{\omega_0^{n(k+1)}}, 1).$$

The assumption of Theorem 2.1 implies that there are l and $w(\underline{j})$, such that this Hom-space is nonzero. In this case, we get, by Frobenius reciprocity, that

$$\mathrm{Hom}_{\mathcal{S}^l_{n,k}(w(\underline{j}))}(\sigma^\vee \otimes \pi^{\omega_0^{n(k+1)}}, \Delta \cdot (\delta \cdot \delta^l_{n,k}(w(\underline{j})) \cdot \chi^l_{n,k}(w(\underline{j})))^{-1}) \neq 0, \tag{5.16}$$

where Δ is the modular function of the subgroup $\mathcal{S}^l_{n,k}(w(\underline{j}))$ (recall that "ind" denotes non-normalized compact induction). The character Δ takes an element, as above, in $\mathcal{S}^l_{n,k}(w(\underline{j}))$ to

$$\prod_{i=1}^n |\det \epsilon_i|^{j_{i-1}+j_i-1} \cdot \prod_{i=1}^n |\det h_i|^{4(n-i)+j_{i-1}+j_i+1}, \tag{5.17}$$

$j_0 = k$, $j_n = k-l$. This and (5.15) show that the character

$$\Delta^{\frac{1}{2}} \cdot (\delta \cdot \delta^l_{n,k}(w_{\underline{k}}) \cdot \chi^l_{n,k}(w_{\underline{k}}))^{-1}$$

takes an element, as above, in $\mathcal{S}^l_{n,k}(w(\underline{j}))$ to

$$\prod_{i=1}^n (\chi_i^{-1} \circ |\det|^{\frac{j_i - j_{i-1}}{2}})(h_i) \cdot \prod_{i=1}^n (\chi_i^{-1} \circ |\det|^{\frac{j_i - j_{i-1}}{2}})(\epsilon_i). \tag{5.18}$$

Thus, if the Hom-space in (5.16) is nonzero, then the irreducible, unitary, generic, unramified representation $\sigma^\vee$ of $\mathrm{SO}_{2k}(F)$ can be embedded inside a normalized induced representation

$$\mathrm{Ind}^{\mathrm{SO}_{2k}}_{Q_{k-j_1, j_1-j_2, \cdots, j_{n-2}-j_{n-1}, j_{n-1}-k+l}}\Big(\bigotimes_{i=1}^n \chi_i^{-1} \circ |\det|^{\frac{j_i - j_{i-1}}{2}} \otimes \eta\Big), \tag{5.19}$$

where η is an irreducible, unramified representation of $\mathrm{SO}_{2(k-l)}(F)$. Recall that

$$Q_{k-j_1, j_1-j_2, \cdots, j_{n-2}-j_{n-1}, j_{n-1}-k+l} = Q^{(2k)}_{k-j_1, j_1-j_2, \cdots, j_{n-2}-j_{n-1}, j_{n-1}-k+l}$$

denotes the standard parabolic subgroup of SO_{2k}, whose Levi part is isomorphic to

$$\mathrm{GL}_{k-j_1} \times \mathrm{GL}_{j_1-j_2} \times \cdots \times \mathrm{GL}_{j_{n-2}-j_{n-1}} \times \mathrm{GL}_{j_{n-1}-k+l} \times \mathrm{SO}_{2(k-l)}.$$

Similarly, $\pi^{\omega_0^{n(k+1)}}$ can be embedded inside a normalized induced representation

$$\mathrm{Ind}_{Q_{k-j_1+2,j_1-j_2+2,\cdots,j_{n-2}-j_{n-1}+2,j_{n-1}-k+l+2,1^{k-l}}}^{\mathrm{SO}_{4n+2k}} \Big(\bigotimes_{i=1}^{n} \chi_i^{-1} \circ |\det|^{\frac{j_i-j_{i-1}}{2}}\Big) \otimes \Big(\bigotimes_{j=1}^{k-l} \xi_j\Big), \tag{5.20}$$

where $\xi_1, \cdots, \xi_{k-l}$ are unramified characters of F^* and

$$Q_{k-j_1+2,j_1-j_2+2,\cdots,j_{n-2}-j_{n-1}+2,j_{n-1}-k+l+2,1^{k-l}}$$

denotes the standard parabolic of SO_{4n+2k}, whose Levi part is isomorphic to

$$\mathrm{GL}_{k-j_1+2} \times \mathrm{GL}_{j_1-j_2+2} \times \cdots \times \mathrm{GL}_{j_{n-2}-j_{n-1}+2} \times \mathrm{GL}_{j_{n-1}-k+l+2} \times \mathrm{GL}_1^{\times(k-l)}.$$

The parabolic induction (5.20) admits a unique maximal orbit (over the algebraic closure of F) of degenerate Whittaker models, namely, the Richardson orbit of

$$Q_{k-j_1+2,j_1-j_2+2,\cdots,j_{n-2}-j_{n-1}+2,j_{n-1}-k+l+2,1^{k-l}}$$

(see the proof of Prop. II.1.3 in [14]). It is easy to see that this orbit corresponds to a partition of the form

$$[(2n+2(k-l)-1)(2n+1)(2u_3)(2u_4)\cdots] \tag{5.21}$$

with u_i nonnegative integers. Thus all partitions corresponding to any degenerate Whittaker model of π are majorized by (5.21). Therefore, if we assume that π has a degenerate Whittaker model corresponding to an orthogonal partition of the form $[(2n+2k-1)\cdots]$ (e.g., $[(2n+2k-1)(2n+1)]$) then it follows that

$$2n+2k-1 \leq 2n+2(k-l)-1,$$

and hence $l=0$.

We reach the same conclusion if we assume that σ is unitary and generic. In this case, η must be generic, and σ is actually equal to the representation (5.19). Recall that $\chi_i(x) = u_i(x)|x|^{\alpha_i}$, where u_i is a unitary character and α_i is real. Recall also that in (2.4), we made the choice that $0 \leq \alpha_i < \frac{1}{2}$, for $1 \leq i \leq n$. If one of the integers

$$k-j_1, j_1-j_2, \cdots, j_{n-2}-j_{n-1}, j_{n-1}-k+l$$

is larger than 1, then the representation (5.19) can not have a generic constituent. Since σ is generic, all these integers must be either 0, or 1, that is,

$$j_i - j_{i-1} = 0, -1, \quad 1 \leq i \leq n.$$

If there exists an index i, such that $j_i - j_{i-1} = -1$, then the corresponding exponent in (5.19) satisfies

$$-\alpha_i + \frac{j_i - j_{i-1}}{2} = -\alpha_i - \frac{1}{2} \leq -\frac{1}{2}.$$

This contradicts the main theorem in [13], which implies, in our case, that the exponents of an irreducible, unitary, generic, unramified representation σ of SO_{2k} lie in the open interval $\left(-\frac{1}{2},\frac{1}{2}\right)$. We conclude that

$$k=j_1=j_2=\cdots=j_{n-1}=k-l.$$

In particular, $l=0$. We proved that either one of the two assumptions (1) or (2) in Theorem 2.1 implies that $l=0$. We conclude that $j_1=j_2=\cdots=j_{n-1}=k$, and $w(\underline{j})$ is determined uniquely. In this case, the subgroup $\mathcal{S}^0_{n,k}$ consists of the elements of the form

$$\left(g',\begin{pmatrix} a_4 & * & * \\ & g & * \\ & & a_4^* \end{pmatrix}\right)\in \mathrm{SO}_{2k}\times\mathrm{SO}_{4n+2k}, \tag{5.22}$$

where for $g\in\mathrm{SO}_{2k}$, $g'=\begin{pmatrix} & I_k \\ -I_k & \end{pmatrix} g \begin{pmatrix} & -I_k \\ I_k & \end{pmatrix}$; $a_4\in\mathrm{GL}_{2n}$ is of the form

$$a_4=\begin{pmatrix} h_1 & * & \cdots & * \\ & h_2 & \cdots & * \\ & & \ddots & \vdots \\ & & & h_n \end{pmatrix},$$

where $h_i\in\mathrm{GL}_2$, for $i=1,2,\cdots,n$. By (5.18), the character $\Delta\cdot(\delta\cdot\delta^0_{n,k}(w(\underline{j}))\cdot\chi^0_{n,k}(w(\underline{j})))^{-1}$ is

$$\delta_{Q_{2^n}}^{\frac{1}{2}}((\mathrm{diag}(h_1,\cdots,h_n),I_{2k}))\cdot\prod_{i=1}^{n}\chi_i^{-1}(\det h_i),$$

where $(\mathrm{diag}(h_1,\cdots,h_n),I_{2k})\in\mathrm{GL}_2^{\times n}\times\mathrm{SO}_{2k}$. Hence the Hom-space condition (5.16) can be written as

$$\mathrm{Hom}_{Q_{2^n}^{(2n+k)}}(\pi^{\omega_0^{n(k+1)}},\delta_{Q_{2^n}}^{\frac{1}{2}}\otimes(\chi_1^{-1}\circ\det)\otimes\cdots\otimes(\chi_n^{-1}\circ\det)\otimes\sigma_1), \tag{5.23}$$

where $\sigma_1(g)=\sigma(g')$. By Frobenius reciprocity, the Hom-space (5.23) is isomorphic to

$$\mathrm{Hom}_{\mathrm{SO}_{4n+2k}}(\pi,\mathrm{Ind}_{Q_{2^n}}^{\mathrm{SO}_{4n+2k}}((\chi_1^{-1}\circ\det)\otimes\cdots\otimes(\chi_n^{-1}\circ\det)\otimes\sigma_1)). \tag{5.24}$$

We proved that if the Hom-space in Theorem 2.1 is nonzero, then the Hom-space in (5.24) is non-zero. When n is even, the representation

$$\mathrm{Ind}_{Q_{2^n}}^{\mathrm{SO}_{4n+2k}}((\chi_1^{-1}\circ\det)\otimes\cdots\otimes(\chi_n^{-1}\circ\det)\otimes\sigma_1) \tag{5.25}$$

shares the same unramified constituent with the representation

$$\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}}\otimes\sigma_1), \tag{5.26}$$

and hence $\pi=\pi^{\omega_0^{n(k+1)}}$ is isomorphic to the spherical sub-quotient of (5.26). Note that $\sigma_1\cong(\sigma)^{\omega_0^k}$. When n is odd, the representation (5.25) shares the same unramified constituent with the representation

$$\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}}\otimes(\sigma_1)^{\omega_0})=\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}}\otimes(\sigma)^{\omega_0^{k+1}}), \tag{5.27}$$

and hence $\pi^{\omega_0^{n(k+1)}} = \pi^{\omega_0^{k+1}}$ is isomorphic to the spherical sub-quotient of (5.27). Therefore π is isomorphic to the spherical sub-quotient of

$$\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}} \otimes \sigma).$$

In both cases, we get that π is isomorphic to the spherical sub-quotient of

$$\mathrm{Ind}_{Q_{2n}}^{\mathrm{SO}_{4n+2k}}(\tau|\det|^{\frac{1}{2}} \otimes (\sigma)^{\omega_0^{k(n+1)}}).$$

This completes the proof of Theorem 2.1.

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