Chin. Ann. Math.
36B(4), 2015, 523–542
DOI: 10.1007/s11401-015-0912-x

Chinese Annals of Mathematics, Series B
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A Result on the Quasi-periodic Solutions of Forced Isochronous Oscillators at Resonance*

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Abstract In this paper, the authors are concerned with the forced isochronous oscillators with a repulsive singularity and a bounded nonlinearity

$$x'' + V'(x) + g(x) = e(t, x, x'),$$

where the assumptions on V, g and e are regular, described precisely in the introduction. Using a variant of Moser's twist theorem of invariant curves, the authors show the existence of quasi-periodic solutions and boundedness of all solutions. This extends the result of Liu to the case of the above system where e depends on the velocity.

Keywords Isochronous oscillators, Repulsive singularity, Invariant curves, Time reversibility, Quasi-periodic solutions, Lazer-Landesman conditions, Boundedness of solutions

2000 MR Subject Classification 70K43, 34C15

1 Introduction

In this paper, we consider the existence of quasi-periodic solutions and the boundedness of all solutions for forced isochronous oscillators with a repulsive singularity. We also assume that the equation we considered depends on the velocity.

Consider the second-order ordinary differential equation

$$x'' + V'(x) = 0,$$

in which the potential function V is continuous. We call $x = 0$ an isochronous center if

$$V'(0) = 0, \quad xV'(x) > 0 \quad \text{for } x \neq 0,$$

and there is a fixed number $T > 0$ such that every solution is periodic with period T. If $x = 0$ is an isochronous center, we call the equation above an isochronous system. A typical example of the isochronous system is

$$x'' + \omega^2 x = 0.$$

Manuscript received December 6, 2013. Revised April 15, 2014.
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*This work was supported by the National Natural Science Foundation of China (No. 10325103) and the Chinese Scholarship Council (No. 201206010092).

It is easy to see that every solution of this equation is $\frac{2\pi}{\omega}$-periodic in t. Another important class of isochronous systems is the asymmetric equation

$$x'' + ax^+ - bx^- = 0,$$

where $x^+ = \max(x, 0),\ x^- = x - x^+$. This is because all solutions are periodic with the period $\pi(\frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}})$. In the above examples, the equations are both defined on the whole real line. People also consider the system

$$x'' + \frac{x+1}{4} - \frac{1}{4(1+x)^3} = 0.$$

Obviously, all solutions are 2π-periodic. The difference between this equation and the first two equations is that, this equation is not defined on $\mathbb{R}$, and the potential tends to infinity as $x \to -1$. More information of isochronous centers can be found in [3].

In 1969, Lazer and Leach studied the equation

$$x'' + m^2 x + g(x) = p(t), \quad m \in \mathbb{Z}^+,$$

with a 2π-periodic function p. They showed in [12] that, if $g(\pm\infty) = \lim_{x \to \pm\infty} g(x)$ exists and

$$2\,|g(+\infty) - g(-\infty)| > \Big| \int_0^{2\pi} p(t) \mathrm{e}^{\mathrm{i}mt} \mathrm{d}t \Big|,$$

then this equation has at least one 2π-periodic solution. The above inequality is called the Lazer-Landesman condition.

Since then, many mathematicians investigated the existence of periodic solutions for the equations

$$x'' + V'(x) + g(x) = p(t), \tag{1.1}$$

where p is periodic with period 2π (see [4–5, 7–11] and the references therein). In their works, they assumed the function V' to be of the form $V'(x) = m^2 x$ or $V'(x) = ax^+ - bx^-$. So the equation (1.1) can be viewed as a perturbation of an isochronous system. They showed that the type of Lazer-Landesman condition always plays a key role for the existence of periodic solutions.

Bonheure, Fabry and Smets [1] studied the forced isochronous oscillators with jumping nonlinearities and a repulsive singularity. The Lazer-Landesman-type condition is a key assumption to guarantee the existence of periodic solutions in their work. In the following, we briefly go over their result.

Assume that the function g is smooth and bounded, and the function V satisfies

$$\lim_{x \to +\infty} \frac{2V(x)}{x^2} = \frac{m^2}{4}, \quad \lim_{x \to a_+} V(x) = +\infty, \tag{1.2}$$

where $m \in \mathbb{Z}^+,\ a \in (-\infty, 0)$ and V is defined on $(a, +\infty)$. We also assume that all solutions of the unperturbed equation

$$x'' + V'(x) = 0 \tag{1.3}$$

are $\frac{2\pi}{m}$-periodic, that is, (1.3) is an isochronous oscillator with period $\frac{2\pi}{m}$. In this case, the equation (1.1) is a bounded perturbation of isochronous oscillators at resonance. The second condition in (1.2) means that the equation (1.3) has a repulsive singularity at a.

Let

$$g_*(\rho) = \int_0^{2\pi} g\Big(\rho\Big|\sin\Big(\frac{mt}{2}\Big)\Big|\Big)\Big|\sin\Big(\frac{mt}{2}\Big)\Big|\mathrm{d}t, \quad p_*(\theta) = \int_0^{2\pi} p(t+\theta)\Big|\sin\Big(\frac{mt}{2}\Big)\Big|\mathrm{d}t.$$

Then (1.1) has at least one 2π-periodic solution if there is $g_0 \in [g_*^-, g_*^+]$, which is a regular value of p_*, and the number of zeros of $p_* - g_0$ in $[0, \frac{2\pi}{m})$ is different from 2, where

$$g_*^- = \liminf_{\rho\to+\infty} g_*(\rho), \quad g_*^+ = \limsup_{\rho\to+\infty} g_*(\rho).$$

In particular, as a corollary, if the limit $\lim\limits_{x\to+\infty} g(x) = g^+$ exists, then the condition of the Lazer-Landesman type

$$4g^+ > \max_\theta p_*(\theta) \tag{1.4}$$

guarantees the existence of 2π-periodic solutions of (1.1).

In [17], Ortega considered the boundedness of solutions and the existence of quasi-periodic solutions for asymmetric oscillators. Following his result, there are several results (see [14, 18–19] and the references therein) on the boundedness of solutions for (1.1). However, in these works, the function V is globally defined in $\mathbb{R}$. That is, they do not include the case of the oscillators with a singularity.

In [18], Ortega also proved a variant of Moser's small twist theorem. Under some reasonable assumptions, he showed that a C^6 small twist area-preserving mapping has invariant curves. Moreover, he used the variant of Moser's small twist theorem to obtain the boundedness of a piecewise linear equation

$$x'' + n^2 x + h_L(x) = p(t),$$

where $p(t)$ is a 2π-periodic function of class C^5, $h_L(x)$ is of the following form:

$$h_L(x) = \begin{cases} L, & \text{if } x \ge 1, \\ Lx, & \text{if } |x| < 1, \\ -L, & \text{if } x \le -1, \end{cases}$$

and $p(t)$ satisfies

$$\frac{1}{2\pi}\Big|\int_0^{2\pi} p(t)\mathrm{e}^{-\mathrm{i}nt}\mathrm{d}t\Big| < \frac{2L}{\pi}.$$

In 2009, Capietto, Dambrosio and Liu [2] studied (1.1) with $g(x) = 0$ and

$$V = \frac{1}{2}x_+^2 + \frac{1}{(1-x_-^2)^\gamma} - 1,$$

where γ is a positive integer. They showed the boundedness of solutions and the existence of quasi-periodic solutions via Moser's twist theorem. Here, V has a singularity -1. As far as we know, this is the first example of the boundedness of solutions for the equations with singularities. However, this equation is not isochronous.

In [15], Liu showed that, under the condition (1.4) and other regular assumptions on V, g and p, the equation (1.1) has many quasi-periodic solutions and all solutions are bounded. It seems that this is the first result on the existence of quasi-periodic solutions and the boundedness of all solutions for isochronous oscillators with a singularity.

In this paper, we extend the results in [15] to the case of the equation where e depends on the velocity. More precisely, we study the equation

$$x'' + V_x(x) + g(x) = e(t, x, x'), \tag{1.5}$$

where the functions V, g and e satisfy the following assumptions:

(1) The function V is defined in the interval $(-1, +\infty)$ and $V(0) = V'(0) = 0$, $V''(x) > 0$ for $x \neq 0$, and the condition (1.2) holds.

(2) The function

$$W(x) := \frac{V(x)}{V'(x)} \tag{1.6}$$

is smooth in $(-1, \infty)$ and the limit $\lim\limits_{x\to -1} W(x)$ exists. Furthermore, we assume that the following estimates hold: For each $1 \le k \le 6$, there is a constant c_0, such that

$$|W(x)| \le c_0(1 + x), \quad |W^{(k)}(x)| \le c_0 \quad \text{for } x \in [-1, \infty).$$

(3) The positive function V is smooth and for $0 \le k \le 6$,

$$|(1 + x)^k V^{(k)}(x)| \le c_0' V(x) \quad \text{for } x \in (-1, +\infty),$$

where c_0' is a positive constant.

(4) The function g is bounded on the interval $[-1, +\infty)$ and $g(x) > 0$ for $x > 0$. Moreover, the following equalities hold:

$$\lim_{x\to+\infty} (1 + x)^k \frac{\mathrm{d}^k}{\mathrm{d}x^k} g(x) = 0 \quad \text{for } k > 0.$$

(5) For $x > 0$, let $\Phi(x) = V(x) - \frac{m^2}{8}x^2$, and the function Φ satisfies

$$\lim_{x\to+\infty} x^{k-2}\Phi^{(k)}(x) = 0,$$

for every positive integer k.

(6) There is a constant $M > 0$, such that $|e(t, x, y)| \le M$, and for $1 \le j + i + l \le 7$,

$$\lim_{\substack{x\to+\infty\\ y\to\infty}} x^i y^l \frac{\partial^{j+i+l} e(t, x, y)}{\partial t^j \partial x^i \partial y^l} = 0.$$

Furthermore, there exists a function $\overline{e}(t)$, such that

$$\lim_{\substack{x\to+\infty\\ y\to\infty}} e(t, x, y) = \overline{e}(t), \qquad \lim_{\substack{x\to+\infty\\ y\to\infty}} e_t'(t, x, y) = \overline{e}'(t).$$

Moreover, the function e is 2π-periodic in t, and

$$e(-t, x, -y) = e(t, x, y).$$

Then we have the following theorem.

Theorem 1.1 *Under the hypotheses* (1)–(6) *above, for a smooth function* $e = e(t, x, y)$, *if the Lazer-Landesman-type condition*

$$4g^+ > \max_\theta \overline{e}_*(\theta)$$

holds, where $\overline{e}_*(\theta) = \int_0^{2\pi} \overline{e}(t+\theta)\left|\sin\left(\frac{mt}{2}\right)\right|\mathrm{d}t$, *then all solutions of* (1.5) *are bounded, i.e., for each solution* x, *we have*

$$\sup_{t\in\mathbb{R}}(|x(t)| + |x'(t)|) < +\infty, \quad \inf_{t\in\mathbb{R}} x(t) > -1.$$

Furthermore, in this case, the equation (1.5) *has infinite many quasi-periodic solutions.*

The idea for proving our theorem is that, under the hypothesis (1)–(6) of our theorem, we can obtain that the Poincaré map of (1.5) satisfies the assumptions of a variant of Moser's twist theorem in [16]. These conditions are analogous to those in [13].

In the following, for simplicity and brevity, we assume that $m = 1$, i.e., the solutions of the equation $x'' + V_x(x) = 0$ are 2π-periodic, and $m = 1$ in (1.2) and the assumption (5). The proof of our statements for general m (the function e is also 2π-periodic in t) can be treated analogously.

The paper is organized as follows. In Section 2, we introduce action and angle variables. After that we state and prove some technical lemmas in Section 3, which are employed in the proof of our main result. In Sections 4–6, we will give an asymptotic expression of the Poincaré map and prove the main result by the twist theorem in [16].

2 Action and Angle Variables

The equation (1.5) can be written in the following form:

$$x' = y, \quad y' = -V'(x) - g(x) + e(t, x, y). \tag{2.1}$$

In order to introduce action and angle variables, we consider the auxiliary autonomous system

$$x' = y, \quad y' = -V'(x). \tag{2.2}$$

From our assumptions, we know that all solutions of this system are 2π-periodic in t. For every $h > 0$, we denote by $I(h)$ the area enclosed by the (closed) curve $\frac{1}{2}y^2 + V(x) = h$. Let $-1 < -\alpha_h < 0 < \beta_h$ be such that $V(-\alpha_h) = V(\beta_h) = h$. Then by (1.2) it follows that

$$\lim_{h\to+\infty} \alpha_h = 1, \quad \lim_{h\to+\infty} \beta_h = +\infty.$$

Moreover, it is easy to see that

$$I(h) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h - V(s))}\,\mathrm{d}s, \quad \forall h > 0. \tag{2.3}$$

Let

$$T_-(h) = 2\int_{-\alpha_h}^{0} \frac{1}{\sqrt{2(h - V(s))}}\mathrm{d}s, \quad T_+(h) = 2\int_0^{\beta_h} \frac{1}{\sqrt{2(h - V(s))}}\mathrm{d}s. \tag{2.4}$$

Then

$$I'(h) = T_-(h) + T_+(h).$$

Because all the solutions of the auxiliary equation (2.2) are 2π-periodic, we have

$$T_-(h) + T_+(h) \equiv 2\pi \tag{2.5}$$

which yields that $I(h) = 2\pi h$.

For every $(x, y) \in (-1, +\infty) \times \mathbf{R}$, let us define the angle and action variables (θ, I) by

$$\theta(x,y) = \begin{cases} \displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}}\,\mathrm{d}s, & \text{if } y \geq 0, \\ \displaystyle -\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h(x,y) - V(s))}}\,\mathrm{d}s, & \text{if } y < 0, \end{cases} \tag{2.6}$$

$$I(x,y) = 2\int_{-\alpha_h}^{\beta_h} \sqrt{2(h(x,y) - V(s))}\mathrm{d}s, \tag{2.7}$$

where

$$h(x,y) = \frac{1}{2}y^2 + V(x).$$

Obviously, we have

$$\frac{\mathrm{d}}{\mathrm{d}t} h(x,y)\Big|_{(2.1)} = y(-g(x) + e(t,x,y))$$

and

$$x(-\theta, I) = x(\theta, I), \quad y(-\theta, I) = -y(\theta, I).$$

In the new variables (θ, I), (2.1) becomes

$$\theta' = 1 + \Psi_1(\theta, I, t), \quad I' = \Psi_2(\theta, I, t), \tag{2.8}$$

where

$$\Psi_1(\theta, I, t) = \begin{cases} \displaystyle -y(\theta, I)(g(x) - e(t,x,y))\frac{\partial}{\partial h}\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h-V)}}\mathrm{d}s & \text{for } y \geq 0, \\ \displaystyle y(\theta, I)(g(x) - e(t,x,y))\frac{\partial}{\partial h}\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h-V)}}\mathrm{d}s & \text{for } y \leq 0, \end{cases}$$

$$\Psi_2(\theta, I, t) = -2\pi y(\theta, I)(g(x(\theta, I)) - e(t, x(\theta, I), y(\theta, I))).$$

We have used the equality

$$2\int_{-\alpha_h}^{\beta_h} \frac{1}{\sqrt{2(h(x,y) - V(s))}}\mathrm{d}s = 2\pi.$$

Obviously, this equation is time-reversible with respect to the involution $(\theta, I) \mapsto (-\theta, I)$.

3 Some Technical Lemmas

The proof of the main theorem 1.1 is based on a variant of the small twist theorem in the reversible system (see [16]). Therefore, we state it first and then give some technical estimates which will be used in the next sections. More precisely, we may use these estimates to obtain an asymptotic expression of the Poincaré map of (2.8).

3.1 A variant of the small twist theorem

In this subsection, we will state a variant of the small twist theorem (see [16]).

Let $A = \mathbb{S}^1 \times [a, b]$ be a finite cylinder with a universal cover $\mathbb{A} = \mathbb{R} \times [a, b]$. The coordinate in $\mathbb{A}$ is denoted by (τ, v). Consider a map

$$\overline{f} : A \to \mathbb{S}^1 \times \mathbb{R}.$$

We assume that the map is reversible with respect to the involution $\mathcal{G} : (\theta, I) \mapsto (-\theta, I)$, that is,

$$\mathcal{G} \circ \overline{f} \circ \mathcal{G} = \overline{f}^{-1}.$$

Suppose that $f : \mathbb{A} \to \mathbb{R} \times \mathbb{R}$, $(\tau_0, v_0) \mapsto (\tau_1, v_1)$ is a lift of $\overline{f}$ and it has the form

$$f : \begin{cases} \tau_1 = \tau_0 + 2N\pi + \delta l_1(\tau_0, v_0) + \delta\varphi_1(\tau_0, v_0, \delta), \\ v_1 = v_0 + \delta l_2(\tau_0, v_0) + \delta\varphi_2(\tau_0, v_0, \delta), \end{cases} \tag{3.1}$$

where N is an integer, $\delta \in (0, 1)$ is a parameter and l_1, l_2, φ_1 and φ_2 are functions satisfying

$$l_1 \in C^6(\mathbb{A}), \quad l_1(\tau_0, v_0) > 0, \quad \frac{\partial l_1}{\partial v_0}(\tau_0, v_0) > 0, \quad \forall (\tau_0, v_0) \in \mathbb{A}, \tag{3.2}$$

$$l_2, \varphi_1, \varphi_2 \in C^5(\mathbb{A}), \quad \varphi_1(\tau_0, v_0, 0) = 0. \tag{3.3}$$

In addition, we assume that there exists a function $\mathcal{I} : \mathbb{A} \to \mathbb{R}$ satisfying

$$\mathcal{I} \in C^6(A), \quad \mathcal{I}(-\tau_0, v_0) = \mathcal{I}(\tau_0, v_0), \quad \frac{\partial \mathcal{I}}{\partial v_0}(\tau_0, v_0) > 0, \quad \forall (\tau_0, v_0) \in \mathbb{A}, \tag{3.4}$$

$$l_1(\tau_0, v_0)\frac{\partial \mathcal{I}}{\partial \tau_0}(\tau_0, v_0) + l_2(\tau_0, v_0)\frac{\partial \mathcal{I}}{\partial v_0}(\tau_0, v_0) = 0, \quad \forall (\tau_0, v_0) \in \mathbb{A}. \tag{3.5}$$

Define the functions

$$\mathcal{I}_{\max}(v_0) = \max_{\tau_0 \in \mathbb{R}} \mathcal{I}(\tau_0, v_0), \quad \mathcal{I}_{\min}(v_0) = \min_{\tau_0 \in \mathbb{R}} \mathcal{I}(\tau_0, v_0), \quad v_0 \in [a, b].$$

Small Twist Theorem (see [16, Theorem 2]) *Let $\overline{f}$ be such that* (3.1)–(3.3) *hold. Assume in addition that there exists a function $\mathcal{I}$ satisfying* (3.4)–(3.5) *and numbers $\widetilde{a}$, $\widetilde{b}$ with*

$$a < \widetilde{a} < \widetilde{b} < b, \quad \mathcal{I}_{\max}(a) < \mathcal{I}_{\min}(\widetilde{a}) \leq \mathcal{I}_{\max}(\widetilde{a}) < \mathcal{I}_{\min}(\widetilde{b}) \leq \mathcal{I}_{\max}(\widetilde{b}) < \mathcal{I}_{\min}(b).$$

Then there exist $\epsilon > 0$ and $\Delta > 0$ such that if $\delta < \Delta$ and $\|\varphi_1\|_{C^5(A)} + \|\varphi_2\|_{C^5(A)} < \epsilon$, the map $\overline{f}$ has an invariant curve Γ. The constant ϵ is independent of δ. Furthermore, if we denote by $\mu(\Gamma, \delta) \in \mathbb{S}^1$ the rotation number of $\overline{f}$, then

$$\lim_{\delta \to 0} \mu(\Gamma, \delta) = 0.$$

Remark 3.1 From the last inequality in (3.2), we know that τ_1 is increasing as v_0 increases. This means that (3.1) is a twist map. By the proof in [16], one can see that the conclusions of this theorem still hold if the condition (3.2) is replaced by

$$l_1 \in C^6(A), \quad l_1(\tau_0, v_0) \neq 0, \quad \frac{\partial l_1}{\partial v_0}(\tau_0, v_0) \neq 0, \quad \forall (\tau_0, v_0) \in \mathbb{A}.$$

Remark 3.2 Note that $l_1(\tau_0, v_0) = l_1(-\tau_0, v_0)$, and $l_2(\tau_0, v_0) = -l_2(-\tau_0, v_0)$. If the function $\mathcal{I}$ does not satisfy $\mathcal{I}(-\tau_0, v_0) = \mathcal{I}(\tau_0, v_0)$, we can choose $\mathcal{J}(\tau_0, v_0) = \frac{1}{2}(\mathcal{I}(\tau_0, v_0) + \mathcal{I}(-\tau_0, v_0))$ instead of $\mathcal{I}(\tau_0, v_0)$.

3.2 Some technical lemmas

In order to obtain an asymptotic expression of the Poincaré map of (2.8), we must give some estimates first. In this subsection, we will deal with some technical estimates. Throughout this subsection, we suppose that the assumptions (1)–(5) stated in Section 1 hold.

Lemma 3.1 *For every positive integer $0 \leq k \leq 6$, there is a constant $c_1 > 0$, such that*

$$\Big| h^k \frac{\mathrm{d}^k T_-(h)}{\mathrm{d}h^k} \Big| \leq c_1 T_-(h).$$

Proof According to [13], we know that

$$\begin{aligned} T_-'(h) &= \frac{2}{h} \int_{-\alpha_h}^{0} \Big(W'(s) - \frac{1}{2} \Big) \cdot \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s, \\ T_+'(h) &= \frac{2}{h} \int_{0}^{\beta_h} \Big(W'(s) - \frac{1}{2} \Big) \cdot \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s, \end{aligned} \tag{3.6}$$

and here and in the rest of this subsection, the function W is defined by (1.6). By the assumption (2) in Section 1, it follows that

$$|hT_-'(h)| \leq 2\Big(c_0 + \frac{1}{2}\Big) \int_{-\alpha_h}^{0} \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s \leq \Big(c_0 + \frac{1}{2}\Big) T_-(h).$$

From (3.6) and the equality (the proof can be found in [13])

$$\frac{\mathrm{d}}{\mathrm{d}h} \int_{-\alpha_h}^{0} K(s) \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s = \frac{1}{h} \int_{-\alpha_h}^{0} \Big(\frac{\mathrm{d}}{\mathrm{d}s}(W(s)K(s)) - \frac{1}{2} K(s) \Big) \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s,$$

where K is a smooth function, it follows that

$$h^2 T_-^{(2)}(h) = -hT_-'(h) + 2 \int_{-\alpha_h}^{0} \Big(\Big(W'(s) - \frac{1}{2} \Big)^2 + W''(s) W(s) \Big) \frac{1}{\sqrt{2(h - V(s))}} \mathrm{d}s,$$

which yields, by the assumption (2) in Section 1 and the estimate on T_-', that

$$|h^2 T_-^{(2)}(h)| \leq \Big(2c_0 + \frac{1}{2} + \Big(c_0 + \frac{1}{2} \Big)^2 \Big) T_-(h).$$

The general case can be obtained by an induction argument and a direct computation.

Lemma 3.2 *There is a constant $c_2 > 0$ such that, for each positive integer $k \le 6$,*

$$|h^k T_+^{(k)}(h)| \le c_2 \cdot \frac{1}{\sqrt{h}}.$$

Proof Let

$$I_-(h) = 2\int_{-\alpha_h}^{0} \sqrt{2(h - V(s))}\mathrm{d}s.$$

Then $T_-(h) = I'_-(h)$. On the other hand, similar to the proof of (3.6), it is not difficult to see that

$$I'_-(h) = \frac{2}{h}\int_{-\alpha_h}^{0} \Big(\frac{1}{2} + W'(s)\Big)\sqrt{2(h - V(s))}\mathrm{d}s.$$

From the assumption (2) in Section 1, it follows that $T_-(h) \le \frac{c_0+1}{\sqrt{h}}$. By Lemma 3.1, we have, for each positive integer $k \le 6$,

$$\Big|h^k \frac{\mathrm{d}^k T_-(h)}{\mathrm{d}h^k}\Big| \le c_1(c_0 + 1)\frac{1}{\sqrt{h}}.$$

The conclusion of this lemma follows from this inequality and the identity $T_-(h) + T_+(h) \equiv 2\pi$.

Define a function F

$$F(x, I) = \int_{-\alpha_h}^{x} \Big(W'(s) - \frac{1}{2}\Big)\frac{1}{\sqrt{2(h - V(s))}}\mathrm{d}s \tag{3.7}$$

and an operator $\mathcal{L}$

$$\mathcal{L}(f) = \frac{h'_I}{h}\Big[\frac{\partial(fW(x))}{\partial x} - \frac{1}{2}f\Big] + \frac{\partial f}{\partial I}, \tag{3.8}$$

where $f = f(x, I)$, $h = h(I)$ and h'_I is the derivative of h with respect to I.

The proof of the following lemma can be found in [13].

Lemma 3.3 *For every smooth function $g(x, I)$, we have*

$$\begin{aligned} &\frac{\partial}{\partial I}\int_{-\alpha_h}^{x} g(s, I)\frac{1}{\sqrt{2(h(I) - V(s))}}\mathrm{d}s \\ &= \int_{-\alpha_h}^{x} \mathcal{L}(g)\frac{1}{\sqrt{2(h(I) - V(s))}}\mathrm{d}s - W(x)g(x, I)\frac{h'_I}{h}\frac{1}{\sqrt{2(h(I) - V(x))}}. \end{aligned} \tag{3.9}$$

Next, we give an estimate of the derivatives of $x = x(\theta, I)$ and $y = y(\theta, I)$ with respect to the action variable I.

Proposition 3.1 *There is a constant $C > 0$ such that, for $1 \le k \le 6$,*

$$\Big|I^k \frac{\partial^k x}{\partial I^k}\Big| \le C(1 + x), \quad \Big|I^k \frac{\partial^k y}{\partial I^k}\Big| \le C|y|, \tag{3.10}$$

where $x = x(\theta, I)$ and $y = y(\theta, I)$ are defined implicitly by (2.6) and (2.7), respectively.

The idea of the proof of this proposition is similar to the corresponding one in [13]. A complete proof can be found in the appendix of [15].

Note that $-1 \le -\alpha_h \le x \le \beta_h$ and the assumption (5) in Section 1, there is a constant $c_3 > 0$ such that for $I \gg 1$, $\beta_h \le c\sqrt{I}$. Hence, by Proposition 3.1, we have

$$\Big|I^k \frac{\partial^k x(\theta, I)}{\partial I^k}\Big|, \Big|I^k \frac{\partial^k y(\theta, I)}{\partial I^k}\Big| \le c_4\sqrt{I} \quad \text{for } 0 \le k \le 6, \tag{3.11}$$

where $c_4 > 0$ is a constant, not depending on I.

4 An Asymptotic Formula of $x(\theta, I)$

In this section, we will give an asymptotic expression of $x(\theta, I)$ when $I \gg 1$.

From the definition of θ (cf. (2.6)), it follows that

$$x_\theta(\theta, I) = y(\theta, I).$$

Since $\frac{1}{2}y^2 + V(x) = h = \frac{I}{2\pi}$, combining with the above equality, we have

$$y_\theta(\theta, I) = -V'(x).$$

That is, the function $x(\theta, I)$ satisfies

$$x_{\theta\theta} + V'(x) = 0.$$

Let

$$\widetilde{x}(\theta) = \frac{1}{\sqrt{2h}} x\Big(\theta + \frac{T_-(h)}{2}, I\Big).$$

Then

$$\widetilde{x}(0) = 0, \quad \widetilde{x}_\theta(0) = 1.$$

Obviously, there is a $\delta > 0$ such that $\widetilde{x}(\theta) > 0$ for $\theta \in (0, \delta)$. By the assumption (5) in Section 1, we know that, if $\widetilde{x} > 0$, then it is the solution of

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\theta^2} + \frac{1}{4}u + \frac{1}{\sqrt{2h}}\Phi'(\sqrt{2h}u) = 0. \tag{4.1}$$

Let $\widetilde{\Theta}_+(I)$ be the subset of the interval $[0, 2\pi]$ such that for $\theta \in \widetilde{\Theta}_+(I)$, $\widetilde{x}(\theta, I) > 0$.

Lemma 4.1 *For $\theta \in \widetilde{\Theta}_+(I)$, the function $\widetilde{x}$ has the following expression*:

$$\widetilde{x}(\theta, I) = 2\sin\frac{\theta}{2} + \widetilde{X}(\theta, I),$$

where the function $\widetilde{X}$ satisfies

$$\lim_{I \to +\infty} \sum_{k=0}^{6} \Big|I^k \frac{\partial^k \widetilde{X}}{\partial I^k}\Big| = 0.$$

Proof In the following, we assume that $\theta \in \widetilde{\Theta}_+(I)$. Since $\widetilde{x}$ is the solution of (4.1) with the initial condition $u(0) = 0,\ u'(0) = 1$, we have

$$\widetilde{x}(\theta, I) = 2\sin\frac{\theta}{2} + 2\int_0^\theta \frac{1}{\sqrt{2h}}\Phi'(\sqrt{2h}\widetilde{x}(\tau))\sin\frac{\theta-\tau}{2}\mathrm{d}\tau,$$

where $h = \frac{I}{2\pi}$. Hence, the function $\widetilde{X}$ is determined implicitly by

$$\widetilde{X}(\theta, I) = 2\int_0^\theta \frac{1}{\sqrt{2h}}\Phi'\Big(\sqrt{2h}\Big(2\sin\frac{\tau}{2} + \widetilde{X}(\tau, I)\Big)\Big)\sin\frac{\theta-\tau}{2}\mathrm{d}\tau.$$

From the hypothesis (5) in Section 1 and the Lebesgue dominated theorem, we have

$$\lim_{I\to+\infty}\widetilde{X}(\theta, I) = 0.$$

Taking the derivative with respect to I in both sides of the above equality, one has

$$\frac{\partial\widetilde{X}}{\partial I}(\theta, I) = 2\int_0^\theta \Big[\Big(-(2h)^{-\frac{3}{2}}\Phi' + (2h)^{-1}\Phi''\cdot\Big(2\sin\frac{\tau}{2} + \widetilde{X}(\tau, I)\Big)\Big)\frac{1}{2\pi} + \Phi''\cdot\frac{\partial\widetilde{X}}{\partial I}(\tau, I)\Big]\sin\frac{\theta-\tau}{2}\mathrm{d}\tau.$$

By the hypothesis (5) in Section 1 and the Gronwall inequality, it follows that

$$\lim_{I\to+\infty}\Big|I\frac{\partial\widetilde{X}}{\partial I}(\theta, I)\Big| = 0.$$

The estimates for the derivatives of higher order can be obtained in a similar way.

By the definition of $\widetilde{x}$ and $h = \frac{I}{2\pi}$, we have

$$x(\theta, I) = 2\sqrt{\frac{I}{\pi}}\sin\Big(\frac{\theta}{2} - \frac{T_-(h)}{4}\Big) + \sqrt{\frac{I}{\pi}}\widetilde{X}\Big(\theta - \frac{T_-(h)}{2}, I\Big) \tag{4.2}$$

and

$$\lim_{I\to+\infty}\sum_{k=1}^{6}\Big|I^{k-\frac{1}{2}}\frac{\partial^k}{\partial I^k}\Big(x(\theta, I) - 2\sqrt{\frac{I}{\pi}}\sin\Big(\frac{\theta}{2} - \frac{T_-(h)}{4}\Big)\Big)\Big| = 0. \tag{4.3}$$

Now we turn to estimate the measure of the set $\widetilde{\Theta}_+(I)$. By the definitions of θ and $\widetilde{x}$, we know that

$$\widetilde{x}(\theta, I) > 0 \iff \theta \in (0, T_+(h)).$$

Hence, $\widetilde{\Theta}_+(I) = (0, T_+(h))$. Because (1.3) is isochronous, we have, by Lemma 3.2, that

$$T_+(h) = 2\pi - T_-(h).$$

So

$$\mu\,\widetilde{\Theta}_+(I) = 2\pi - T_-(h) = 2\pi - O(I^{-\frac{1}{2}}),$$

where μ denotes the Lebesgue measure.

Let

$$\Theta_+(I) = \widetilde{\Theta}_+(I) + \frac{T_-(h)}{2} = \Big(\frac{T_-(h)}{2}, \frac{T_-(h)}{2} + T_+(h)\Big) = \Big(\frac{T_-(h)}{2}, 2\pi - \frac{T_-(h)}{2}\Big).$$

Then

$$\mu\ \Theta_+(I) = 2\pi - T_-(h) = 2\pi - O(I^{-\frac{1}{2}}) \tag{4.4}$$

and $\theta \in \Theta_+(I) \iff x(\theta, I) > 0$.

In the next section, we introduce a canonical transformation such that the transformed system is a perturbation of an integrable system.

5 Another Set of Action and Angle Variables

Now we consider the system (2.8). Note that

$$\Psi_1(\theta, I, t) = \begin{cases} -y(\theta, I)(g(x) - e(t, x, y))\dfrac{\partial}{\partial h}\displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h-V)}}\mathrm{d}s \quad \text{for } y \geq 0, \\ y(\theta, I)(g(x) - e(t, x, y))\dfrac{\partial}{\partial h}\displaystyle\int_{-\alpha_h}^{x} \frac{1}{\sqrt{2(h-V)}}\mathrm{d}s \quad \text{for } y \leq 0. \end{cases}$$

We have, by Lemma 3.3,

$$\begin{aligned} |\Psi_1(\theta, I, t)| &\leq (|g(x)| + |e(t, x, y)|)\frac{\sqrt{2(h - V(x))}}{h}\int_{-\alpha_h}^{x}\Big|W'(s) - \frac{1}{2}\Big|\frac{1}{\sqrt{2(h - v(s))}}\mathrm{d}s \\ &\quad + W(x)\frac{(|g(x)| + |e(t, x, y)|)}{h} \\ &\leq C\Big(h^{-\frac{1}{2}}\int_{-\alpha_h}^{\beta_h}\frac{1}{\sqrt{2(h - v(s))}}\mathrm{d}s + h^{-\frac{1}{2}}\Big) \\ &\leq Ch^{-\frac{1}{2}}. \end{aligned}$$

Hence, from (2.8), we know that

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = 1 + \Psi_1(\theta, I, t) \to 1 \quad \text{as } I \to +\infty.$$

Instead of (2.8), we will consider the following system:

$$\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{1}{1 + \Psi_1(\theta, I, t)}, \quad \frac{\mathrm{d}I}{\mathrm{d}\theta} = \frac{\Psi_2(\theta, I, t)}{1 + \Psi_1(\theta, I, t)}. \tag{5.1}$$

The relation between (2.8) and (5.1) is that if $(I(t), \theta(t))$ is a solution of (2.8) and the inverse function $t(\theta)$ of $\theta(t)$ exists, then $(I(t(\theta)), t(\theta))$ is a solution of (5.1) and vice versa. Hence in order to find quasi-periodic solutions of (2.8) and to obtain the boundedness of the solutions, it is sufficient to prove the existence of quasi-periodic solutions and the boundedness of solutions of (5.1). This trick was used in [13] in the proof of boundedness for superquadratic potentials.

From the definition of θ, we have, for $y > 0$,

$$\theta = \int_{-\alpha_h}^{x}\frac{1}{\sqrt{2(h - V(s))}}\mathrm{d}s.$$

Since $I = 2\pi h$, take the derivative with respect to the action variable I in both sides of the above equality (the angle variable θ is independent of I), it follows that

$$\frac{\partial}{\partial I}\int_{-\alpha_h}^{x}\frac{1}{\sqrt{2(h - V(s))}}\mathrm{d}s = 0,$$

which yields that

$$\frac{x_h}{\sqrt{2(h-V(x))}} = -\frac{\partial}{\partial h}\int_{-\alpha_h}^{x}\frac{1}{\sqrt{2(h-V(s))}}\mathrm{d}s.$$

Hence, we obtain that

$$\Psi_1(\theta, I, t) = 2\pi x_I(g(x(\theta, I)) - e(t, x(\theta, I), y(\theta, I))).$$

Definition 5.1 *We say a function $g(t,\rho,\theta,\epsilon) \in O_k(1)$ if g is smooth in (t,ρ) and for $k_1 + k_2 \le k$,*

$$\Big|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial \rho^{k_2}} g(t,\rho,\theta,\epsilon)\Big| \le C$$

for some constant $C > 0$ which is independent of the arguments t, ρ, θ and ϵ. Similarly, we say a function $g(t,\rho,\theta,\epsilon) \in o_k(1)$ if g is smooth in (t,ρ) and for $k_1 + k_2 \le k$,

$$\lim_{\epsilon\to 0}\Big|\frac{\partial^{k_1+k_2}}{\partial t^{k_1}\partial \rho^{k_2}} g(t,\rho,\theta,\epsilon)\Big| = 0, \quad \textit{uniformly in } (t,\rho,\theta).$$

Now we introduce a new action variable $\rho \in [1,2]$ and a parameter $\epsilon > 0$ by $I = \epsilon^{-2}\rho$. Then, $I \gg 1 \iff 0 < \epsilon \ll 1$. Under this transformation, the system (5.1) is changed into the form

$$\begin{cases} \dfrac{\mathrm{d}t}{\mathrm{d}\theta} = 1 - \widetilde{\Psi}_1(\theta,\rho,t,\epsilon) + \epsilon^2 O_6(1), \\ \dfrac{\mathrm{d}\rho}{\mathrm{d}\theta} = \widetilde{\Psi}_2(\theta,\rho,t,\epsilon) + \epsilon^2 O_6(1), \end{cases} \tag{5.2}$$

where

$$\widetilde{\Psi}_1(\theta,\rho,t,\epsilon) = 2\pi x_I(\theta,\epsilon^{-2}\rho)(g(x(\theta,\epsilon^{-2}\rho)) - e(t, x(\theta,\epsilon^{-2}\rho), y(\theta,\epsilon^{-2}\rho))),$$
$$\widetilde{\Psi}_2(\theta,\rho,t,\epsilon) = -2\pi\epsilon^2 y(\theta,\epsilon^{-2}\rho)(g(x(\theta,\epsilon^{-2}\rho)) - e(t, x(\theta,\epsilon^{-2}\rho), y(\theta,\epsilon^{-2}\rho))).$$

Obviously, if $\epsilon \ll 1$, the solution $(t(\theta,t_0,\rho_0), \rho(\theta,t_0,\rho_0))$ of (5.2) with the initial data $(t_0,\rho_0) \in \mathbb{R}\times[1,2]$ is defined in the interval $\theta \in [0,2\pi]$ and $\rho(\theta,t_0,\rho_0) \in [\frac{1}{2},3]$. So the Poincaré map of (5.2) is well defined in the domain $\mathbb{R}\times[1,2]$.

Lemma 5.1 *The Poincaré map of* (5.2) *is reversible with respect to the involution $(t,\rho) \mapsto (-t,\rho)$.*

By (4.4) and Lemma 3.1, we know that, there is a function η such that

$$T_-(h) = T_-\Big(\frac{I}{2\pi}\Big) = 2\eta(t_0,\rho_0,\theta;\epsilon),$$

where $\eta \in \epsilon O_6(1)$. By the definition of Θ_+ and Θ_-, we have

$$\Theta_+(I) = 2\pi - 2\eta, \quad \Theta_-(I) = 2\eta. \tag{5.3}$$

6 Proof of the Main Result

In this section, firstly, using the estimates in Subsection 3.2, we will obtain an asymptotic expression of the Poincaré map of (5.2) as $\epsilon \ll 1$. After that, we can prove the main result using a variant of Moser's small twist theorem in [16].

We make the ansatz that the solution of (5.2) with the initial condition $(t(0), \rho(0)) = (t_0, \rho_0)$ is of the form

$$t = t_0 + \theta + \epsilon\Sigma_1(t_0, \rho_0, \theta; \epsilon), \quad \rho = \rho_0 + \epsilon\Sigma_2(t_0, \rho_0, \theta; \epsilon).$$

Then, the Poincaré map of (5.2) is

$$\Phi: \ t_1 = t_0 + 2\pi + \epsilon\Sigma_1(t_0, \rho_0, 2\pi; \epsilon), \quad \rho_1 = \rho_0 + \epsilon\Sigma_2(t_0, \rho_0, 2\pi; \epsilon). \tag{6.1}$$

The functions Σ_1 and Σ_2 satisfy

$$\begin{cases} \Sigma_1 = -2\pi\epsilon^{-1} \displaystyle\int_0^\theta \frac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho) \big(g(x(\theta, \epsilon^{-2}\rho)) - e(t, x, y)\big) \mathrm{d}\theta + \epsilon O_6(1), \\ \Sigma_2 = -2\pi\epsilon \displaystyle\int_0^\theta y(\theta, \epsilon^{-2}\rho)(g(x(\theta, \epsilon^{-2}\rho)) - e(t, x, y))\mathrm{d}\theta + \epsilon O_6(1), \end{cases} \tag{6.2}$$

where $\rho = \rho_0 + \epsilon\Sigma_2$ and $t = t_0 + \theta + \epsilon\Sigma_1$.

By Proposition 3.1 and the assumptions (1)–(5) in Section 1, we know that the terms in the right-hand side of the above equations are bounded, so we have

$$|\Sigma_1| + |\Sigma_2| \le c_8, \quad \text{for } \theta \in [0, 2\pi] \tag{6.3}$$

where $c_8 > 0$ is a constant. Hence, for $\rho_0 \in [1, 2]$, we may choose ϵ sufficiently small such that

$$\rho_0 + \epsilon\Sigma_2 \ge \frac{\rho_0}{2} \ge \frac{1}{2} \tag{6.4}$$

for $(t_0, \theta) \in [0, 2\pi] \times [0, 2\pi]$. Similar to the proof in [6], one can obtain

$$\Sigma_1, \Sigma_2 \in O_6(1). \tag{6.5}$$

Lemma 6.1 *The following estimates hold*:

$$x(\theta, \epsilon^{-2}\rho) - x(\theta, \epsilon^{-2}\rho_0) \in O_6(1), \quad \frac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho) - \frac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho_0) \in \epsilon^2 O_6(1).$$

Proof Let

$$\Delta(t_0, \rho_0, \theta) := x(\theta, \epsilon^{-2}\rho) - x(\theta, \epsilon^{-2}\rho_0) = \int_0^1 \frac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2)\epsilon^{-1}\Sigma_2 \mathrm{d}s. \tag{6.6}$$

By (3.11) and (6.4), we have

$$|\Delta| \le \frac{c_4}{\sqrt{\epsilon^{-2}\rho_0 + s\epsilon^{-1}\Sigma_2}} \cdot \epsilon^{-1} c_8 \le 2c_4 c_8.$$

Taking the derivative with respect to ρ_0 in the both sides of (6.6), we have

$$\frac{\partial \Delta}{\partial \rho_0} = \int_0^1 \Big[\frac{\partial^2 x}{\partial I^2} \cdot \frac{1 + s\epsilon\dfrac{\partial \Sigma_2}{\partial \rho_0}}{\epsilon^2} \cdot \frac{\Sigma_2}{\epsilon} + \frac{\partial x}{\partial I} \cdot \frac{\partial \Sigma_2}{\partial \rho_0} \cdot \frac{1}{\epsilon}\Big] \mathrm{d}s.$$

Using (3.11) and (6.5), one may find a constant $c_9 > 0$ such that

$$\Big|\frac{\partial \Delta}{\partial \rho_0}\Big| \le c_9.$$

Analogously, one may obtain, by a direct but cumbersome computation, that

$$\Big|\frac{\partial^{k+l} \Delta}{\partial \rho_0^k \partial t_0^l}\Big| \le c_{10}.$$

The estimates for $\dfrac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho) - \dfrac{\partial x}{\partial I}(\theta, \epsilon^{-2}\rho_0)$ follow from a similar argument, and we omit it here.

Now we turn to give an asymptotic expression of the Poincaré map of (5.2), that is, we study the behavior of the functions Σ_1 and Σ_2 at $\theta = 2\pi$ as $\epsilon \to 0$.

By (6.2) and Lemma 6.1, it follows that

$$\begin{aligned}
\Sigma_1(t_0, \rho_0, 2\pi; \epsilon) &= -2\pi\epsilon^{-1}\int_0^{2\pi} \frac{\partial x}{\partial I}\cdot (g(x) - e(t, x, y))\,\mathrm{d}\theta + \epsilon O_6(1) \\
&= -2\pi\epsilon^{-1}\int_0^{2\pi} \frac{\partial x}{\partial I}\cdot (g(x) - e(t_0+\theta, x, y))\,\mathrm{d}\theta + \epsilon O_6(1) \\
&= -2\pi\epsilon^{-1}\int_{\Theta_+(I)} \frac{\partial x}{\partial I}\cdot (g(x) - e(t_0+\theta, x, y))\,\mathrm{d}\theta \\
&\quad -2\pi\epsilon^{-1}\int_{\Theta_-(I)} \frac{\partial x}{\partial I}\cdot (g(x) - e(t_0+\theta, x, y))\,\mathrm{d}\theta + \epsilon O_6(1)
\end{aligned}$$

and

$$\begin{aligned}
\Sigma_2(t_0, \rho_0, 2\pi; \epsilon) &= -2\pi\epsilon\int_0^{2\pi} y\cdot (g(x) - e(t, x, y))\mathrm{d}\theta + \epsilon O_6(1) \\
&= -2\pi\epsilon\int_0^{2\pi} y\cdot e(t_0+\theta, x, y)\mathrm{d}\theta + \epsilon O_6(1) \\
&= -2\pi\epsilon\int_{\Theta_+(I)} y\cdot e(t_0+\theta, x, y)\mathrm{d}\theta \\
&\quad -2\pi\epsilon\int_{\Theta_-(I)} y\cdot e(t_0+\theta, x, y)\mathrm{d}\theta + \epsilon O_6(1)
\end{aligned}$$

with $x = x(\theta, \epsilon^{-2}\rho_0),\ y = y(\theta, \epsilon^{-2}\rho_0)$. Here we have used that $y(-\theta, \epsilon^{-2}\rho_0) = -y(\theta, \epsilon^{-2}\rho_0)$ and $x(-\theta, \epsilon^{-2}\rho_0) = x(\theta, \epsilon^{-2}\rho_0)$. By Proposition 3.1, we know that when $\theta \in \Theta_-(I)$,

$$x(\theta, \epsilon^{-2}\rho_0) \in O_6(1), \quad y(\theta, \epsilon^{-2}\rho_0) \in \epsilon^{-1}O_6(1),$$

which yield that

$$\begin{cases}
\Sigma_1(t_0, \rho_0, 2\pi; \epsilon) = -2\pi\epsilon^{-1}\displaystyle\int_{\Theta_+(I)} \frac{\partial x}{\partial I}\cdot (g(x) - e(t_0+\theta, x, y))\,\mathrm{d}\theta + \epsilon O_6(1), \\
\Sigma_2(t_0, \rho_0, 2\pi; \epsilon) = -2\pi\epsilon\displaystyle\int_{\Theta_+(I)} y\cdot e(t_0+\theta, x, y)\mathrm{d}\theta + \epsilon O_6(1)
\end{cases}$$

with $x = x(\theta, \epsilon^{-2}\rho_0),\ y = y(\theta, \epsilon^{-2}\rho_0)$.

Our next task is to estimate the above two integrals.

Lemma 6.2 *If* $\lim\limits_{x\to+\infty} g(x)=g^+$ *and the assumption* (4) *in Section* 1 *holds, then, for any function* $f\in o_6(1)$,

$$\int_0^{2\pi} g\Big(2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}+\epsilon^{-1}f(\rho_0,t_0,\theta;\epsilon)\Big)\sin\frac{\theta}{2}\mathrm{d}\theta=\int_0^{2\pi} g^+\sin\frac{\theta}{2}\mathrm{d}\theta+o_6(1)=4g^++o_6(1).$$

Proof Let

$$\overline{g}_0(\rho_0;\epsilon)=\int_0^{2\pi} g\Big(2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}\Big)\sin\frac{\theta}{2}\mathrm{d}\theta.$$

Note that $\sin\frac{\theta}{2}>0$ for $\theta\in(0,2\pi)$, so by the Lebesgue dominated theorem, we have

$$\lim_{\epsilon\to0}\overline{g}_0(\rho_0;\epsilon)=\int_0^{2\pi} g^+\sin\frac{\theta}{2}\mathrm{d}\theta=4g^+.$$

Since

$$\frac{\partial\overline{g}_0(\rho_0;\epsilon)}{\partial\rho_0}=\frac{1}{2\rho_0}\int_0^{2\pi}\Big[g'\Big(2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}\Big)\cdot2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}\Big]\cdot\sin\frac{\theta}{2}\mathrm{d}\theta,$$

by the assumption (4) in Section 1 and the Lebesgue dominated theorem, it follows that

$$\lim_{\epsilon\to0}\frac{\partial\overline{g}_0(\rho_0;\epsilon)}{\partial\rho_0}=0.$$

The estimates for the derivatives of higher order can be obtained in a similar way. Hence, we have proved the conclusion when $f\equiv0$. In the general case, let

$$\overline{g}_f(\rho_0,t_0;\epsilon)=\int_0^{2\pi} g\Big(2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}+\epsilon^{-1}f(\rho_0,t_0,\theta;\epsilon)\Big)\sin\frac{\theta}{2}\mathrm{d}\theta.$$

Then

$$\overline{g}_f-\overline{g}_0=\int_0^{2\pi}\int_0^1 g'\Big(2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}+s\epsilon^{-1}f(\rho_0,t_0,\theta;\epsilon)\Big)\epsilon^{-1}f(\rho_0,t_0,\theta;\epsilon)\sin\frac{\theta}{2}\mathrm{d}s\mathrm{d}\theta.$$

The conclusion follows from the Lebesgue dominated theorem and the assumption (4) in Section 1.

Lemma 6.3 *If the assumption* (6) *holds, then, for any function* $f_1,f_2\in o_6(1)$,

$$\begin{aligned}&\int_0^{2\pi} e\Big(t_0+\theta,2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}+\epsilon^{-1}f_1(\rho_0,t_0,\theta;\epsilon),\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}+\epsilon^{-1}f_2(\rho_0,t_0,\theta;\epsilon)\Big)\sin\frac{\theta}{2}\mathrm{d}\theta\\&=\int_0^{2\pi}\overline{e}(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta+o_6(1).\end{aligned}$$

Proof Let

$$\overline{e}_0(t_0,\rho_0;\epsilon)=\int_0^{2\pi} e\Big(t_0+\theta,2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2},\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}\Big)\sin\frac{\theta}{2}\mathrm{d}\theta.$$

Note that $\sin\frac{\theta}{2}>0$ for $\theta\in(0,2\pi)$, so by Lebesgue dominated theorem, we have

$$\lim_{\epsilon\to0}\overline{e}_0(t_0,\rho_0;\epsilon)=\int_0^{2\pi}\overline{e}(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta.$$

Since

$$\begin{aligned}\frac{\partial \overline{e}_0(t_0,\rho_0;\epsilon)}{\partial \rho_0} &= \frac{1}{2\rho_0}\int_0^{2\pi}\Big[e'_x\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}\Big)\cdot 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}\Big]\cdot\sin\frac{\theta}{2}\mathrm{d}\theta\\
&\quad+\frac{1}{2\rho_0}\int_0^{2\pi}\Big[e'_y\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}\Big)\\
&\quad\cdot\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}\Big]\cdot\sin\frac{\theta}{2}\mathrm{d}\theta,\\
\frac{\partial \overline{e}_0(t_0,\rho_0;\epsilon)}{\partial t_0} &= \int_0^{2\pi} e'_t\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}\Big)\sin\frac{\theta}{2}\mathrm{d}\theta,\end{aligned}$$

by the assumption (6) in Section 1 and the Lebesgue dominated theorem, it follows that

$$\lim_{\epsilon\to 0}\frac{\partial \overline{e}_0(t_0,\rho_0;\epsilon)}{\partial \rho_0}=0,\quad \lim_{\epsilon\to 0}\frac{\partial \overline{e}_0(t_0,\rho_0;\epsilon)}{\partial t_0}=0.$$

The estimates for the derivatives of higher order can be obtained in a similar way. Hence, we have proved the conclusion when $f_1=f_2\equiv 0$. In the general case, let

$$\overline{e}_f(t_0,\rho_0;\epsilon)=\int_0^{2\pi} e\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}+\epsilon^{-1}f_1, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}+\epsilon^{-1}f_2\Big)\sin\frac{\theta}{2}\mathrm{d}\theta.$$

Then

$$\begin{aligned}\overline{e}_f-\overline{e}_0 &= \int_0^{2\pi}\int_0^1\Big[e'_x\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}\\
&\quad+s\epsilon^{-1}f_1, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}+\epsilon^{-1}f_2\Big)\cdot\epsilon^{-1}f_1\Big]\sin\frac{\theta}{2}\mathrm{d}s\mathrm{d}\theta\\
&\quad+\int_0^{2\pi}\int_0^1\Big[e'_y\Big(t_0+\theta, 2\sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\sin\frac{\theta}{2}, \sqrt{\frac{\rho_0}{\pi}}\epsilon^{-1}\cos\frac{\theta}{2}+s\epsilon^{-1}f_2\Big)\cdot\epsilon^{-1}f_2\Big]\sin\frac{\theta}{2}\mathrm{d}s\mathrm{d}\theta.\end{aligned}$$

The conclusion follows from the Lebesgue dominated theorem and the assumption (6) in Section 1.

Similarly, we have the following Lemma.

Lemma 6.4 *If the assumption* (6) *in Section* 1 *holds, then*

$$\begin{aligned}&\int_0^{2\pi} e'_t(t_0+\theta, x(\theta,\epsilon^{-2}\rho_0), y(\theta,\epsilon^{-2}\rho_0))\sin\frac{\theta}{2}\mathrm{d}\theta=\int_0^{2\pi}\overline{e}'(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta+o_6(1),\\
&\int_0^{2\pi} e'_x(t_0+\theta, x(\theta,\epsilon^{-2}\rho_0), y(\theta,\epsilon^{-2}\rho_0))\frac{\partial x}{\partial\theta}(\theta,\epsilon^{-2}\rho_0)\sin\frac{\theta}{2}\mathrm{d}\theta=o_6(1),\\
&\int_0^{2\pi} e'_y(t_0+\theta, x(\theta,\epsilon^{-2}\rho_0), y(\theta,\epsilon^{-2}\rho_0))\frac{\partial y}{\partial\theta}(\theta,\epsilon^{-2}\rho_0)\sin\frac{\theta}{2}\mathrm{d}\theta=o_6(1).\end{aligned}$$

From these lemmas, we have the following lemma.

Lemma 6.5 *The following estimates hold*:

$$\begin{cases}\Sigma_1(t_0,\rho_0,2\pi;\epsilon)=-2\sqrt{\dfrac{\pi}{\rho_0}}\displaystyle\int_0^{2\pi}\big(g^+-\overline{e}(t_0+\theta)\big)\sin\frac{\theta}{2}\mathrm{d}\theta+o_6(1),\\
\Sigma_2(t_0,\rho_0,2\pi;\epsilon)=4\sqrt{\pi\rho_0}\displaystyle\int_0^{2\pi}\overline{e}'(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta+o_6(1).\end{cases}$$

Proof By (4.2)–(4.3), the definition of Θ_+ and (5.3), it follows that

$$
\begin{aligned}
&\Sigma_1(t_0,\rho_0,2\pi;\epsilon)\\
&= -2\pi\epsilon^{-1}\int_{\Theta_+(I)}\frac{\partial x}{\partial I}(\theta,\epsilon^{-2}\rho_0)\cdot g(x(\theta,\epsilon^{-2}\rho_0))\mathrm{d}\theta - 2\pi\epsilon^{-1}\int_{\Theta_+(I)}\frac{\partial x}{\partial I}(\theta,\epsilon^{-2}\rho_0)\\
&\quad\cdot(-e(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0)))\mathrm{d}\theta + \epsilon O_6(1)\\
&= -2\sqrt{\frac{\pi}{\rho_0}}\int_{\Theta_+(I)}(g(x(\theta,\epsilon^{-2}\rho_0)) - e(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0)))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= -2\sqrt{\frac{\pi}{\rho_0}}\Big(\int_0^{2\pi}-\int_0^{\eta}-\int_{2\pi-\eta}^{2\pi}\Big)g(x(\theta,\epsilon^{-2}\rho_0))\sin\frac{\theta}{2}\mathrm{d}\theta - 2\sqrt{\frac{\pi}{\rho_0}}\Big(\int_0^{2\pi}-\int_0^{\eta}-\int_{2\pi-\eta}^{2\pi}\Big)\\
&\quad\cdot(-e(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0)))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= -2\sqrt{\frac{\pi}{\rho_0}}\int_0^{2\pi}(g(x(\theta,\epsilon^{-2}\rho_0)) - e(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0)))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= -2\sqrt{\frac{\pi}{\rho_0}}\int_0^{2\pi}(g^+ - \overline{e}(t_0+\theta))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1),\\
&\Sigma_2(t_0,\rho_0,2\pi;\epsilon)\\
&= 2\pi\epsilon\int_{\Theta_+(I)}2\sqrt{\frac{\rho_0}{\pi\epsilon^2}}\sin\Big(\frac{\theta}{2}-\frac{\eta}{2}\Big)\frac{\partial e}{\partial\theta}(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0))\mathrm{d}\theta + o_6(1)\\
&= 2\pi\epsilon\int_{\Theta_+(I)}2\sqrt{\frac{\rho_0}{\pi\epsilon^2}}\sin\Big(\frac{\theta}{2}\Big)\frac{\partial e}{\partial\theta}(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0))\mathrm{d}\theta + o_6(1)\\
&= 4\sqrt{\pi\rho_0}\int_{\Theta_+(I)}\sin\Big(\frac{\theta}{2}\Big)\frac{\partial e}{\partial\theta}(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0))\mathrm{d}\theta + o_6(1)\\
&= 4\sqrt{\pi\rho_0}\Big(\int_0^{2\pi}-\int_0^{\eta}-\int_{2\pi-\eta}^{2\pi}\Big)\frac{\partial e}{\partial\theta}(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= 4\sqrt{\pi\rho_0}\int_0^{2\pi}\frac{\partial e}{\partial\theta}(t_0+\theta,x(\theta,\epsilon^{-2}\rho_0),y(\theta,\epsilon^{-2}\rho_0))\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= 4\sqrt{\pi\rho_0}\int_0^{2\pi}\Big[e_t' + e_x'\cdot\frac{\partial x}{\partial\theta}(\theta,\epsilon^{-2}\rho_0) + e_y'\cdot\frac{\partial y}{\partial\theta}(\theta,\epsilon^{-2}\rho_0)\Big]\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1)\\
&= 4\sqrt{\pi\rho_0}\int_0^{2\pi}\overline{e}'(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta + o_6(1).
\end{aligned}
$$

Let

$$
\begin{aligned}
\Psi_1(t_0,\rho_0) &= -2\sqrt{\frac{\pi}{\rho_0}}\int_0^{2\pi}\big(g^+ - \overline{e}(t_0+\theta)\big)\sin\frac{\theta}{2}\mathrm{d}\theta,\\
\Psi_2(t_0,\rho_0) &= 4\sqrt{\pi\rho_0}\int_0^{2\pi}\overline{e}'(t_0+\theta)\sin\frac{\theta}{2}\mathrm{d}\theta.
\end{aligned}
$$

Then there are two functions ϕ_1 and ϕ_2, such that the Poincaré map of (5.2), given by (6.1), is of the form

$$
\Phi:\quad t_1 = t_0 + 2\pi + \epsilon\Psi_1(t_0,\rho_0) + \epsilon\phi_1(t_0,\rho_0;\epsilon),\quad \rho_1 = \rho_0 + \epsilon\Psi_2(t_0,\rho_0) + \epsilon\phi_2(t_0,\rho_0;\epsilon),
$$

where $\phi_1,\phi_2\in o_6(1)$.

Note that, by the Lazer-Landesman condition $4g^+ > \max_\theta \overline{e}_*(\theta)$, we know that

$$\Psi_1 < 0, \quad \frac{\partial \Psi_1}{\partial \rho_0} > 0.$$

Let

$$L(t_0, \rho_0) = \frac{\rho_0^{-\frac{1}{2}}}{\displaystyle\int_0^{2\pi} (g^+ - \overline{e}(t_0 + \theta)) \sin \frac{\theta}{2} \mathrm{d}\theta}.$$

Then

$$\frac{\partial L}{\partial t_0} \Psi_1(t_0, \rho_0) + \frac{\partial L}{\partial \rho_0} \Psi_2(t_0, \rho_0) \equiv 0.$$

The other assumptions of Ortega's theorem are verified directly. Hence, for sufficiently small ϵ, there is an invariant curve of Φ in the annulus $(t_0, \rho_0) \in S^1 \times [1, 2]$. The boundedness of the solutions to our original equation (1.5) can be obtained by the existence of such invariant curves, and the precise proof can be found in [14].

Moreover, the solutions starting from such curves are quasi-periodic solutions. Using the Poincaré-Birkhoff fixed point theorem, there is a positive integer n_0, such that, for any $n \geq n_0$, there are at least two periodic solutions of (1.5) with the minimal period $2n\pi$ (see [6]).

Since then, we are done with the proof of the existence of the quasi-periodic solutions and boundedness of all solutions for reversible forced isochronous oscillators with a repulsive singularity.

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